Approximation of Random Wave Phenomenon in Terms of Heat Conductive Model

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Summary

In this paper we consider approximating random wave phenomenon in terms of heat conductive model. As a matter of fact, random wave phenomenon is described by a hyperbolic type PDE driven by noise and a heat conductive model is expressed by a parabolic type PDE driven by noise. We prove that a solution to a type of stochastic evolution equation corresponding to the former PDE converges in probability to a solution to another type of stochastic evolution equation associated with the latter PDE. Moreover, the existence and uniqueness of solutions to those stochastic evolution equations are also derived.

Key Words: random wave phenomenon, heat conductive model, stochastic partial differential equation, Smoluchowski-Kramers approximation.

1. Introduction

We are very interested in random wave phenomena, especially when they are formulated by some stochastic wave equations [5], [6]. On the other hand, mathematical physical phenomena involving heat conduction also do interest us so much, and we have been studied several types of physical models related to stochastic parabolic partial differential equations [7], [9], [11]. In this article we consider a certain approximation method of random wave phenomenon in terms of heat conductive model. As a matter of fact, in the case we have treated here, a random wave phenomenon is described by a hyperbolic type partial differential equation (PDE) driven by some noise [4], [19], [21], and a heat conductive model is expressed by a parabolic type PDE driven by some noise [22], [23], [24]. Our goal of this paper is to establish some approximation results of stochastic wave equations by a stochastic heat equation. In fact we prove that a solution to a type of stochastic evolution equation corresponding to the former hyperbolic PDE converges in probability to a solution to another type of stochastic evolution equation associated with the latter parabolic PDE [1], [2], [3]. In particular, [1] and [3] treat stochastic equations with the Laplacian $\varDelta$, while in this article we deal with stochastic partial differential equations with a general second order differential operator L instead of $\varDelta$, so that the approximation result obtained in this paper is a generalization of the results of [1] and [3]. Moreover, the existence and uniqueness results of solutions to those related stochastic evolution equations are proven as well.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered basic complete probability space with filtration $(\mathcal{F}_t)$ satisfying the usual conditions [20]. The element ω taken from Ω is called a sample, and as for a stochastic process $X = \{X_t\}$, $t \in \mathbb{R}_+ = [0, \infty)$, the symbol $X_t(\omega) \equiv X(t, \omega)$ indicates a sample path or its realization [17]. Let D be a bounded regular domain in $\mathbb{R}^d$, and we set $H = L^2(D)$.

Let us consider the following mixed problem for stochastic wave equation driven by a random noise:

$$\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}(t,x,\omega) = Lu^\varepsilon(t,x,\omega) - \frac{\partial u^\varepsilon}{\partial t}(t,x,\omega) + G(u^\varepsilon)(t,x,\omega) + \frac{\partial W}{\partial t}(t,x,\omega), \quad \text{a.a.-}\omega \in \Omega \quad \text{for} \quad t > 0, x \in D, \tag{1}$$

$$u^\varepsilon(0,x,\omega) = u_0(x), \quad \frac{\partial u^\varepsilon}{\partial t}(0,x,\omega) = u_1(x), \quad \text{a.e.} \quad \text{for} \quad x \in D, \tag{2}$$

$$\text{and} \qquad u^\varepsilon(t,\xi,\omega) = 0, \quad \text{a.e.} \quad \text{for} \quad t > 0, \xi \in \partial D, \tag{3}$$

where $0 < \varepsilon \ll 1$, L is the second order differential operator, and $W_t(x,\omega) \equiv W(t,x,\omega)$ is a cylindrical Wiener process [18] which is a white noise in time and is a colored noise in space, with covariance operator Q^2 for some $Q \in \mathcal{L}(H)$. While, G is defined by

$$G(x) := -Q^2 \cdot DF(x), \qquad \text{for} \quad x \in H \tag{4}$$

for some function $F : H \to \mathbb{R}$ satisfying suitable conditions with derivative DF of F [23]. More precisely, we assume the following gradient structure for the non-linearity of G: there exists $F : H \to \mathbb{R}$ of class C^1, with $F(0) = 0$, $F(x) \geq 0$ and

$$\langle DF(x), x \rangle \geq 0 \quad \text{for all} \quad x \in H,$$

such that $G(x) = -Q^2 \cdot DF(x)$ holds for any $x \in H$. Moreover, there exists a positive constant $C > 0$ such that

$$\|DF(x) - DF(y)\|_H \leqslant C\|x - y\|_H, \qquad \text{for} \quad \forall x, y \in H. \tag{5}$$

On the other hand, we consider the heat conductive model with a random noise:

$$\frac{\partial w}{\partial t^2}(t,x,\omega) = Lw^\varepsilon(t,x,\omega) + G(w)(t,x,\omega) + \frac{\partial W}{\partial t}(t,x,\omega), \quad \text{a.a.-}\omega \in \Omega \quad \text{for} \quad t > 0, x \in D, \tag{6}$$

$$w(0,x,\omega) = u_0(x), \quad \text{a.e.} \quad \text{for} \quad x \in D, \tag{7}$$

$$\text{and} \qquad w(t,\xi,\omega) = 0, \quad \text{a.e.} \quad \text{for} \quad t > 0, \xi \in \partial D. \tag{8}$$

Let A be the realization of L with Dirichlet boundary condition in a Hilbert space H. Let (e_k), $k \in \mathbb{N}$, be the complete orthonormal basis of eigenfunctions of A, and let $(-\alpha_k)$, $k \in \mathbb{N}$, be the corresponding sequence of positive eigenvalues $\alpha_k > 0$, with monotone property

$$\alpha_k \leqslant \alpha_{k+1}, \qquad \text{for any} \quad k \in \mathbb{N}. \tag{9}$$

It is interesting to note that the cylindrical Wiener process $W_t(x,\omega)$ has a more explicit representation

$$W_t(x,\omega) = \sum_{k=1}^{\infty} Qe_k(x)B^k(t,\omega), \tag{10}$$

where (e_k), $k \in \mathbb{N}$, is the complete orthonormal basis in H which diagonalizes A, and $\{B^k(t)\} \equiv \{B^k(t,\omega)\}$, $k \in \mathbb{N}$, is a sequence of mutually independent standard Brownian motions [17] defined on the same complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. In addition, we assume that the linear operator Q is bounded in H, and diagonal with respect to the basis (e_k), $k \in \mathbb{N}$, which diagonalizes A. Moreover, if (λ_k), $k \in \mathbb{N}$, is the corresponding sequence of eigenvalues, we are supposed to have

$$\frac{\lambda_{k+1}}{\lambda_k} < \sqrt{\frac{\alpha_{k+1}}{\alpha_k}} \qquad \text{for} \quad \forall k. \tag{11}$$

We denote by the symbol $C_0^{\infty}(D)$ the totality of all infinitely times differentiable functions defined on D with compact support. For any $p \in \mathbb{R}$, we denote by H^p the completion of $C_0^{\infty}(D)$ in the norm

$$\|u\|_{H^p} := \left\{ \sum_{k=1}^{\infty} \alpha_k^p \langle u, e_k \rangle_H \right\}^{1/2}. \tag{12}$$

Notice that this H^p is nothing but a Hilbert space with the scalar product

$$\langle u, v \rangle_{H^p} := \sum_{k=1}^{\infty} \alpha_k^p \langle u, e_k \rangle_H \cdot \langle v, e_k \rangle_H. \tag{13}$$

In what follows, we shall use the symbol

$$\mathcal{H}_p := H^p \times H^{p-1}, \tag{14}$$

and we set $\mathcal{H} := \mathcal{H}_0$ for simplicity.

