

An analytic approach to O'Hara's energy:
Discretization, variational formulae and estimates
(解析的手法による O'Hara エネルギーの研究：
離散化，変分公式と評価)

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1 Introduction

In his papers [26, 27, 28], O'Hara proposed several energies for knots to determine the *canonical shape* in a given knot class. In order to describe this energy, let $\mathbf{f} : \mathbb{R}/\mathbb{L}\mathbb{Z} \ni s \mapsto \mathbf{f}(s)$ be an arc-length parametrization of a knot, or more generally, of a closed curve in \mathbb{R}^n without self-intersections. For positive constants α and p , the O'Hara (α, p) -energy is defined as

$$\mathcal{E}_{(\alpha, p)}(\mathbf{f}) = \iint_{(\mathbb{R}/\mathbb{L}\mathbb{Z})^2} \mathcal{M}_{(\alpha, p)}(\mathbf{f}) ds_1 ds_2,$$

where

$$\begin{aligned} \mathcal{M}_{(\alpha, p)}(\mathbf{f}) &= \mathcal{M}_{(\alpha, p)}(\mathbf{f})(s_1, s_2) := (\mathcal{M}_\alpha(\mathbf{f}))^p, \\ \mathcal{M}_\alpha(\mathbf{f}) &:= \frac{1}{\|\mathbf{f}(s_1) - \mathbf{f}(s_2)\|_{\mathbb{R}^n}^\alpha} - \frac{1}{\mathcal{D}(\mathbf{f}(s_1), \mathbf{f}(s_2))^\alpha}. \end{aligned}$$

Here, $\|\mathbf{f}(s_1) - \mathbf{f}(s_2)\|_{\mathbb{R}^n}$ and $\mathcal{D}(\mathbf{f}(s_1), \mathbf{f}(s_2))$ are the extrinsic and the intrinsic distances between two points $\mathbf{f}(s_1)$ and $\mathbf{f}(s_2)$ on the curve, respectively. Freedman-He-Wang [13] showed that $\mathcal{E}_{(2,1)}$ is invariant under Möbius transformations, and therefore it is often referred to as the *Möbius energy*.

The purpose of this thesis is three-fold. First, we consider a discretization of O'Hara's energy, which was studied in [20, 21]. Several discrete versions of the Möbius energy have been already introduced; one was given by Kim-Kusner [23], and another was given by Simon [33]. Their convergence was shown by Rawdon-Simon [30] and Scholtes [32], respectively. The Möbius invariance was not used for the proof of result in [32]. Here, we extend the results of Kim-Kusner [23] and Scholtes [32] to the case $\mathcal{E}_{(\alpha, p)}$, and improve the rate of convergence of $\mathcal{E}_{(2,1)}$. Moreover we present the outcomes of some related numerical experiments.

The second aim is to study a generalized energy

$$\mathcal{E}_{(\Phi, p)}(\mathbf{f}) := \iint_{(\mathbb{R}/\mathbb{L}\mathbb{Z})^2} \left(\frac{1}{\Phi(\|\mathbf{f}(s_1) - \mathbf{f}(s_2)\|_{\mathbb{R}^n})} - \frac{1}{\Phi(\mathcal{D}(\mathbf{f}(s_1), \mathbf{f}(s_2)))} \right)^p ds_1 ds_2$$

under suitable assumptions on Φ , and we should see that such a generalization brings out certain properties of $\mathcal{E}_{(\alpha, p)}$ in a clearer manner. It is known that the finiteness of $\mathcal{E}_{(\alpha, p)}(\mathbf{f})$ implies bi-Lipschitz continuity and some regularity of \mathbf{f} ; see [3]. We generalize this fact to the case $\mathcal{E}_{(\Phi, p)}$, and we clarify what properties of Φ give rise to these properties of \mathbf{f} . In particular, we define a function space $W^{k+\Phi, p}$, which is a generalization of the Sobolev-Slobodeckii space, and discuss the relation between our new space and the domain of $\mathcal{E}_{(\Phi, p)}$. These results were announced in [21].

Thirdly, we consider the variational formulae of O'Hara's energy, which was a topic studied in [22]. The first variational formula of $\mathcal{E}_{(\alpha, 1)}$ ($2 \leq \alpha < 3$) was derived in [13, 31]. However, in these papers, the absolute integrability of the first variational formula was not clear and they derived the first variational formula using Cauchy's principle value. The absolute integrability of the first and second variational formulae of the Möbius energy was shown by Ishizeki-Nagasawa [17] in the energy class. They used the decomposition of the $(2, 1)$ -energy shown in [16], which gives an expression of the energy density without using the intrinsic distance. Furthermore, in [17], they derived other

estimates of variational formulae in several function spaces and a similar approach is applicable to other $(\alpha, 1)$ -energies; see [19]. The (α, p) -case of the first variational formulae was considered in [9]. Here, we extend the results of [17] for the (α, p) -energy. We, however, do not have a decomposition like the $(\alpha, 1)$ case, and therefore the technique in [17, 19] cannot be used. Hence, we give another expression of $\mathcal{M}_{(\alpha, p)}(\mathbf{f})$, and using this, we calculate and estimate variational formulae of O'Hara's energy.

In § 2, we summarize the known results concerning O'Hara's energy. Next, we study a discretization of O'Hara's energy in § 3. The finiteness of the generalized energy introduced above is considered in § 4. Finally, we study the variational formulae of O'Hara's energy in § 5.

For simplicity of notation, we will use $\Delta_{s_j}^{s_i} \mathbf{u}$ to mean $\mathbf{u}(s_i) - \mathbf{u}(s_j)$ for a function $\mathbf{u} : \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^d$, where $d = 1$ or n , and $\Delta_j^i s$ to mean $s_i - s_j$ for $s_i, s_j \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. Furthermore, we rewrite $\Delta_{s_2}^{s_1} \mathbf{u}$ and $\mathcal{D}(\mathbf{f}(s_1), \mathbf{f}(s_2))$ as $\Delta \mathbf{u}$ and $\mathcal{D}(\mathbf{f})$, respectively.

2 Known results

In this section, we summarize the known results concerning O'Hara's energy.

2.1 Fundamental properties: Scale invariance, self-repulsiveness, and the pull-tight phenomena

O'Hara's energy is scale invariant only when $\alpha p = 2$. Indeed, it holds that

$$\mathcal{E}_{(\alpha, p)}(\mathbf{f}_\lambda) = \lambda^{2-\alpha p} \mathcal{E}_{(\alpha, p)}(\mathbf{f}),$$

when $\mathbf{f}_\lambda(s) = \lambda \mathbf{f}(\frac{s}{\lambda})$ for $\lambda > 0$. When we study the regularity of critical knots and the gradient flow, the aspect changes depending on whether scale invariance holds or not, see § 2.5 and 2.6.

O'Hara's energy was introduced to determine the canonical shape in a given knot class. Therefore, it should be the case that if a knot has self-intersection, then the energy value diverges. This property is called *self-repulsiveness*. In [27], O'Hara studied conditions on α and p for which $\mathcal{E}_{(\alpha, p)}$ is self-repulsive with respect to the C^2 -topology.

Theorem 2.1 ([27]). *$\mathcal{E}_{(\alpha, p)}$ is well-defined, i.e., any smooth knot has bounded energy if*

$$\alpha \leq 2$$

or

$$2 < \alpha < 4 \text{ and } p < \frac{1}{\alpha - 2}.$$

Moreover, if $\mathcal{E}_{(\alpha, p)}$ is well defined and if $\alpha p \geq 2$, then $\mathcal{E}_{(\alpha, p)}$ is self-repulsive.

Even if O'Hara's energy is self-repulsive, we cannot prevent deformation of knots which changes knot types, e.g., the *pull-tight* phenomena. The pull-tight phenomena is the deformation which shrinks a tangle to a point. In [27], O'Hara studied the energy behavior along the pull-tight.

Theorem 2.2 ([27]). *There exists a knot K_ε which is a connected sum of a knot K and small tangle T_ε such that*

$$\mathcal{E}_{(\alpha,p)}(K_\varepsilon) - \mathcal{E}_{(\alpha,p)}(K) \rightarrow \begin{cases} \infty & \text{for } \alpha p > 2, \\ a \text{ positive constant} & \text{for } \alpha p = 2, \\ 0 & \text{for } \alpha p < 2 \end{cases}$$

in a pull-tight process $T_\varepsilon \rightarrow \{\text{a point}\}$ as $\varepsilon \rightarrow +0$.

2.2 Minimizers and Möbius invariance of the Möbius energy

A study on minimizers of O'Hara's energy among all knots under length constraint was carried out in Abrams et al. [1]. More precisely, the more general energy

$$\mathcal{E}_F(\mathbf{f}) := \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} F(\|\Delta \mathbf{f}\|_{\mathbb{R}^n}, \mathcal{D}(\mathbf{f})) ds_1 ds_2$$

was studied in [1], where $F = F(x, y)$ is increasing and convex in $x \in (0, y]$ for $y \in (0, \frac{\mathcal{L}}{2})$.

Theorem 2.3 ([1]). *The minimizers of \mathcal{E}_F among closed curves parametrized by arc-length with length \mathcal{L} are round circles, and the minimizer is unique up to translations.*

Let $F(x, y) = (x^{-\alpha} - y^{-\alpha})^p$. Then, F satisfies the assumption above when $p \in [1, \infty)$ and $0 < \alpha < 2 + \frac{1}{p}$.

Corollary 2.4 ([1]). *Let $\alpha \in (0, \infty)$, $p \in [1, \infty)$ with $0 < \alpha < 2 + \frac{1}{p}$. Then, the minimizers of $\mathcal{E}_{(\alpha,p)}$ among closed curves parametrized by arc-length with length \mathcal{L} are round circles, and the minimizer is unique up to translations.*

The existence of minimizers in a given knot class was studied by O'Hara [28]. He used the direct method of calculus of variations.

Theorem 2.5 ([28]). *If $\alpha p > 2$, then $\mathcal{E}_{(\alpha,p)}$ has a minimizer in any knot class.*

Since the energy is scale-invariant when $\alpha p = 2$, we cannot employ the direct method of calculus of variations for finding minimizers. The Möbius energy $\mathcal{E}_{(2,1)}$ is one of the cases where $\alpha p = 2$. Freedman-He-Wang [13] established the Möbius invariance of $\mathcal{E}_{(2,1)}$, and this result allows us to show the existence of minimizers in *prime* knot classes and the trivial knot class.

Theorem 2.6 ([13]). *Let T be a Möbius transformation. Then, if $T\mathbf{f} \cap \{\infty\} = \emptyset$, it holds that*

$$\mathcal{E}_{(2,1)}(T\mathbf{f}) = \mathcal{E}_{(2,1)}(\mathbf{f}),$$

and if $T\mathbf{f} \cap \{\infty\} \neq \emptyset$, we have

$$\mathcal{E}_{(2,1)}(T\mathbf{f}) = \mathcal{E}_{(2,1)}(\mathbf{f}) + 4.$$

Theorem 2.7 ([13]). *$\mathcal{E}_{(2,1)}$ has a minimizer in prime knot classes and the trivial knot class.*

The existence of the Möbius energy in a given composite class is an open problem, however, Kusner-Sullivan [24] predicted the non-existence of minimizers of the Möbius energy.

Conjecture 2.8 (The Kusner-Sullivan conjecture [24]). 1. *There does not exist minimizers of the Möbius energy in a given composite knot class.*

2. *Let $[K_1], [K_2]$ be given knot classes. Then, it holds that*

$$\mathcal{E}_{(2,1)}([K_1 \# K_2]) = \mathcal{E}_{(2,1)}([K_1]) + \mathcal{E}_{(2,1)}([K_2]) + 4,$$

where $\mathcal{E}_{(2,1)}([K_j]) := \inf_{\mathbf{f} \in [K_j]} \mathcal{E}_{(2,1)}(\mathbf{f})$, and $[K_1 \# K_2]$ is the knot type of the connected sum of K_1 and K_2 .

2.3 Finiteness of O'Hara's energy

For a closed curve \mathbf{f} in \mathbb{R}^n without self-intersections, it was shown in [27] that if $\mathcal{E}_{(\alpha,p)}(\mathbf{f}) < \infty$, then \mathbf{f} has the *bi-Lipschitz continuity* property, i.e., there exists $C_b > 0$ such that

$$C_b^{-1} \|\Delta \mathbf{f}\|_{\mathbb{R}^n} \leq \mathcal{D}(\mathbf{f}) \leq C_b \|\Delta \mathbf{f}\|_{\mathbb{R}^n}$$

for $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. This property suggests that \mathbf{f} with bounded energy cannot bend sharply, see also [3].

The energy class of O'Hara's energy was determined by Blatt [3] using the *Sobolev-Slobodeckij space*. For $k \in \mathbb{N} \cup \{0\}$, $0 < \sigma < 1$, and $q \leq 1$, the Sobolev-Slobodeckij space $W^{k+\sigma,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ is defined by

$$W^{k+\sigma,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) := \{\mathbf{f} \in W^{k,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \mid [\mathbf{f}^{(k)}]_{\sigma,q} < \infty\}.$$

We equip this space with the norm

$$\|\mathbf{f}\|_{W^{k+\sigma,q}} := \left(\|\mathbf{f}\|_{W^{k,q}}^q + [\mathbf{f}^{(k)}]_{\sigma,q}^q \right)^{\frac{1}{q}},$$

where

$$[\mathbf{f}^{(k)}]_{\sigma,q} := \left(\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\Delta \mathbf{f}^{(k)}\|_{\mathbb{R}^n}^q}{|\Delta s|^{1+\sigma q}} ds_1 ds_2 \right)^{\frac{1}{q}}.$$

Theorem 2.9 ([3]). *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ with $2 \leq \alpha p < 2p + 1$ and set $\sigma := \frac{\alpha p - 1}{2p}$. Then, $\mathcal{E}_{(\alpha,p)}(\mathbf{f}) < \infty$ if and only if \mathbf{f} is bi-Lipschitz and $\mathbf{f} \in W^{1+\sigma,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$.*

2.4 Variational formulae of $\mathcal{E}_{(\alpha,1)}$ ($2 \leq \alpha < 3$)

Let δ be the first variation, i.e. for a geometric quantity $\mathcal{F}(\mathbf{f})$ of the closed curve \mathbf{f} , and for functions $\phi : \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^n$, δ is given by

$$\delta \mathcal{F}(\mathbf{f})[\phi] = \left. \frac{d}{d\varepsilon} \mathcal{F}(\mathbf{f} + \varepsilon \phi) \right|_{\varepsilon=0}.$$

The first variational formula of the $(\alpha,1)$ -energy was given in [13, 31]. The obvious difficulty in the analysis is that the energy density has a singularity on

the diagonal set $\{(s_1, s_2) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \mid s_1 = s_2 \pmod{\mathcal{L}}\}$. To avoid this, the variational formula was first derived using Cauchy's principle value given by

$$\text{p.v.} \iint := \lim_{\varepsilon \rightarrow +0} \iint_{|\Delta s| \geq \varepsilon}.$$

Theorem 2.10 ([13]). *Let $\mathbf{f}, \phi \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. Then, it holds that*

$$\delta \mathcal{E}_{(2,1)}(\mathbf{f}) = 2 \text{p.v.} \iint \left(\frac{\mathbf{f}'(s_1) \cdot \phi'(s_1)}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} - \frac{\Delta \mathbf{f} \cdot \Delta \phi}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^4} \right) ds_1 ds_2.$$

Theorem 2.11 ([31]). *Let $\mathbf{f}, \phi \in W^{2,2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. For $\alpha \in [2, 3)$, it holds that*

$$\begin{aligned} \delta \mathcal{E}_{(\alpha,1)}(\mathbf{f}) = \text{p.v.} \iint & \left\{ (\alpha - 2) \frac{\mathbf{f}'(s_1) \cdot \phi'(s_1)}{|s_2|^\alpha} \right. \\ & \left. + 2 \frac{\mathbf{f}'(s_1) \cdot \phi'(s_1)}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} - \alpha \frac{\Delta \mathbf{f} \cdot \Delta \phi}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha+2}} \right\} ds_1 ds_2. \end{aligned}$$

For the Möbius energy, see also § 2.7.

2.5 Gradient flow

The L^2 -gradient $\delta_{L^2} \mathcal{E}_{(\alpha,p)}(\mathbf{f})$ of O'Hara's energy is given by

$$\delta \mathcal{E}_{(\alpha,p)}(\mathbf{f})[\phi] = \langle \delta_{L^2} \mathcal{E}_{(\alpha,p)}(\mathbf{f}), \phi \rangle_{L^2}.$$

If a one-parameter family of curves $\mathbf{f}(t) = \mathbf{f}(t, s) : [0, \infty) \times \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^n$ satisfies

$$\partial_t \mathbf{f}(t) = -\delta_{L^2} \mathcal{E}_{(\alpha,p)}(\mathbf{f}(t)), \quad (2.1)$$

the energy decreases along the flow; indeed

$$\frac{d}{dt} \mathcal{E}_{(\alpha,p)}(\mathbf{f}(t)) = -\|\partial_t \mathbf{f}(t)\|_{L^2}^2 \leq 0.$$

The equation (2.1) is called the *gradient flow*.

The gradient flow of the Möbius energy was first considered in He [15].

Theorem 2.12 ([15]). *Let \mathbf{f}_0 be a smooth and injective regular curve. Then, there exists a unique local solution of the gradient flow of the Möbius energy with initial curve \mathbf{f}_0 .*

Blatt [2] weakened the assumption on the initial curve.

Theorem 2.13 ([2]). *For $k > 0$, $k \notin \mathbb{N}$, let $\mathbf{f}_0 \in h^{2+k}$ be an injective regular curve, where h^{2+k} is the little Hölder space of order $2+k$. Then, there exists a unique local solution of the gradient flow of the Möbius energy with initial curve \mathbf{f}_0 .*

Moreover, Blatt showed global existence of the gradient flow of the Möbius energy near local minimizers in [2].

Theorem 2.14 ([2]). *Let $\mathbf{f}_0 \in C^\infty$ be a local minimizer of the Möbius energy in C^k for some $k \in \mathbb{N} \cup \{0\}$. Then, there exists a global solution of the gradient flow of the Möbius energy with the initial curve \mathbf{f}_0 . Moreover, there exists a limit curve $\mathbf{f}(\infty)$ such that*

$$\mathcal{E}_{(2,1)}(\mathbf{f}(\infty)) = \mathcal{E}_{(2,1)}(\mathbf{f}_0).$$

Blatt [4] considered the gradient flow of the $(\alpha, 1)$ -energy for $\alpha \in (2, 3)$. When $\alpha \in (2, 3)$, $\mathcal{E}_{(\alpha,1)}$ is not scale invariant, therefore Blatt studied the gradient flow of $\mathcal{E}_{(\alpha,1)} + \lambda\mathcal{L}$.

