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Non-Perturbative Evaluation of the Effective Potential of $\lambda\phi^4$ Theory at Finite Temperature in the Super-Daisy Approximation

Jiro ARAFUNE,[*] Kenzo OGURE[**] and Joe SATO[*,***]

Institute for Cosmic Ray Research
University of Tokyo, Tanashi 188, Japan
[*]Department of Physics, School of Science
University of Tokyo, Tokyo 113, Japan

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We calculate the effective potential of $\lambda\phi^4$ theory at finite temperature in the super-daisy approximation, after expressing its derivative with respect to the mass square in terms of the full propagator. This expression becomes a self-consistent equation for the derivative of the effective potential. We find the phase transition is first order in this approximation. We compare our result with those of previous studies.

§1. Introduction

The symmetry restoration, or the phase transition, at high temperature is today regarded as playing an important role in particle physics and cosmology.[1] The investigation of the phase transition, however, is quite often found to be difficult because of the unreliability of the perturbation theory at high temperature. In case of the electroweak phase transition, which may be important for baryogenesis,[2] for example, such a breakdown of the ordinary perturbative expansion occurs when the Higgs boson mass m_H is equal to or larger than the weak boson mass M_W.[3] Experimental evidence,[4] in fact, suggests that m_H is not smaller than M_W, indicating that we cannot examine the electroweak phase transition using the perturbation theory.

The difficulty is caused by Bose condensation in the thermal bath. This is a well-known feature inherent to finite temperature field theory.[5),6] The traditional method to improve the perturbation theory is to re-sum the daisy diagrams[3),7)-13] and use the mass m_T^2, the sum of the thermal mass and the zero temperature mass, instead of the latter only. But this improvement is not sufficient. When one uses the loop expansion, one can estimate the size of a contribution of a loop at a given order in comparison with that of the previous order loop.[3),9] The ratio of these is called a 'loop expansion parameter'. After re-summing the daisy diagrams, we find the loop expansion parameter is $\frac{\lambda T}{m_T}$ in the $\lambda\phi^4$ theory for the temperature T. This is $O(1)$ near the critical temperature, and the analysis with the daisy diagram indicates that the phase transition is first order, though it is well known to be of second order.[3),14] This shows the perturbation theory breaks down near the critical temperature and is not reliable. Many non-perturbative approaches, such as lattice

*) E-mail: arafune@icrhp3.icrr.u-tokyo.ac.jp
**) E-mail: ogure@icrhp3.icrr.u-tokyo.ac.jp
***) E-mail: joe@hep-th.phys.s.u-tokyo.ac.jp

Monte Carlo calculation,[15] epsilon expansion,[16] the CJT method,[17)-19] effective three-dimensional theory,[20] the gap equation method,[21] etc., have been used to investigate the phase transition at high temperature. However, these approaches have not reached a consensus in predicting the parameter region of m_H and M_W in which the phase transition is first order.

Recently, two of the present authors (K. O. and J. S.) proposed a new non-perturbative method to calculate the effective potential V.[22] We express its derivative with respect to the mass square, $\partial V/\partial m^2$, in terms of the full propagator. We calculate the effective potential by integrating the derivative of the effective potential with an initial condition at $m^2 = m_0^2$ given in the region where the perturbation theory is reliable:

$$V(m^2 = -\mu^2) = V(m^2 = m_0^2) + \int_{m_0^2}^{-\mu^2} \frac{\partial V}{\partial m^2} dm^2. \tag{1}$$

Here $-\mu^2$ is the mass parameter of the theory and m_0 is the mass scale at which we give the initial condition $V(m^2 = m_0^2)$. In principle we can obtain the effective potential non-perturbatively with this procedure. The main problem in this context is to determine how to approximate the full propagator in this method. In our previous paper we ignored the momentum dependence of the self-energy graphs and replaced it by the second derivative of the effective potential with respect to the expectation value of the field. In this paper we try another approximation of the full propagator. We can sum up all the super-daisy diagram contributions correctly in this method.*) We then evaluate the effective potential numerically with this approximation.