2. Main results: Smoluchowski-Kramers approximation

For a sufficiently small positive parameter $\varepsilon > 0$, the operator $A_\varepsilon : \mathrm{Dom}(A_\varepsilon) \longrightarrow \mathcal{H}_p$ is defined by

$$A_\varepsilon(u,v) := (v, \frac{1}{\varepsilon}Au - \frac{1}{\varepsilon}v) \tag{15}$$

for all element $(u, v) \in \mathrm{Dom}\,(A_\varepsilon) = \mathcal{H}_{p+1} = H^{p+1} \times H^p$, and $S_\varepsilon(t)$ denotes the semigroup on $\mathcal{H} = \mathcal{H}_0$, that is generated by the infinitesimal generator A_ε. That is to say, for such an operator A_ε, the operator $e^{tA\varepsilon}$ is defined as a formal Taylor expansion

$$e^{tA_\varepsilon} = \sum_{n=1}^{\infty} \frac{t^n}{n!} A_\varepsilon^n \tag{16}$$

with the n-th power $A^n{}_\varepsilon = A_\varepsilon \times A_\varepsilon \times \cdots (n \text{ times}) \cdots \times A_\varepsilon$. If we put $S_\varepsilon(t) = \mathrm{e}^{tA\varepsilon}$ for $t > 0$, then $S_\varepsilon(0) = I$ (the identity) and $S_\varepsilon(t)\, S_\varepsilon(s) = S_\varepsilon(t+s)$ holds for any pair $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Suggested by the argument in [1] and taking discussion in [3] into account, we suppose that the semigroup $S_\varepsilon(t)$ forms a class of bounded semigroup. Indeed, we suppose that $S_\varepsilon\,(t)$ is a C_0-semigroup of negative type: namely, there exist positive constants $M_\varepsilon > 0$ and $\beta(\varepsilon) > 0$ such that

$$|||S_\varepsilon(t)|||_{\mathcal{L}(H^p\times H^{p-1})} \leqslant M_\varepsilon \cdot e^{-\beta(\varepsilon)t} \qquad \text{for} \quad t \geq 0. \tag{17}$$

Then it is easy to see that

$$A_\varepsilon u = \lim_{h\downarrow 0} \frac{S_\varepsilon(h)u - u}{h} \qquad \text{for} \quad u \in \mathcal{D}(A_\varepsilon) \tag{18}$$

where $D(A_\varepsilon)$ is determined as a subspace of $\mathcal{H}_p$ such that the limit in the right-hand side of (18) exists. On this account, for $u \in D(A_\varepsilon)$, $S_\varepsilon(t)u \in \mathrm{D}(A_\varepsilon)$ holds for any $t > 0$, and $S_\varepsilon(t)A_\varepsilon u = A_\varepsilon S_\varepsilon(t)u$ also holds for $t \geq 0$. Moreover, since $S_\varepsilon(t)u$ is continuously differentiable, we have $\frac{d}{dt} S_\varepsilon(t)u = A_\varepsilon S_\varepsilon(t)u$ for any $t > 0$. Especially, note that A is a closed operator.

LEMMA 1. (Stochastic evolution equation equivalent to (1)) *For any* $\varepsilon > 0$, *the operator* $Q_\varepsilon : H^{p-1} \to \mathcal{H}_p$ *is defined by*

$$Q_\varepsilon v := \frac{1}{\varepsilon}(0, Qv) \qquad \text{for} \quad v \in H^{p-1}. \tag{19}$$

Moreover, if we put

$$\hat{F}(u,v) = F(u), \quad \hat{Q}(u,v) = Qu \qquad \text{for} \quad (u,v) \in \mathcal{H}, \tag{20}$$

then (1) *can be rewritten into a stochastic partial differential equation of evolution type in the space* $\mathcal{H}$. *As a matter of fact, when we set*

$$z_t^\varepsilon(\omega) = z^\varepsilon(t) = (u^\varepsilon(t), v^\varepsilon(t)) = \left(u^\varepsilon(t), \frac{d}{dt}u^\varepsilon(t)\right) = (u^\varepsilon(t), \dot{u}^\varepsilon(t)), \tag{21}$$

then we have the following stochastic evolution equation :

$$dz_t^\varepsilon(\omega) = \{A_\varepsilon z_t^\varepsilon(\omega) - Q_\varepsilon \hat{Q} D\hat{F}(z_t^\varepsilon(\omega))\}dt + Q_\varepsilon d\tilde{W}_t(\omega) \quad \text{with} \quad z^\varepsilon(0) = (u_0, u_1). \tag{22}$$

Proof. The stochastic evolution equation (22) should be interpreted naturally as the following integral equation

$$z_t^\varepsilon(\omega) = z^\varepsilon(0) + \int_0^t A_\varepsilon z_s^\varepsilon(\omega)ds - \int_0^t Q_\varepsilon \hat{Q} D\hat{F}(z_s^\varepsilon(\omega))ds + \int_0^t Q_\varepsilon d\tilde{W}_t(\omega),$$

where the last term is a stochastic integral of Itô type. It is quite easy to see the following equalities.

$$\begin{aligned} dz_t^\varepsilon &= d(u^\varepsilon(t), v^\varepsilon(t)) = d(u^\varepsilon(t), \dot{u}^\varepsilon(t)), \\ \int_0^t A_\varepsilon z_s^\varepsilon ds &= \int_0^t (\dot{u}^\varepsilon(s), \frac{1}{\varepsilon}Au^\varepsilon(s) - \frac{1}{\varepsilon}\dot{u}^\varepsilon(s))ds, \\ \hat{F}(z_t^\varepsilon) &= \hat{F}(u^\varepsilon(t), \dot{u}^\varepsilon(t)) = F(u^\varepsilon(t)), \quad \text{and} \quad \hat{Q}D\hat{F}(z_t^\varepsilon) = QDF(u^\varepsilon(t)). \end{aligned}$$

Furthermore, we can get

$$Q_\varepsilon \hat{Q} D\hat{F}(z_t^\varepsilon) = \frac{1}{\varepsilon}(0, Q(\hat{Q}D\hat{F}(z_t^\varepsilon))) = \frac{1}{\varepsilon}(0, Q^2 DF(u^\varepsilon(t)))$$

and $Q_\varepsilon d\tilde{W}_t = \frac{1}{\varepsilon}(0, Qd\tilde{W}_t)$. When we decompose the stochastic evolution equation (22) and regard it as a simultaneous equation, then the first component of (22) leads to a trivial identity:

$$du^\varepsilon(t,\omega) = \dot{u}^\varepsilon(t,\omega)dt = \frac{d}{dt}u^\varepsilon(t,\omega)dt, \quad \text{a.s.}$$

