GENERALIZED PARASUPERSYMMETRIC QUANTUM MECHANICS

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Abstract:

We construct the arbitrary order parasupersymmetric quantum mechanics of one boson and one parafermion degrees of freedom. The parasupersymmetry algebra is $Q^{2j}Q^{+}+Q^{2j-1}Q^{+}Q+\cdots+Q^{+}Q^{2j}=\alpha_{j}Q^{2j-1}H$, $Q^{2j+1}=0$, $[Q,H]=0$, where $2j$ represents the order of parasupersymmetry, H is the Hamiltonian and Q is the parasupercharge. Superpotentials of any order are introduced without breaking the above algebra.

§1. Introduction

In recent years statistically exotic behavior of particles has been discussed intensively in various fields in physics. For example the anyon system is a candidate for the carrier of high temperature superconductivity[1]. In this paper parasupersymmetry is considered and applied to quantum mechanics. Parasupersymmetry is generalized supersymmetry and its fermionic degrees of freedom is parastatistical[2]. Its potentiality is seriously examined in the context of the conformal field theory[3]. Roughly speaking, the parasupersymmetry enhances the "fermionic" degrees of freedom. Therefore it could reduce the critical dimension of the string theories.

In the ordinary N=1 supersymmetric quantum mechanics[4], it is fundamental that the Hamiltonian H and the supercharge Q make the superalgebra

$$\begin{gathered}\{Q^{+},Q\}=H\ , \\ Q^{2}=0\ , \\ [Q,H]=0\ .\end{gathered} \tag{1-1}$$

Introducing a parafermion which obeys parastatistics, Rubakov and Spiridonov[5] generalized the superalgebra for the case that the order of parafermion is two (hereafter denoted as p, i.e. p=2)

$$\begin{gathered}Q^{2}Q^{+}+QQ^{+}Q+Q^{+}Q^{2}=4QH\ , \\ Q^{3}=0\ , \\ [Q,H]=0\ .\end{gathered} \tag{1-2}$$

In our previous paper[6] we proposed a generalization of this algebra (1-2). We constructed the parafermions as the higher dimensional representation of the algebra *su(2)*. Its three generators are the creation, annihilation operator of the parafermion and the fermionic part of the Hamiltonian. The algebra can be generalized in a natural way following the treatment of Rubakov and Spiridonov[4]. We can introduce up to p bosonic degrees of freedom. The relation between our construction of the parafermion and another representations under the Green-Cusson ansätze and the Green ansätze was also discussed.

In this paper we will prove the superalgebra of the parasupersymmetric quantum mechanics of arbitrary parafermionic order with appropriate superpotentials. Its slightly restricted version was already mentioned in our previous paper[6]. But superpotentials were not treated explicitly and the parasuperalgebra was not yet proved wholly. We will also discuss physical contents of the system.

In §2 we first show a relation between the parasupercharge and the Hamiltonian. This relation was proved in our last paper[6]. Here this result is reproduced in another way. In §3 we introduce superpotentials without breaking the superalgebra. Superpotentials can be naturally introduced in the manner of Rubakov and Spiridonov[5]. Discussion of spectra, superpotentials and some mathematical aspects of the parasuperalgebra is given in §4.

§2. Superalgebra with One Boson and One Parafermion

We define the Hamiltonian H_j and the supercharge $Q_j^{(+)}$ as

$$Q_j = a^+ f_j,\quad Q_j^+ = a f_j^+,$$
$$H_j = \frac{1}{2}\{a^+, a\} + \frac{1}{2}[f_j^+, f_j]\ , \tag{2-1}$$

where $a^{(+)}$ stands for a bosonic annihilation(creation) operator and $f_j^{(+)}$ a parafermionic annihilation(creation) operator of order:$p=2j(\in Z^+)$. In the following we shall prove that the generalized parasupersymmetric quantum mechanics of parafermion's order $p=2j$ satisfies the following superalgebra

$$Q_j^{2j}Q_j^+ + Q_j^{2j-1}Q_j^+Q_j + Q_j^{2j-2}Q_j^+Q_j^2 + \cdots + Q_j^+Q_j^{2j} = \alpha_j Q_j^{2j-1}H_j\ ,$$
$$Q_j^{2j+1} = 0\ , \tag{A}$$
$$[Q_j, H_j] = 0\ ,$$

where

$$\alpha_j = \frac{2}{3}j(j+1)(2j+1)\ .$$

The second and last eqns of (A) are automatically satisfied by definition. Thus the first one remains to be proved.