The paper is organized as follows. In §2 we discuss the structure of the evolution equation that we must solve. In §3 we introduce the new approximation. In §4 we give the numerical result. In §5 we summarize and discuss our result, comparing our method with those of previous works.

§2. The structure of the evolution equation

In this section we discuss the evolution of the effective potential with respect to the mass. We consider the $\lambda\phi^4$ theory which is defined by the Lagrangian density

$$\mathcal{L}_E = -\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi + \text{c.t.} \tag{2}$$

In a previous paper[22] we derived the evolution equation of the effective potential,

$$\begin{aligned}\frac{\partial V}{\partial m^2} = \frac{1}{2}\bar{\phi}^2 &+ \frac{1}{4\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{-p_0^2 + \boldsymbol{p}^2 + m^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi} \frac{1}{e^{\beta p_0} - 1} \\ &+ \frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{-p_0^2 + \boldsymbol{p}^2 + m^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi}\end{aligned}$$

*) One can also do this using the CJT method.[17] We will discuss this point in §5.

$$
\begin{aligned}
&+(Z_m Z_\phi - 1)\left[\frac{1}{2}\bar{\phi}^2\right. \\
&+\frac{1}{4\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi} \frac{1}{e^{\beta p_0}-1} \\
&\left.+\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\right],
\end{aligned}
\tag{3}
$$

where $\Pi = \Pi(\boldsymbol{p}^2, -p_0^2, \bar{\phi}, m^2, \tau)$ is the sum of all the one particle irreducible (1PI) self-energy graphs.

The first term of Eq. (3) is the simple background field contribution. The second term gives the finite temperature contribution. The third term expresses the effect which remains finite at zero temperature. The last term is the counter term contribution.

§3. Super-daisy approximation

The main problem encountered in the application of our method is how to approximate Π in Eq. (3). The function Π is expressed by the full propagator, the full three- and the full four-point functions (Fig. 1) according to the Schwinger-Dyson equation. The Schwinger-Dyson equation expresses the inverse of the full propagator by the relation expressed by Fig. 2. On the right-hand side of Fig. 1, the first and the fourth terms are independent of the external momentum, and the second and the third terms depend on them. In this paper we assume the simplest case,

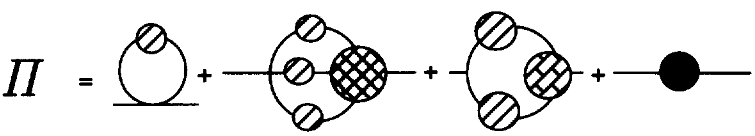

Fig. 1. 1 PI self-energy graphs of Π. The line with the *striped blob* represents the full propagator. The *circle with a net* expresses the full 4-point vertex (the second term on the RHS). The *circle with bricks* represents the full 3-point vertex (the third term on the RHS). The *black blob* represents the counter term (the fourth term on the RHS).

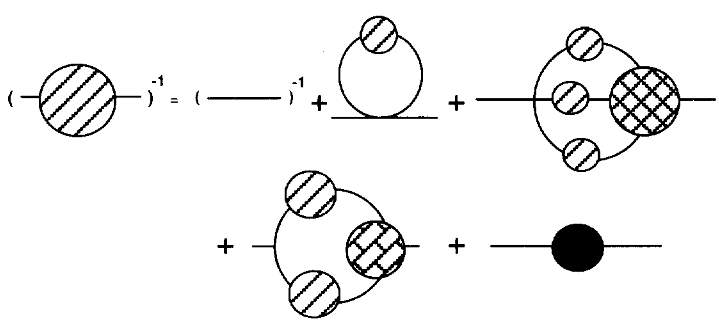

Fig. 2. Schwinger-Dyson equation for the full propagator in $\lambda\phi^4$ theory. The symbols are the same as in Fig. 1 .

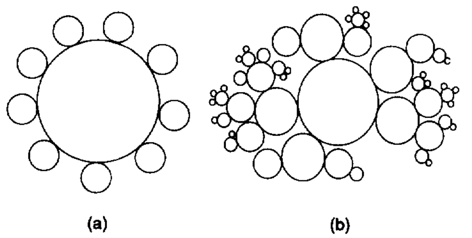

Fig. 3. Examples of super-daisy diagrams. The graph (a) is often simply referred to as a 'daisy diagram'.