On the other hand, comparison of the second components in both hand sides provides with a stochastic evolution equation

$$dv^\varepsilon(t,\omega) = \left\{\frac{1}{\varepsilon}Au^\varepsilon(t,\omega) - \frac{1}{\varepsilon}\dot{u}^\varepsilon(t,\omega) - \frac{1}{\varepsilon}Q^2 DF(u^\varepsilon(t,\omega))\right\}dt + \frac{1}{\varepsilon}Qd\tilde{W}_t(\omega).$$

On this account, we can easily rewrite it into another form in a natural way

$$\varepsilon\frac{d}{dt}\left(\frac{d}{dt}u^\varepsilon(t,\omega)\right) = Au^\varepsilon(t,\omega) - \frac{d}{dt}u^\varepsilon(t,\omega) - Q^2 DF(u^\varepsilon(t,\omega)) + Q\frac{d\tilde{W}_t}{dt}(\omega).$$

Then, taking the definition $G(x) = -Q^2 DF(x)$ into consideration, it is clear that the above expression is nothing but the reinterpretation of the original hyperbolic stochastic partial differential equation (1):

$$\varepsilon\frac{\partial^2 u^\varepsilon}{\partial t^2}(t,x,\omega) = Au^\varepsilon(t,x,\omega) - \frac{\partial u^\varepsilon}{\partial t}(t,x,\omega) + G(u^\varepsilon)(t,x,\omega) + \frac{\partial W}{\partial t}(t,x,\omega), \quad \text{a.a.-}\omega. \qquad \square$$

LEMME 2. (Stochastic evolution equation equivalent to (6)) *The stochastic heat conductive model* (6) *can be rewritten into a stochastic partial differential equation of evolution type in H, namely,*

$$dw_t(\omega) = \{Aw_t(\omega) - Q^2 DF(w_t(\omega))\}dt + Qd\tilde{W}_t(\omega) \qquad \text{with} \quad w(0) = u_0. \tag{23}$$

Proof. We have only to consider the first component this time. It goes almost similarly as in the proof of Lemma 1. $\square$

Now we are in a position to state the notion of mild solution to the evolution type equation.

DEFINITION 3. A predictable process $z_t^\varepsilon = z_t^\varepsilon(\omega) = z^\varepsilon(t,\omega)$ in the Hilbert space $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H}) = L^2(\Omega, C([0,T];\mathcal{H}))$ is a mild solution to the stochastic evolution equation (22) if $z_t^\varepsilon(\omega)$ satisfies, for $\forall t \in [0,T]$

$$z_t^\varepsilon(\omega) = S_\varepsilon(t)(u_0,u_1) - \int_0^t S_\varepsilon(t-s)Q_\varepsilon\hat{Q}D\hat{F}(z_s^\varepsilon(\omega))ds + \int_0^t S_\varepsilon(t-s)Q_\varepsilon d\tilde{W}_s(\omega), \quad \mathbb{P}-\text{a.s.} \tag{24}$$

where the space $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H})$ is the totality of all square integrable random functions consisting of Hilbert space $\mathcal{H}$-valued continuous paths.

DEFINITION 4. A predictable process $w_t = w_t(\omega) = w(t,\omega)$ in the Hilbert space $\mathcal{L}^2\mathcal{C}_t^{0,T}(H) = L^2(\Omega, C([0,T];H))$ is a *mild solution* to the stochastic evolution equation (23) if $w_t(\omega)$ satisfies, for $\forall t \in [0,T]$

$$w_t(\omega) = e^{tA}u_0 - \int_0^t e^{(t-s)A}Q^2 DF(w_s(\omega))ds + \int_0^t e^{(t-s)A}Qd\tilde{W}_s(\omega), \quad \mathbb{P}-\text{a.s.} \tag{25}$$

where the space $\mathcal{L}^2\mathcal{C}_t^{0,T}(H)$ is the totality of all square integrable random functions consisting of Hilbert space $\mathcal{H}$-valued continuous paths.

Remark 5. In the above Definition 4, the operator e^{tA} is defined as a Taylor expansion

$$e^{tA} := \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \tag{26}$$

with the n-th power $A^n := A \cdot A^{n-1}$ ($\forall n \in \mathbb{N}$) and $n \geq 2$. Moreover, it follows that $e^{0A} = I$ (the identity) and $e^{tA}e^{sA} = e^{(t+s)\,A}$ holds for any pair $(t,\, s) \in \mathbb{R}^+ \times \mathbb{R}^+$, and we assume that the semi-group e^{tA} forms a class of bounded semigroup, and also that e^{tA} is a C_0-semigroup of negative type, and there exist some positive constants $C_0 > 0$ and $\beta_0 > 0$ such that

$$|||e^{tA}|||_{\mathcal{L}(H)} \leqslant C_0 e^{-\beta_0 t} \qquad \text{for} \quad t \geq 0. \tag{27}$$

Then it is easy to see that $Au = \lim_{h\downarrow 0} \frac{1}{h}(e^{hA}u - u)$ holds for any $u \in \text{Dom}\,(A)$. □

Now we shall introduce our main results in this paper. The first result (Theorem 6) treats the existence and uniqueness of solutions to the stochastic evolution equation (22), which just corresponds to the original stochastic hyperbolic partial differential equation (1) describing the random wave phenomenon. The second result (Theorem 7) deals with the existence and uniqueness theorem for solutions to the stochastic evolution equation (23), which is associated with the stochastic parabolic partial differential equation (6) expressing the so-called heat conductive model with a random noise. The last main result (Theorem 8) is devoted to a Smoluchowski-Kramers approximation problem, where the random function expressing a random wave phenomenon converges in probability sense to the random function associated with a heat conductive model.

THEOREM 6. *If $DF : H \to H$ is Lipschitz continuous, then the stochastic evolution equation* (22) *of gradient type has a unique solution $z^{\varepsilon}(t, x, \omega)$* in $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H})$.

COROLLARY 6. *If $DF : H \to H$ is Lipschitz continuous, then the stochastic evolution equation* (22) *of gradient type has a unique mild solution $z^{\varepsilon}(t, x, \omega)$* in $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H})$.

THEOREM 7. *If $DF : H \to H$ is Lipschitz continuous, then the stochastic evolution equation* (23) *of gradient type has a unique solution $w(t, x, \omega)$* in $\mathcal{L}^2\mathcal{C}_t^{0,T}(H)$.

COROLLARY 7. *If $DF : H \to H$ is Lipschitz continuous, then the stochastic evolution equation* (23) *of gradient type has a unique mild solution $w(t, x, \omega)$* in $\mathcal{L}^2\mathcal{C}_t^{0,T}(H)$.

THEOREM 8. (Smoluchowski-Kramers approximation) *Let $u^{\varepsilon}(t, x, \omega)$ be the solution to the stochastic partial differential equation* (1), *and let $w(t, x, \omega)$ be the solution to the stochastic partial differential equation* (6) *respectively. Then, as ε tends to zero, the $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H})$-valued predictable process $u^{\varepsilon}(t, x, \omega)$ satisfying the random wave phenomenon* (1) *converges in probability to the $\mathcal{L}^2\mathcal{C}_t^{0,T}$*(H)*-valued predictable process $w(t, x, \omega)$ satisfying the random heat conductive model* (6), *namely, for any $T > 0$ and $\eta > 0$*

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|u^{\varepsilon}(t,\omega)-w(t,\omega)\|_H>\eta\right)=0 \tag{28}$$

holds, where $u^{\varepsilon}(t,\omega)=\Pi_1 z^{\varepsilon}(t,\omega)$ and Π_1 *is the projection operator of* $(u,v)\in H\times H^{-1}$ *into the first component.*

Remark. If the stochastic equation (22) has a unique solution in $\mathcal{L}^2\mathcal{C}_t^{0,T}(H)$, then it is easily proven that the stochastic equation (22) has a unique mild solution in the same function space $\mathcal{L}^2\mathcal{C}_t^{0,T}(H)$. That is to say, Corollary 6 yields directly from Theorem 6. Likewise, a similar situation is true about the relation between Theorem 7 and Corollary 7.