In the first place we derive an operator identity of *su(2)*

$$f_j^{2j}f_j^+ + f_j^{2j-1}f_j^+f_j + f_j^{2j-2}f_j^+f_j^2 + \cdots + f_j^+f_j^{2j} = \alpha_j f_j^{2j-1}\ . \tag{B}$$

The generators of *su(2)*(J_3, $f_j = J_1 + \sqrt{-1}J_2$, $f_j^+ = J_1 - \sqrt{-1}J_2$) satisfy the following algebra

$$[f_j^+, f_j] = 2J_3,\quad [J_3, f_j^+] = f_j^+,\quad [J_3, f_j] = -f_j\ . \tag{2-2}$$

The Casimir invariant is defined as

$$J^2=\frac{1}{2}(f_j f_j^\dagger+f_j^\dagger f_j)+J_3^2 \ . \qquad (2\text{-}3)$$

Using the following properties,

a) $f_j^m J^2=J^2 f_j^m$, $(m\in Z^+)$

b) $f_j^m J_3=(J_3+m) f_j^m$,

c) $f_j f_j^\dagger=J^2-J_3^2-J_3$, $f_j^\dagger f_j=J^2-J_3^2+J_3$,

the following operator X_p becomes

$$\begin{aligned}X_p&=f_j^p f_j^\dagger+f_j^{p-1}f_j^\dagger f_j+f_j^{p-2}f_j^\dagger f_j^2+\cdots+f_j^\dagger f_j^p\\&=\sum_{m=0}^{p-1} f_j^m(f_j f_j^\dagger)f_j^{p-m-1}+f_j^\dagger f_j^p \ .\\&=\{p(J^2-J_3^2)-p^2J_3-\frac{1}{3}(p-1)p(p+1)\}f_j^{p-1}+(J^2-J_3^2+J_3)f_j^{p-1}\\&=(p+1)\{(J^2-J_3^2)-(p-1)J_3-\frac{1}{3}(p-1)p\}f_j^{p-1} \ .\end{aligned} \qquad (2\text{-}4)$$

After putting p=2j, the left hand side of (B) is

$$X_{2j}=(2j+1)\{J^2-\frac{1}{3}j(j+1)-(J_3+j)(J_3+(j-1))\}f^{2j-1}. \qquad (2\text{-}5)$$

Let us take representations which diagonalize J^2 and J_3 simultaneously. Namely the basis $|j,k\rangle$ is defined as

$$\begin{aligned}J^2|j,k\rangle&=j(j+1)|j,k\rangle,\\J_3|j,k\rangle&=k|j,k\rangle,\end{aligned} \qquad (2\text{-}6)$$

where 2k represents the weight of the basis. Considering the property of the state $j,k\rangle$

$$f_j^{2j-1}|j,k\rangle=0,\ (k\neq j,j-1)\ , \tag{2-7}$$

we have an identity

$$(J_3+j)(J_3+(j-1))f_j^{2j-1}=0\ \ \text{(for all states)}\ . \tag{2-8}$$

Thus

$$\begin{aligned} X_{2j} &= (2j+1)\{J^2-\frac{1}{3}j(j+1)\}f_j^{2j-1} \\ &= (2j+1)\{j(j+1)-\frac{1}{3}j(j+1)\}f_j^{2j-1} \\ &= \frac{2}{3}j(j+1)(2j+1)f_j^{2j-1} \\ &= \alpha_j f_j^{2j-1}\quad . \end{aligned} \tag{2-9}$$