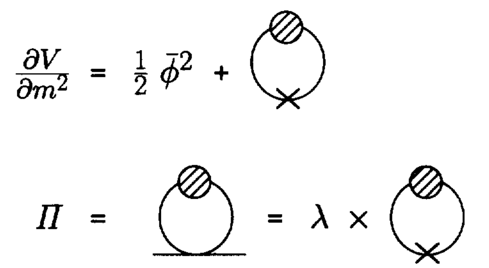

Fig. 4. The relation between the self-energy Π and the derivative of the effective potential V with respect to the mass square. (See the first two terms of Eq. (3).)

that the momentum-independent term (the first term in Fig. 1) is dominant. This corresponds to the well-known super-daisy approximation [7] which sums up all the graphs like Fig. 3. We stress that the effective potential consists of all the super-daisy diagrams without overcounting by this approximation.

In this approximation we have relation (see Fig. 4)

$$\Pi = \lambda\left(\frac{\partial V}{\partial m^2} - \frac{1}{2}\bar{\phi}^2\right). \qquad (4)$$

In the following we ignore the third and fourth terms in Eq. (3), assuming that only the first two terms are important in regard to the phase transition. [22] Due to this approximation, we neglect the loop contribution remaining finite at $T = 0$. Using relation (4) and integrating Eq. (3) over p_0, [10),11),24] we obtain the approximate evolution equation,

$$\frac{\partial V}{\partial m^2} = \frac{1}{2}\bar{\phi}^2 + \frac{1}{2\pi^2}\int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2+\lambda\frac{\partial V}{\partial m^2}}} \frac{1}{\exp\left(\frac{1}{T}\sqrt{p^2+m^2+\lambda\frac{\partial V}{\partial m^2}}\right)-1}. \qquad (5)$$

§4. Numerical result

In the previous section we obtained the evolution equation for $\frac{\partial V}{\partial m^2}$ within the super-daisy approximation. In this section we explain the details of the numerical calculation and present its result.

4.1. *Details of the numerical calculation*

We can calculate the effective potential by solving Eq. (5) and integrating it according to Eq. (1) numerically.

In Eq. (1) we set the initial scale m_0 as large as T so that we can evaluate the effective potential very well by the loop expansion. We use the same initial condition as was used in our previous paper, [22]

$$V(m^2 = m_0^2) = \frac{1}{2}m_0^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4$$

$$+\frac{T^2}{2\pi^2}\int_0^\infty dp p^2 \log\left[1-\exp\left(-\frac{1}{T}\sqrt{p^2+m_0^2+\frac{\lambda}{2}\bar{\phi}^2}\right)\right]. \tag{6}$$

The integral in the evolution equation, (5), is well-defined when the effective mass square, $m^2+\lambda\frac{\partial V}{\partial m^2}$, is real and positive. Below the critical temperature, however, the effective mass square can be negative or complex. We need an analytic continuation of the integral in Eq. (5). This can be done by rewriting Eq. (5) as in the previous paper,[22)]

$$\frac{\partial V}{\partial m^2}=\frac{1}{2}\bar{\phi}^2+\frac{T^2}{4\pi^2}\int_0^\infty dz\frac{\sqrt{z(z+2M)}}{e^{z+M}-1}, \tag{7}$$

where $M=\frac{1}{T}\sqrt{m^2+\lambda\frac{\partial V}{\partial m^2}}$. To find $\partial V/\partial m^2$ we solve Eq. (7).

As is well-known, the effective potential becomes complex at small $\bar{\phi}$ below the critical temperature. This indicates the instability of the state, and the imaginary part of the effective potential is interpreted as being related to the decay rate of the state.[23)] The imaginary part arises from the integral of the evolution equation in our method (see Eq. (7)). It is natural to suppose that the imaginary part of the effective potential is negative in order to interpret it as representing the decay rate. In order for the imaginary part of the effective potential be negative, the imaginary part of $\frac{\partial V}{\partial m^2}$ must be positive (see Eq. (1)). We find that there are two solutions for Eq. (7) and that their imaginary parts are of the opposite sign. We choose the solution of $\frac{\partial V}{\partial m^2}$ with positive imaginary part.