3. Sketch of proofs

First of all, we shall consider proving Theorem 6. By virtue of the fundamental theory on resolvent and spectrum of the linear operator in Functional Analysis, when A is a linear operator of negative type on a Banach space, i.e., $A\in\mathcal{L}(X)$, $\rho(A)$ denotes the resolvent set of A (where the operator $zI-A$ is a one-to-one mapping and $\exists\ (zI-A)^{-1}$ is a linear bounded operator for $z\in\rho(A)$), and $R_A(z)=(zI-A)^{-1}$ is the resolvent of A, then we have

$$\frac{1}{\lambda-A}=\int_0^{\infty}e^{-\lambda t}e^{tA}dt.$$

Hence, it follows immediately that the expression

$$(-A_{\varepsilon})^{-1}=\int_0^{\infty}S_{\varepsilon}(t)dt \tag{29}$$

is valid. Moreover, since we have $A_{\varepsilon}(u,v)=(v,\frac{1}{\varepsilon}Au-\frac{1}{\varepsilon}v)$, it is easy to see that A_{ε} can be expressed formally as

$$A_{\varepsilon}=\begin{pmatrix}0 & I\\ \frac{1}{\varepsilon}A & -\frac{1}{\varepsilon}I\end{pmatrix},$$

consequently, we can have

$$(-A_{\varepsilon})^{-1}=\int_0^{\infty}S_{\varepsilon}(t)dt=\begin{pmatrix}(-A)^{-1} & \varepsilon(-A)^{-1}\\ -I & 0\end{pmatrix}.$$

Noting that $(AB)^*=B^*A^*$ holds for operators A and B, we have $S_{\varepsilon}(t)Q_{\varepsilon}Q_{\varepsilon}^*S_{\varepsilon}^*(t)=S_{\varepsilon}(t)Q_{\varepsilon}(S_{\varepsilon}(t)(t)Q_{\varepsilon})^*$. Suggested by [1], when we define

$$C_{\varepsilon}:=\int_0^{\infty}S_{\varepsilon}(t)Q_{\varepsilon}Q_{\varepsilon}^*S_{\varepsilon}^*(t)dt, \tag{30}$$

then we have $C_{\varepsilon}(u,v)=\frac{1}{2}((-A)^{-1}Q^2u,\frac{1}{\varepsilon}(-A)^{-1}Q^2v)$ for $(u,v)\in\mathcal{H}$. Therefore, we get a new expression

$$2A_{\varepsilon}C_{\varepsilon}D\hat{F}(u,v)=(0,-\frac{1}{\varepsilon}Q^2DF(u))=-Q_{\varepsilon}\hat{Q}D\hat{F}(u,v)\qquad\text{for}\quad(u,v)\in\mathcal{H}. \tag{31}$$

When we employ the above-mentioned expression, then our stochastic evolution equation (22) can be rewritten into another form

$$dz_t^\varepsilon(\omega) = \{A_\varepsilon z_t^\varepsilon(\omega) + 2A_\varepsilon C_\varepsilon D\hat{F}(z_t^\varepsilon(\omega))\}dt + Q_\varepsilon d\tilde{W}_t(\omega) \quad \text{with} \quad z^\varepsilon(0) = (u_0, u_1). \tag{32}$$

Consequently, it suffices to verify the following proposition, in order to prove Theorem 6.

PROPOSITION 9. *If $DF : H \to H$ is Lipschitz continuous, then the stochastic evolution equation* (32) *has a unique solution $z^\varepsilon(t, x, \omega)$* in $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H})$.

Proof. The stochastic evolution equation (32) should be interpreted as an integral equation like this:

$$z_t^\varepsilon(\omega) = z^\varepsilon(0) + \int_0^t A_\varepsilon z_s^\varepsilon(\omega)ds + 2\int_0^t A_\varepsilon C_\varepsilon \cdot D\hat{F}(z_s^\varepsilon(\omega))ds + \int_0^t Q_\varepsilon d\tilde{W}_s(\omega). \tag{33}$$

We are going to make use of the Picard iteration method and resort to the fixed point theorem in order to show the existence of the solution to (33). So we consider the following iteration scheme: that is to say, in what follows, as abuse of notation, we shall omit the superscript ε for simplicity, and define

$$z_0^\varepsilon(\omega) := z^\varepsilon(0) = (u_0, u_1) \in \mathcal{H} \tag{34}$$

and

$$z_t^{n+1} := z^\varepsilon(0) + \int_0^t A_\varepsilon z_s^n ds + 2\int_0^t A_\varepsilon C_\varepsilon \cdot D\hat{F}(z_s^n)ds + \int_0^t Q_\varepsilon d\tilde{W}_s(\omega) \tag{35}$$

for $n \in \mathbb{N}0 := \mathbb{N} \cup \{0\}$. For $z = (u, v) \in \mathcal{H}$, we use the norm and symbols

$$\|z\|_{\mathcal{H}} := \|z\|_{0,-1} = \|(u, v)\|_{H\times H^{-1}} = \sqrt{\|u\|_H^2 + \|v\|_{H^{-1}}^2} = \left\{\|u\|_0^2 + \|v\|_{-1}^2\right\}^{1/2}. \tag{36}$$

We may apply the triangular inequality together with the Minkowskii inequality to obtain

$$\begin{aligned} &\|z_t^{n+1} - z_t^n\|_{\mathcal{H}} \\ &\leqslant \int_0^t \|A_\varepsilon(z_s^n - z_s^{n-1})\|_{\mathcal{H}} ds + 2\int_0^t \|A_\varepsilon C_\varepsilon(D\hat{F}(z_s^n) - D\hat{F}(z_s^{n-1}))\|_{\mathcal{H}} ds. \end{aligned} \tag{37}$$

Next we shall estimate below each integrand term in the above (37). In so doing, we shall use the notation

$$z_t^n(\omega) = z^n(t, \omega) = (u^n(t, \omega), v^n(t, \omega)) = (u^n(t, \omega), \dot{u}^n(t, \omega)) \in \mathcal{H} = H \times H^{-1}$$

just for convention to proceed computing. To estimate the integrand $\|A_\varepsilon(z_s^n - z_s^{n-1})\|_{\mathcal{H}}$ in the first integral of (37), it suffices to estimate the operator norm $|||A_\varepsilon|||_{\mathcal{L}(\mathcal{H})}$. And also the second integrand in (37) can be estimated dominantly by

$$|||A_\varepsilon|||_{\mathcal{L}(\mathcal{H})} |||C_\varepsilon|||_{\mathcal{L}(\mathcal{H})} \|D\hat{F}(z_s^n - z_s^{n-1})\|_{\mathcal{H}}. \tag{38}$$

