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Spaces having a generator for a homeomorphism

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Abstract

In this paper we construct a non-metrizable Lindelöf space X having a generator for a homeomorphism on X. We also construct a locally compact, infinite-dimensional metric space X having an expansive homeomorphism on X.

Keywords and phrases. generator, expansive, metrizable, dimension.

1 Introduction and preliminaries.

All spaces are assumed to be completely regular and T_1 unless otherwise stated. In this paper we study spaces having a generator for a homeomorphism. For standard results and notation in Topological Dynamics we refer to [1].

H. B. Keynes and J. B. Robertson [2] proved that every compact space X having a generator for some homeomorphism is metrizable. In Section 2 we prove that this result is true in the case when X is locally compact. However, we construct a non-metrizable Lindelöf space having a generator for some homeomorphism.

Every homeomorphism f on a compact metric space is expansive if and only if f has a generator. R. Mañé [3] proved that if a compact metric space X has an expansive homeomorphism, then X is finite-dimensional. In Section 3 we construct a locally compact, infinite-dimensional metric space having an expansive homeomorphism.

Let $\mathcal{U}$ and $\mathcal{V}$ be covers of a space X and f be a homeomorphism on X. Let us set

$$\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\} \text{ and}$$
$$f^{-n}(\mathcal{U}) = \{f^{-n}(U) : U \in \mathcal{U}\},$$

where $f^{-n}(U)$ is the inverse image of U under the n-fold composition f^n of f.

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2 Generators.

Let f be a homeomorphism on a space X. A finite open cover $\mathcal{U}$ of X is said to be a *generator* for f if for every bisequence $\{U_n : n \in \mathbb{Z}\}$ of members of $\mathcal{U}$ the intersection

$$\bigcap\{f^{-n}(\mathrm{Cl}\,U_n) : n \in \mathbb{Z}\}$$

contains at most one point.

H. B. Keynes and J. B. Robertson [2] proved that if a compact space X has a generator for some homeomorphism on X, then X is metrizable. By using the same method, we can prove the following theorem, which is a generalization of the above result.

Theorem. *Let X be a locally compact space. If a homeomorphism f on X has a generator $\mathcal{U}$, then the space X is metrizable.*

Proof. For every $n \in \mathbb{Z}$ we set

$$B_n = \bigwedge\{f^{-i}(\mathcal{U}) : -n \leq i \leq n\} \text{ and}$$
$$B = \bigcup\{B_n : n \in \mathbb{N}\}.$$

Since $\mathcal{U}$ is finite, so is B_n. Thus B is countable. We shall show that B is a base for X. To this end, let x be a point of X and O a neighborhood of x in X. We can assume that Cl O is compact. For every $n \in \mathbb{Z}$ we take $U_n \in \mathbb{Z}$ with $f^n(x) \in U_n$. Obviously, we have $x \in f^{-n}(U_n)$. Since $\mathcal{U}$ is a generator for f, we have

$$\bigcap\{f^{-n}(\mathrm{Cl}\,U_n) : n \in \mathbb{Z}\} = \{x\} \subset O.$$

From compactness of Cl O it follows that

$$\bigcap\{f^{-i}(\mathrm{Cl}\,U_i) : -n \leq i \leq n\} \subset O$$

for some $n \in \mathbb{N}$. Let us set

$$V = \bigcap\{f^{-i}(U_i) : -n \leq i \leq n\}.$$

Then we have $x \in V \subset O$ and $V \in B_n \subset B$. This implies that B is a base for X. Hence X is second-countable, therefore X is metrizable.

As shown in the above theorem, every locally compact space X having a generator for some homeomorphism on X is metrizable. However, in this theorem the local compactness can not be omitted.

Example. There exist a non-metizable Lindelöf space X and a homeomorphism f on X such that f has a generator.

Let us set

$$X = \{z\} \cup \{x_{i,j} : i, j < \omega \text{ with } i \geq j\} \cup \{y_{i,j} : i, j < \omega \text{ with } i \geq j\}.$$

For every pair (i, j) with $i \geq j$ we set

$$B(x_{i,j}) = \{\{x_{i,j}\}\} \text{ and } B(y_{i,j}) = \{\{y_{i,j}\}\}.$$

Let Φ be the set of all mappings $\varphi : \omega \to \omega$ such that $\varphi(j) \geq j$ for every $j < \omega$. For every $\varphi \in \Phi$ we set

$$U(\varphi) = \{z\} \cup \{x_{i,j} : i, j < \omega \text{ with } i \geq \varphi(j)\} \cup \{y_{i,j} : i, j < \omega \text{ with } i \geq \varphi(j)\}.$$

Let us set

$$B(z) = \{U(\varphi) : \varphi \in \Phi\}.$$

We give X the topology by taking

$$\{B(x) : x \in X\}$$

as a neighborhood system. Since X is countable, X is a Lindelöf space.