Theorem 2.15 ([4]). *Let \mathbf{f}_0 be a smooth curve. Then, there exists a unique global solution of the gradient flow of $\mathcal{E}_{(\alpha,1)} + \lambda\mathcal{L}$ with initial curve \mathbf{f}_0 .*

Note that in [4], Blatt showed the short-time existence of the gradient flow of $\mathcal{E}_{(\alpha,1)} + \lambda\mathcal{L}$ when the initial curve belongs to h^β for some $\beta > \alpha$, $\beta \notin \mathbb{N}$.

2.6 Regularity of critical knots

We state results concerning the regularity of critical knots for O'Hara's energy in [7, 8, 9, 10, 13, 15, 31, 34].

The regularity of local minimizers of the Möbius energy was first considered by Freedman-He-Wang [13].

Theorem 2.16 ([13]). *Any local minimizers of the Möbius energy belong to $C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^3)$.*

He [15] studied critical knots of the Möbius energy and strengthened the result of Freedman-He-Wang [13].

Theorem 2.17 ([15]). *Any critical knots of the Möbius energy $\mathbf{f} \in W^{2,2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^3)$ such that*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \|\mathbf{f}''(s)\|_{\mathbb{R}^3}^3 ds < \infty$$

are smooth. Moreover, any local minimizers of the Möbius energy are smooth.

Blatt-Reiter-Schikorra [8] weakened the assumption on the curves and considered the case of general dimension n .

Theorem 2.18 ([8]). *Any critical knots of the Möbius energy belonging to $W^{\frac{3}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ are smooth.*

Blatt-Vorderobermeier [10] finally settled the problem of the regularity of critical knots of the Möbius energy.

Theorem 2.19 ([10]). *Any critical knots of the Möbius energy belonging to $W^{\frac{3}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ are analytic.*

The $(\alpha, 1)$ -case was studied in [31, 7, 34].

Theorem 2.20 ([31]). *For $2 \leq \alpha < 3$, if $\mathbf{f} \in W^{\alpha,2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \|\mathbf{f}''(s)\|_{\mathbb{R}^n}^3 ds < \infty$$

and is a critical knot of $\mathcal{L}^{\alpha-2}\mathcal{E}_{(\alpha,1)}$, then it is smooth.

Theorem 2.21 ([7]). *For $2 < \alpha < 3$, any critical knots of $\mathcal{E}_{(\alpha,1)} + \lambda\mathcal{L}$ belonging to $W^{\frac{\alpha+1}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ are smooth.*

Vorderobermeier [34] has very recently announced the following result.

Theorem 2.22 ([34]). *For $2 < \alpha < 3$, any critical knots of $\mathcal{E}_{(\alpha,1)} + \lambda\mathcal{L}$ belonging to $W^{\frac{\alpha+1}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ are analytic.*

By comparison, the case $p > 1$ is rather less well studied. However, in recent years, Blatt-Reiter-Schikorra [9] considered the critical knots in the case of scale invariance $\alpha p = 2$. They state the following.

Theorem 2.23 ([9]). *Any critical knots of $\mathcal{E}_{(\alpha, \frac{2}{\alpha})}$ for $\alpha \leq 2$ belong to $C^{1,k}$ for some $k > 0$.*

2.7 Decomposition of the Möbius energy

2.7.1 Decomposition theorem

In [16], the Möbius energy is decomposed into parts measuring the degree of bending and twisting, and a constant. Let $\boldsymbol{\tau} = \boldsymbol{f}'$ be the tangent vector of \boldsymbol{f} , and let

$$\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{u} \wedge \boldsymbol{v} \rangle = \langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{u} \wedge \boldsymbol{v} \rangle_{\wedge^2 \mathbb{R}^n} := \det \begin{pmatrix} \boldsymbol{x} \cdot \boldsymbol{u} & \boldsymbol{x} \cdot \boldsymbol{v} \\ \boldsymbol{y} \cdot \boldsymbol{u} & \boldsymbol{y} \cdot \boldsymbol{v} \end{pmatrix}.$$

Theorem 2.24 ([16]). *If $\boldsymbol{f} \in W^{\frac{3}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ is bi-Lipschitz, then it holds that*

$$\mathcal{E}_{(2,1)}(\boldsymbol{f}) = \mathcal{E}_{(2,1),1}(\boldsymbol{f}) + \mathcal{E}_{(2,1),2}(\boldsymbol{f}) + 4,$$

where

$$\begin{aligned} \mathcal{E}_{(2,1),1}(\boldsymbol{f}) &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} ds_1 ds_2 \\ \mathcal{E}_{(2,1),2}(\boldsymbol{f}) &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\langle \boldsymbol{\tau}(s_1) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}, \boldsymbol{\tau}(s_2) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right\rangle ds_1 ds_2. \end{aligned}$$

Moreover, each of the decomposed energies is Möbius invariant.

Theorem 2.25 ([16, 18]). *Let \boldsymbol{f} be bi-Lipschitz, and let T be a Möbius transformation.*

1. *If $\boldsymbol{f} \in W^{\frac{3}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and $T\boldsymbol{f} \cap \{\infty\} = \emptyset$, then it holds that*

$$\mathcal{E}_{(2,1),1}(T\boldsymbol{f}) = \mathcal{E}_{(2,1),1}(\boldsymbol{f}), \quad \mathcal{E}_{(2,1),2}(T\boldsymbol{f}) = \mathcal{E}_{(2,1),2}(\boldsymbol{f}).$$

2. *If $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and $T\boldsymbol{f} \cap \{\infty\} \neq \emptyset$, then it holds that*

$$\mathcal{E}_{(2,1),1}(T\boldsymbol{f}) = \mathcal{E}_{(2,1),1}(\boldsymbol{f}) - 2\pi^2, \quad \mathcal{E}_{(2,1),2}(T\boldsymbol{f}) = \mathcal{E}_{(2,1),2}(\boldsymbol{f}) + 2\pi^2.$$

A similar decomposition theorem holds not only in the case of the Möbius energy but also a generalized O'Hara energy, see [19].

2.7.2 Variational formulae

The decomposed Möbius energies are useful calculating variational formulae of the Möbius energy because each decomposed energy is absolutely integrable on the energy class.

Although the definition of first variation δ was defined in § 2.4, we recall the definition of the first and second variation δ , δ^2 . For a geometric quantity $\mathcal{F}(\mathbf{f})$ of the closed curve \mathbf{f} , and for functions $\phi, \psi : \mathbb{R}/\mathbb{L}\mathbb{Z} \rightarrow \mathbb{R}^n$, let δ and δ^2 be given by

$$\begin{aligned}\delta \mathcal{F}(\mathbf{f})[\phi] &= \left. \frac{d}{d\varepsilon} \mathcal{F}(\mathbf{f} + \varepsilon \phi) \right|_{\varepsilon=0}, \\ \delta^2 \mathcal{F}(\mathbf{f})[\phi, \psi] &= \left. \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2} \mathcal{F}(\mathbf{f} + \varepsilon_1 \phi + \varepsilon_2 \psi) \right|_{\varepsilon_1=\varepsilon_2=0}.\end{aligned}$$

Let $\mathcal{M}_{(2,1),i}(\mathbf{f})$ be the integrand of $\mathcal{E}_{(2,1),i}(\mathbf{f})$. Moreover, $\mathcal{G}_{(2,1),i}(\mathbf{f})[\phi]$ and $\mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi]$ are given by

$$\begin{aligned}\mathcal{G}_{(2,1),i}(\mathbf{f})[\phi] ds_1 ds_2 &:= \delta(\mathcal{M}_{(2,1),i}(\mathbf{f}) ds_1 ds_2)[\phi], \\ \mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi] ds_1 ds_2 &:= \delta^2(\mathcal{M}_{(2,1),i}(\mathbf{f}) ds_1 ds_2)[\phi, \psi].\end{aligned}$$

Theorem 2.26 ([18]). *Let $X = W^{\frac{3}{2},2}(\mathbb{R}/\mathbb{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}/\mathbb{L}\mathbb{Z}, \mathbb{R}^n)$ and $Y = W^{\frac{1}{2},2}(\mathbb{R}/\mathbb{L}\mathbb{Z}, \mathbb{R}^n) \cap L^\infty(\mathbb{R}/\mathbb{L}\mathbb{Z}, \mathbb{R}^n)$. Assume that \mathbf{f} is bi-Lipschitz, i.e. there exists $C_b > 0$ such that $\mathcal{D}(\mathbf{f}(s_1), \mathbf{f}(s_2)) \leq C_b \|\mathbf{f}(s_1) - \mathbf{f}(s_2)\|_{\mathbb{R}^n}$.*

1. *If $\mathbf{f}, \phi, \psi \in X$, then $\mathcal{M}_{(2,1),i}(\mathbf{f}), \mathcal{G}_{(2,1),i}(\mathbf{f})[\phi], \mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi] \in L^1((\mathbb{R}/\mathbb{L}\mathbb{Z})^2)$. Moreover there exists $C = C(\|\tau\|_Y, C_b) > 0$ such that*

$$\begin{aligned}\|\mathcal{M}_{(2,1),i}(\mathbf{f})\|_X &\leq C \\ \|\mathcal{G}_{(2,1),i}(\mathbf{f})[\phi]\|_X &\leq C \|\phi'\|_Y \\ \|\mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi]\|_X &\leq C \|\phi'\|_Y \|\psi'\|_Y.\end{aligned}$$

2. *If $\mathbf{f}, \phi, \psi \in C^{1,1}(\mathbb{R}/\mathbb{L}\mathbb{Z}, \mathbb{R}^n)$, then $\mathcal{M}_{(2,1),i}(\mathbf{f}), \mathcal{G}_{(2,1),i}(\mathbf{f})[\phi], \mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi] \in L^\infty((\mathbb{R}/\mathbb{L}\mathbb{Z})^2)$. Moreover there exists $C = C(\|\tau\|_{C^{0,1}}, C_b) > 0$ such that*

$$\begin{aligned}\|\mathcal{M}_{(2,1),i}(\mathbf{f})\|_{L^\infty} &\leq C \\ \|\mathcal{G}_{(2,1),i}(\mathbf{f})[\phi]\|_{L^\infty} &\leq C \|\phi'\|_{C^{0,1}} \\ \|\mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi]\|_{L^\infty} &\leq C \|\phi'\|_{C^{0,1}} \|\psi'\|_{C^{0,1}}.\end{aligned}$$

3. *If $\mathbf{f}, \phi, \psi \in C^2(\mathbb{R}/\mathbb{L}\mathbb{Z}, \mathbb{R}^n)$, then $\mathcal{M}_{(2,1),i}(\mathbf{f}), \mathcal{G}_{(2,1),i}(\mathbf{f})[\phi], \mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi]$ can be extended to the diagonal set $\{(s, s) | s \in \mathbb{R}/\mathbb{L}\mathbb{Z}\}$ such that these functions are continuous everywhere on $(\mathbb{R}/\mathbb{L}\mathbb{Z})^2$. In each case, the limit function on the diagonal set is 0, i.e.*

$$\begin{aligned}\lim_{(s_1, s_2) \rightarrow (s, s)} (\mathcal{M}_{(2,1),1}(\mathbf{f}) + \mathcal{M}_{(2,1),2}(\mathbf{f})) &= 0, \\ \lim_{(s_1, s_2) \rightarrow (s, s)} (\mathcal{G}_{(2,1),1}(\mathbf{f})[\phi] + \mathcal{G}_{(2,1),2}(\mathbf{f})[\phi]) &= 0, \\ \lim_{(s_1, s_2) \rightarrow (s, s)} (\mathcal{H}_{(2,1),1}(\mathbf{f})[\phi, \psi] + \mathcal{H}_{(2,1),2}(\mathbf{f})[\phi, \psi]) &= 0.\end{aligned}$$

Moreover there exists $C = C(\|\tau\|_{C^1}, C_b) > 0$ such that

$$\begin{aligned}\|\mathcal{M}_{(2,1),i}(\mathbf{f})\|_{C^0} &\leq C \\ \|\mathcal{G}_{(2,1),i}(\mathbf{f})[\phi]\|_{C^0} &\leq C\|\phi'\|_{C^1} \\ \|\mathcal{H}_{(2,1),i}(\mathbf{f})[\phi, \psi]\|_{C^0} &\leq C\|\phi'\|_{C^1}\|\psi'\|_{C^1}.\end{aligned}$$

2.8 Discretization of the Möbius energy

It is difficult to calculate values of O'Hara's energy directly, and therefore, it is not easy to evaluate well-balancedness. Therefore, it is desirable to numerically calculate these energies. In the mid-1990s, a discretization of the Möbius energy was proposed by Kim-Kusner [23] and Simon [33]. Let $\mathbf{p}_m : \mathbb{R}/\mathcal{L}_m\mathbb{Z} \rightarrow \mathbb{R}^n$ be an m -gon, i.e. a polygon with m edges, parametrized by arc-length and embedded in \mathbb{R}^n with length \mathcal{L}_m . Let a_i be the value of the arc-length parameter at the i -th vertex of \mathbf{p}_m , and note that \mathbf{p}_m is made by connecting $\{\mathbf{p}_m(a_i)\}$ in turn. Then, Kim-Kusner's polygonal discrete energy $\mathcal{E}_{\text{KK}}^m(\mathbf{p}_m)$ is defined by

$$\mathcal{E}_{\text{KK}}^m(\mathbf{p}_m) := \sum_{\substack{i,j=1 \\ i \neq j}}^m \mathcal{M}^m(\mathbf{p}_m) \|\Delta_{a_{i+1}}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n} \|\Delta_{a_{j+1}}^{a_j} \mathbf{p}_m\|_{\mathbb{R}^n},$$

where

$$\mathcal{M}^m(\mathbf{p}_m) = \mathcal{M}^m(\mathbf{p}_m)(a_i, a_j) := \frac{1}{\|\Delta_{a_j}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n}^2} - \frac{1}{\mathcal{D}(\mathbf{p}_m(a_i), \mathbf{p}_m(a_j))^2}.$$

Using this discrete energy, Kim-Kusner [23] calculated values of the Möbius energy of torus knots by numerical experiments.

In order to define the discretization introduced by Simon, let X_j be i -th edge of \mathbf{p}_m , $|X_j|$ be length of X_j , and $\text{dist}(X_i, X_j)$ be the minimum distance between edges X_i and X_j . Then, Simon's discrete energy $\mathcal{E}_S^m(\mathbf{p}_m)$ is defined by

$$\mathcal{E}_S^m(\mathbf{p}_m) := \tilde{\mathcal{E}}_S^m(\mathbf{p}_m) - \tilde{\mathcal{E}}_S^m(\mathbf{g}_m) + 4,$$

where

$$\tilde{\mathcal{E}}_S^m(\mathbf{p}_m) := \sum_{|i-j|>1} \frac{|X_i||X_j|}{\text{dist}(X_i, X_j)}$$

and \mathbf{g}_m is a regular m -gon.

Rawdon-Simon [30] and Scholtes [32] considered the convergence of $\mathcal{E}_S^m(\mathbf{p}_m)$ and $\mathcal{E}_{\text{KK}}^m(\mathbf{p}_m)$, respectively. In [32], Scholtes showed the Γ -convergence of $\mathcal{E}_{\text{KK}}^m(\mathbf{p}_m)$. The merit of Γ -convergence is that the minimum value of the discrete energy converges to the minimum value of O'Hara's energy as $m \rightarrow \infty$.

Theorem 2.27 ([32]). 1. For a curve $\mathbf{f} : \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^n$, let \mathbf{p}_m be an inscribed polygon in \mathbf{f} , and suppose that the vertices correspond to parameters $t_j \in \mathbb{R}/\mathcal{L}\mathbb{Z}$; that is, \mathbf{p}_m is made by connecting $\{\mathbf{f}(t_j)\}$ in turn. If $\mathbf{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and there exist $c, \bar{c} > 0$ such that

$$\frac{c}{m} \leq \min_{j=1, \dots, m} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \leq \max_{j=1, \dots, m} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \leq \frac{\bar{c}}{m},$$

then the following holds. For all $\varepsilon > 0$, there exist $C_\varepsilon > 0$ depending on \mathbf{f} , c , and \bar{c} such that

$$|\mathcal{E}_{(2,1)}(\mathbf{f}) - \mathcal{E}_{\text{KK}}^m(\mathbf{p}_m)| \leq C_\varepsilon \frac{1}{m^{1-\varepsilon}}.$$

Moreover, if $\mathcal{E}_{(2,1)}(\mathbf{f}) < \infty$, then it holds that

$$\lim_{m \rightarrow \infty} \mathcal{E}_{\text{KK}}^m(\mathbf{p}_m) = \mathcal{E}_{(2,1)}(\mathbf{f}).$$

2. $\mathcal{E}_{\text{KK}}^m$ converges to $\mathcal{E}_{(2,1)}$ in the sense of Γ -convergence on metric spaces. Here, these metric spaces contain C^1 curves and equilateral polygons with length 1 belonging to a given tame knot class equipped with the metric induced by the L^r -norm and $W^{1,r}$ -norm with $r \in [1, \infty]$.

Theorem 2.28 ([30]). For $\mathbf{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, if \mathbf{p}_m is an inscribed polygon and m is sufficiently large, then there exists $C = C(\mathbf{f}) > 0$ such that

$$|(\mathcal{E}_{(2,1)}(\mathbf{f}) + 4) - \mathcal{E}_S^m(\mathbf{p}_m)| \leq C \frac{1}{m^{\frac{1}{4}}}.$$

In [32], it was shown that the minimal values of $\mathcal{E}_{\text{KK}}^m$ converge to the minimal value of $\mathcal{E}_{(2,1)}$. Moreover, it was shown that minimizers of $\mathcal{E}_{\text{KK}}^m$ in the set of equilateral polygons are regular polygons and that the minimizers are unique up to congruent transformations and similar transformations.

In recent years, Blatt-Ishizeki-Nagasawa [5, 6] proposed a Möbius invariant discretization of the Möbius energy.

Remark 2.29. Okamoto [29] produced a discretization of O'Hara's energy using a sequence of random variables $\{X_i\}_{i \in \mathbb{N}}$ on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ and showed the Γ -convergence and the compact convergence of his discrete energy.

3 A discretization of O'Hara's energy

Scholtes [32] did not use the Möbius invariance of $\mathcal{E}_{(2,1)}$ for proving his result, and thus it is natural to believe that this argument may be applicable to all of O'Hara's energies; we prove this here. More precisely, in this section, we propose a discretization of (α, p) -O'Hara energies by using the idea of [32], and we discuss approximation of the discrete energies to O'Hara energies and the Γ -convergence.

Definition 3.1 (A discretization of (α, p) -O'Hara energies). Let $\alpha, p \in (0, \infty)$, and let $\mathbf{p}_m : \mathbb{R}/\mathcal{L}_m\mathbb{Z} \rightarrow \mathbb{R}^n$ be a polygon parametrized by arc-length with m vertices whose total length is $\mathcal{L}_m > 0$. Let a_j be the value of arc-length parameters corresponding to its vertices and assume

$$0 \leq a_1 < a_2 < \cdots < a_m < \mathcal{L}_m \pmod{\mathcal{L}_m}.$$

Then, we define $\mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m)$ by

$$\mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) := \sum_{\substack{i,j=1 \\ i \neq j}}^m \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \|\Delta_{a_{i+1}}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n} \|\Delta_{a_{j+1}}^{a_j} \mathbf{p}_m\|_{\mathbb{R}^n},$$

where

$$\begin{aligned}\mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) &= \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m)(a_i, a_j) := (\mathcal{M}_\alpha^m(\mathbf{p}_m))^p, \\ \mathcal{M}_\alpha^m(\mathbf{p}_m) &:= \frac{1}{\|\Delta_{a_j}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n}^\alpha} - \frac{1}{\mathcal{D}(\mathbf{p}_m(a_i), \mathbf{p}_m(a_j))^\alpha}.\end{aligned}$$

Our main theorems in this section are as follows.