As a matter of fact, it follows immediately that

$$
\begin{aligned}
|||A_\varepsilon|||_{\mathcal{L}(\mathcal{H})} &= \sup_{\|(u,v)\|_{\mathcal{H}}\neq 0} \frac{\|A_\varepsilon(u,v)\|_{\mathcal{H}}}{\|(u,v)\|_{\mathcal{H}}} = \sup_{\|(u,v)\|_{\mathcal{H}}=1} \|A_\varepsilon(u,v)\|_{\mathcal{H}} \\
&= \sup \left\| \left(v, \frac{1}{\varepsilon}Au - \frac{1}{\varepsilon}v \right) \right\|_{H\times H^{-1}} = \sup \sqrt{\|v\|_H^2 + \left\| \frac{1}{\varepsilon}Au - \frac{1}{\varepsilon}v \right\|_{H^{-1}}^2} \\
&\leqslant \sup \sqrt{C_0\|v\|_{H^{-1}}^2 + \left(\|\frac{1}{\varepsilon}Au\|_{H^{-1}} + \|\frac{1}{\varepsilon}v\|_{H^{-1}} \right)^2}. \qquad (39)
\end{aligned}
$$

While, since we have

$$
\|\frac{1}{\varepsilon}Au\|_{H^{-1}} \leqslant \frac{1}{\varepsilon}|||A|||_{\mathcal{L}(H;H^{-1})}\|u\|_H \leqslant \frac{1}{\varepsilon}M\|u\|_0
$$

by the boundedness of the operator A, we would estimate (39) further and can get easily

$$
\begin{aligned}
\|A_\varepsilon(u,v)\|_{\mathcal{H}} &\leqslant \sqrt{\frac{2M^2}{\varepsilon^2}\|u\|_0^2 + \left(C_0 + \frac{2}{\varepsilon^2} \right)\|v\|_{-1}^2} \\
&\leqslant \sqrt{K} \cdot \sqrt{\|u\|_0^2 + \|v\|_{-1}^2} = \sqrt{K}\|(u,v)\|_{\mathcal{H}}, \qquad (40)
\end{aligned}
$$

where we made use of a trivial inequality : $(a + b)^2 \leqslant 2\,(a^2 + b^2)$ and put $K := \max\,(2M^2/\varepsilon^2,\, C_0 + 2/\varepsilon^2)$. Next we are going to estimate the operator norm of C_ε. To do so, we need several lemmas.

LEMMA 10. *There exists some positive constant $M_q > 0$ such that*

$$
\|(-A)^{-1}Q^2u\|_0 \leqslant \frac{C_0M_q^2}{\beta_0}\|u\|_0 < \infty \qquad (41)
$$

holds as long as u lies in the space H.

Proof of Lemma 10. By a similar reason for the integral representation (29), we have

$$
(-A)^{-1} = \int_0^\infty e^{tA}dt, \qquad (42)
$$

hence it follows immediately form (27) that

$$
|||(-A)^{-1}|||_{\mathcal{L}(\mathcal{H})} \leqslant \int_0^\infty |||e^{tA}|||_{\mathcal{L}(\mathcal{H})}dt \leqslant C_0\int_0^\infty e^{-\beta_0 t}dt = \frac{C_0}{\beta_0} < \infty. \qquad (43)
$$

Moreover, since the operator Q is bounded in H, we have $|||Q|||_{\mathcal{L}(H)} \leqslant M_q < \infty$ for some positive constant M_q . Therefore we obtain

$$
\|(-A)^{-1}Q^2u\|_0 \leqslant |||(-A)^{-1}|||_{\mathcal{L}(\mathcal{H})}|||Q|||_{\mathcal{L}(H)}^2\|u\|_0 \leqslant \frac{C_0M_q^2}{\beta_0}\|u\|_0 < \infty,
$$

as far as $u \in H$. □

LEMMA 11. *Similarly, as far as v lives in H^{-1}, we have*

$$
\|\frac{1}{\varepsilon}(-A)^{-1}Q^2v\|_{-1} \leqslant \frac{C_0M_q^2}{\varepsilon\beta_0}\|v\|_{-1} < \infty. \qquad (44)
$$

Proof. It goes almost similarly as in the proof of Lemma 10 and is also easy, hence omitted. □

LEMMA 12. *There exists a positive constant $K_0 > 0$ such that*

$$|||C_\varepsilon|||_{\mathcal{L}(\mathcal{H})} \leqslant \frac{K_0 C_0 M_q^2}{2\beta_0} < \infty. \tag{45}$$

Proof. By the definition (30) of C_ε, an application of Lemma 10 and Lemma 11 yields to

$$\begin{aligned}
|||C_\varepsilon|||_{\mathcal{L}(\mathcal{H})} &= \sup_{\|(u,v)\|_{\mathcal{H}}=1} \|C_\varepsilon(u,v)\|_{\mathcal{H}} = \left|\left|\left| \int_0^\infty S_\varepsilon(t) Q_\varepsilon Q_\varepsilon^* S_\varepsilon^*(t) dt \right|\right|\right|_{\mathcal{L}(\mathcal{H})} \\
&= \sup_{\|(u,v)\|_{\mathcal{H}}=1} \left\| \frac{1}{2}\left((-A)^{-1} Q^2 u, \frac{1}{\varepsilon}(-A)^{-1} Q^2 v \right) \right\|_{\mathcal{H}} \\
&\leqslant \frac{1}{2} \sup_{\|(u,v)\|_{\mathcal{H}}=1} \sqrt{\|(-A)^{-1}Q^2 u\|_0^2 + \|\frac{1}{\varepsilon}(-A)^{-1}Q^2 v\|_{-1}^2} \\
&\leqslant \frac{1}{2} \sup_{\|(u,v)\|_{\mathcal{H}}=1} \sqrt{\left(\frac{C_0 M_q^2}{\beta_0}\right)^2 \|u\|_0^2 + \left(\frac{C_0 M_q^2}{\varepsilon\beta_0}\right)^2 \|v\|_{-1}^2} \\
&= \frac{1}{2}\frac{C_0 M_q^2}{\beta_0} \sup_{\|(u,v)\|_{\mathcal{H}}=1} \sqrt{\|u\|_0^2 + \frac{1}{\varepsilon^2}\|v\|_{-1}^2} \\
&\leqslant \frac{K_0 C_0 M_q^2}{2\beta_0} \sup \|(u,v)\|_{\mathcal{H}},
\end{aligned} \tag{46}$$

where we put $K_0 := \sqrt{\max(1, \varepsilon^{-2})}$. □

By taking advantage of the estimation result obtained in Lemma 12 we can proceed computing the inequality (37). In fact, we obtain

$$\begin{aligned}
&\|z_t^{n+1} - z_t^n\|_{\mathcal{H}} \\
&\leqslant \int_0^t \|A_\varepsilon(z_s^n - z_s^{n-1})\|_{\mathcal{H}} ds + 2\int_0^t \|A_\varepsilon C_\varepsilon(D\hat{F}(z_s^n) - D\hat{F}(z_s^{n-1}))\|_{\mathcal{H}} ds \\
&\leqslant \int_0^t |||A_\varepsilon||| \cdot \|(z_s^n - z_s^{n-1})\|_{\mathcal{H}} ds + 2\int_0^t |||A_\varepsilon||| \cdot |||C_\varepsilon||| \cdot \|D\hat{F}(z_s^n) - D\hat{F}(z_s^{n-1})\|_{\mathcal{H}} ds.
\end{aligned} \tag{47}$$