Finally we derive the formula (A). Making use of the identity (B), the left hand side of (A) becomes

$$\begin{aligned} & a^{+2j}af_j^{2j}f_j^{+}+a^{+2j-1}aa^{+}f_j^{2j-1}f_j^{+}f_j+\cdots\cdots+aa^{+2j}f_j^{+}f_j^{2j} \\ &= a^{+2j-1}H_B(f_j^{2j}f_j^{+}+f_j^{2j-1}f_j^{+}f_j+\cdots\cdots+f_j^{+}f_j^{2j}) \\ &\quad + a^{+2j-1}\frac{1}{2}\{-f_j^{2j}f_j^{+}+f_j^{2j-1}f_j^{+}f_j+\cdots \\ &\quad \cdots+2(k-\frac{1}{2})f_j^{2j-k}f_j^{+}f_j^{k}+\cdots+2(2j-\frac{1}{2})f_j^{+}f_j^{2j}\} \\ &= a^{+2j-1}H_B\alpha_j f_j^{2j-1}+a^{+2j-1}\frac{1}{2}\frac{4}{3}j(j+1)(2j+1)f_j^{2j-1}H_{F_j} \\ &= \alpha_j(a^{+2j-1}f_j^{2j-1})(H_B+H_{F_j}) \\ &= \alpha_j Q^{2j-1}H_j \quad , \end{aligned} \tag{2-10}$$

where

$$H_B=\frac{1}{2}\{a^{+},a\}\ ,\ H_{F_j}=\frac{1}{2}[f_j^{+},f_j]\ \text{ and }\ H_j=H_B+H_{F_j}\ .$$

Note that J_3 turns out to become the parafermionic part of the Hamiltonian. Therefore the proof is completed.

In the previous paper[6], the proof of (A) heavily relied on the structure of the states. However, we realize that the identity (2-8)(or (2-7)) plays the essential role in the proof.

The identity removes the terms which depend upon J_3.

Since we now have the superalgebra of arbitrary order, we can introduce superpotentials as Rubakov and Spiridonov did in the case of p=2[5]. They will be introduced in the next section.

§3. Generalized Supersymmetric Quantum Mechanics

In Rubakov and Spiridonov's paper[5], a generalization without considering the first eq. of (A) was already discussed. They took the supercharge as

$$(Q^{+})_{i,i+1}=(p-\sqrt{-1}W_i),$$
$$(Q)_{i+1,i}=(p+\sqrt{-1}W_i)\quad({}^{\forall}i=1,2,\cdots,2j),$$

where p is the conjugate momentum of a bosonic coordinate x, W_i are arbitrary superpotentials and $(Q^{(+)})_{i,j}$ represents the (i,j) element of the matrix $Q^{(+)}$. Even though they claimed that W_i were meaningfully constructed in their paper, only $Q^{2j+1}=0$, $[Q,H]=0$ were satisfied. Some authors might say that this generalization slightly deviate from the treatment of parafermion as in the previous section, because the supercharge cannot be written in the direct product of the bosonic and parafermionic degrees of freedom. But it opens the possibility of the introduction of the non-trivial superpotentials and the positive semi-definiteness of the spectrum as in the following. In the trivial case $W_i(x)=x$ it is apparent that all eq.s of (A) are satisfied, because this is the consequence of the previous section. It will become clear, however, that the system cannot avoid negative energy states.

In this section we want to generalize the supercharge so that all the equations of the parasuperalgebra (A) are retained, even after introducing superpotentials.

We define the supercharge $Q^{(+)}$ with the superpotentials $W_i(x)$ as

$$(Q^{+})_{i,i+1}=\gamma_i(p-\sqrt{-1}W_i)\ ,$$
$$(Q)_{i+1,i}=\gamma_i(p+\sqrt{-1}W_i)\ , \qquad \text{(D-a)}$$
where $\gamma_i=\sqrt{\frac{i}{2}(2j-i+1)}$ and $p=-\sqrt{-1}\frac{\partial}{\partial x}$

and the Hamiltonian as

$$H=\frac{1}{2}p^2+\frac{1}{\alpha_j}\cdot\sum_{i=1}^{2j}(\gamma_i W_i)^2 + \frac{1}{\alpha_j}\cdot diag(g_1,g_2,\cdots,g_{2j+1}) \qquad \text{(D-b)}$$