Theorem 3.2 (The rate of convergence of discretization via the approximation by inscribed polygons). *Assume that $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ satisfy $2 \leq \alpha p < 2p + 1$. Let $\mathbf{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ be a curve parametrized by arc-length embedded in \mathbb{R}^n , where \mathcal{L} is the length of \mathbf{f} . Let $c, \bar{c} > 0$, and set $K := \|\mathbf{f}''\|_{L^\infty}$.*

In addition, for $m \in \mathbb{N}$, let $\{t_j\}_{j=1}^m$ be a division of $\mathbb{R}/\mathcal{L}\mathbb{Z}$ satisfying

$$\frac{c\mathcal{L}}{m} \leq \min_{j=1,\dots,m} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \leq \max_{j=1,\dots,m} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \leq \frac{\bar{c}\mathcal{L}}{m}, \quad (3.1)$$

and let \mathbf{p}_m be the inscribed polygon in \mathbf{f} with vertices $\mathbf{f}(t_1), \dots, \mathbf{f}(t_m)$. Then, if the number m of points of the division is sufficiently large, there exists $C > 0$ depending on $c, \bar{c}, \mathcal{E}_{(\alpha,p)}(\mathbf{f})$ such that

$$|\mathcal{E}_{(\alpha,p)}(\mathbf{f}) - \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m)| \leq C \frac{(\mathcal{L}K)^{2p} + (\mathcal{L}K)^{2p+2}}{\mathcal{L}^{\alpha p-2}} \frac{1}{m^{2p-\alpha p+1}}.$$

Furthermore, if $\alpha \leq 2$, then there exists $C > 0$ depending on $c, \bar{c}, \mathcal{E}_{(\alpha,p)}(\mathbf{f})$ such that

$$|\mathcal{E}_{(\alpha,p)}(\mathbf{f}) - \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m)| \leq C \frac{(\mathcal{L}K)^{\alpha p} + (\mathcal{L}K)^{\alpha p-\alpha+2} + (\mathcal{L}K)^{\alpha p+2} \log m}{\mathcal{L}^{\alpha p-2}} \frac{1}{m}.$$

Theorem 3.3 (The convergence of the discrete energy of inscribed polygons). *Assume that $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ satisfy $2 \leq \alpha p < 2p + 1$. Let $\mathbf{f} \in W^{1+\sigma, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, and let \mathbf{p}_m be the inscribed polygon as in Theorem 3.2. Then, we have*

$$\lim_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) = \mathcal{E}_{(\alpha,p)}(\mathbf{f}).$$

Theorem 3.4 (cf. Theorem 3.28). *For $\alpha \in (0, \infty)$, $p \in [1, \infty)$ satisfying $2 \leq \alpha p < 2p + 1$, $\mathcal{E}_{(\alpha,p)}^m$ converges to $\mathcal{E}_{(\alpha,p)}$ in the sense of Γ -convergence on a metric space X .*

Remark 3.5. A metric function on X , $d_X : X \times X \rightarrow \mathbb{R}$, satisfies

$$C_1 \|\mathbf{u} - \mathbf{v}\|_{L^1} \leq d_X(\mathbf{u}, \mathbf{v}) \leq C_2 \|\mathbf{u} - \mathbf{v}\|_{W^{1,\infty}}$$

for $\mathbf{u}, \mathbf{v} \in X$, where $C_1, C_2 > 0$ are constants. The full definition of X is given in § 3.2.1.

In addition, we discuss minimizers of the discrete energies $\mathcal{E}_{(\alpha,p)}^m$ of the set of all *equilateral* m -gons. There do not exist minimizers of $\mathcal{E}_{(\alpha,p)}^m$ in the set of *all* m -gons which are not necessarily equilateral. Let us consider an $(m-1)$ -gon as a degenerate m -gon. Here “degenerate” means that two vertices of the m -gon coincide. Note that it does *not* mean the degeneracy of parametrization. Then, we have

$$0 \leq \inf_{m\text{-gon}} \mathcal{E}_{(\alpha,p)}^m \leq \inf_{(m-1)\text{-gon}} \mathcal{E}_{(\alpha,p)}^{m-1} \leq \dots \leq \inf_{3\text{-gon}} \mathcal{E}_{(\alpha,p)}^3,$$

and because $\mathcal{E}_{(\alpha,p)}^3(\mathbf{p}_3) = 0$ for all 3-gons \mathbf{p}_3 , we obtain

$$\inf_{m\text{-gon}} \mathcal{E}_{(\alpha,p)}^m = 0.$$

That is reason why we consider their minimizers in the set of all equilateral m -gons.

Theorem 3.6 (cf. Theorem 3.34). *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$. Then, minimizers of $\mathcal{E}_{(\alpha,p)}^m$ are regular polygons in the set of equilateral m -gons. In particular, a regular m -gon is the only minimizer, except for congruent transformations and similar transformations.*

Remark 3.7. Only in this section, for simplicity, we write $\mathcal{D}(\mathbf{f}(s_i), \mathbf{f}(s_j))$, $\mathcal{D}(\mathbf{f}(t_i), \mathbf{f}(t_j))$, $\mathcal{D}(\mathbf{p}_m(a_i), \mathbf{p}_m(a_j))$ as $|\Delta_j^i s|$, $|\Delta_j^i t|$, $|\Delta_j^i a|$ respectively.

3.1 Approximation of O'Hara's energy by inscribed polygons

In this subsection, we show that the discrete energy defined previously converges to O'Hara's energy under certain conditions.

From now on, we write $\sigma = \frac{\alpha p - 1}{2p}$. For a given regular curve \mathbf{f} with total length \mathcal{L} , we say that a polygon \mathbf{p} is *inscribed* in \mathbf{f} if \mathbf{p} satisfies

- (i) the number of vertices is finite,
- (ii) the set of vertices is $\{\mathbf{f}(s_1), \mathbf{f}(s_2), \dots, \mathbf{f}(s_m)\}$ with $s_1 < s_2 < \dots < s_m (< s_1 + \mathcal{L})$,
- (iii) the j -th edge is the segment jointing $\mathbf{f}(s_j)$ and $\mathbf{f}(s_{j+1})$, where we interpret $s_{m+1} = s_1$.

The aim of this subsection is to prove Theorems 3.2 and 3.3.

3.1.1 Lemmas

In this subsubsection, we prove estimates and properties of parameters of curves and polygons in preparation for our proofs of Theorems 3.2 and 3.3.

First, we give the parametrization of an inscribed polygon. For a division $\{t_j\}_{j=1}^m$ on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, let \mathbf{p}_m be the inscribed polygon in \mathbf{f} with vertices $\mathbf{f}(t_j)$ ($j = 1, \dots, m$). We extend the notation $\mathbf{f}(t_j)$ to all $j \in \mathbb{Z}$ in the natural way via congruency modulo m ; i.e., $\mathbf{f}(t_0) = \mathbf{f}(t_m)$, $\mathbf{f}(t_1) = \mathbf{f}(t_{m+1})$, and so on. Let

$$\mathcal{L}_m = \sum_{j=1}^m \|\Delta_{t_{k+1}}^{t_k} \mathbf{f}\|_{\mathbb{R}^n}$$

be the length of \mathbf{p}_m . Set

$$a_j = \sum_{i=0}^{j-1} \|\Delta_{t_{k+1}}^{t_k} \mathbf{f}\|_{\mathbb{R}^n}$$

as the value of the arc-length parameter of the j -th vertex of \mathbf{p}_m . Then, note that

$$|\Delta_j^i t| \geq |\Delta_j^i a| \geq \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \geq C_b^{-1} |\Delta_j^i t|. \quad (3.2)$$

In what follows, we set $N := 4C_b \frac{\bar{c}}{c}$. We get the following lemma by the triangle inequality.

Lemma 3.8. *Let $s_j \in [t_j, t_{j+1}]$, $s_i \in [t_i, t_{i+1}]$.*

1. *It holds that*

$$|\Delta_j^i s| \leq \left(1 + 2C_b \frac{\bar{c}}{c}\right) |\Delta_j^i t|.$$

2. *If $|j - i| \geq N$, we have*

$$|\Delta_j^i s| \geq C_b^{-1} \frac{c}{2\bar{c}} |\Delta_j^i t|.$$

In the next lemma, we calculate the difference between the arc-length and the distance of two points.

Lemma 3.9. 1. *Let $\mathbf{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, and $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. Then, we have*

$$0 \leq |\Delta s|^2 - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2 \leq \frac{K^2}{2} |\Delta s|^4.$$

2. *Let $\alpha \in (0, \infty)$, $p \in [1, \infty)$, $\mathbf{f} \in C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and let $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. Then, we have*

$$|\Delta s| - \|\Delta \mathbf{f}\|_{\mathbb{R}^n} \leq \frac{1}{2} |\Delta s|^{\alpha+1-\frac{2}{p}} \left(\int_{s_2}^{s_1} \int_{s_2}^{s_1} \frac{\|\Delta_{s_4}^{s_3} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{|\Delta_4^3 s|^{\alpha p}} ds_3 ds_4 \right)^{\frac{1}{p}}.$$

Proof. We only prove 2. In the case where $p = 1$, we get

$$\begin{aligned} |\Delta s| - \|\Delta \mathbf{f}\|_{\mathbb{R}^n} &\leq \frac{|\Delta s|^2 - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}{|\Delta s|} \\ &= \frac{1}{2|\Delta s|} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\Delta_{s_4}^{s_3} \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \\ &\leq \frac{1}{2} |\Delta s|^{\alpha-1} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \frac{\|\Delta_{s_4}^{s_3} \boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{|\Delta_4^3 s|^\alpha} ds_3 ds_4. \end{aligned}$$

On the other hand, in the case where $p \in (1, \infty)$, we get

$$\begin{aligned} |\Delta s| - \|\Delta \mathbf{f}\|_{\mathbb{R}^n} &\leq \frac{1}{2|\Delta s|} \left(\int_{s_2}^{s_1} \int_{s_2}^{s_1} \frac{\|\Delta_{s_4}^{s_3} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{|\Delta_4^3 s|^{\alpha p}} ds_3 ds_4 \right)^{\frac{1}{p}} \left(\int_{s_2}^{s_1} \int_{s_2}^{s_1} |\Delta_4^3 s|^{\frac{\alpha p}{p-1}} ds_3 ds_4 \right)^{1-\frac{1}{p}} \\ &\leq \frac{1}{2} |\Delta s|^{\alpha+1-\frac{2}{p}} \left(\int_{s_2}^{s_1} \int_{s_2}^{s_1} \frac{\|\Delta_{s_4}^{s_3} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{|\Delta_4^3 s|^{\alpha p}} ds_3 ds_4 \right)^{\frac{1}{p}} \end{aligned}$$

by Hölder's inequality. \square

The following lemma is proved by simple calculations, hence, we omit the proof.

Lemma 3.10. 1. Let $0 < a \leq 2$. Then we have

$$1 - x^a \leq (1 - x^2)^{\frac{a}{2}}$$

for all $0 \leq x \leq 1$.

2. Let $a > 0$. Then, we have

$$1 - x^a \leq (a + 1)(1 - x).$$

for all $0 \leq x \leq 1$.

Finally, we have the following lemma, which may be proved by using Lemmas 3.9.1 and 3.10.

Lemma 3.11. 1. Let $\alpha > 0$. Then we have

$$|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha \leq \left(\frac{\alpha}{2} + 1\right) \frac{K^2}{2} |\Delta s|^{\alpha+2}$$

for all $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$.

2. Let $0 < \alpha \leq 2$. Then we have

$$|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha \leq \frac{K^\alpha}{2^{\frac{\alpha}{2}}} |\Delta s|^{2\alpha}$$

for all $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$.

In subsubsections 3.1.2 and 3.1.3, unless otherwise noted, we assume that $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ satisfy $2 \leq \alpha p < 2p + 1$

3.1.2 Proof of Theorem 3.2

Firstly, we have

$$\begin{aligned} & |\mathcal{E}_{(\alpha,p)}(\mathbf{f}) - \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m)| \\ & \leq \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \left| \mathcal{M}_{(\alpha,p)}(\mathbf{f}) - \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \right| ds_1 ds_2 \\ & \quad + \sum_{i=1}^m \sum_{|j-i| > N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \left| \mathcal{M}_{(\alpha,p)}(\mathbf{f}) - \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \right| ds_1 ds_2, \end{aligned}$$

where $\sum_{|j-i| \leq N}$ and $\sum_{|j-i| > N}$ are summations with respect to j with $|j - i| \leq N$ and $|j - i| > N$ for each $i = 1, \dots, n$ respectively. In what follows, we estimate each of them.

Estimates for the case where $|j - i| \leq N$

Proposition 3.12. We have

$$\begin{aligned} & \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2 \\ & \leq \frac{(\alpha + 2)^p C_b^{\frac{\alpha p}{2}} \{\bar{c}(N + 1)\}^{2p - \alpha p + 2} (2N + 1)}{4^p (2p - \alpha p + 1)(2p - \alpha p + 2)} \mathcal{L}^{2p - \alpha p + 2} K^{2p} \frac{1}{m^{2p - \alpha p + 1}}. \end{aligned}$$

Moreover, if $\alpha \leq 2$, then we have

$$\sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2 \leq \frac{C_b^{\alpha p+1} (2N+1) \bar{c}}{2^{\frac{\alpha p}{2}}} \mathcal{L}^2 K^{\alpha p} \frac{1}{m}.$$

Proof. Using the bi-Lipschitz continuity of \mathbf{f} , Lemma 3.11, and (3.1), we have

$$\begin{aligned} & \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2 \\ & \leq C_b^{\alpha p} \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{(|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha)^p}{|\Delta s|^{2\alpha p}} ds_1 ds_2 \\ & \leq C_b^{\alpha p} \left(\frac{\alpha}{2} + 1\right)^p \frac{K^{2p}}{2^p} \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} |\Delta s|^{2p-\alpha p} ds_1 ds_2 \\ & \leq \frac{(\alpha+2)^q C_b^{\alpha p} \{\bar{c}(N+1)\}^{2p-\alpha p+2} (2N+1)}{4^p (2p-\alpha p+1)(2p-\alpha p+2)} \mathcal{L}^{2p-\alpha p+2} K^{2p} \frac{1}{m^{2p-\alpha p+1}}, \end{aligned}$$

and in the case where $\alpha \leq 2$, we have

$$\begin{aligned} & \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2 \\ & \leq C_b^{\alpha p} \sum_{i=1}^m \sum_{|j-i| \leq N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{(|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha)^p}{|\Delta s|^{2\alpha p}} ds_1 ds_2 \\ & \leq C_b^{\alpha p} \frac{K^{\alpha p}}{2^{\frac{\alpha p}{2}}} \sum_{i=1}^m \sum_{|j-i| \leq N} |\Delta_{i+1}^i t| |\Delta_{j+1}^j t| \\ & \leq \frac{C_b^{\alpha p+1} (2N+1) \bar{c}}{2^{\frac{\alpha p}{2}}} \mathcal{L}^2 K^{\alpha p} \frac{1}{m}. \end{aligned}$$

□

The following proposition is proved by the same calculations as those in the proof of Proposition 3.12.

Proposition 3.13. *We have*

$$\begin{aligned} & \sum_{i=1}^m \sum_{|j-i| \leq N} \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \|\Delta_{t_{i+1}}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \\ & \leq \frac{(\alpha+2)^p C_b^{\alpha p} c^{2p-\alpha p} \bar{c}^2 (2N+1)}{4^p} \mathcal{L}^{2p-\alpha p+2} K^{2p} \frac{1}{m^{2p-\alpha p+1}}. \end{aligned}$$

Moreover, if $\alpha \leq 2$, then we have

$$\sum_{i=1}^m \sum_{|j-i| \leq N} \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \|\Delta_{t_{i+1}}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \leq \frac{C_b^{\alpha p} (2N+1) \bar{c}}{2^{\frac{\alpha p}{2}}} \mathcal{L}^2 K^{\alpha p} \frac{1}{m}.$$

Estimates for the case where $|j-i| > N$ In what follows, C_g is a positive constant that may change from line to line.

In order to determine how to prove Theorem 3.2, set

$$X_N := \sum_{i=1}^m \sum_{|j-i|>N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2,$$

$$Y_N := \sum_{i=1}^m \sum_{|j-i|>N} \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \|\Delta_{t_{i+1}}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n}.$$

It is sufficient to estimate $|X_N - Y_N|$.

Next, set

$$A_{i,j} := \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \left| \frac{|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha} - \frac{|\Delta_j^i t|^\alpha - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha} \right| ds_1 ds_2,$$

$$B_{i,j} := \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \left| \frac{|\Delta_j^i t|^\alpha - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha} - \frac{|\Delta_j^i t|^\alpha - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta_j^i t|^\alpha} \right| ds_1 ds_2,$$

$$C_{i,j} := \left| \frac{1}{|\Delta_j^i a|^\alpha} - \frac{1}{|\Delta_j^i t|^\alpha} \right| |\Delta_{i+1}^i t| |\Delta_{j+1}^j t|,$$

$$D_{i,j} := \|\Delta_{i+1}^i t| |\Delta_{j+1}^j t| - \|\Delta_{t_{i+1}}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} |.$$

Then, we have the following key lemma.

Lemma 3.14. *There exists a positive constant C_g such that we have*

$$|X_N - Y_N| \leq C_g K^{2(p-1)} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} (A_{i,j} + B_{i,j} + C_{i,j})$$

$$+ C_g K^{2p} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)p} D_{i,j}.$$

Moreover, if $\alpha \leq 2$, we have

$$|X_N - Y_N| \leq C_g K^{\alpha(p-1)} \sum_{i=1}^m \sum_{|j-i|>N} (A_{i,j} + B_{i,j} + C_{i,j})$$

$$+ C_g K^{\alpha p} \sum_{i=1}^m \sum_{|j-i|>N} D_{i,j}.$$

Proof. Using the bi-Lipschitz continuity and Lemmas 3.8.1, 3.10.2, and 3.11.1,

We have

$$\begin{aligned}
|X_N - Y_N| &\leq C_g K^{2(p-1)} \sum_{i=1}^m \sum_{|j-i|>N} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} |\mathcal{M}_\alpha(\mathbf{f}) - \mathcal{M}_\alpha^m(p_n)| \\
&\quad \times \max \left\{ \frac{|\Delta s|^2}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}, \frac{|\Delta_j^i t|^2}{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\}^{p-1} ds_1 ds_2 \\
&\quad + C_g K^{2p} \sum_{i=1}^m \sum_{|j-i|>N} \frac{|\Delta_j^i t|^{2p}}{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^{\alpha p}} \\
&\quad \times \left| \|\Delta_{i+1}^i t\| |\Delta_{j+1}^j t| - \|\Delta_{i+1}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{j+1}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \right| \\
&\leq C_g K^{2(p-1)} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} (A_{i,j} + B_{i,j} + C_{i,j}) \\
&\quad + C_g K^{2p} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)p} D_{i,j}.
\end{aligned}$$

In the case where $\alpha \leq 2$, we get the claim in a similar way using Lemma 3.11.2 instead of Lemma 3.11.1. \square

Before we estimate the summations appearing in the statement of Lemma 3.14, we state inequalities used later. The following lemma is proved by using the bi-Lipschitz continuity, and Lemmas 3.9.2 and 3.10.2.