While, noting together with (5) that

$$\begin{aligned}
&\|D\hat{F}(z_s^n) - D\hat{F}(z_s^{n-1})\|_{\mathcal{H}} = \|DF(u_s^n) - DF(u_s^{n-1})\|_H \\
&\leqslant C\|u_s^n - u_s^{n-1}\|_0 \leqslant C\|z_s^n - z_s^{n-1}\|_{\mathcal{H}}
\end{aligned} \tag{48}$$

where we have employed the definition (20). Hence we readily obtain

$$\begin{aligned}
\|z_t^{n+1} - z_t^n\|_{\mathcal{H}} \leqslant &\int_0^t \sqrt{K}\|z_t^n - z_t^{n-1}\|_{\mathcal{H}} ds \\
&+ 2\int_0^t \sqrt{K}\frac{K_0 C_0 M_q^2}{2\beta_0} \cdot C\|z_t^n - z_t^{n-1}\|_{\mathcal{H}} ds
\end{aligned} \tag{49}$$

Thus we finally attain that $\sup_{0\leqslant t\leqslant T} \|z_t^{n+1} - z_t^n\|_{\mathcal{H}} \leqslant C_1 \sup_{0\leqslant t\leqslant T} \|z_t^n - z_t^{n-1}\|_{\mathcal{H}}$. Here we put

$C_1 = \sqrt{K}T(1 + K_0C_0M_q^2C/\beta_0)$. Note that when we carefully select the parameter ε as a number large enough such that $K \ll 1$, we can choose a sufficiently large number $\beta_0 \gg 1$ such that $1 > \sqrt{K}T$ and $C_1 < 1$. This gives more or less a restriction on A, but these choices of parameters are possible. So that, it is guaranteed that our iteration scheme provides with a contraction mapping Φ in $C([0, T]; \mathcal{H})$. This means that the stochastic equation (32) has a solution

$$z_t(\omega) \in C([0,T];\mathcal{H}), \quad \mathbb{P}-\text{a.s.} \tag{50}$$

such that $\sup_{0\leqslant t\leqslant T}\|z_t(\omega)\|_{\mathcal{H}} < \infty$ with probability one. Furthermore, we have

$$\mathbb{E}\|z^\varepsilon(0)\|_{\mathcal{H}}^2 = \mathbb{E}(\|u_0\|_0^2 + \|u_1\|_{-1}^2) < \infty \tag{51}$$

as far as $z^\varepsilon(0) = (u_0, u_1)$ lies in the space $\mathcal{H}$. While, for the stochastic integral equation (33), by an easy inequality $(a + b + c + d)^2 \leqslant 4(a^2 + b^2 + c^2 + d^2)$ and the triangular inequality we can rewrite the term $\mathbb{E}\|z_t^\varepsilon(\omega)\|_{\mathcal{H}}^2$ into

$$\begin{aligned}\mathbb{E}\|z_t\|_{\mathcal{H}}^2 \leqslant 4\mathbb{E}\|z(0)\|_{\mathcal{H}}^2 + 4\mathbb{E}\left\|\int_0^t A_\varepsilon z_s ds\right\|_{\mathcal{H}}^2 \\ + 4\mathbb{E}\left\|\int_0^t 2A_\varepsilon C_\varepsilon D\hat{F}(z_s)ds\right\|_{\mathcal{H}}^2 + 4\mathbb{E}\left\|\int_0^t Q_\varepsilon d\tilde{W}_s\right\|_{\mathcal{H}}^2.\end{aligned} \tag{52}$$

In addition, noting that

$$\begin{aligned}\mathbb{E}\left\|\int_0^t A_\varepsilon z_s ds\right\|_{\mathcal{H}}^2 &\leqslant \mathbb{E}\left(\int_0^t \|A_\varepsilon z_s\|_{\mathcal{H}} ds\right)^2 \leqslant T^2 \cdot \mathbb{E}\int_0^t |||A_\varepsilon|||_{\mathcal{L}(\mathcal{H})}^2 \|z_s\|_{\mathcal{H}}^2 ds \\ &\leqslant T^2K \cdot \mathbb{E}\int_0^t \|z_s\|_{\mathcal{H}}^2 ds \leqslant T^2K \cdot \mathbb{E}[\sup_{0\leqslant t\leqslant T}\|z_t\|_{\mathcal{H}}^2],\end{aligned} \tag{53}$$

$$\begin{aligned}&\mathbb{E}\left\|\int_0^t 2A_\varepsilon C_\varepsilon D\hat{F}(z_s)ds\right\|_{\mathcal{H}}^2 \leqslant 4\mathbb{E}\left(\int_0^t \|A_\varepsilon C_\varepsilon \cdot D\hat{F}(z_s)\|_{\mathcal{H}} ds\right)^2 \\ &\leqslant 4\mathbb{E}\left(\int_0^t |||A_\varepsilon||| \cdot |||C_\varepsilon||| \cdot \|D\hat{F}(z_s)\|_{\mathcal{H}} ds\right)^2 \leqslant 4K\left(\frac{K_0C_0M_q^2}{2\beta_0}\right)^2 M_1^2C^2 \cdot \mathbb{E}\left(\int_0^t \|z_s\|_{\mathcal{H}} ds\right)^2 \\ &\leqslant K\left(\frac{K_0C_0M_q^2M_1CT}{2\beta_0}\right)^2 \mathbb{E}[\sup_{0\leqslant t\leqslant T}\|z_t\|_{\mathcal{H}}^2],\end{aligned} \tag{54}$$

and taking Itô's isometry [18] into consideration in

$$\mathbb{E}\left\|\int_0^t Q_\varepsilon d\tilde{W}_s\right\|_{\mathcal{H}}^2 = \mathbb{E}\int_0^t \|Q_\varepsilon Q_\varepsilon^*\| ds, \tag{55}$$

we can verify

$$\mathbb{E}[\sup_{0\leqslant t\leqslant T}\|z_t^{n+1}(\omega) - z_t^n(\omega)\|_{\mathcal{H}}^2] \leqslant C_1\mathbb{E}[\sup_{0\leqslant t\leqslant T}\|z_t^n(\omega) - z_t^{n-1}(\omega)\|_{\mathcal{H}}^2] \tag{56}$$

and

$$
\begin{aligned}
\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|z_s(\omega)-\tilde{z}_s(\omega)\|_{\mathcal{H}}^2]\leqslant C_2\int_0^t \mathbb{E}[\sup_{0\leqslant q\leqslant s}\|z_q(\omega)-\tilde{z}_q(\omega)\|_{\mathcal{H}}^2]ds \\
+C_3\mathbb{E}\int_0^t \sup_{0\leqslant q\leqslant s}\|z_q(\omega)-\tilde{z}_q(\omega)\|_{\mathcal{H}}^2 ds,
\end{aligned}
\tag{57}
$$

where we have employed the Cauchy-Schwarz inequality in deriving (53) and (54). Consequently, we conclude from (56) that the solution $z_t(\omega)$ exists in the Hilbert space $\mathcal{L}^2\mathcal{C}_t^{0,T}(\mathcal{H})=L^2(\Omega,C([0,T];\mathcal{H}))$, and also from (57) that the uniqueness of solutions $z_t(\omega)$ is valid, by making use of the Gronwall inequality [17] for (57). This finishes the proof of Proposition 9. On this account, this also completes the proof of Theorem 6.