where

$$g_1=(\gamma_1^2+2\sum_{i=2}^{2j}\gamma_i^2)W_1'+(\gamma_2^2+2\sum_{i=3}^{2j}\gamma_i^2)W_2'+\cdots+\gamma_{2j}^2W_{2j}'\ ,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$g_{k+1}=g_k-\alpha_j W_k'$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$g_{2j+1}=-\gamma_1^2W_1'-(2\gamma_1^2+\gamma_2^2)W_2'-\cdots-(2\sum_{i=1}^{2j-1}\gamma_i^2+\gamma_{2j}^2)W_{2j}'=g_{2j}-\alpha_jW_{2j}'\ ,$$

and $W_i'(x)$ indicates the derivative of $W_i(x)$ with respect to x. Here we have used

$$\alpha_j=2\sum_{i=1}^{2j}\gamma_i^2=\frac{2}{3}j(j+1)(2j+1)\ .$$

Then we shall show that introduction of the super-potentential preserve the superalgebra

$$Q^{2j}Q^{+}+Q^{2j-1}Q^{+}Q+\cdots+Q^{+}Q^{2j}=\alpha_j Q^{2j-1}H, \tag{T-a}$$

$$Q^{2j+1}=0, \tag{T-b}$$

$$[Q,H]=0, \tag{T-c}$$

provided that

$$(W_i^2-W_{i+1}^2)'-(W_i+W_{i+1})''=0 \quad \forall\ i=1,2,\cdots,2j-1 \ . \tag{C}$$

The condition (C) is exactly a generalization of Rubakov and Spiridonov's constraints in the case of p=2.

From the definition of the supercharge, (T-b) is automatically satisfied. Thus (T-a) and (T-c) are to be proved. Defining $P_i=p+\sqrt{-1}W_i$, $P_i^*=p-\sqrt{-1}W_i$ and $\rho_i = \gamma_i P_i$, $\rho_i^* = \gamma_i P_i^*$, we have the properties

$$\begin{aligned}&[P_k,P_l]=W_l'-W_k' , \quad [P_k^*,P_l^*]=-W_l'+W_k' ,\\ &[P_k,P_l^*]=-W_l'-W_k' , \quad [P_k^*,P_l]=W_l'+W_k' .\end{aligned} \tag{3-1}$$

First we shall derive (T-a). Actually the left hand side of (T-a) has only two non-zero components,

$$A\equiv(L.H.S\ of\ (T\text{-}a))_{2j,1}$$
$$= \rho_{2j-1}\cdots\rho_1\cdot\rho_1^*\rho_1 + \rho_{2j-1}\cdots\rho_2\cdot\rho_2^*\rho_2\cdot\rho_1 + \cdots\cdots + \rho_{2j}^*\rho_{2j}\cdot\rho_{2j-1}\cdots\rho_1 ,$$

$$B\equiv(L.H.S\ of\ (T\text{-}a))_{2j+1,2}$$
$$= \rho_{2j}\cdots\rho_2\cdot\rho_1\rho_1^* + \rho_{2j}\cdots\rho_2\cdot\rho_2^*\rho_2 + \rho_{2j}\cdots\rho_3\cdot\rho_3^*\rho_3\cdot\rho_2 + \cdots$$
$$\cdots + \rho_{2j}\cdot\rho_{2j}^*\rho_{2j}\cdot\rho_{2j-1}\cdots\rho_2 ,$$

where we have used the property that the r-th power of the supercharge becomes

$$(Q^r)_{i+r,i}=\rho_{i+r-1}\cdots\rho_i \quad (^\forall i=1,\cdots,2j-r+1) \ .$$

All the other components of Q^r vanish. Two non-zero components are

$$\begin{aligned}
A= &\ \rho_{2j-1}\cdots\rho_1\left(\frac{1}{2}\{\rho_1^*,\rho_1\}+\frac{1}{2}\{\rho_2^*,\rho_2\}+\cdots+\frac{1}{2}\{\rho_{2j}^*,\rho_{2j}\}\right)\\
&+\rho_{2j-1}\cdots\rho_1\left(\frac{1}{2}[\rho_1^*,\rho_1]+\frac{1}{2}[\rho_2^*,\rho_2]+\cdots+\frac{1}{2}[\rho_{2j}^*,\rho_{2j}]\right)\\
&+\sum_{i=2}^{2j}\rho_{2j-1}\cdots\rho_i\cdot\sum_{k=i}^{2j}[\rho_k^*,\rho_{i-1}]\,\rho_k\cdot\rho_{i-2}\cdots\rho_1\\
&+\sum_{i=2}^{2j}\rho_{2j-1}\cdots\rho_i\cdot\sum_{k=i}^{2j}\rho_k^*[\rho_k,\rho_{i-1}]\cdot\rho_{i-2}\cdots\rho_1 \ ,
\end{aligned}$$