Lemma 3.15. *For $s_1 \in [t_i, t_{i+1}]$, $s_2 \in [t_j, t_{j+1}]$, the following estimates hold.*

1. $|\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha| \leq C_g |\Delta_j^i t|^{\alpha-1} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|.$
2. $|\|\Delta_j^i t\|^\alpha - |\Delta s|^\alpha| \leq C_g |\Delta_j^i t|^{\alpha-1} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|.$

Using Lemma 3.15, we estimate the summations appearing in Lemma 3.14.

Proposition 3.16. *We have*

$$\sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(q-1)} A_{i,j} \leq C_g \mathcal{L}^{2p-\alpha p+2} K^2 \frac{1}{m^{2p-\alpha p+1}}.$$

Moreover, if $\alpha \leq 2$, then we have

$$\sum_{i=1}^m \sum_{|j-i|>N} A_{i,j} \leq C_g \left(\mathcal{L}^2 K^\alpha \frac{\log m}{m} + \mathcal{L}^{4-\alpha} K^2 \frac{1}{m} \right).$$

Proof. Fix $s_1 \in [t_i, t_{i+1}]$ and $s_2 \in [t_j, t_{j+1}]$. Without loss of generality, we may assume $s_1 < s_2$. By Lemmas 3.9, 3.11.1, and 3.15.2, we have

$$\begin{aligned}
\left| |\Delta s|^\alpha - |\Delta_j^i t|^\alpha \right| \left(1 - \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha} \right) &\leq C_g \frac{\max_{k=1, \dots, m} |\Delta_{k+1}^k t|}{|\Delta_j^i t|} (|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha) \\
&\leq C_g K^2 |\Delta_j^i t|^{\alpha+1} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|. \tag{3.3}
\end{aligned}$$

Also, we have

$$\begin{aligned} & |(|\Delta s|^2 - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2) - (|\Delta_j^i t|^2 - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^2)| \\ &= \left| \iint_A (1 - \tau(s_3) \cdot \tau(s_4)) ds_3 ds_4 - \iint_B (1 - \tau(s_3) \cdot \tau(s_4)) ds_3 ds_4 \right|, \end{aligned}$$

where

$$\begin{aligned} A &:= ([s_1, s_2] \times [t_j, s_2]) \cup ([t_j, s_2] \times [s_1, s_2]), \\ B &:= ([t_i, t_j] \times [t_i, s_1]) \cup ([t_i, s_1] \times [t_i, t_j]). \end{aligned}$$

The integral over $[t_j, s_2] \times [s_1, s_2]$ is estimated as

$$\left| \int_{t_j}^{s_2} \int_{s_1}^{s_2} (1 - \tau(s_3) \cdot \tau(s_4)) ds_3 ds_4 \right| \leq \frac{K^2}{2} |\Delta s|^3 |s_2 - t_j|.$$

We can dominate the integrals over $[s_1, s_2] \times [t_j, s_2]$ and B similarly. Then, we get

$$\begin{aligned} & |(|\Delta s|^2 - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2) - (|\Delta_j^i t|^2 - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^2)| \\ & \leq 2K^2 (|\Delta s|^3 |s_2 - t_j| + |\Delta_j^i t|^3 |s_1 - t_i|). \end{aligned}$$

By this equality, the bi-Lipschitz continuity, and Lemmas 3.8.2, 3.10.2, 3.11.2, and 3.15.2, it holds that

$$\begin{aligned} |\Delta_j^i t|^\alpha \left| \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha} - \frac{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta_j^i t|^\alpha} \right| & \leq C_g K^2 |\Delta_j^i t|^{\alpha-2} \frac{(|\Delta s|^3 |s_2 - t_j| + |\Delta_j^i t|^3 |s_1 - t_i|)}{|\Delta s|^2} \\ & \quad + C_g K^2 |\Delta_j^i t|^{\alpha+4} \left| \frac{1}{|\Delta s|^2} - \frac{1}{|\Delta_j^i t|^2} \right| \\ & \leq C_g K^2 |\Delta_j^i t|^{\alpha+1} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|. \end{aligned} \quad (3.4)$$

Using (3.3) and (3.4), we have

$$\begin{aligned} & |(|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha) - (|\Delta_j^i t|^\alpha - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha)| \\ &= \left| (|\Delta s|^\alpha - |\Delta_j^i t|^\alpha) \left(1 - \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha} \right) - |\Delta_j^i t|^\alpha \left(\frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha} - \frac{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta_j^i t|^\alpha} \right) \right| \\ & \leq C_g K^2 |\Delta_j^i t|^{\alpha+1} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|. \end{aligned}$$

Hence, we have

$$A_{i,j} \leq C_g K^2 |\Delta_j^i t|^{1-\alpha} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|^3,$$

and therefore, we get

$$\begin{aligned} & \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} A_{i,j} \\ & \leq C_g K^2 \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)+1-\alpha} \max_{k=1, \dots, m} |\Delta_{k+1}^k t|^3 \\ & \leq C_g L^{2p-\alpha p+2} K^2 \frac{1}{m^{2p-\alpha p+1}} \end{aligned}$$

using (3.1) and the bi-Lipschitz continuity.

Next, assume that $\alpha \leq 2$. Using Lemmas 3.8, 3.11.2, and 3.15.2, we have

$$||\Delta s|^\alpha - |\Delta_j^i t|^\alpha| \left(1 - \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha}\right) \leq C_g K^\alpha |\Delta_j^i t|^{2\alpha-1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|.$$

Therefore, using in addition Lemmas 3.8 and 3.15, it holds that

$$\begin{aligned} & |(|\Delta s|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha) - (|\Delta_j^i t|^\alpha - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha)| \\ &= \left| (|\Delta s|^\alpha - |\Delta_j^i t|^\alpha) \left(1 - \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha}\right) - |\Delta_j^i t|^\alpha \left(\frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha} - \frac{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta_j^i t|^\alpha}\right) \right| \\ &\leq C_g K^\alpha |\Delta_j^i t|^{2\alpha-1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t| + C_g K^2 |\Delta_j^i t|^{\alpha+1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|. \end{aligned}$$

Moreover, by (3.1) and the bi-Lipschitz continuity, we get

$$A_{i,j} \leq C_g \left(\mathcal{L}^2 K^\alpha \frac{1}{|j-i|m^2} + \mathcal{L}^{4-\alpha} K^2 \frac{1}{|j-i|^{\alpha-1} m^{4-\alpha}} \right).$$

Hence, we obtain

$$\sum_{i=1}^m \sum_{|j-i|>N} A_{i,j} \leq C_g \left(\mathcal{L}^2 K^\alpha \frac{\log m}{m} + \mathcal{L}^{4-\alpha} K^2 \frac{1}{m} \right).$$

□

Proposition 3.17. *We have*

$$\sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} B_{i,j} \leq C_g \mathcal{L}^{2p-\alpha p+2} K^2 \frac{1}{m^{2p-\alpha p+1}}.$$

Moreover, if $\alpha \leq 2$, then we have

$$\sum_{i=1}^m \sum_{|j-i|>N} B_{i,j} \leq C_g \mathcal{L}^2 K^\alpha \frac{\log m}{m}.$$

Proof. Since we have

$$||\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta_j^i t|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha| \leq C_g |\Delta_j^i t|^{2\alpha-1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|$$

using Lemma 3.15, we have

$$\begin{aligned} & \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{||\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta_j^i t|^\alpha - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha|}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta_j^i t|^\alpha} ds_1 ds_2 \\ &\leq C_g \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{|\Delta_j^i t|^{\alpha-1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha |\Delta s|^\alpha \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^\alpha} ds_1 ds_2 \\ &\leq C_g \frac{\max_{k=1,\dots,m} |\Delta_{k+1}^k t|^3}{|\Delta_j^i t|^{2\alpha+1}} \end{aligned}$$

by (3.1), the bi-Lipschitz continuity, and Lemma 3.8, and therefore, using Lemma 3.11.1, we have

$$B_{i,j} \leq C_g K^2 |\Delta_j^i t|^{1-\alpha} \max_{k=1,\dots,n} |\Delta_{k+1}^k t|^3.$$

Hence, by (3.1) and the bi-Lipschitz continuity, we get

$$\sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} B_{i,j} \leq C_g \mathcal{L}^{2p-\alpha p+2} K^2 \frac{1}{m^{2p-\alpha p+1}}.$$

If $\alpha \leq 2$, using Lemma 3.11.2 instead of Lemma 3.11.1, similarly we have

$$B_{i,j} \leq C_g |\Delta_j^i t|^{-1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|.$$

Therefore, by (3.1) and the bi-Lipschitz continuity, we get

$$\sum_{i=1}^m \sum_{|j-i|>N} B_{i,j} \leq C_g \mathcal{L}^2 K^\alpha \frac{\log m}{m}.$$

□

Proposition 3.18. *We have*

$$\sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} C_{i,j} \leq C_g \mathcal{L}^{2p-\alpha p+2} K^2 \frac{1}{m^{2p-\alpha p+1}}.$$

Moreover, if $\alpha \leq 2$, then we have

$$\sum_{i=1}^m \sum_{|j-i|>N} C_{i,j} \leq \begin{cases} C_g \mathcal{L}^{4-\alpha} K^2 \frac{1}{m^2} & \text{for } 0 < \alpha < 1, \\ C_g \mathcal{L}^3 K^2 \frac{\log m}{m^2} & \text{for } \alpha = 1, \\ C_g \mathcal{L}^{4-\alpha} K^2 \frac{1}{m^{3-\alpha}} & \text{for } 1 < \alpha \leq 2. \end{cases}$$

Proof. We may assume $j > i$ because of the symmetry of i and j . Also, since

$$|\Delta_j^i t| = \min \left\{ \sum_{k=i}^{j-1} |\Delta_{k+1}^k t|, \sum_{k=j}^{i+n-1} |\Delta_{k+1}^k t| \right\},$$

we may assume

$$|\Delta_j^i t| = \sum_{k=i}^{j-1} |\Delta_{k+1}^k t|.$$

Otherwise, we reduce to the above case by changing $\{j, i+n\}$ with $\{i, j\}$. In this situation, we have

$$\begin{aligned} |\Delta_j^i t|^\alpha - |\Delta_j^i a|^\alpha &\leq \left(\frac{\alpha}{2} + 1 \right) |\Delta_j^i t|^{\alpha-2} (|\Delta_j^i t| + |\Delta_j^i a|) ||\Delta_j^i t| - |\Delta_j^i a|| \\ &\leq 2 \left(\frac{\alpha}{2} + 1 \right) K^2 |\Delta_j^i t|^\alpha \max_{k=1,\dots,m} |\Delta_{k+1}^k t|^2 \end{aligned}$$

by Lemmas 3.9.2 and 3.10.2. Using the bi-Lipschitz continuity and (3.2), we have

$$C_{i,j} \leq C_g K^2 |\Delta_j^i t|^{-\alpha} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|^4.$$

By (3.1) and the bi-Lipschitz continuity, we get

$$\sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} C_{i,j} \leq C_g \mathcal{L}^{2p-\alpha p+2} K^2 \frac{1}{m^{2p-\alpha p+1}},$$

and if $\alpha \leq 2$, since we have

$$\sum_{i=1}^m \sum_{|j-i|>N} C_{i,j} \leq C_g \mathcal{L}^{4-\alpha} K^2 \frac{1}{m^{3-\alpha}} \sum_{k=1}^m \frac{1}{k^\alpha},$$

we get the claim by estimating $\sum_{k=1}^m k^{-\alpha}$. \square

Proposition 3.19. *We have*

$$\sum_{i=1}^m \sum_{|j-i|<N} |\Delta_j^i t|^{-(\alpha-2)p} D_{i,j} \leq C_g \mathcal{L}^{2p-\alpha p+4} K^2 \frac{1}{m^2}.$$

Moreover, if $\alpha \leq 2$, then we have

$$\sum_{i=1}^m \sum_{|j-i|<N} D_{i,j} \leq C_g \mathcal{L}^4 K^2 \frac{1}{m^2}.$$

Proof. Since we have

$$D_{i,j} \leq C_g K^2 \max_{k=1,\dots,m} |\Delta_{k+1}^k t|^4$$

using by Lemma 3.9, we get

$$\begin{aligned} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)p} D_{i,j} &\leq C_g K^2 \max_{k=1,\dots,m} |\Delta_{k+1}^k t|^4 \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)q} \\ &\leq C_g \mathcal{L}^{2p-\alpha p+4} K^2 \frac{1}{m^2}, \end{aligned}$$

and we get

$$\sum_{i=1}^m \sum_{|j-i|<N} D_{i,j} \leq C_g K^2 \max_{k=1,\dots,m} |\Delta_{k+1}^k t|^4 \sum_{i=1}^m \sum_{|j-i|<N} 1 \leq C_g \mathcal{L}^4 K^2 \frac{1}{m^2}$$

if $\alpha \leq 2$ by (3.1) and the bi-Lipschitz continuity. \square

Using Propositions 3.16–3.19, we get

$$\begin{aligned} |X_N - Y_N| &\leq C_g K^{2(p-1)} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)(p-1)} (A_{i,j} + B_{i,j} + C_{i,j}) \\ &\quad + C_g K^{2p} \sum_{i=1}^m \sum_{|j-i|>N} |\Delta_j^i t|^{-(\alpha-2)p} D_{i,j} \\ &\leq C_g (\mathcal{L}^{2p-\alpha p+2} K^{2p} + \mathcal{L}^{2p-\alpha p+4} K^{2p+2}) \frac{1}{m^{2p-\alpha p+1}}. \end{aligned}$$

Moreover, in the case where $\alpha \leq 2$, we have

$$\begin{aligned}
& |X_N - Y_N| \\
& \leq C_g K^{\alpha(p-1)} \sum_{i=1}^m \sum_{|j-i|>N} (A_{i,j} + B_{i,j} + C_{i,j}) + C_g K^{\alpha p} \sum_{i=1}^m \sum_{|j-i|>N} D_{i,j} \\
& \leq \begin{cases} C_g \left\{ \mathcal{L}^2 K^{\alpha p} \frac{\log m}{m} + \mathcal{L}^{4-\alpha} K^{\alpha p - \alpha + 2} \left(\frac{1}{m} + \frac{\log m}{m^2} \right) + \mathcal{L}^4 K^{\alpha p + 2} \frac{1}{m^2} \right\} & \text{for } 0 < \alpha < 1, \\ C_g \left\{ \mathcal{L}^2 K^p \frac{\log m}{m} + \mathcal{L}^3 K^{p+1} \left(\frac{1}{m} + \frac{1}{m^2} \right) + \mathcal{L}^4 K^{p+2} \frac{1}{m^2} \right\} & \text{for } \alpha = 1, \\ C_g \left\{ \mathcal{L}^2 K^{\alpha p} \frac{\log m}{m} + \mathcal{L}^{4-\alpha} K^{\alpha p - \alpha + 2} \left(\frac{1}{m} + \frac{1}{m^{3-\alpha}} \right) + \mathcal{L}^4 K^{\alpha p + 2} \frac{1}{m^2} \right\} & \text{for } 1 < \alpha \leq 2. \end{cases}
\end{aligned}$$

Thus, we get

$$|X_N - Y_N| \leq C_g (\mathcal{L}^2 K^{\alpha p} + \mathcal{L}^{4-\alpha} K^{\alpha p - \alpha + 2} + \mathcal{L}^4 K^{\alpha p + 2}) \frac{\log m}{m}.$$

This completes our proof of Theorem 3.2. \square

3.1.3 Proof of Theorem 3.3

Set

$$\varepsilon_m := \sum_{k=1}^m \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \frac{\|\Delta \tau\|_{\mathbb{R}^n}^{2p}}{|\Delta s|^{\alpha p}} ds_1 ds_2$$

for $\mathbf{f} \in W^{1+\sigma, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. Note that $\varepsilon_m < \infty$ because $\alpha p = 1 + 2\sigma p$. Since

$$\mu \left(\bigcup_{k=1}^m [t_k, t_{k+1}]^2 \right) \rightarrow 0$$

as $m \rightarrow \infty$, we have $\varepsilon_m \rightarrow 0$ from the absolute continuity of integrals for absolutely integrable functions, where μ is the Lebesgue measure on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$.

Using ε_m , set $N_m := m \max \left\{ \varepsilon_m^{-\frac{1}{4p}}, m^{-\frac{1}{6p}} \right\}$.

Estimates for the case where $|j - i| \leq N_m$ Set $\mathbf{c} = \left(1 + 2\frac{\bar{c}}{c}\right)^{-1}$. Let $m \in \mathbb{N}$ be sufficiently large such that $\{t_k\}$ satisfies

$$|\Delta_{k+1}^k t| \leq (1 + \mathbf{c}) \|\Delta_{t_{k+1}}^{t_k} \mathbf{f}\|_{\mathbb{R}^n}.$$

Then, since we have

$$\begin{aligned}
|\Delta_{k+1}^k t| & \leq (1 - \mathbf{c} + 2\mathbf{c}) \|\Delta_{t_{k+1}}^{t_k} \mathbf{f}\|_{\mathbb{R}^n} \\
& \leq (1 - \mathbf{c}) (|\Delta_{k+1}^k t| + |\Delta_k^{k+1} t|) \\
& = (1 - \mathbf{c}) |\Delta_{k+1}^{k-1} t|
\end{aligned}$$

using (3.1), we have

$$\mathbf{c} |\Delta_{j+1}^i t| \leq |\Delta_{j+1}^i t| - (1 - \mathbf{c}) |\Delta_{j+1}^{j-1} t| \leq |\Delta_j^i t|. \quad (3.5)$$

Therefore, we get

$$\begin{aligned}
& \sum_{i=1}^m \sum_{|j-i| \leq N_m} \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{t_{i+1}}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \\
& \leq C_b^{\alpha p} \frac{\bar{c}}{c} \left(\frac{\alpha}{2} + 1\right)^p \\
& \quad \times \sum_{i=1}^m \sum_{|j-i| \leq N_m} \left(\frac{|\Delta_j^i t|^2 - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^2}{|\Delta_j^i t|^{\alpha+2}} \right)^p |\Delta_i^{i-1} t| |\Delta_{j+1}^j t| \\
& \leq C_g \mathbf{c}^{-2p(\alpha+2)} \\
& \quad \times \sum_{i=1}^m \sum_{|j-i| \leq N_m} \int_{t_j}^{t_{j+1}} \int_{t_{i-1}}^{t_i} \left(\frac{\int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\Delta_{s_4}^{s_3} \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 ds_3 ds_4}{2|\Delta_s|^{\alpha+2}} \right)^p ds_1 ds_2 \\
& \leq C_g \mathbf{c}^{-2p(\alpha+2)} \sum_{i=1}^m \sum_{|j-i| \leq N_m} \int_{t_j}^{t_{j+1}} \int_{t_{i-1}}^{t_i} \mathcal{M}_{(\alpha,p)}(\mathbf{f})^p ds_1 ds_2 \\
& \rightarrow 0
\end{aligned}$$

as $m \rightarrow \infty$ by (3.2) and Lemmas 3.8.1 and 3.10.2. Here, we have used

$$\mu \left(\bigcup_{|j-i| \leq N_m} [t_{i-1}, t_i] \times [t_j, t_{j+1}] \right) \leq 2C_b^2 \bar{c}^2 \frac{1}{m} (N_m + 1) \rightarrow 0$$

and the absolute continuity of the integral. Also, we have

$$\sum_{i=1}^m \sum_{|j-i| \leq N_m} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2 \rightarrow 0,$$

which follows easily from the absolute continuity of the integral.