4.2. *Result*

Let us explain the numerical result. For the graphs Figs. (5) $\sim$ (7) we set*) the four-point coupling as $\lambda=1$.

We display the real part of the effective potential near the critical temperature in Fig. 5. We see a small barrier between the symmetric vacuum ($\bar{\phi}=0$) and the symmetry-broken vacuum ($\bar{\phi}\neq 0$). This clearly shows that a first order phase transition occurs here.

The presence of a first order phase transition is also indicated by the behavior of the field expectation value at the stable point, since Fig. 6 shows it possesses a finite jump.

We obtain a negative imaginary part of the effective potential below the critical temperature. It is shown in Fig. 7. We can see that the magnitude of the imaginary part increases as the field expectation value decreases. This implies that as the field expectation value approaches the origin, the state below the critical temperature becomes less stable.

In the above discussion we used the initial condition $m=m_0=T$. It is required that m_0 be the same order as the temperature in order to allow for the initial condition to be evaluated by the perturbation. We have calculated the effective

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*) We varied the value of λ and found no qualitative change.

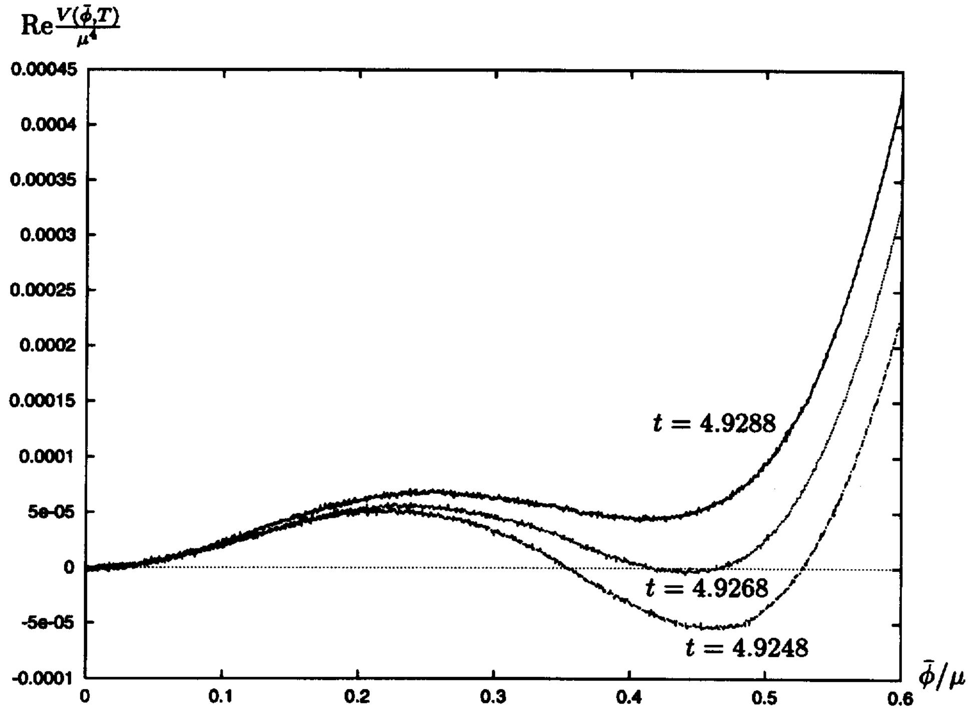

Fig. 5. Real part of the effective potential at $T = \mu t$. We see a small barrier to indicate a first order phase transition.