Let $f : X \to X$ be the mapping defined by

$$f(z) = z,$$

$$f(x_{i,j}) = \begin{cases} x_{i,j+1} & \text{if } i > j \\ x_{i+1,0} & \text{if } i = j, \text{ and} \end{cases}$$

$$f(y_{i,j}) = \begin{cases} x_{0,0} & \text{if } i = j = 0 \\ y_{i,j-1} & \text{if } i \neq 0, j \neq 0 \\ y_{i-1,i-1} & \text{if } i \neq 0, j = 0 \end{cases}$$

We shall show that the point z has no countable neighborhood base. Assume that there exists a countable neighborhood base $\{U_n : n < \omega\}$ at z. For every $n < \omega$ we take $\varphi_n \in \Phi$ such that $U(\varphi_n) \subset U_n$. Let $\psi : \omega \to \omega$ be the mapping defined by $\psi(n) = \varphi_n(n) + 1$ for every $n < \omega$. Since $\psi(n) > \varphi_n(n) \geq n$, we have $\psi \in \Phi$. Take U_n such that $U_n \subset U(\psi)$. Then we have $x_{\varphi_n(n),n} \in U(\varphi_n) \subset U_n$. However, since $\varphi_n(n) < \psi(n)$, we have $x_{\varphi_n(n),n} \notin U(\psi)$. This is a contradiction. Hence the point z has no countable neighborhood base. This implies that X is not metrizable.

Next, we shall show that f has a generator. Let us set

$$U = \{x_{0,0}\}, V = X - U \text{ and } \mathcal{U} = \{U, V\}.$$

It suffices to show that the cover $\mathcal{U}$ is a generator for f. Suppose that $\{U_n : n \in \mathbb{Z}\}$ is a bisequence of members of $\mathcal{U}$. If $U_i = U$ for some $i \in \mathbb{Z}$, then, obviously,

$$\bigcap\{f^{-n}(\mathrm{Cl}U_n) : n \in \mathbb{Z}\} \subset f^{-i}(\mathrm{Cl}U_i) = \{f^{-i}(x_{0,0})\}.$$

Thus the intersection $\bigcap\{f^{-n}(\mathrm{Cl}U_n) : n \in \mathbb{Z}\}$ contains at most one point. If $U_i = V$ for every $i \in \mathbb{Z}$, then we have $\bigcap\{f^{-n}(\mathrm{Cl}U_n) : n \in \mathbb{Z}\} = \{z\}$. Hence the cover $\mathcal{U}$ is a generator for f.

3 Expansive homeomorphisms.

A homeomorphism f on a metric space (X, d) is *expansive* if there exists a positive number $c > 0$ such that for every pair of distinct points $x, y \in X$ there is an integer $n \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) \geq c$. Every positive number having this last property is called an *expansive constant* for f.

For every expansive homeomorphism f on a compact metric space X a finite sub-cover of

$\{B(x, \frac{c}{2}) : x \in X\}$ is a generator for f, where c is an expansive constant for f. Thus every expansive homeomorphism on a compact metric space has a generator. Conversely, let f be a homeomorphism on a compact space such that f has a generator $\mathcal{U}$. Then it is easy to see that a Lebesgue number of $\mathcal{U}$ is an expansive constant for f. Hence for every homeomorphism f on a compact metric space f is expansive if and only if f has a generator.

R. Mañé[3] proved that if a compact metric space X has an expansive homeomorphism on X, then X is finite-dimensional. In this section we shall show that the compactness can not be omitted.

Example. Let $X = \bigcup\{\{n\}\times[0, 2^n]^\omega : n \in \mathbb{Z}\}$ be the subspace of the metric space $\mathbb{R}\times\mathbb{R}^\omega$. Since the space X is a topological sum of compact metric spaces, X is locally compact. Let $f : X \to X$ be the mapping defined by

$$f((n,x)) = (n+1, 2x)$$

for every $(n, x) \in X$. Then f is a homeomorphism on X. We shall show that the number 1 is an expansive constant for f. To this end, let $((n, x), (m, y))$ be a pair of distinct points of X. If $n \neq m$, then, obviosly, $d((n, x), (m, y)) \geq |n - m| \geq 1$. Suppose that $n = m$. Then we have $x \neq y$. We put $\varepsilon = d'(x, y)$, where d' is the metric on $\mathbb{R}^\omega$. Since $\varepsilon > 0$, we can take $k \in \mathbb{Z}$ with $2^k\varepsilon \geq 1$. Then we have

$$d(f^k(n,x), f^k(m,y)) = d'(2^k x, 2^k y) = 2^k d'(x,y) = 2^k\varepsilon \geq 1.$$

Hence the homeomorphism f ia expansive. On the other hand, X contains the Hilbert cube as a subspace. Hence X is infinite-dimensional.

References

[1] M. Denker, C. Grillenberger and K. Sigmund, Ergodic Theory on Compact Spaces Lecture Notes in Mathematics 527, 1976.

[2] H. B. Keynes and J. B. Robertson, *Generators for topological entropy and expansiveness,* Math. Syst. Theory 3 (1969), 51-59.

[3] R. Mañé, *Expansive homeomorphism and topological dimension,* Trans. Amer. Math. Soc. 252 (1979), 313-319.

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