As for the proof of Theorem 7, likewise as in the above proof of Theorem 6, we shall consider rewriting our stochastic evolution equation (23). If we define an operator C as

$$
C:=\int_0^\infty e^{tA}QQ^*e^{tA^*}dt=\frac{1}{2}(-A)^{-1}Q^2,
\tag{58}
$$

then we have $2ACDF\,(w_t(\omega))=-Q^2DF\,(w_t(\omega))$. An application of the above newly-derived expression for the evolution equation (23) leads to

$$
dw_t(\omega)=\{Aw_t(\omega)+2AC\cdot DF(w_t(\omega))\}dt+Qd\tilde{W}_t(\omega)\quad\text{with}\quad w(0)=u_0.
\tag{59}
$$

Hence it suffices to verify the following proposition, in order to prove Theorem 7.

PROPOSITION 13. *If $DF: H\to H$ is Lipschitz continuous, then the stochastic evolution equation* (59) *has a unique solution $w(t,x,\omega)$* in $\mathcal{L}^2\mathcal{C}_t^{0,T}(H)$.

Proof. It goes almost similarly as the proof of Proposition 9. □

The rest of the paper is devoted to the proof of Theorem 8. We shall use the solutions to stochastic evolution equations (32) and (59). In particular, for $z^{\varepsilon}{}_t(\omega)=(u^{\varepsilon}{}_t(\omega),v^{\varepsilon}{}_t(\omega))=(u^{\varepsilon}{}_t(\omega),\dot{u}^{\varepsilon}{}_t(\omega))$, we consider the equation

$$
du_t^\varepsilon=\left(\frac{1}{\varepsilon}Au_t^\varepsilon-\frac{1}{\varepsilon}v_t^\varepsilon\right)dt-\frac{1}{\varepsilon}Q^2DF(u_t^\varepsilon)dt+\frac{1}{\varepsilon}Qd\tilde{W}_t.
\tag{60}
$$

Hence, if we think of a difference term $u^{\varepsilon}{}_t-w_t$ for $u^{\varepsilon}{}_t$ as the projected term of $z^{\varepsilon}{}_t$ by the projection Π_1 onto the first component, then we have the following equation

$$
d(u_t^\varepsilon-w_t)=\left(\frac{1}{\varepsilon}Au_t^\varepsilon-\frac{1}{\varepsilon}\dot{u}_t^\varepsilon-Aw_t\right)dt+\left(-\frac{1}{\varepsilon}Q^2DF(u_t^\varepsilon)+Q^2DF(w_t)\right)dt+\left(\frac{1}{\varepsilon}Q-Q\right)d\tilde{W}_t.
\tag{61}
$$

We need the following lemma.

LEMMA 14. (Partial derivatives [18]) *Let $p\geq 1$. The function F on a Hilbert space H is defined by $F(\cdot):=\|\cdot\|^{2}{}_{H}{}^{p}: H\to\mathbb{R}$. Then F is continuous and twice differentiable, and there exist the first partial derivative F_x and the second partial derivative F_{xx}. Actually, those derivatives are allowed to possess the following explicit representations* :

$$(F_x(x))(h) = 2p\|x\|^{2(p-1)}\langle x, h\rangle_H \qquad (h \in H) \tag{62}$$

$$(F_{xx}(x))(h, g) = 4p(p-1)\|x\|_H^{2(p-2)}\langle x, h\rangle_H\langle x, g\rangle_H + 2p\|x\|_H^{2(p-1)}\langle h, g\rangle_H \quad (h, g \in H). \tag{63}$$

LEMMA 15. (A version of Itô's formula) *Let $X = (X_t)$ be a Hilbert space-valued Itô process defined by the stochastic differential equation in infinite dimensions $dX_t = A(X_t)dt + G(X_t)$ $dW^Q{}_t$. If we set $F(x) = \|x\|^2{}_H$ for any $x \in H$, then we can get*

$$dF(X_t) \equiv d\|X_t\|_H^2 = 2\langle X_t, A(X_t)\rangle_H dt + 2\langle X_t, G(X_t)dW_t^Q\rangle_H + \langle G(X_t), G(X_t)\rangle_H \mathrm{tr}(QQ^*)dt. \tag{64}$$

Proof. By virtue of the Itô formula for functions on a Hilbert space [18], we have $dF(X_t)=$ $\langle F_x(X_t), dX_t\rangle_H + \frac{1}{2}F_{xx}(X_t)(G(X_t), G^*(X_t))\mathrm{tr}(QQ^*)dt$. If we substitute the stochastic differential equation of X_t for the above stochastic integral part $\langle F_x(X_t), dX_t\rangle_H$, then the result equation (64) yields from Lemma 14, where we have only to note that $F_x(X_t)(h) = 2\langle X_t, h\rangle_H$ and $F_{xx}(X_t)$ $(h, g) = 2\langle h, g\rangle_H$. □

By employing the Chebyshev inequality we readily obtain

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|u_t^\varepsilon(\omega) - w_t(\omega)\|_H > \eta\right) \leqslant \frac{1}{\eta^2}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\|u_t^\varepsilon - w_t\|_H^2\right]. \tag{65}$$

That's why we need to estimate the term $\|u^\varepsilon{}_t - w_t\|^2{}_H$. We may apply Lemma 15 for the term to obtain

$$\begin{aligned}\|u_t^\varepsilon - w_t\|_H^2 = {} & 2\int_0^t \langle u_s^\varepsilon - w_s, \frac{1}{\varepsilon}(A - \frac{d}{dt})u_s^\varepsilon - Aw_s\rangle ds \\ & + 2\int_0^t \langle u_s^\varepsilon - w_s, Q^2DF(w_s) - \frac{1}{\varepsilon}Q^2DF(u_s^\varepsilon)\rangle ds \\ & + \frac{2(1-\varepsilon)}{\varepsilon}\int_0^t \langle u_s^\varepsilon - w_s, Qd\tilde{W}_t\rangle + \left(\frac{1-\varepsilon}{\varepsilon}\right)^2\int_0^t \mathrm{tr}(QQ^*)dt.\end{aligned} \tag{66}$$

Each integral term will be estimated by the following series of lemmas.