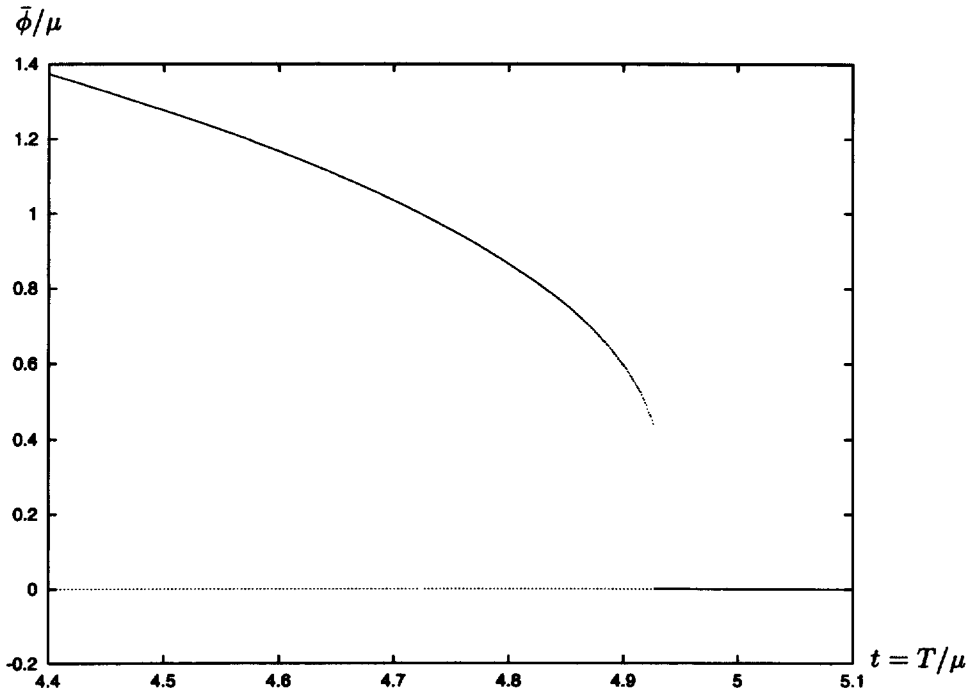

Fig. 6. The field expectation value near the critical temperature at the stable point. We see a finite jump of $\bar{\phi}/\mu$ at $t = 4.93\mu$ to indicate a first order phase transition.