Estimates for the case where $|j - i| > N_m$ Note that, by (3.1) and the bi-Lipschitz continuity, we have

$$\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \geq C_b c \mathcal{L} \frac{N_m}{m}. \tag{3.6}$$

In order to prove Theorem 3.3, it suffices to prove

$$|X_{N_m} - Y_{N_m}| \rightarrow 0$$

as $m \rightarrow \infty$. To this end, observe that we have

$$\begin{aligned}
& |X_{N_m} - Y_{N_m}| \\
& \leq (p+1) \sum_{i=1}^m \sum_{|j-i| > N_m} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \max\{\mathcal{M}_\alpha(\mathbf{f}), \mathcal{M}_\alpha^m(\mathbf{p}_m)\}^{p-1} \\
& \quad \times |\mathcal{M}_\alpha(\mathbf{f}) - \mathcal{M}_\alpha^m(\mathbf{p}_m)| ds_1 ds_2 \\
& \quad + \sum_{i=1}^m \sum_{|j-i| > N_m} \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m) |\Delta_{i+1}^i t| |\Delta_{j+1}^j t| - \|\Delta_{t_{i+1}}^{t_i} \mathbf{f}\|_{\mathbb{R}^n} \|\Delta_{t_{j+1}}^{t_j} \mathbf{f}\|_{\mathbb{R}^n}| \\
& \leq C_g \sum_{i=1}^m \sum_{|j-i| > N_m} \frac{1}{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^{\alpha(p-1)}} (A_{i,j} + B_{i,j} + C_{i,j}) \\
& \quad + C_g \sum_{i=1}^m \sum_{|j-i| > N_m} \frac{1}{\|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^{\alpha p}} D_{i,j} \\
& \leq C_g \mathcal{L}^{-\alpha(p-1)} \sum_{i=1}^m \sum_{|j-i| > N_m} \max\left\{\varepsilon_m^{\frac{1}{4p}}, m^{-\frac{1}{6p}}\right\}^{-\alpha(p-1)} (A_{i,j} + B_{i,j} + C_{i,j}) \\
& \quad + C_g \mathcal{L}^{-\alpha p} \sum_{i=1}^m \sum_{|j-i| > N_m} \max\left\{\varepsilon_m^{\frac{1}{4p}}, m^{-\frac{1}{6p}}\right\}^{-\alpha p} D_{i,j}
\end{aligned}$$

by Lemma 3.10.2 and 3.6. We estimate these summations.

Proposition 3.20. *We have*

$$\sum_{i=1}^m \sum_{|j-i| > N_m} \max\left\{\varepsilon_m^{\frac{1}{4p}}, m^{-\frac{1}{6p}}\right\}^{-\alpha(p-1)} (A_{i,j} + B_{i,j}) \leq C_g \mathcal{L}^{2-\alpha} \frac{1}{m^{1-\frac{\alpha p+1}{6p}}}.$$

Proof. Because

$$\begin{aligned}
& |(|\Delta s|^2 - \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2) - (|\Delta_j^i t|^2 - \|\Delta_{t_j}^{t_i} \mathbf{f}\|_{\mathbb{R}^n}^2)| \\
& = \left| \int_A (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) ds_3 ds_4 - \int_B (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) ds_3 ds_4 \right| \\
& \leq C_g |\Delta s| |s_2 - t_j| + |\Delta_j^i t| |s_1 - t_i|,
\end{aligned}$$

where

$$\begin{aligned}
A &:= ([s_1, s_2] \times [t_j, s_2]) \cup ([t_j, s_2] \times [s_1, s_2]), \\
B &:= ([t_i, t_j] \times [t_i, s_1]) \cup ([t_i, s_1] \times [t_i, t_j]),
\end{aligned}$$

it holds that

$$\begin{aligned}
A_{i,j} &\leq C_g \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{|\Delta_j^i t|^{\alpha-1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|}{|\Delta s|^\alpha} ds_1 ds_2 \\
&\quad + C_g \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{|\Delta_j^i t|^\alpha \max_{k=1,\dots,m} |\Delta_{k+1}^k t|}{|\Delta s|^{2\alpha+1}} ds_1 ds_2 \\
&\quad + C_g \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \frac{|\Delta_j^i t|^{\alpha+1} \max_{k=1,\dots,m} |\Delta_{k+1}^k t|}{|\Delta s|^{2\alpha+2}} ds_1 ds_2 \\
&\leq C_g \frac{\max_{k=1,\dots,m} |\Delta_{k+1}^k t|^3}{|\Delta_j^i t|^{\alpha+1}}
\end{aligned}$$

by Lemmas 3.8, 3.10.2, and 3.15.2. Also, we have

$$B_{i,j} \leq C_g \frac{\max_{k=1,\dots,m} |\Delta_{k+1}^k t|^3}{|\Delta_j^i t|^{\alpha+1}}$$

using Lemmas 3.8.2 and 3.15, and the bi-Lipschitz continuity. Then, we get

$$A_{i,j} + B_{i,j} \leq C_g \mathcal{L}^{2-\alpha} \frac{1}{m^{3-\frac{\alpha+1}{6p}}}$$

by (3.6) and the bi-Lipschitz continuity. Therefore, we obtain

$$\sum_{i=1}^m \sum_{|j-i|>N_m} \max \left\{ \varepsilon_m^{\frac{1}{4p}}, m^{-\frac{1}{6p}} \right\}^{-\alpha(p-1)} (A_{i,j} + B_{i,j}) \leq C_g \mathcal{L}^{2-\alpha} \frac{1}{m^{1-\frac{\alpha p+1}{6p}}}.$$

□

Using (3.6), we obtain

$$C_{i,j} \leq C_g \mathcal{L}^{2-\alpha} \frac{1}{m^{\alpha+2-\frac{2}{p}}} \varepsilon_m^{\frac{4-\alpha}{4p}}, \quad D_{i,j} \leq C_g \mathcal{L}^2 \frac{1}{m^{\alpha+2-\frac{2}{p}}} \varepsilon_m^{\frac{1}{p}}.$$

Therefore, we can show the next lemma in a similar manner to the proof of Proposition 3.20.

Proposition 3.21. *We have*

$$\sum_{i=1}^m \sum_{|j-i|>N_m} \max \left\{ \varepsilon_m^{\frac{1}{4p}}, m^{-\frac{1}{6p}} \right\}^{-\alpha(p-1)} C_{i,j} \leq C_g \mathcal{L}^{2-\alpha} \varepsilon_m^{\frac{5\alpha p-8}{4p}},$$

and

$$\sum_{i=1}^m \sum_{|j-i|>N_m} \max \left\{ \varepsilon_m^{\frac{1}{4p}}, m^{-\frac{1}{6p}} \right\}^{-\alpha p} D_{i,j} \leq C_g \mathcal{L}^2 \varepsilon_m^{\frac{5\alpha p-8}{4p}}.$$

Using Propositions 3.20 and 3.21, we get

$$|X_{N_m} - Y_{N_m}| \leq C_g \mathcal{L}^{2-\alpha p} \left\{ \frac{1}{m^{1-\frac{\alpha p+1}{6p}}} + \varepsilon_m^{\frac{5\alpha p-8}{4p}} \right\} \rightarrow 0$$

as $m \rightarrow \infty$, and this proves Theorem 3.3. □

3.2 Γ -convergence

In this subsection, we prove that $\mathcal{E}_{(\alpha,p)}^m$ converges to $\mathcal{E}_{(\alpha,p)}$ in the sense of Γ -convergence. When we consider Γ -convergence, it is necessary that we consider the functionals $\mathcal{E}_{(\alpha,p)}^m$ and $\mathcal{E}_{(\alpha,p)}$ on a common set of simply closed curves. Hence, we need to extend their domains.

3.2.1 Preparation

In this subsubsection, we give the definition of Γ -convergence and introduce its fundamental property, and we extend the domains of $\mathcal{E}_{(\alpha,p)}^m$ and $\mathcal{E}_{(\alpha,p)}$.

Definition 3.22 (Γ -convergence). Let X be a metric space. If $\mathcal{F}^m : X \rightarrow \overline{\mathbb{R}}$ and $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$ satisfy the following two properties for all $x \in X$, we say that \mathcal{F}^m Γ -converges to \mathcal{F} on X and denote this by $\mathcal{F}^m \xrightarrow{\Gamma} \mathcal{F}$ on X .

1. (lim inf inequality) For all $\{x_m\} \subset X$ converging to x in X , we have

$$\mathcal{F}(x) \leq \liminf_{m \rightarrow \infty} \mathcal{F}^m(x_m).$$

2. (lim sup inequality) There exists $\{x_m\} \subset X$ converging to x in X and we have

$$\mathcal{F}(x) \geq \limsup_{m \rightarrow \infty} \mathcal{F}^m(x_m).$$

The following lemma states a sufficient condition under which the minimum of \mathcal{F} is less than that of \mathcal{F}^m . This lemma is useful for the investigation of minimality of functionals.

Lemma 3.23. Let (X, d_X) be a metric space, and let Y be a subspace of X . Assume that $\mathcal{F}^m, \mathcal{F} : X \rightarrow \overline{\mathbb{R}}$ satisfy the following.

1. We have

$$\mathcal{F}(x) \leq \liminf_{m \rightarrow \infty} \mathcal{F}^m(x_m)$$

for all $\{x_m\} \subset X$ such that $d_X(x_m, x) \rightarrow 0$ ($x \in X$) as $m \rightarrow \infty$.

2. For all $y \in Y$, there exists $\{y_m\} \subset X$ such that $d_X(y_m, y) \rightarrow 0$ as $m \rightarrow \infty$ and

$$\mathcal{F}(y) \geq \limsup_{m \rightarrow \infty} \mathcal{F}^m(y_m).$$

Then, for $z_m, z \in X$ satisfying

$$d_X(z_m, z) \rightarrow 0, \quad \left| \mathcal{F}^m(z_m) - \inf_X \mathcal{F}^m \right| \rightarrow 0$$

as $m \rightarrow \infty$, we have

$$\mathcal{F}(z) \leq \liminf_{m \rightarrow \infty} \inf_X \mathcal{F}^m \leq \inf_Y \mathcal{F}.$$

Next, we extend the domains of $\mathcal{E}_{(\alpha,p)}^m$ and $\mathcal{E}_{(\alpha,p)}$. For a given tame knot class \mathcal{K} , let $\mathcal{C}(\mathcal{K})$ be the set of simply closed curves of length 1 belonging to \mathcal{K} ,

and let $\mathcal{P}_m(\mathcal{K})$ be the set of equilateral m -gons with total length 1 belonging to \mathcal{K} . Also, we set

$$\mathcal{X}(\mathcal{K}) := (\mathcal{C}(\mathcal{K}) \cap C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \cup \bigcup_{m \in \mathbb{N}} \mathcal{P}_m(\mathcal{K}).$$

Furthermore, let $d_{L^1}, d_{W^{1,\infty}} : \mathcal{X}(\mathcal{K}) \times \mathcal{X}(\mathcal{K}) \rightarrow \mathbb{R}$ be two metric functions induced from the L^1 -norm or $W^{1,\infty}$ -norm, respectively. Then, we consider a metric function $d_X : \mathcal{X}(\mathcal{K}) \times \mathcal{X}(\mathcal{K}) \rightarrow \mathbb{R}$ for which there exist two constants $C_1, C_2 > 0$ such that

$$C_1 d_{L^1}(\mathbf{u}, \mathbf{v}) \leq d_X(\mathbf{u}, \mathbf{v}) \leq C_2 d_{W^{1,\infty}}(\mathbf{u}, \mathbf{v}) \quad (3.7)$$

for $\mathbf{u}, \mathbf{v} \in \mathcal{X}(\mathcal{K})$. For example, $d_X(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\|_{L^r}$ or $\|\mathbf{u} - \mathbf{v}\|_{W^{1,r}}$ ($r \in [1, \infty]$) satisfies (3.7) because \mathbb{R}/\mathbb{Z} is a bounded set. In what follows, we put

$$X := (\mathcal{X}(\mathcal{K}), d_X).$$

Moreover, let

$$Y := (\mathcal{C}(\mathcal{K}) \cap C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap W^{1+\sigma, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), d_X).$$

We extend the domain of $\mathcal{E}_{(\alpha, p)}^m$ to X as follows. For $\tilde{m} \neq m$, $p_m \in \mathcal{P}_m(\mathcal{K})$, and a simply closed curve \mathbf{f} , we *define*

$$\mathcal{E}_{(\alpha, p)}^{\tilde{m}}(p_m) := \infty, \quad \mathcal{E}_{(\alpha, p)}^{\tilde{m}}(\mathbf{f}) := \infty.$$

Concerning the extension of the domain of $\mathcal{E}_{(\alpha, p)}$, we obtain the following proposition.

Proposition 3.24. *Let \mathbf{p}_m be a polygon of length 1 with m edges and vertices $\mathbf{p}_m(a_i) \in \mathbb{R}^n$ ($i = 1, \dots, m$). Suppose $\alpha \in (0, \infty)$, $p \in [1, \infty)$ with $2 \leq \alpha p < 2 + \frac{1}{p}$. Then, we have $\mathbf{p}_m \notin W^{1+\sigma, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, that is, $\mathcal{E}_{(\alpha, p)}(\mathbf{p}_m) = \infty$.*

Proof. It is sufficient to prove

$$\mathbf{p}_m'' \notin W^{\frac{\alpha-3}{2}, 2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \quad (3.8)$$

for $2 \leq \alpha < 3$ because we have $W^{\sigma-1, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \subset W^{\frac{\alpha-3}{2}, 2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Note that there exist constant vectors $\mathbf{c}_j = {}^t(c_{j1} \ c_{j2} \ \dots \ c_{jn})$ ($1 \leq j \leq m$) such that

$$\mathbf{p}_m'' = \sum_{j=1}^m \mathbf{c}_j \delta_{a_j},$$

where δ_{a_j} is the Dirac measure supported at a_j .

In order to prove (3.8), we show

$$\sum_{k \in \mathbb{Z}} |k|^{\alpha-3} |(\mathbf{p}_m'')^\wedge(k)|^2 = \sum_{\ell=1}^n \sum_{k \in \mathbb{Z}} |k|^{\alpha-3} \left| \sum_{j=1}^m c_{j\ell} e^{-2\pi i k a_j} \right|^2 = \infty, \quad (3.9)$$

where $(\mathbf{p}_m'')^\wedge(k) = \mathcal{D}' \langle \mathbf{p}_m'', e^{-2\pi i k \cdot} \rangle_{\mathcal{D}'}$, and $i = \sqrt{-1}$. Fix $\ell = 1, \dots, n$. Then, we have

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} |k|^{\alpha-3} \left| \sum_{j=1}^m c_{j\ell} e^{-2\pi i k a_j} \right|^2 \\ &= \sum_{k \in \mathbb{Z}} |k|^{\alpha-3} \sum_{j=1}^m |c_{j\ell}|^2 + 2 \sum_{k \in \mathbb{Z}} |k|^{\alpha-3} \sum_{1 \leq j_1 < j_2 \leq m} c_{j_1\ell} c_{j_2\ell} \cos 2\pi k(a_{j_2} - a_{j_1}). \end{aligned}$$

It is obvious that the first term diverges to infinity, and the second term is bounded because the infinite series $\sum_{k=1}^{\infty} k^a \cos(ks)$ converges for $s \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and $a < 0$. Therefore, we get (3.9). \square

3.2.2 The Γ -convergence of $\mathcal{E}_{(\alpha,p)}^m$

Note that we prove the liminf inequality with respect to L^1 -topology and the limsup inequality with respect to $W^{1,\infty}$ -topology because we have to consider the liminf inequality for *all* polygonal sequences $\{\mathbf{p}_m\}$ and the limsup inequality for *a* polygonal sequence $\{\mathbf{p}_m\}$.

First, we prove the liminf inequality needed for proof of the Γ -convergence of $\mathcal{E}_{(\alpha,p)}^m$.

Theorem 3.25 (The liminf inequality). *Let $\alpha \in (0, \infty)$, $p \in [1, \infty)$. Assume that $\mathbf{p}_m, \mathbf{f} \in \mathcal{C}(\mathcal{K})$ satisfy*

$$\|\mathbf{p}_m - \mathbf{f}\|_{L^1} \rightarrow 0$$

as $m \rightarrow \infty$. Then, we have

$$\mathcal{E}_{(\alpha,p)}(\mathbf{f}) \leq \liminf_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m).$$

Proof. We may assume $\liminf_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) < \infty$. Note that $\mathbf{p}_m \in \mathcal{P}_m(\mathcal{K})$ by the way we extended the domain of $\mathcal{E}_{(\alpha,p)}^m$. Now, there exists $\{m_k\}_{k=1}^{\infty}$ such that

$$m_1 < m_2 < \dots \rightarrow \infty, \quad \liminf_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) = \lim_{k \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^{m_k}(\mathbf{p}_{m_k}).$$

Thus, there exists $\{\mathbf{p}_{m_{k(\nu)}}\}_{\nu=1}^{\infty}$ which is a subsequence of $\{m_k\}_{k=1}^{\infty}$ such that $\mathbf{p}_{m_{k(\nu)}} \rightarrow \mathbf{f}$ as $\nu \rightarrow \infty$ a.e. on \mathbb{R}/\mathbb{Z} . It is sufficient to prove the claim for $\{\mathbf{p}_{m_{k(\nu)}}\}_{\nu=1}^{\infty}$.