$$\begin{aligned}
B= &\ \rho_{2j}\cdots\rho_2\left(\frac{1}{2}\{\rho_1^*,\rho_1\}+\frac{1}{2}\{\rho_2^*,\rho_2\}+\cdots+\frac{1}{2}\{\rho_{2j}^*,\rho_{2j}\}\right)\\
&+\rho_{2j}\cdots\rho_2\left(-\frac{1}{2}[\rho_1^*,\rho_1]+\frac{1}{2}[\rho_2^*,\rho_2]+\frac{1}{2}[\rho_3^*,\rho_3]+\cdots+\frac{1}{2}[\rho_{2j}^*,\rho_{2j}]\right)\\
&+\sum_{i=3}^{2j}\rho_{2j}\cdots\rho_i\cdot\sum_{k=i}^{2j}[\rho_k^*,\rho_{i-1}]\,\rho_k\cdot\rho_{i-2}\cdots\rho_2\\
&+\sum_{i=3}^{2j}\rho_{2j}\cdots\rho_i\cdot\sum_{k=i}^{2j}\rho_k^*[\rho_k,\rho_{i-1}]\cdot\rho_{i-2}\cdots\rho_2 \ .
\end{aligned}$$

Calculations are simplified by the properties (3-1). Using the constraints (C) we have

$$\begin{aligned}
&A=\alpha_j\rho_{2j-1}\cdots\rho_1\times\\
&\times\left[\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_iW_i)^2+(\gamma_1^2+2\sum_{i=2}^{2j}\gamma_i^2)\,W_1'+(\gamma_2^2+2\sum_{i=3}^{2j}\gamma_i^2)\,W_2'+\cdots+\gamma_{2j}^2W_{2j}'\right)\right] \ ,
\end{aligned}$$

$$B=\alpha_j\rho_{2j}\cdots\rho_2\left[\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_iW_i)^2-\gamma_1^2W_1'+(\gamma_2^2+2\sum_{i=3}^{2j}\gamma_i^2)\,W_2'+\cdots+\gamma_{2j}^2W_{2j}'\right)\right] \ .$$

Thus (T-a) is satisfied.

Next we shall show (T-c). Only 2j components need to be considered,

$$([Q,H])_{2,1}$$

$$=\rho_1\left\{\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_i W_i)^2+(\gamma_1^2+2\sum_{i=2}^{2j}\gamma_i^2)W_1'+(\gamma_2^2+2\sum_{i=3}^{2j}\gamma_i^2)W_2'+\cdots+\gamma_{2j}^2 W_{2j}'\right)\right\}$$

$$-\left\{\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_i W_i)^2-\gamma_1^2 W_1'+(\gamma_2^2+2\sum_{i=3}^{2j}\gamma_i^2)W_2'+\cdots+\gamma_{2j}^2 W_{2j}'\right)\right\}\rho_1 \quad ,$$

$$([Q,H])_{3,2}$$

$$=\rho_2\left\{\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_i W_i)^2-\gamma_1^2 W_1'+(\gamma_2^2+2\sum_{i=3}^{2j}\gamma_i^2)W_2'+\cdots+\gamma_{2j}^2 W_{2j}'\right)\right\}$$

$$-\left\{\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_i W_i)^2\right.\right.$$

$$\left.\left.-\gamma_1^2 W_1'-(2\gamma_1^2+\gamma_2^2)W_2'+(\gamma_3^2+2\sum_{i=4}^{2j}\gamma_i^2)W_3'+\cdots+\gamma_{2j}^2 W_{2j}'\right)\right\}\rho_2 \quad ,$$

. . . .