LEMMA 16. *There exist some constants $C_1 \equiv C_1(\varepsilon)$, $C_2 \equiv C_2(\varepsilon, T)$ and $C_3 \equiv C_3(\varepsilon, T)$ such that*

$$\begin{aligned}&\int_0^t \langle u_s^\varepsilon - w_s, \frac{1}{\varepsilon}(A - \frac{d}{dt})u_s^\varepsilon - Aw_s\rangle ds \\ &\qquad \leqslant C_1\int_0^t \|u_s^\varepsilon - w_s\|_H^2 ds + C_2\sup_{0\leqslant t\leqslant T}\|v_t^\varepsilon\|_{-1}^2 + C_3\sup_{0\leqslant t\leqslant T}\|w_t\|_H^2.\end{aligned} \tag{67}$$

Proof. A direct computation leads to the desired result. Actually, by employing the Cauchy-Schwarz-Bunyakovskii type inequality and the triangular inequality we can estimate the integral term dominantly by

$$
\begin{aligned}
&\int_0^t \langle u_s^\varepsilon - w_s, \frac{1}{\varepsilon}(A - \frac{d}{dt})u_s^\varepsilon - Aw_s\rangle ds \\
&\leqslant \int_0^t \|u_s^\varepsilon - w_s\|_H \cdot \left\| \frac{1}{\varepsilon}(A - \frac{d}{dt})u_s^\varepsilon - Aw_s \right\|_{H^{-1}} ds \\
&\leqslant \int_0^t \|u_s^\varepsilon - w_s\|_H \cdot \frac{1}{\varepsilon}\|A(u_s^\varepsilon - w_s)\|_{H^{-1}} ds + \int_0^t \|u_s^\varepsilon - w_s\|_H \cdot \left(\frac{1}{\varepsilon} - 1\right) |||A||| \cdot \|w_s\| ds \\
&\qquad\qquad + \frac{1}{\varepsilon}\int_0^t \frac{1}{2}\left(\|u_s^\varepsilon - w_s\|_H^2 + \left\| \frac{d}{dt}u_s^\varepsilon \right\|_{H^{-1}}^2 \right) ds \\
&\leqslant C_1 \int_0^t \|u_s^\varepsilon - w_s\|_H^2 ds + C_2 \sup_{0 \leqslant t \leqslant T} \|v_t^\varepsilon\|_{-1}^2 + C_3 \sup_{0 \leqslant t \leqslant T} \|w_t\|_H^2,
\end{aligned}
$$

where we made use of a trivial inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$, took advantage of an estimate $|||\mathrm{A}|||_{\mathcal{L}(H^{-1})} \leqslant M_1$, and put

$$
C_1 := \frac{3M_1 + 1}{2\varepsilon} - \frac{M_1}{2}, \quad C_2 := \frac{T}{2\varepsilon} \quad \text{and} \quad C_3 := \frac{M_1 T(1-\varepsilon)}{2\varepsilon}.
$$

□

LEMMA 17. *There exist some constants $C_4 \equiv C_4(\varepsilon)$ and $C_5 \equiv C_5(\varepsilon, T)$* such that

$$
\begin{aligned}
&\int_0^t \langle u_s^\varepsilon - w_s, Q^2 DF(w_s) - \frac{1}{\varepsilon}Q^2 DF(u_s^\varepsilon)\rangle ds \\
&\qquad\qquad \leqslant C_4 \int_0^t \|u_s^\varepsilon - w_s\|_H^2 ds + C_5 \sup_{0 \leqslant t \leqslant T} \|u_t^\varepsilon\|_H^2.
\end{aligned} \tag{68}
$$

Proof. We may apply the Schwarz inequality for the integrand to obtain

$$
\begin{aligned}
&\int_0^t \langle u_s^\varepsilon - w_s, Q^2 DF(w_s) - \frac{1}{\varepsilon}Q^2 DF(u_s^\varepsilon)\rangle ds \\
&\qquad\qquad \leqslant \int_0^t \|u_s^\varepsilon - w_s\|_H \cdot \|Q^2 DF(w_s) - \frac{1}{\varepsilon}Q^2 DF(u_s^\varepsilon)\|_H ds.
\end{aligned} \tag{69}
$$

While, we decompose the second norm of the integrand in (69) and rewrite it into another form by using the Lipschitz continuity (5):

$$
\begin{aligned}
&\|Q^2 DF(w_s) - \frac{1}{\varepsilon}Q^2 DF(u_s^\varepsilon)\|_H \\
&\leqslant \|Q^2 DF(w_s) - Q^2 DF(u_s^\varepsilon)\|_H + \|Q^2 DF(u_s^\varepsilon) - \frac{1}{\varepsilon}Q^2 DF(u_s^\varepsilon)\|_H \\
&\leqslant |||Q|||^2 \cdot \|DF(w_s) - DF(u_s^\varepsilon)\| + \left\| \left(1 - \frac{1}{\varepsilon}\right) Q^2 DF(u_s^\varepsilon) \right\|_H \\
&\leqslant MC \cdot \|u_s^\varepsilon - w_s\|_H + \left(\frac{1}{\varepsilon} - 1\right) MC \cdot \|u_s^\varepsilon\|_H.
\end{aligned} \tag{70}
$$

By substituting (70) for (69), an application of the easy inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$ provides with the desired inequality (68), where we have only to put

$$C_4 := \frac{MC}{2}\left(1+\frac{1}{\varepsilon}\right), \qquad C_5 := \frac{(1-\varepsilon)MCT}{2\varepsilon} \quad \text{with} \quad M := M_q^2.$$

□

We apply Lemma 16 and Lemma 17 to the equality (66) and rearrange those terms. After all, summing up, we readily obtain

$$\begin{aligned} \|u_t^\varepsilon - w_t\|_H^2 \leqslant K_1(\varepsilon)\int_0^t \|u_s^\varepsilon - w_s\|_H^2 ds + K_2(\varepsilon)\cdot \sup_{0\leqslant t\leqslant T}\|v_t^\varepsilon\|_{-1}^2 + K_3(\varepsilon)\cdot \sup_{0\leqslant t\leqslant T}\|w_t\|_H^2 \\ + K_4(\varepsilon)\cdot \sup_{0\leqslant t\leqslant}\|u_t^\varepsilon\|_H^2 + \left(\frac{1-\varepsilon}{\varepsilon}\right)^2\int_0^t \mathrm{tr}(QQ^*)ds + \frac{2(1-\varepsilon)}{\varepsilon}\int_0^t \langle u_s^\varepsilon - w_s, Qd\tilde{W}_s\rangle. \end{aligned} \tag{71}$$

Then, taking the supremum over the interval [0, T] and also taking the expectation $\mathbb{E}[\cdot]$ over the probability space (Ω, $\mathcal{F}$, $\mathbb{P}$), we can finally get

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\|u_t^\varepsilon - w_t\|_H^2\right] \leqslant K_5(\varepsilon)\int_0^T \mathbb{E}\left[\sup_{0\leqslant t\leqslant s}\|u_t^\varepsilon - w_t\|_H^2\right] ds + K_6(\varepsilon) \tag{72}$$

where the stochastic integral term vanishes under the expectation because the stochastic integral of Itô type is a ($\mathcal{F}_t$)-martingale and the expectation of martingale becomes null by the theory of martingales. Hence it follow immediately that

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\|u_t^\varepsilon - w_t\|_H^2\right] \leqslant K_6(\varepsilon)e^{K_5(\varepsilon)T}, \qquad \text{for} \quad \forall T > 0. \tag{73}$$

There may be two cases: one is the case where the constant $K_5(\varepsilon)$ diverges to minus infinity as the parameter ε tends to zero, and the other is the case where $K_6(\varepsilon)$ vanishes by the passage to the limit $\varepsilon \to 0$. However, here the first case is impossible, and the second case is possible. This implies that the assertion of Theorem 8 is established and the expression (28) is valid. This finishes the proof of Theorem 8.

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