Now, we write $\mathbf{p}_{m_{k(\nu)}}$ as \mathbf{p}_m for simplicity. Let

$$s_1, s_2 \in \left\{ s \in \mathbb{R}/\mathbb{Z} \mid \lim_{m \rightarrow \infty} \mathbf{p}_m(s) = \mathbf{f}(s) \right\}, \quad s_1 \neq s_2 \pmod{\mathcal{L}\mathbb{Z}}.$$

For all $m \in \mathbb{N}$, we can put consecutive points $a_1^{(m)}, \dots, a_m^{(m)} \in \mathbb{R}/\mathbb{Z}$ which satisfy $|a_{k+1}^{(m)} - a_k^{(m)}| = \frac{1}{m}$ for $k = 1, \dots, m$ and such that there exists $i_m, j_m \in \{1, \dots, m\}$ satisfying

$$(s_1, s_2) \in [a_{i_m}^{(m)}, a_{i_m+1}^{(m)}] \times [a_{j_m}^{(m)}, a_{j_m+1}^{(m)}].$$

Then, we have

$$\begin{aligned} \sum_{\substack{i,j=1 \\ i \neq j}}^m \mathcal{M}_{(\alpha,p)}^m(\mathbf{p}_m)(a_i^{(m)}, a_j^{(m)}) \chi_{[a_{i_m}^{(m)}, a_{i_m+1}^{(m)}] \times [a_{j_m}^{(m)}, a_{j_m+1}^{(m)}]}(s_1, s_2) \\ \rightarrow \mathcal{M}_{(\alpha,p)}(\mathbf{f})(s_1, s_2) \end{aligned}$$

as $m \rightarrow \infty$. Using Fatou's lemma, we have

$$\begin{aligned} \mathcal{E}_{(\alpha,p)}(\mathbf{f}) &= \iint_{(\mathbb{R}/\mathbb{Z})^2} \mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2 \\ &= \iint_{(\mathbb{R}/\mathbb{Z})^2} \lim_{m \rightarrow \infty} \sum_{\substack{i,j=1 \\ i \neq j}}^m \mathcal{M}_{(\alpha,p)}(\mathbf{p}_m)(a_i^{(m)}, a_j^{(m)}) \chi_{[a_{i_m}^{(m)}, a_{i_m+1}^{(m)}] \times [a_{j_m}^{(m)}, a_{j_m+1}^{(m)}]}(s_1, s_2) ds_1 ds_2 \\ &\leq \liminf_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) \end{aligned}$$

because of the definition of $\{a_k^{(m)}\}_{k=1}^m$. \square

Furthermore, by Ascoli-Arzelà's theorem, we get the following corollary.

Corollary 3.26. *Assume that $\mathbf{p}_m \in \mathcal{P}_m(\mathcal{K})$ satisfy that*

$$\sup_{m \in \mathbb{N}} \|\mathbf{p}_m\|_{L^\infty} < \infty, \quad \sup_{m \in \mathbb{N}} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) < \infty.$$

Then, there exists a subsequence $\{\mathbf{p}_{m_j}\}$ and $\mathbf{f} \in W^{1+\sigma, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that $\|\mathbf{p}_{m_j} - \mathbf{f}\|_{L^1} \rightarrow 0$ as $j \rightarrow \infty$ for $\alpha \in (0, \infty)$, $p \in [1, \infty)$ with $2 \leq \alpha p < 2p + 1$.

The following claim is a strong version of the limsup inequality for $\mathbf{f} \in W^{1+\sigma, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. We can prove it using the method of proof of [32, Proposition 4.1].

Theorem 3.27 (A strong version of the limsup equality). *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ with $2 \leq \alpha p < 2p + 1$, and let $\mathbf{f} \in \mathcal{C}(\mathcal{K}) \cap C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap W^{1+\sigma, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then, there exists $\mathbf{p}_m \in \mathcal{P}_m(\mathcal{K})$ such that*

$$\lim_{m \rightarrow \infty} \|\mathbf{p}_m - \mathbf{f}\|_{W^{1,\infty}} = 0, \quad \lim_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) = \mathcal{E}_{(\alpha,p)}(\mathbf{f}).$$

Next, we show that $\mathcal{E}_{(\alpha,p)}^m$ Γ -converges to $\mathcal{E}_{(\alpha,p)}$ using previous results.

Theorem 3.28 (Γ -convergence of $\mathcal{E}_{(\alpha,p)}^m$). *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ with $2 \leq \alpha p < 2p + 1$. Then, we have*

$$\mathcal{E}_{(\alpha,p)}^m \xrightarrow{\Gamma} \mathcal{E}_{(\alpha,p)} \text{ on } X. \quad (3.10)$$

Proof. Put $\mathbf{f} \in X$. If $\mathbf{p}_m \in X$ satisfies $d_X(\mathbf{p}_m, \mathbf{f}) \rightarrow 0$, we have $\|\mathbf{p}_m - \mathbf{f}\|_{L^1} \leq C_1^{-1} d_X(\mathbf{p}_m, \mathbf{f}) \rightarrow 0$. Then, we have

$$\mathcal{E}_{(\alpha,p)}(\mathbf{f}) \leq \liminf_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m)$$

using Theorem 3.25. This implies that $\mathcal{E}_{(\alpha,p)}^m$ satisfies the lim inf inequality.

Now, we prove the lim sup inequality. The claim is obvious in the case where $\mathbf{f} \in X \setminus Y$. Therefore, let $\mathbf{f} \in Y$. Then, there exists $\mathbf{p}_m \in \mathcal{P}_m(\mathcal{K})$ such that

$$\lim_{m \rightarrow \infty} d_X(\mathbf{p}_m, \mathbf{f}) = 0, \quad \lim_{m \rightarrow \infty} \mathcal{E}_{(\alpha, p)}^m(\mathbf{p}_m) = \mathcal{E}_{(\alpha, p)}(\mathbf{f}) \quad (3.11)$$

by Theorem 3.27 and (3.7). In particular, we have

$$\mathcal{E}_{(\alpha, p)}(\mathbf{f}) \geq \limsup_{m \rightarrow \infty} \mathcal{E}_{(\alpha, p)}^m(\mathbf{p}_m).$$

□

Remark 3.29. (3.11) implies that $\mathcal{E}_{(\alpha, p)}^m$ not only Γ -converges to $\mathcal{E}_{(\alpha, p)}$ but also satisfies the assumption of Lemma 3.23.

The following corollary suggests the following: assume that a polygonal sequence has values of the discrete energy are sufficiently close to the minimum value for all numbers of vertices. Then, this sequence converges to a curve, which is a right circle by [1].

Corollary 3.30. *If $\mathbf{p}_m \in \mathcal{P}_m(\mathcal{K})$ and $\mathbf{f} \in \mathcal{C}(\mathcal{K})$ satisfy*

$$\left| \inf_{\mathcal{P}_m(\mathcal{K})} \mathcal{E}_{(\alpha, p)}^m - \mathcal{E}_{(\alpha, p)}^m(\mathbf{p}_m) \right| \rightarrow 0, \quad d_X(\mathbf{p}_m, \mathbf{f}) \rightarrow 0,$$

then \mathbf{f} is the minimizer of $\mathcal{E}_{(\alpha, p)}$ in $\mathcal{C}(\mathcal{K})$, and we have

$$\lim_{m \rightarrow \infty} \mathcal{E}_{(\alpha, p)}^m(\mathbf{p}_m) = \mathcal{E}_{(\alpha, p)}(\mathbf{f}).$$

3.3 Minimizers of $\mathcal{E}_{(\alpha, p)}^m$

In this subsection, we consider minimizers of a generalized discrete energy using techniques of [1]. In what follows, we set $\Omega := \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq y\}$.

Theorem 3.31. *Let $F : \Omega \rightarrow \mathbb{R}$ be a function such that, if we set $g_y(u) = F(\sqrt{u}, y)$ for $u \in (0, y^2]$ and $y \in (0, 1/2)$, then g_y is decreasing and convex. For an m -gon with total length 1, set*

$$\mathcal{E}_F^m(\mathbf{p}_m) := \sum_{\substack{i, j=1 \\ i \neq j}}^m F(\|\Delta_{a_j}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n}, |\Delta_j^i a|) \|\Delta_{a_{i+1}}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n} \|\Delta_{a_{j+1}}^{a_j} \mathbf{p}_m\|_{\mathbb{R}^n}.$$

Moreover, for $0 < a < b$, set $[a]_b := \min\{a, b - a\}$. Then, if $\mathbf{p}_m \in \mathcal{P}_m(\mathcal{K})$, we have

$$\mathcal{E}_F^m(\mathbf{p}_m) \geq \frac{1}{m} \sum_{k=1}^{m-1} F\left(\frac{1}{m} \frac{\sin([k]_m \pi / m)}{\sin(\pi / m)}, |\Delta_0^k a|\right),$$

and the minimizers of \mathcal{E}_F^m are regular m -gons.

The proof of Theorem 3.31 makes use of the following lemma.

Lemma 3.32 ([14, Theorem II], [1, Lemma 7]). *Let $m \geq 4$, and put $k = 1, \dots, m$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing and concave function. Then, there exists $c > 0$ with $|\mathbf{x}_{i+1} - \mathbf{x}_i| \leq c$ such that*

$$\frac{1}{m} \sum_{i=1}^m f(\|\mathbf{x}_{i+k} - \mathbf{x}_i\|_{\mathbb{R}^n}^2) \leq f\left(c^2 \frac{\sin^2([k]_m \pi/m)}{\sin^2(\pi/m)}\right)$$

for all $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ with $\mathbf{x}_{m+i} = \mathbf{x}_i$ for $i = 1, \dots, m$. Equality holds in the above inequality only when the polygon which is made by joining $\mathbf{x}_1, \dots, \mathbf{x}_m$ by segments in turn is a regular m -gon.

Proof of Theorem 3.31. Since \mathbf{p}_m is an equilateral polygon, we have

$$\mathcal{E}_F^m(\mathbf{p}_m) = \frac{1}{m^2} \sum_{k=1}^{m-1} \sum_{i=1}^m F(\|\Delta_{a_{i+k}}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n}, |\Delta_0^k a|).$$

For $k = 1, \dots, m$, set

$$f_k(x) = \begin{cases} -F(\sqrt{x}, |\Delta_0^k a|) & \text{for } 0 < x < |\Delta_0^k a|^2, \\ -F(|\Delta_0^k a|, |\Delta_0^k a|) & \text{for } x \geq |\Delta_0^k a|^2. \end{cases}$$

Then, $f_k(x)$ is an increasing and concave function on $0 < x < |a_k - a_0|^2$. Hence, using Lemma 3.32, we have

$$\begin{aligned} \frac{1}{m} \sum_{i=1}^m F(\|\Delta_{a_{i+k}}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n}, |\Delta_0^k a|) &= -\frac{1}{m} \sum_{i=1}^m f_k(\|\Delta_{a_{i+k}}^{a_i} \mathbf{p}_m\|_{\mathbb{R}^n}^2) \\ &\geq -f_k\left(\frac{1}{m^2} \frac{\sin^2([k]_m \pi/m)}{\sin^2(\pi/m)}\right), \end{aligned}$$

where the equality holds only when \mathbf{p}_m is a regular m -gon by the condition of equality in Lemma 3.32.

Let $\mathbf{g}_m \in \mathcal{P}_m(\mathcal{K})$ be a regular m -gon, and suppose $1 \leq k \leq m$. Then, we have

$$\begin{aligned} \frac{1}{m} \frac{\sin([k]_m \pi/m)}{\sin(\pi/m)} &= \|\Delta_{a_0}^{a_k} \mathbf{g}_m\|_{\mathbb{R}^n} = \|\Delta_{a_{i+k}}^{a_i} \mathbf{g}_m\|_{\mathbb{R}^n}, \\ |\Delta_0^k a| &= |\Delta_{i+k}^i a| \end{aligned}$$

for all $i = 1, \dots, m-1$. Hence, we obtain

$$\begin{aligned} \mathcal{E}_F^m(\mathbf{p}_m) &\geq -\frac{1}{m} \sum_{k=1}^{m-1} f_k\left(\frac{1}{m^2} \frac{\sin^2([k]_m \pi/m)}{\sin^2(\pi/m)}\right) \\ &= \frac{1}{m} \sum_{k=1}^{m-1} F\left(\frac{1}{m} \frac{\sin([k]_m \pi/m)}{\sin(\pi/m)}, |\Delta_0^k a|\right) = \mathcal{E}_F^m(\mathbf{g}_m). \end{aligned}$$

Therefore, minimizers of \mathcal{E}_F^m are regular m -gons. \square

Applying Theorem 3.31 to $\mathcal{E}_{(\alpha, p)}^m$, we obtain the following corollary.

Corollary 3.33. *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$. Then, for all equilateral m -gons \mathbf{p}_m , we have*

$$\mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) \geq m^{\alpha p - 1} \sum_{k=1}^{m-1} \left(\frac{\sin^\alpha(\pi/m)}{\sin^\alpha([k]_m \pi/m)} - \frac{1}{[k]_m^\alpha} \right)^p$$

with equality if and only if \mathbf{p}_m is a regular m -gon.

Proof. For $(x, y) \in \Omega$, set

$$F(x, y) := \left(\frac{1}{x^\alpha} - \frac{1}{y^\alpha} \right)^p$$

Then, we have $F(\sqrt{u}, y)$ is decreasing and convex on $u \in (0, y^2]$ whenever $y \in (0, 1/2)$. Therefore, F satisfies the assumption of Theorem 3.31.

Using Theorem 3.31, we obtain

$$\mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) \geq m^{\alpha p - 1} \sum_{k=1}^{m-1} \left(\frac{\sin^\alpha(\pi/m)}{\sin^\alpha([k]_m \pi/m)} - \frac{1}{[k]_m^\alpha} \right)^p$$

for all equilateral polygons \mathbf{p}_m with m edges. By the condition of equality in Lemma 3.32, equality holds in the above inequality only when \mathbf{p}_m is a regular m -gon. \square

By Corollary 3.33, we obtain the following claim about the minimizers of $\mathcal{E}_{(\alpha,p)}^m$.

Theorem 3.34 (Minimizers of $\mathcal{E}_{(\alpha,p)}^m$). *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$. Then, minimizers of $\mathcal{E}_{(\alpha,p)}^m$ in the set of equilateral m -gons are regular polygons. Especially, a regular m -gon is the only minimizer except for congruent transformations and similar transformations.*

From Theorem 3.34, we immediately obtained the following property of minimizers of $\mathcal{E}_{(\alpha,p)}^m$.

Corollary 3.35. *Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$, and let \mathbf{p}_m satisfy $\mathcal{E}_{(\alpha,p)}^m(\mathbf{p}_m) = \inf_{\mathcal{P}_m(\mathcal{K})} \mathcal{E}_{(\alpha,p)}^m$. Then, there exists a similar transformation such that $\{\mathbf{p}_m\}$ converges to a right circle in the sense of $W^{1,\infty}$ as $m \rightarrow \infty$.*

3.4 Numerical experiments of the values of O'Hara's energy of a right circle

In this subsection, we show some examples of numerical experiments. Let \mathbf{g}_m be a regular n -gon. By the property of Γ -convergence and Corollary ??, we have

$$\inf \mathcal{E}_{(\alpha,p)} = \lim_{m \rightarrow \infty} \mathcal{E}_{(\alpha,p)}^m(\mathbf{g}_m),$$

where $2 \leq \alpha p < 2p + 1$, and the infimum in the left-hand side is taken over the space of all embedded curves in \mathbb{R}^n . Therefore, considering [1], we can calculate O'Hara's energy of a right circle numerically by increasing the number of vertices m in $\mathcal{E}_{(\alpha,p)}^m(\mathbf{g}_m)$. Moreover, we calculate energies $\mathcal{L}(\mathbf{g}_m)^{\alpha p - 2} \mathcal{E}_{(\alpha,p)}^m(\mathbf{g}_m)$, where

	Number of vertices m	α					
		2	2.1	2.3	2.5	2.7	2.9
D	4	1	1.147365	1.500936	1.949372	2.516555	3.232177
	8	2.325253	2.739102	3.780728	5.187945	7.085586	9.640817
	16	3.134412	3.754475	5.372714	7.672833	10.95137	15.64031
	32	3.562332	4.320470	6.363289	9.408493	13.99728	20.99456
	64	3.780229	4.626457	6.969742	10.61781	16.42130	25.87401
	128	3.889916	4.790718	7.341313	11.46626	18.37252	30.38526
	256	3.944913	4.878765	7.569466	12.06415	19.95194	34.58121
	512	3.972446	4.925946	7.709746	12.48634	21.23325	38.49223
	1024	3.986220	4.951228	7.796054	12.78472	22.27356	42.14019
	2048	3.993109	4.964776	7.849171	12.99567	23.11844	45.54354
	4096	3.996555	4.972036	7.881865	13.14482	23.80467	48.71889
	8192	3.998277	4.975926	7.901990	13.25028	24.36205	51.68157
	16384	3.999139	4.978011	7.914378	13.32485	24.81478	54.44584
	32768	3.999569	4.979129	7.922004	13.37758	25.18251	57.02499
	65536	3.999785	4.979727	7.926698	13.41487	25.48120	59.43143
	131072	3.999892	4.980048	7.929588	13.44124	25.72381	61.67671
	262144	3.999946	4.980220	7.931366	13.45988	25.92087	63.77161
	524288	3.999973	4.980312	7.932461	13.47306	26.08094	65.72639
	1048576	3.999987	4.980362	7.933135	13.48238	26.21093	67.55013
	2097152	4.000004	4.980401	7.933568	13.48900	26.31651	69.25143
	4194304	3.999997	4.980402	7.933807	13.49362	26.40257	70.84417
	Exact values	4	4.980419	7.934215	13.50489	26.77342	92.95965
	D/A	0.999999	0.999997	0.999949	0.999166	0.986148	0.762096

Table 1: Numerical calculation of $\mathcal{L}(\mathbf{f}_0)^{\alpha-2}\mathcal{E}_{(\alpha,1)}(\mathbf{f}_0)$ when $2 \leq \alpha < 3$ (D: Values of discretization, D/A: Divisions of value of discretization when $m = 4194304$ by exact value)

$\mathcal{L}(\mathbf{g}_m)$ is the total length of \mathbf{g}_m , because the factor $\mathcal{L}(\mathbf{g}_m)^{\alpha p-2}$ makes these energies scale invariant. Note that in [11], the values of O'Hara's energy $\mathcal{E}_{(\alpha,1)}$ ($2 \leq \alpha < 3$) of a right circle \mathbf{f}_0 are obtained and expressed by

$$\mathcal{E}_{(\alpha,1)}(\mathbf{f}_0) = \frac{1}{(\alpha-1)\mathcal{L}(\mathbf{f}_0)^{\alpha-2}} \left\{ \frac{(\alpha-2)\pi^{\alpha-\frac{1}{2}}\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{4-\alpha}{2})} + 2^\alpha \right\}.$$

Here, we compare $\mathcal{L}(\mathbf{g}_m)^{\alpha-2}\mathcal{E}_{(\alpha,1)}^m(\mathbf{g}_m)$ with $\mathcal{L}(\mathbf{f}_0)^{\alpha-2}\mathcal{E}_{(\alpha,1)}(\mathbf{f}_0)$, and we tabulate the result of numerical calculation when $\alpha = 2, 2.1, 2.3, 2.5, 2.7, 2.9$ in Table 1. It follows from Theorem 3.2 that the convergence becomes slow, when α approaches to 3. We can see this fact from Table 1. Moreover, we investigate the behavior of

$$e_\alpha(m) := m^{\alpha-2} \left| \mathcal{L}(\mathbf{f}_0)^{\alpha-2}\mathcal{E}_{(\alpha,1)}(\mathbf{f}_0) - \mathcal{L}(\mathbf{g}_m)^{\alpha-2}\mathcal{E}_{(\alpha,1)}^m(\mathbf{g}_m) \right|$$

when number of vertices m increases, where $2 \leq \alpha < 3$. We expect that $e_\alpha(m)$ converges to a constant if the order of convergence in Theorem 3.2 is optimal, and we can see that this conjecture seems to be true in Figure 1.