$$([Q,H])_{2j+1,2j}$$

$$=\rho_{2j}\left\{\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_i W_i)^2\right.\right.$$

$$\left.\left.-\gamma_1^2 W_1'-(2\gamma_1^2+\gamma_2^2)W_2'-\cdots-(2\sum_{i=1}^{2j-2}\gamma_i^2+\gamma_{2j-1}^2)W_{2j-1}'+\gamma_{2j}^2 W_{2j}'\right)\right\}$$

$$-\left\{\frac{1}{2}p^2+\frac{1}{\alpha_j}\left(\sum_{i=1}^{2j}(\gamma_i W_i)^2-\gamma_1^2 W_1'-(2\gamma_1^2+\gamma_2^2)W_2'-\cdots-(2\sum_{i=1}^{2j-1}\gamma_i^2+\gamma_{2j}^2)W_{2j}'\right)\right\}\rho_{2j}$$

They all vanish using the constraints(C). Other elements are zero by definition. Therefore the proof is accomplished.

As Rubakov and Spiridonov have already shown[5], we can give some simple superpotentials. In the simplest case: $W_i'(x)=W'(x)$ ($^\forall i=1,2,\cdots,2j$) with constraints (C) the superpotential is given as

$$W=\omega_1 x+\omega_2 \tag{3-2}$$

or

$$\begin{aligned} W_1&=\omega_1\exp(-ax)+\omega_2 \\ W_{i+1}&=W_i+a \end{aligned} \tag{3-3}$$

where ω_1, ω_2 and a are arbitrary constants. If we put $\omega_1=1$ and $\omega_2=0$ in eq.(3-2), it becomes the trivial case which has been discussed in §2. In two cases (3-2) and (3-3) the Hamiltonian is written as

$$H=\frac{1}{2}p^2+V(x)+B(x)\cdot J_3 ,$$

$$\text{where } V(x)=\frac{1}{\alpha_j}\sum_{i=1}^{2j}(\gamma_i W_i)^2, \quad B(x)=\frac{1}{2}W'(x)$$
$$\text{and } J_3=diag(j,j-1,\cdots,-j) .$$

Especially in the case (3-2) V(x)=W. It can be interpreted as the Hamiltonian describing one-dimensional motion of a spin j particle in a magnetic field B(x) directed along the third axis. The case (3-2) corresponds to an oscillator in a homogeneous magnetic field, while the case (3-3) corresponds to the Morse potential and inhomogeneous (exponential) magnetic field.

Integration of the constraints (C) is

$$W_{i+1}^2+W_{i+1}'=W_i^2-W_i'+c_i , \tag{3-5}$$

where c_i ($\forall i=1,2,\cdots,2j-1$) are integration constants. Superpotentials are related by eq.(3-5), which allows us to find W_i+W_{i+1}, if $W_{i+1}-W_i$ is fixed. Therefore we can obtain a general expression for W_i and W_{i+1} parametrized by 2j arbitrary functions

$$V_i(x) = \exp\int^x (W_{i+1} - W_i)\,dx \tag{3-6}$$

and by two kinds of arbitrary constants c_i and d_i

$$W_{i(i+1)} = \frac{1}{2V_i}\left(-(+)\,V_i' + d_i + c_i\int^x V_i(x)\,dx\right)\ . \tag{3-7}$$

Of course the existence of the constraints (3-5) remarkably reduces the degrees of freedom of the arbitrary functions W_i. Still the functions V_i remain arbitrary. Plenty of the degrees of freedom are left to study.

§4. Discussion

In this paper we have generalized the parasupersymmetric quantum mechanics by introducing superpotentials. In this generalization the superalgebra is naturally constructed and we have meaningful constraints for the superpotentials.

The spectrum of the parasupersymmetric quantum mechanics has interesting structure. It originates from the fact that the superpotentials must obey the conditions(C). The simplest example has been given in eq.(3-2): $W_i(x)=W(x)=\omega_1 x+\omega_2$, $^\forall i$. All $W_i(x)$ must satisfy eq.(3-5), where $c_i=2\omega_1$. Providing $\omega_1>0$, there exists the unique vacuum $|0>=(0,\cdots,0,\psi(x))$, where $\psi(x)=\exp(-\int^x W(x)dx)$ and $Q^{(+)}|0>=0$. (If $\omega_1<0$, unique vacuum is $|0>=(\psi(x),0,\cdots,0)$ and $\psi(x)=\exp(\int^x W(x)dx)$.) The first excited states are two-fold degenerate: $f^+|0>$, $a^+|0>$; the i-th excited states are (i+1)-fold degenerate. Above the 2j-th excited levels, the states are (2j+1)-fold degenerate.

