Analysis of Affine Algebraic Threefolds from a point of view of Log Minimal Model Program
(ログ極小モデル理論の視点からの３次元アフィン代数多様体の双正則的構造解析)

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This is a report concerning our research entitled *"Analysis of Affine Algebraic Threefolds from a point of view of Log Minimal Model Program"* under the financial support from *"sôgô kenkyû kikô kenkyû project"*.

1 Affine Algebraic Threefolds

Our main interest lies in the investigation concerning algebro-geometric properties on affine threefolds defined over the filed of complex numbers $\mathbb{C}$. For the solutions of plenty of problems concerning affine threefolds, it is essential to find a good characterization of the affine 3-space $\mathbb{C}^3$ as an affine algebraic variety. Although $\mathbb{C}^3$ seems to be a simple variety, it is, in general, very hard to distinguish it from other smooth contractible affine threefolds, so-called, exotic threefolds. This kind of subject is one of the main streams of the contemporary research on affine algebraic geometry.

Traditionally, it is normal to investigate affine algebraic threefolds by means of purely algebraic argument. Certainly, it is useful to produce certain kinds of counter-examples, but it depends heavily on the case-by-case argument, so that such a method seems not to be suitable in order to establish a general theory of affine threefolds.

That is why it is desirable to develop some techniques which are more algebro-geometric and does not depend on the forms of defining equations of affine threefolds. In what follows, we shall state the feature of our approach to the analysis of affine algebraic varieties and the associated results.

2 The feature of our approach and the recent results

In order to investigate from a point of view of algebro-geometric method, we think that it is the best way to embed affine threefolds into suitable projective varieties, and make use of the theory of *Birational Geometry*. Birational Geometry is interested in algebro-geometric properties up to birational equivalence. Roughly speaking, *small* difference between varieties are neglected in Birational Geometry. Fortunately, Birational Geometry has been developing drastically in the past two decades, and we have strong enough theory, so-called *Minimal Model Program* (or *Mori Theory*) to investigate higher-dimensional varieties (having certain kinds of singularities). Due to Minimal Model Program, in principal, we are able to, at least, look into projective threefolds (having terminal singularities and boundaries). By making use of Minimal Model Program, our approach to investigate affine algebraic threefolds is stated as in the following fashion:

$\boxed{\textbf{Our method}}$

Let X be a given smooth affine algebraic threefold defined over $\mathbb{C}$. In order to analyze X, we shall embed X into a smooth projective threefold V in such a way that the boundary part $D := V \backslash X$ is a divisor with only simple normal crossings (SNC). (Note that we are free to take such a variety V as far as D is SNC.) Then the pair (V, D) has a certain kind of singularities, namely *dlt* (=divisorial log terminal) singularities in terms of Minimal Model Program (MMP). Although we do not mention the definition of dlt singularities here, MMP works in this category of singularities. So, let us perform MMP for (V, D) to obtain the following diagram:

$$(*) \qquad \phi : (V, D) \overset{\phi^0}{\cdots\to} (V^1, D^1) \overset{\phi^1}{\cdots\to} \cdots\cdots \to (V^{s-1}, D^{s-1}) \overset{\phi^{s-1}}{\cdots\to} (V^s, D^s),$$

where the final object $(V', D') := (V^s, D^s)$ is either a Log Mori fiber space or a log minimal model, i.e., $K_{V'} + D'$ is nef, according to the value of log Kodaira dimension $\overline{\kappa}(X)$ of X. Set $X' := V' \backslash \mathrm{Supp}\,(D')$. Usually, since (V', D') has a distinguished simpler structure compared with that of the original pair (V, D), we expect that we are able to analyze the structure of X' in detail. Hence, *if we were able to compare X with X' in an explicit manner, then we obtain the data on X from those on X'*. This expectation is, in general, a little bit far from being actual and, in fact, we cannot succeed in a full generality at present. However, under a certain kind of condition concerning compactifications, we have succeeded in the investigation in detail (cf. [8, 9, 11, 12]).

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