Now, we show some interesting examples of $\mathcal{E}_{(\alpha,p)}(\mathbf{g}_m)$ when the number of vertices m is not so large. As we can see in Figure 2, $\mathcal{L}(\mathbf{g}_{2^k})^{58}\mathcal{E}_{(2,30)}^{2^k}(\mathbf{g}_{2^k})$ for $k \in \mathbb{N}$ takes the maximum value at $k = 4$, and the larger the value that p takes, the larger the maximum value is. Therefore, we show a figure of $\mathcal{E}_{2,30}^m(\mathbf{g}_m)$ for $m \geq 100$ in Figure 3. Note that $\mathcal{L}(\mathbf{g}_{2^\ell+1})^{58}\mathcal{E}_{(2,30)}^{2^\ell+1}(\mathbf{g}_{2^\ell+1})$ for $\ell \geq 2$ is monotonically increasing. However, $\mathcal{L}(\mathbf{g}_{2^\ell})^{58}\mathcal{E}_{(2,30)}^{2^\ell}(\mathbf{g}_{2^\ell})$ for $\ell \geq 2$ takes the maximum at $\ell = 10$

($m = 20$) and is decreasing to the value of $\mathcal{L}(\mathbf{f}_0)^{58}\mathcal{E}_{(2,30)}(\mathbf{f}_0)$ when $\ell \geq 10$. The cause of this phenomena we think is as follows: when m is much less than 20, the discrete energy is a summation which consists of a small number of terms with large value. On the other hand, when m is much larger than 20, the discrete energy is a summation which consists of a large number of terms with small value. If m is around 20, then the number of terms and the size of each term might make the energy large. This phenomena will be remarkable when p becomes large. To the author, the reason seems to be as follows: when p is large, the difference of the size of the terms becomes bigger. Moreover, we observe from Figure 3 that the energy with even m is larger than that with odd m . The energy density becomes large when the difference between the intrinsic distance and the extrinsic distance is large. The difference maximizes when two points are antipodal, which is a situation that occurs only when m is even.

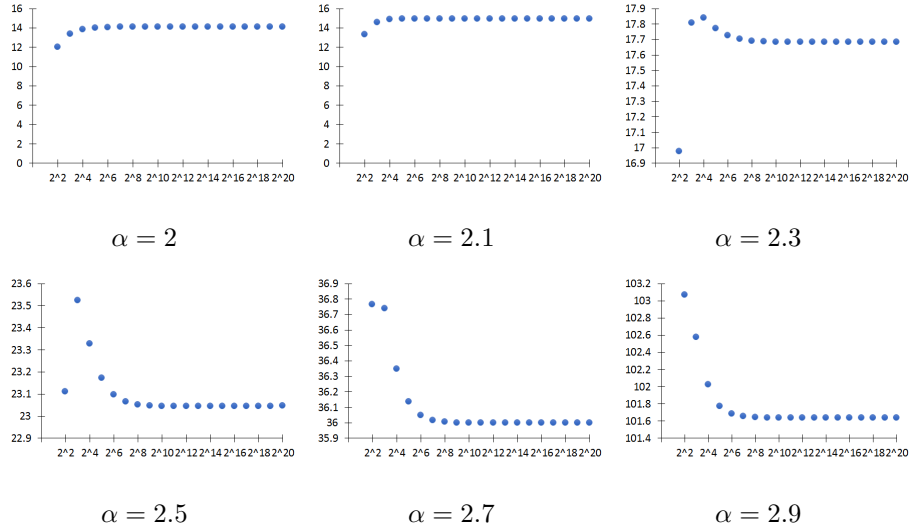


Figure 1: Graphs of $e_\alpha(m)$ (The vertical and horizontal axes show values of $e_\alpha(m)$ and numbers of vertices $m = 2^k$ ($k = 2, 3, \dots, 20$), respectively)

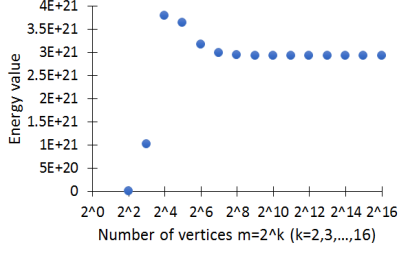


Figure 2: Values of $\mathcal{E}_{(2,30)}^{2^k}(\mathbf{g}_{2^k})$

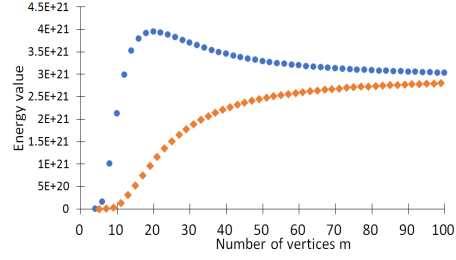


Figure 3: Values of $\mathcal{E}_{(2,30)}^m(\mathbf{g}_m)$ when $m \leq 100$ (Blue, round points and orange, diamond points show values when m is even and odd, respectively)

4 Finiteness of a generalization of O'Hara's energy

Although minimizers of \mathcal{E}_F were obtained in [1], cf. Theorem 2.3, other fundamental properties of \mathcal{E}_F have not been investigated in the existing literature. Here, we consider the problem of characterizing the finiteness of generalized energies. At the level of generality of \mathcal{E}_F , this seems to be a very difficult problem so we restrict ourself to the special case.

Therefore, in this section, we study a generalized energy

$$\mathcal{E}_{(\Phi,p)}(\mathbf{f}) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{\Phi(\|\Delta \mathbf{f}\|_{\mathbb{R}^n})} - \frac{1}{\Phi(\mathcal{D}(\mathbf{f}))} \right)^p ds_1 ds_2$$

under suitable assumptions on Φ , and we should see that such a generalization brings out certain properties of $\mathcal{E}_{(\alpha,p)}$ in a clearer manner. It is known that the finiteness of $\mathcal{E}_{(\alpha,p)}(\mathbf{f})$ implies bi-Lipschitz continuity and some regularity of \mathbf{f} , see [3]. We generalize this fact to the case $\mathcal{E}_{(\Phi,p)}$, and we clarify what properties of Φ give rise to these properties of \mathbf{f} . In particular, we define a function space $W^{k+\Phi,p}$ which is a generalization of the Sobolev-Slobodeckii space, and discuss the relation between our new space and the domain of $\mathcal{E}_{(\Phi,p)}$.

Definition 4.1. Let Ω be a non-empty subset of \mathbb{R} . For $p \in [1, \infty)$, $k \in \mathbb{N} \cup \{0\}$, and measurable function $\Psi : [0, \infty) \rightarrow [0, \infty)$, we define

$$W^{k+\Psi,p}(\Omega, \mathbb{R}^n) := \{\mathbf{f} \in W^{k,p}(\Omega, \mathbb{R}^n) \mid [\mathbf{f}^{(k)}]_{\Psi,p} < \infty\},$$

where

$$[\mathbf{f}^{(k)}]_{\Psi,p} := \left(\iint_{\Omega \times \Omega} \frac{\|\Delta \mathbf{f}^{(k)}\|_{\mathbb{R}^n}^p}{\Psi(|\Delta s|)^p} \frac{1}{|\Delta s|} ds_1 ds_2 \right)^{\frac{1}{p}}.$$

We equip the space $W^{k+\Psi,p}$ with the norm

$$\|\mathbf{f}\|_{W^{k+\Psi,p}} := \|\mathbf{f}\|_{W^{k,p}} + [\mathbf{f}^{(k)}]_{\Psi,p},$$

in which case it becomes a Banach space. Moreover, the dual space of $W^{\Psi,p}(\Omega, \mathbb{R}^n)$ is characterized by the following proposition which is proven by using the argument of [25, pp. 38–42]. The author gives its proof below, since [25] is written in Japanese, and since he is unable to find suitable references in English.

Proposition 4.2. *Let Ω be a non-empty subset of \mathbb{R} , and let $\Psi : [0, \infty) \rightarrow [0, \infty)$ be a measurable function. For $p \in [1, \infty)$, let $q \in (1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $T \in (W^{\Psi,p}(\Omega, \mathbb{R}^n))'$, there exists $(\phi, \psi) \in L^q(\Omega, \mathbb{R}^n) \times L^q(\Omega \times \Omega, \mathbb{R}^n)$ such that*

$$\|T\|_{(W^{\Psi,p}(\Omega))'} = \max\{\|\phi\|_{L^q(\Omega)}, \|\psi\|_{L^q(\Omega \times \Omega)}\}$$

and

$$T(\mathbf{f}) = \int_{\Omega} \mathbf{f}(s) \cdot \phi(s) ds + \iint_{\Omega \times \Omega} \left(\frac{\Delta \mathbf{f}}{\Psi(|\Delta s|)} \cdot \psi(s_1, s_2) \right) \frac{1}{|\Delta s|^{\frac{1}{p}}} ds_1 ds_2$$

for any $\mathbf{f} \in W^{\Psi,p}(\Omega, \mathbb{R}^n)$. In particular, if $1 < p < \infty$, then $W^{\Psi,p}(\Omega, \mathbb{R}^n)$ is reflexive.

Proof. The map $\tau : W^{\Psi,p}(\Omega, \mathbb{R}^n) \rightarrow L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n)$ is defined by

$$\tau(\mathbf{f}) = \left(\mathbf{f}, \frac{\Delta \mathbf{f}}{\Psi(|\Delta s|)} \frac{1}{|\Delta s|^{1/p}} \right)$$

for $\mathbf{f} \in W^{\Psi,p}(\Omega, \mathbb{R}^n)$. Then, it holds that τ is isometric and isomorphic. Therefore, $\tau(W^{\Psi,p}(\Omega, \mathbb{R}^n))$ is equal to a closed subspace $\mathcal{W} \subset L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n)$. Hence, $T \circ \tau^{-1}$ is a linear functional on \mathcal{W} , and we have $\|T \circ \tau^{-1}\|_{\mathcal{W}'} = \|T\|_{(W^{\Psi,p}(\Omega, \mathbb{R}^n))'}$. By the Hahn-Banach theorem, we obtain a bounded linear functional $G \in (L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n))'$ such that

$$(T \circ \tau^{-1})(\mathbf{f}) = G(\mathbf{f})$$

for $\mathbf{f} \in \mathcal{W}$.

The spaces $(L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n))'$ and $(L^p(\Omega, \mathbb{R}^n))' \times (L^p(\Omega \times \Omega, \mathbb{R}^n))'$ are isometric and isomorphic. Indeed, the map $\tau_1 : (L^p(\Omega, \mathbb{R}^n))' \times (L^p(\Omega \times \Omega, \mathbb{R}^n))' \rightarrow (L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n))'$ given by

$$(\tau_1(\varphi, \psi))(\mathbf{u}, \mathbf{v}) = \varphi(\mathbf{u}) + \psi(\mathbf{v})$$

for $(\varphi, \psi) \in (L^p(\Omega, \mathbb{R}^n))' \times (L^p(\Omega \times \Omega, \mathbb{R}^n))'$ and $(\mathbf{u}, \mathbf{v}) \in L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n)$ is isometric and isomorphic. Note that

$$\|(\varphi, \psi)\|_{(L^p(\Omega))' \times (L^p(\Omega \times \Omega, \mathbb{R}^n))'} = \max\{\|\varphi\|_{(L^p(\Omega, \mathbb{R}^n))'}, \|\psi\|_{(L^p(\Omega \times \Omega, \mathbb{R}^n))'}\}.$$

Moreover, the map $\tau_2 : L^q(\Omega, \mathbb{R}^n) \rightarrow (L^p(\Omega, \mathbb{R}^n))'$ defined by

$$(\tau_2(\phi))(\mathbf{u}) = \int_{\Omega} \mathbf{u}(s) \cdot \phi(s) ds$$

for $\phi \in L^q(\Omega, \mathbb{R}^n)$ and $\mathbf{u} \in L^p(\Omega, \mathbb{R}^n)$ is isometric and isomorphic. Therefore, we have $(L^p(\Omega, \mathbb{R}^n))' \cong L^q(\Omega, \mathbb{R}^n)$.

Hence, there exists $(\phi, \psi) \in L^q(\Omega, \mathbb{R}^n) \times L^q(\Omega \times \Omega, \mathbb{R}^n)$ such that

$$\begin{aligned} \|G\|_{(L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n))'} &= \max\{\|\phi\|_{L^q(\Omega, \mathbb{R}^n)}, \|\psi\|_{L^q(\Omega \times \Omega, \mathbb{R}^n)}\}, \\ G((\mathbf{u}, \mathbf{v})) &= \int_{\Omega} \mathbf{u}(s) \cdot \phi(s) ds + \iint_{\Omega \times \Omega} \mathbf{v}(s_1, s_2) \cdot \psi(s_1, s_2) ds_1 ds_2 \end{aligned}$$

for $(\mathbf{u}, \mathbf{v}) \in L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n)$. Since $G \circ \tau = T$ and from the definition of τ , it follows that

$$\begin{aligned} \|T\|_{(W^{\Psi,p}(\Omega, \mathbb{R}^n))'} &= \max\{\|\phi\|_{L^q(\Omega, \mathbb{R}^n)}, \|\psi\|_{L^q(\Omega \times \Omega, \mathbb{R}^n)}\}, \\ T(\mathbf{f}) &= \int_{\Omega} \mathbf{f}(s) \cdot \phi(s) ds + \iint_{\Omega \times \Omega} \left(\frac{\Delta \mathbf{f}}{\Psi(|\Delta s|)} \cdot \psi(s_1, s_2) \right) \frac{1}{|\Delta s|^{1/p}} ds_1 ds_2 \end{aligned}$$

for $\mathbf{f} \in W^{\Psi,p}(\Omega, \mathbb{R}^n)$.

As mentioned previously, $W^{\Psi,p}(\Omega, \mathbb{R}^n)$ and the closed subspace $\mathcal{W} \subset L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n)$ are isometric and isomorphic. Moreover, if $1 < p < \infty$, \mathcal{W} is reflexive because of the reflexivity of $L^p(\Omega, \mathbb{R}^n) \times L^p(\Omega \times \Omega, \mathbb{R}^n)$. Therefore, $W^{\Psi,p}(\Omega, \mathbb{R}^n)$ is reflexive when $1 < p < \infty$. \square

In [3], it was shown that \mathbf{f} is *bi-Lipschitz continuous* for all embedded regular curves $\mathbf{f} \in C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1+\Psi,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, which suggests that \mathbf{f} does not bend sharply. It is natural to expect that all embedded regular curves belonging to the generalized Sobolev space are bi-Lipschitz; we confirm this expectation with the following theorem which we establish by modifying the argument of Blatt [3].

Theorem 4.3 (The bi-Lipschitz continuity). *Let an increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ satisfy $\Phi(0) = 0$ and $\Phi(x) = O(x^{\frac{2}{p}})$ as $x \rightarrow +0$ for $p \in [1, \infty)$. Set $\Psi(x) := (x^{-\frac{1}{p}} \Phi(x))^{\frac{1}{2}}$. Assume that \mathbf{f} belongs to $C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1+\Psi,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ whose image is a closed embedded curve in \mathbb{R}^n parametrized by arc-length. Then, \mathbf{f} is bi-Lipschitz continuous.*

Remark 4.4. For $\Phi(x) = x^\alpha$ with $2 \leq \alpha p < 2p + 1$, the assertion in Theorem 4.3 corresponds to [3, Lemma 2.1].

Proof of Theorem 4.3. We only have to prove that there exists $C_b > 0$ such that

$$\|\Delta \mathbf{f}\|_{\mathbb{R}^n} \geq C_b \mathcal{D}(\mathbf{f})$$

for $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$.

Let $s_1, s_2, s_3 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ with $|\Delta s| = 2r$ and $\mathcal{D}(\mathbf{f}(s_1), \mathbf{f}(s_3)) = \mathcal{D}(\mathbf{f}(s_3), \mathbf{f}(s_2))$. Then, we have

$$\begin{aligned} \|\Delta \mathbf{f}\|_{\mathbb{R}^n} &= \sup_{\|\mathbf{x}\|_{\mathbb{R}^n} \leq 1} \int_{s_3-r}^{s_3+r} \boldsymbol{\tau}(s) \cdot \mathbf{x} ds \\ &= 2r + \sup_{\|\mathbf{x}\|_{\mathbb{R}^n} \leq 1} \int_{s_3-r}^{s_3+r} \boldsymbol{\tau}(s) \cdot (\boldsymbol{\tau}(s) - \mathbf{x}) ds \\ &\geq 2r - \inf_{\|\mathbf{x}\|_{\mathbb{R}^n} \leq 1} \int_{s_3-r}^{s_3+r} \|\boldsymbol{\tau}(s) - \mathbf{x}\|_{\mathbb{R}^n} ds \\ &= \left(1 - \inf_{\|\mathbf{x}\|_{\mathbb{R}^n} \leq 1} \frac{1}{2r} \int_{s_3-r}^{s_3+r} \|\boldsymbol{\tau}(s) - \mathbf{x}\|_{\mathbb{R}^n} ds \right) |\Delta s|. \end{aligned}$$

Note that there exists $M, \delta > 0$ such that if $x < \delta$, then we have $\Phi(x) \leq Mx^{2/p}$ because $\Phi(x) = O(x^{2/p})$ as $x \rightarrow +0$. By the assumption $\mathbf{f} \in W^{1+\Psi,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, we have

$$[\boldsymbol{\tau}]_{\Psi,2p}^{2p} = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \frac{\|\Delta_{s_1+s_2}^{2p} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Phi(|s_1|)^p} ds_1 ds_2 < \infty.$$

Using Lebesgue's dominated convergence theorem, we have

$$\lim_{r \rightarrow +0} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-r}^r \frac{\|\Delta_{s_1+s_2} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Phi(|s_1|)^p} ds_1 ds_2 = 0.$$

Therefore, there exists $\eta \in (0, \min\{\delta, 1, \mathcal{L}\}/2)$ such that

$$\sup_{s \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s-r}^{s+r} \int_{-r}^r \frac{\|\Delta_{s_2} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Phi(|s_1|)^p} ds_1 ds_2 \leq \frac{1}{2^{2p} M^p}$$

if $r \leq \eta$. Putting

$$\boldsymbol{x} = \frac{1}{2r} \int_{s_3-r}^{s_3+r} \boldsymbol{\tau}(s_1) ds_1,$$

we get

$$\begin{aligned} & \frac{1}{2r} \int_{s_3-r}^{s_3+r} \|\boldsymbol{\tau}(s_2) - \boldsymbol{x}\|_{\mathbb{R}^n} ds_2 \\ &= \frac{1}{2r} \int_{s_3-r}^{s_3+r} \left\| \boldsymbol{\tau}(s_2) - \frac{1}{2r} \int_{s_3-r}^{s_3+r} \boldsymbol{\tau}(s_1) ds_1 \right\|_{\mathbb{R}^n} ds_2 \\ &\leq \frac{1}{4r^2} \int_{s_3-r}^{s_3+r} \int_{s_3-r}^{s_3+r} \|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n} ds_1 ds_2 \\ &\leq \left(\frac{\Phi(2r)^p}{4r^2} \int_{s_3-r}^{s_3+r} \int_{s_3-r}^{s_3+r} \frac{\|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Phi(|\Delta s|)^p} ds_1 ds_2 \right)^{1/2p} \leq \frac{1}{2}. \end{aligned}$$

Therefore, it holds that

$$\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n} \geq \frac{1}{2} |\Delta s|.$$

Next, we consider the case where $\mathcal{D}(\boldsymbol{f}) \geq 2\eta$. Let

$$I_\eta := \{(s_1, s_2) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \mid \mathcal{D}(\boldsymbol{f}) \geq 2\eta\}.$$

Then, we have

$$C_\eta := \inf_{(s_1, s_2) \in I_\eta} \frac{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}{\mathcal{D}(\boldsymbol{f})} > 0$$

because \boldsymbol{f} has no self-intersection. Therefore, we obtain

$$\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n} \geq C_\eta \mathcal{D}(\boldsymbol{f}).$$

□

Using the space $W^{k+\Psi, 2p}$, we establish the following theorem concerning the finiteness of the energies $\mathcal{E}_{(\Phi, p)}$.