We assume that the sequence of states $|i>$ ($^\forall i=1,\cdots,2j+1$) exists, where $Q^+|i+1>=|i>$ or $Q|i>=|i+1>$. All the states have the same energy $H|i>=E|i>$, $^\forall i$. The state $|i>$ which is an eigen-state of h_i is also assumed. Here h_i is the diagonal component of the Hamiltonian $H=\mathrm{diag}(h_1,h_2,\cdots,h_{2j+1})$ where

$$\begin{aligned} h_i &= \frac{1}{2}(p-\sqrt{-1}W_i)(p+\sqrt{-1}W_i)+\beta_i \quad , \\ &= \frac{1}{2}(p+\sqrt{-1}W_{i-1})(p-\sqrt{-1}W_{i-1})+\beta_{i-1} \quad , \end{aligned} \tag{4-1}$$

and $$\beta_i=\frac{1}{\alpha_j}\left\{\sum_{k=i}^{2j-1}\left(\sum_{l=k+1}^{2j}\gamma_l^2\right)c_k-\sum_{k=1}^{i-1}\left(\sum_{l=1}^{k}\gamma_l^2\right)c_k\right\} .$$

In the previous simplest example $W_i=W(x)$, $\beta_i=(j-i+1/2)\omega_1$ ($^\forall i=1,2,$

$\cdots,2j$). Using the constraints (C) every pair of diag(h_i,h_{i+1}) can be interpreted as the ordinary SUSY Hamiltonian

$$H^{(i)}=\begin{pmatrix} \frac{1}{2}(p-\sqrt{-1}W_i)(p+\sqrt{-1}W_i) & 0 \\ 0 & \frac{1}{2}(p+\sqrt{-1}W_i)(p-\sqrt{-1}W_i) \end{pmatrix} \qquad (4-2)$$

$$=\begin{pmatrix} h_i & 0 \\ 0 & h_{i+1} \end{pmatrix}-\beta_i \quad .$$

This $H^{(i)}$ has positive semi-definiteness. The state $|i>$ should belong to $H^{(i-1)}$ and $H^{(i)}$. Therefore the energy of $|i>$ must be greater than or equal to $\max(\beta_{i-1},\beta_i)$. The state $|i>$ has para-superpartners $|1>,|2>,\cdots,|2j+1>$, thus the energy of the states must be greater or equal to the maximum of $^{\forall}\beta_i$. It is not necessary that the sequence has (2j+1) states. In general the sequence consists of (m-n+1) states $|i>$ ($1\le n\le i\le m\le 2j+1$, for $^{\exists}n,m$), where $Q^{+}|i+1> =|i>$ or $Q|i>=|i+1>$. This time the energy of the states is greater than or equal to the maximum of $\beta_i(i=n-1,\cdots,m)$. Note that the system has the symmetry in the Hamiltonian H, that is, h_i and h_{2j+2-i} interchange each other under the change of $W_i <=> -W_{2j+1-i}$, $c_i <=> -c_{2j-i}$.

The positive semi-definiteness of the energy of the whole system needs some more physical conditions, which is contrary to the ordinary SUSY case. Actually the previous simplest example has no positive semi-definiteness. The energy of its ground state is $E_0=-(j-1/2)|\omega_1|$. It is equal to $\beta_1(\beta_{2j})$, when $\omega_1<0(\omega_1>0)$. The spectrum depends on the values of the constants β_i. All of β_i are not independent, but they have an identity

$$\sum_{i=1}^{2j} \gamma_i{}^2 \beta_i = 0 \quad . \tag{4-3}$$

Therefore there must exit non-negative β_k. It means that the energy of the state, which has the k-th component of the column vector representation, is non-negative. The energy of the states which have an eigenstate of $H^{(k)}$ as a non-trivial parasuperpartner must also be non-negative. In general we have the positive semi-definiteness, provided that the ground state has a non-trivial component in the k-th row. The example will be illustrated later.