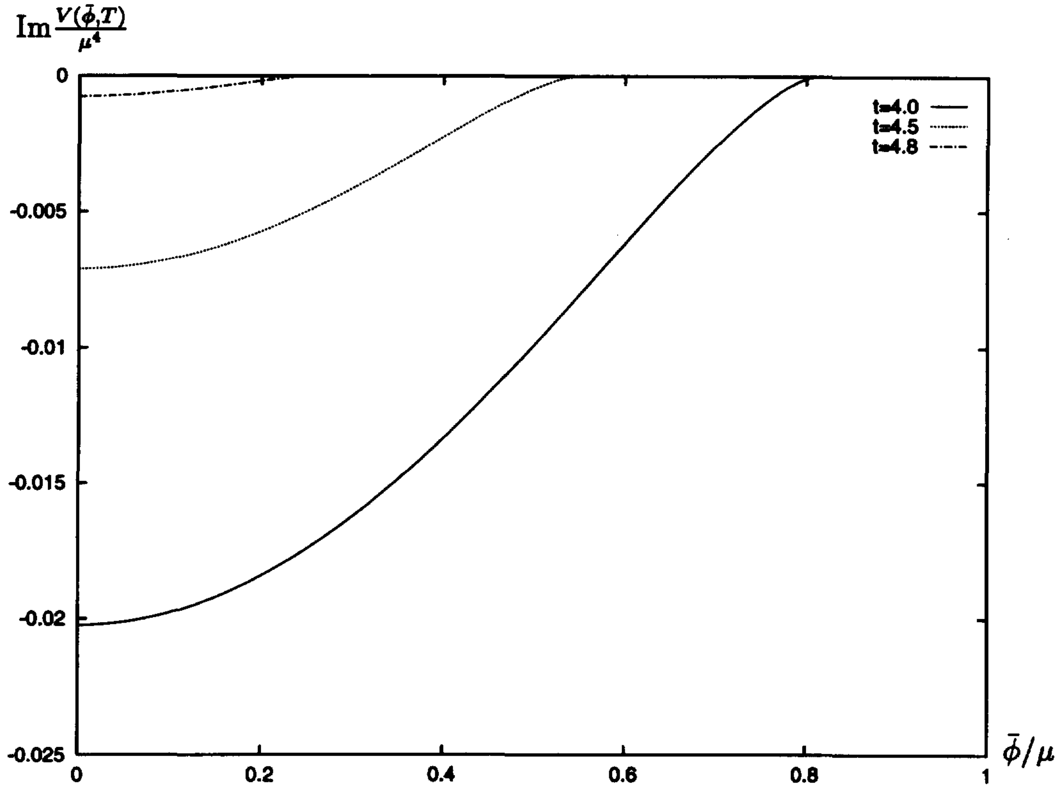

Fig. 7. Imaginary part of the effective potential at $T = \mu t$. We see that as $\bar{\phi}/\mu$ becomes smaller, the imaginary part becomes larger.

potential with other initial conditions, $m_0 = 2T$ and $m_0 = \frac{T}{2}$, but have not found any appreciable change of the effective potential. This ensures the consistency of our calculation.

§5. Summary and discussion

In this paper we have proposed a new method to calculate the effective potential of $\lambda\phi^4$ theory in the super-daisy approximation without overcounting. We have numerically evaluated the real part and imaginary part of the effective potential without using a high temperature expansion. The real part indicates a first order phase transition, though this should actually be second order. The imaginary part indicates that the instability is larger for smaller expectation value of the field below the critical temperature.

Now we compare our method with other approaches. First we compare it with the CJT method.[17] One can also gather up the super-daisy graphs without overcounting. It is carried out by truncating the CJT expansion at $O(\lambda)$, that is, taking into account only Fig. 8(a). A first order phase transition is also indicated in this model with the CJT method.[19] This is naturally consistent with our result. Though both methods can correct super-daisy diagrams without overcounting, there

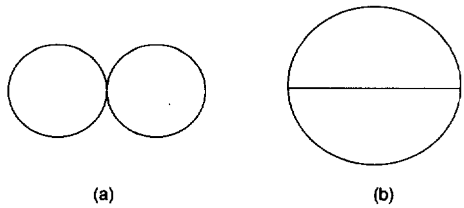

Fig. 8. Two-loop diagrams that contribute to the effective potential. (a) is daisy-like and (b) is not daisy-like.

are several points on which they differ. First, we do not use a high-temperature approximation. Second, we calculate both the imaginary and real parts of the effective potential below the critical temperature. Third, we can make another approximation which cannot be made by the CJT method.[22]

Next let us compare our method with the daisy-improved perturbation theory. In this method the effective potential indicates a first order phase transition at the one-loop level.[13] There are two graphs at the two-loop level (see Fig. 8). If one includes the contribution of Fig. 8(a) alone,[13] the phase transition is still first order.[*] A second order phase transition is indicated when the contributions of both Figs. 8(a) and (b) are included.[**] This indicates that the contribution of Fig. 8(b), which is not daisy-like, seems to play an important role in determining the effective potential.

Finally we compare our result with our approximation of our previous paper.[22] In the previous paper we obtained an effective potential that indicates a second order phase transition. The contributions of the second and third terms of Fig. 1 with zero external momentum are taken into account by the previous approximation. The difference between the two approximations is whether we include the contributions of the daisy-like diagrams only or not. This also indicates that non-daisy-like diagrams are important for the second order phase transition to be derived correctly.

According to the above comparisons, the diagrams which are not daisy like seem to give important contributions. We must take into account the contribution of the second and third terms in Fig. 1. On the other hand, these graphs are negligible at zero temperature because they appear in higher order in perturbative expansions. By comparing the results of the previous and the current paper, we confirm that thermal effects destroy the perturbation theory. This method with $\frac{\partial V}{\partial m^2}$ seems attractive because it may present a chance to step into the region where the perturbation theory breaks down.

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*) In Ref. 13) only Fig. 8(a) was included.

**) The author of Ref. 3) they calculated the effective potential including both Figs. 8(a) and (b). They did not, however, calculate the expectation value of the field nor given argument concerning the order of the phase transition with their 2-loop result. We calculated the expectation value of the field using their effective potential and found the phase transition to be second order.

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