If all β_i are non-negative, the system has the positive semi-definiteness of the energy. An example is obtained, assuming ${}^\forall c_i=0$, thus ${}^\forall \beta_i=0$;

$$W_{i+1}=-W_i \quad (i=1,2,\cdots,2j-1) \quad and \quad W_1=W(x) \quad . \tag{4-4}$$

It is already represented by Beckers and Debergh[7] in the case of W(x)=x and the order p=2. Its vacuum energy is $E_0=0$.

Non-trivial examples are difficult to find analytically because of the condition (C). Further the discussion of spectrum of the non-trivial case is more complicated. The classification of vacua is already complex enough. It depends on the normalizability of the characteristic functions $\psi_\pm^{(i)}=\exp(\pm\int^x W_i(x)dx)$ (${}^\forall i=1,2,\cdots,2j$). It is clear that both of $\psi_+^{(k)}$ and $\psi_-^{(k)}$ cannot be normalizable. In the above simplest example all $\psi_-^{(i)}$ are normalizable. In general only several of $\psi_\pm^{(i)}$'s are normalizable simultaneously.

For the sake of exposition we give an example of p=2j=3 case

$$W_1=-\cosh(x)+\frac{1}{2},\ W_2=-\sinh(x)-\frac{1}{2},\ W_3=\cosh(x)+\frac{1}{2},\quad (c_1=-1,\ c_2=1)\ . \tag{4-5}$$

In this case $\psi_+^{(2)}$ is only normalizable. The ground state is two-fold degenerate: $|0>_1=(0,\psi_+^{(2)},0,0)$ and $|0>_2=Q^+|0>_1=(\sqrt{3}(p-\sqrt{-1}W_1)\psi_+^{(2)},0,0,0)(\neq 0)$, but $Q|0>_1=0$. By definition the state $|0>_1$ is an eigenstate of h_1 and the state $|0>_2$ is an eigenstate of h_2. From the above discussion the eigenstate of h_1 is expected to be greater than or equal to $\beta_1=(7c_1+3c_2)/20=-1/5$, and the eigenstate of h_2 is greater than or equal to $\beta_2=(-3c_1+3c_2)/20=3/10(>\beta_1)$. It implies that the vacuum energy should be $E_0=3/10(=\beta_2)$. Actual calculation tells us that it is correct: $H|0>_1=3/10|0>_1$. More detailed discussion of the spectrum will need further study both analytically and numerically.

In the Lie algebraic point of view, the constraints (C) can be rewritten as

$$\begin{aligned}
[P^*_{i+1}P_{i+1},P_i]&=[P_iP^*_i,P_i]=P_i\cdot 2W'_i\ ,\\
[P^*_{i+1}P_{i+1},P^*_i]&=[P_iP^*_i,P^*_i]=-2W'_i\cdot P^*_i\ ,\\
[P^*_{i+1}P_{i+1},P_{i+1}]&=[P_iP^*_i,P_{i+1}]=2W'_i\cdot P_{i+1}\ ,\\
[P^*_{i+1}P_{i+1},P^*_{i+1}]&=[P_iP^*_i,P^*_{i+1}]=P^*_{i+1}\cdot(-2W'_i)\ .
\end{aligned} \tag{4-6}$$

Their forms reminds us of the relation with the Cartan sub-algebras and the roots system of the super Lie algebras. The operators P_i are very similar to the basis of the roots, if we admit that $P_iP^*_i+P^*_{i+1}P_{i+1}$ generate the Cartan subalgebra. It is strongly suggested from the fact that $P_iP^*_i+P^*_{i+1}P_{i+1}$ is the (i+1)-th diagonal component of $\{Q,Q^+\}$. But complete discussion needs more study, because we still have some delicate problems, e.g.

operating order of W_i' and $P_i^{(*)}$ in the R.H.S. of eq.(4-6) Another interpretation of the constraints (C) can be seen from (3-5)

$$P^*_{i+1}P_{i+1}=P_iP^*_i+c_i \quad . \tag{4-7}$$

where c_i's are arbitrary constants. They would also have a strong connection with the soliton and inverse scattering theory[8,9]. The constraints(C) are a bunch of the Riccati type equations. Especially in the case of $^{\forall}c_i=0$ we might gain deep insight through the inverse scattering method[10].

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