Theorem 4.5 (Finiteness of $\mathcal{E}_{(\Phi, p)}(\boldsymbol{f})$). *Let $p \in [1, \infty)$, and let $\boldsymbol{f} \in C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ be a function whose image is a closed curve parametrized by arc-length embedded in \mathbb{R}^n with total length \mathcal{L} . Assume that a measurable function $\Phi : [0, \infty) \rightarrow [0, \infty)$ satisfies the following.*

(A0) $\Phi(0) = 0$, $\Phi \in C^1$, and $\Phi'(x) > 0$ for $x > 0$.

(A1) There exists $K > 0$ such that $\lim_{x \rightarrow +0} G(x) = K$, where $G(x) := \frac{x\Phi'(x)}{\Phi(x)}$.

(A2) There exists a measurable function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that

(A2-1) $\Phi(kx) \leq \varphi(k)\Phi(x)$ for $k, x \geq 0$,

and $M(a) := \int_0^a \frac{\varphi(t)^p}{t} dt$ ($a > 0$) satisfies

(A2-2) $M(\varepsilon) = o(\varepsilon)$ as $\varepsilon \rightarrow +0$,

(A2-3) $M(a) < \infty$ for $a > 0$.

(A3) $\int_0^a \frac{t^{2p}}{\Phi(t)^p} dt < \infty$ for $a > 0$.

Set $\Psi(x) := \left(\frac{\Phi(x)}{x^{\frac{1}{p}}} \right)^{\frac{1}{2}}$ for $x > 0$. Then, we have the following two properties.

1. If $\mathbf{f} \in W^{1+\Psi, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and \mathbf{f} is bi-Lipschitz continuous, then we have $\mathcal{E}_{(\Phi, p)}(\mathbf{f}) < \infty$.
2. If $\mathcal{E}_{(\Phi, p)}(\mathbf{f}) < \infty$, then \mathbf{f} belongs to $W^{1+\Psi, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$.

Moreover, there exists $C > 0$ depending only p, \mathcal{L} , and Φ such that

$$\|\tau\|_{W^{\Psi, 2p}} \leq C(\mathcal{E}_{(\Phi, p)}(\mathbf{f}) + \|\tau\|_{L^{2p}}). \quad (4.1)$$

Remark 4.6. Suppose that

(A2-2)' $\varphi(x) = O(x^{\frac{2}{p}})$ as $x \rightarrow \infty$

instead of (A2-2) in Theorem 4.5. Then, we have $M(\varepsilon) = o(\varepsilon)$ as $\varepsilon \rightarrow +0$, and using the argument of [28], we can prove that \mathbf{f} is bi-Lipschitz continuous if $\mathcal{E}_{(\Phi, p)}(\mathbf{f}) < \infty$. Thus, it holds that $\mathcal{E}_{(\Phi, p)}(\mathbf{f}) < \infty$ if and only if $\mathbf{f} \in W^{1+\Psi, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and \mathbf{f} is bi-Lipschitz continuous.

Before we proceed to the proof of Theorem 4.5, we consider the concrete examples given by $\Phi(x) = x^\alpha$, $x^\alpha \log(x+1)$, $1 - e^{-x^\alpha} + x^{2\alpha}/2$. The following table shows the allowable range of the exponent α corresponding to the assumptions in Theorems 4.3 and 4.5. The row “Remark 4.6” shows the range of α corresponding to (A0), (A1), (A2-1), (A2-2)', (A2-3), and (A3).

	$\Phi(x) = x^\alpha$	$\Phi(x) = x^\alpha \log(x+1)$	$\Phi(x) = 1 - e^{-x^\alpha} + x^{2\alpha}/2$ ($x \in [0, C_b \mathcal{L}]$)
Theorem 4.3	$[2/p, \infty)$	$[2/p - 1, \infty)$	$[1/p, \infty)$
Theorem 4.5	$(1/p, 2 + 1/p)$	$(1/p, 1/p + 1)$	$(1/p, 2 + 1/p)$
Remark 4.6	$[2/p, 2 + 1/p)$	$[2/p, 1/p + 1)$ ($p > 1$)	$[2/p, 2 + 1/p)$

Table 2: Examples of Φ (Ranges of α)

Notation. For $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ and $\mathbf{v} : \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^n$, we write $\Delta_{s_1}^{s_2} \mathbf{v} := \mathbf{v}(s_2) - \mathbf{v}(s_1)$.

The proof of Theorem 4.5 is based on an argument by Blatt [3]. Before proving Theorem 4.5, we establish the following lemma which is used in proof of inequality (4.1). Let

$$\tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g}) := \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{\left(\int_0^1 \int_0^1 \|\Delta_{s_1+s_4s_2}^{s_1+s_3s_2} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2$$

for $\mathbf{g} : \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^n$.

Lemma 4.7. *There exists $C = C(p, \mathcal{L}, \Phi) > 0$ such that*

$$[\mathbf{g}]_{\Psi, 2p}^{2p} \leq C \left(\tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g}) + \|\mathbf{g}\|_{L^{2p}}^{2p} \right)$$

for all almost-everywhere continuous functions $\mathbf{g} : \mathbb{R}/\mathcal{L}\mathbb{Z} \rightarrow \mathbb{R}^n$.

Proof. First, we consider the case where $\mathbf{g} \in C^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. For $\varepsilon \in (0, 1)$, we decompose

$$[\mathbf{g}]_{\Psi, 2p}^{2p} = J_\varepsilon^1(\mathbf{g}) + J_\varepsilon^2(\mathbf{g}),$$

where

$$\begin{aligned} J_\varepsilon^1(\mathbf{g}) &:= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{|s_1| \geq \frac{\varepsilon\mathcal{L}}{2}} \frac{\|\Delta_{s_2}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^{2p}}{\Phi(|s_1|)^p} ds_1 ds_2, \\ J_\varepsilon^2(\mathbf{g}) &:= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}}{2}} \frac{\|\Delta_{s_2}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^{2p}}{\Phi(|s_1|)^p} ds_1 ds_2. \end{aligned}$$

Now, we have

$$J_\varepsilon^1(\mathbf{g}) \leq \frac{2^{2p}\mathcal{L}}{\Phi(\varepsilon\mathcal{L})} \|\mathbf{g}\|_{L^{2p}}^{2p}$$

because Φ is an increasing function. As in [3], it is not difficult to see

$$\varepsilon^{2p} J_\varepsilon^2(\mathbf{g}) \leq 3^p (K_\varepsilon^1(\mathbf{g}) + K_\varepsilon^2(\mathbf{g}) + K_\varepsilon^3(\mathbf{g})),$$

where

$$\begin{aligned} K_\varepsilon^1(\mathbf{g}) &:= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_0^\varepsilon \int_{1-\varepsilon}^1 \|\Delta_{s_2+s_3s_1}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2, \\ K_\varepsilon^2(\mathbf{g}) &:= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_0^\varepsilon \int_{1-\varepsilon}^1 \|\Delta_{s_2+s_4s_1}^{s_2+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2, \\ K_\varepsilon^3(\mathbf{g}) &:= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_0^\varepsilon \int_{1-\varepsilon}^1 \|\Delta_{s_2}^{s_2+s_4s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2. \end{aligned}$$

By the definition of $\tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g})$, we have $K_\varepsilon^2(\mathbf{g}) \leq \tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g})$. Moreover, we have

$$\begin{aligned}
K_\varepsilon^3(\mathbf{g}) &= \varepsilon^p \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_0^\varepsilon \|\Delta_{s_2+s_4s_1}^{s_2+s_4s_1} \mathbf{g}\|_{\mathbb{R}^n}^{2p} ds_4 \right)^p}{\Phi(|s_1|)^p} ds_2 ds_1 \\
&\leq \varepsilon^{2p-1} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_0^\varepsilon \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}}{2}} \frac{\|\Delta_{s_2+s_4s_1}^{s_2+s_4s_1} \mathbf{g}\|_{\mathbb{R}^n}^{2p}}{\Phi(|s_1|)^p} ds_1 ds_4 ds_2 \\
&= \varepsilon^{2p-1} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_0^\varepsilon \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}s_4}{2}} \frac{\|\Delta_{s_2+\tilde{s}_1}^{s_2+\tilde{s}_1} \mathbf{g}\|_{\mathbb{R}^n}^{2p}}{\Phi(|\tilde{s}_1|/s_4)^p s_4} d\tilde{s}_1 ds_4 ds_2 \\
&\leq \varepsilon^{2p-1} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_0^\varepsilon \int_{|s_1| \leq \frac{\varepsilon\mathcal{L}s_4}{2}} \frac{\varphi(s_4)^p}{s_4} \frac{\|\Delta_{s_2+\tilde{s}_1}^{\tilde{s}_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^{2p}}{\Phi(|\tilde{s}_1|)^p} d\tilde{s}_1 ds_4 ds_2 \\
&\leq M(\varepsilon) \varepsilon^{2p-1} J_\varepsilon^2(\mathbf{g})
\end{aligned}$$

by Hölder's inequality, Fubini's Theorem, and (A2-1). Also, we have

$$\int_{1-\varepsilon}^1 \|\Delta_{s_2+s_3s_1}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 = \int_0^\varepsilon \|\Delta_{s_2+s_1-\tilde{s}_3s_1}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^2 d\tilde{s}_3$$

by the change of variable $\tilde{s}_3 = 1 - s_3$. Therefore, we obtain

$$\begin{aligned}
K_\varepsilon^1(\mathbf{g}) &= \varepsilon^p \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon\mathcal{L}}{2}}^{\frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_{1-\varepsilon}^1 \|\Delta_{s_2+s_3s_1}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2 \\
&= \varepsilon^p \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon\mathcal{L}}{2}}^{\frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_0^\varepsilon \|\Delta_{s_2+s_1-\tilde{s}_3s_1}^{s_1+s_2} \mathbf{g}\|_{\mathbb{R}^n}^2 d\tilde{s}_3 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2 \\
&= \varepsilon^p \int_{-\frac{\varepsilon\mathcal{L}}{2}}^{\frac{\varepsilon\mathcal{L}}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\left(\int_0^\varepsilon \|\Delta_{\tilde{s}_2-\tilde{s}_3s_1}^{\tilde{s}_2} \mathbf{g}\|_{\mathbb{R}^n}^2 d\tilde{s}_3 \right)^p}{\Phi(|s_1|)^p} d\tilde{s}_2 ds_1 \\
&= \varepsilon^p \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon\mathcal{L}}{2}}^{\frac{\varepsilon\mathcal{L}}{2}} \frac{\left(\int_0^\varepsilon \|\Delta_{\tilde{s}_2+\tilde{s}_3\tilde{s}_1}^{\tilde{s}_2} \mathbf{g}\|_{\mathbb{R}^n}^2 d\tilde{s}_3 \right)^p}{\Phi(|s_1|)^p} d\tilde{s}_1 d\tilde{s}_2 \\
&= K_\varepsilon^3(\mathbf{g})
\end{aligned}$$

by Fubini's theorem and the change of variables $\tilde{s}_2 = s_1 + s_2$ and $\tilde{s}_1 = -s_1$. Hence, we get

$$J_\varepsilon^2(\mathbf{g}) \leq \frac{3^p}{\varepsilon^{2p}} \tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g}) + 3^p \cdot 2 \frac{M(\varepsilon)}{\varepsilon} J_\varepsilon^2(\mathbf{g}).$$

Now, we can take ε sufficiently small satisfying

$$3^p \cdot 2 \frac{M(\varepsilon)}{\varepsilon} < 1$$

by (A2-2). Then, we get

$$J_\varepsilon^2(\mathbf{g}) \leq C(p, \varepsilon, \varphi) \tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g})$$

because $J_\varepsilon^2(\mathbf{g}) < \infty$ by $\mathbf{g} \in C^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and because of (A3), where $C(p, \varepsilon, \varphi)$ is a positive constant. Therefore, we obtain

$$[\mathbf{g}]_{\Psi, 2p}^{2p} \leq \frac{2^{2p}\mathcal{L}}{\Phi(\frac{\varepsilon\mathcal{L}}{2})} \|\mathbf{g}\|_{L^{2p}}^{2p} + C(p, \varepsilon, \varphi) \tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g}).$$

Next, we consider the case where \mathbf{g} is an almost everywhere continuous function. Let $\rho \in C_0^\infty(\mathbb{R})$ with $\text{supp } \rho \subset [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho(x) dx = 1,$$

and define $\rho_\lambda(x) := \lambda^{-1} \rho(\frac{x}{\lambda})$ for $x \in \mathbb{R}$. Set

$$\mathbf{g}_\lambda(s) := \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \mathbf{g}(s-x) dx.$$

Then, we have

$$[\mathbf{g}_\lambda]_{\Psi, 2p}^{2p} \leq \frac{2^{2p} \mathcal{L}}{\Phi(\frac{\varepsilon \mathcal{L}}{2})} \|\mathbf{g}_\lambda\|_{L^{2p}}^{2p} + C(p, \varepsilon, \varphi) \tilde{\mathcal{E}}_{(\Phi, p)}(\mathbf{g}_\lambda)$$

because $\mathbf{g}_\lambda \in C^\infty$. From properties of the mollifier, it holds that

$$\|\mathbf{g}_\lambda\|_{L^{2p}} \leq \|\mathbf{g}\|_{L^{2p}}.$$

Next, we show $\tilde{\mathcal{E}}^{\Phi, p}(\mathbf{g}_\lambda) \leq \tilde{\mathcal{E}}^{\Phi, p}(\mathbf{g})$. By Hölder's inequality, we have

$$\begin{aligned} \|\Delta_{s_2+s_4s_1}^{s_2+s_3s_1} \mathbf{g}_\lambda\|_{\mathbb{R}^n}^2 &= \left\| \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \Delta_{s_1+x+s_4s_2}^{s_1+x+s_3s_2} \mathbf{g} dx \right\|_{\mathbb{R}^n}^2 \\ &\leq \left(\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \|\Delta_{s_2+x+s_4s_1}^{s_2+x+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 dx \right) \\ &\leq \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \|\Delta_{s_2+x+s_4s_1}^{s_2+x+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 dx. \end{aligned}$$

Also, it holds that

$$\begin{aligned} &\left(\int_0^1 \int_0^1 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \|\Delta_{s_2+x+s_4s_1}^{s_2+x+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 dx ds_3 ds_4 \right)^p \\ &= \left(\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \int_0^1 \int_0^1 \|\Delta_{s_2+x+s_4s_1}^{s_2+x+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 dx \right)^p \\ &\leq \left(\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) dx \right)^{p-1} \left\{ \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \left(\int_0^1 \int_0^1 \|\Delta_{s_2+x+s_4s_1}^{s_2+x+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p dx \right\} \\ &\leq \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \left(\int_0^1 \int_0^1 \|\Delta_{s_2+x+s_4s_1}^{s_2+x+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p dx \end{aligned}$$

by Fubini's theorem and Hölder's inequality. Therefore, we get

$$\begin{aligned}
\tilde{\mathcal{E}}^{\Phi,p}(\mathbf{g}_\lambda) &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\left(\int_0^1 \int_0^1 \|\Delta_{s_2+s_4s_1}^{s_2+s_3s_1} \mathbf{g}_\lambda\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2 \\
&\leq \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\left(\int_0^1 \int_0^1 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \|\Delta_{s_2+x+s_4s_1}^{s_2+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 dx ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_1 ds_2 \\
&\leq \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \left(\int_0^1 \int_0^1 \|\Delta_{s_2+x+s_4s_1}^{s_2+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p dx}{\Phi(|s_1|)^p} ds_1 ds_2 \\
&= \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\left(\int_0^1 \int_0^1 \|\Delta_{s_2+x+s_4s_1}^{s_2+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} ds_2 ds_1 dx \\
&= \left(\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \rho_\lambda(x) dx \right) \left\{ \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\left(\int_0^1 \int_0^1 \|\Delta_{s_2+s_4s_1}^{\tilde{s}_2+s_3s_1} \mathbf{g}\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)^p}{\Phi(|s_1|)^p} d\tilde{s}_2 ds_1 \right\} \\
&= \tilde{\mathcal{E}}^{\Phi,p}(\mathbf{g})
\end{aligned}$$

by the change of variable $\tilde{s}_2 = s_2 + x$.

Hence, we obtain

$$\|\mathbf{g}_\lambda\|_{W^{\Psi,2p}}^{2p} \leq 2^{2p-1} \left\{ \left(1 + \frac{2^{2p}\mathcal{L}}{\Phi(\frac{\varepsilon\mathcal{L}}{2})} \right) \|\mathbf{g}\|_{L^{2p}}^{2p} + C(p, \varepsilon, \varphi) \tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g}) \right\},$$

and we can see $\{\mathbf{g}_\lambda\}_{\lambda>0}$ is a $W^{\Psi,2p}$ -bounded set. By reflexivity of $W^{\Psi,2p}$, there exists a subsequence $(\mathbf{g}_{\lambda_j})_{j=0}^\infty$ such that

$$\mathbf{g}_{\lambda_j} \rightharpoonup \mathbf{g}$$

as $j \rightarrow \infty$. Therefore, we obtain

$$\begin{aligned}
[\mathbf{g}]_{\Psi^{2p}}^{2p} &\leq \|\mathbf{g}\|_{W^{\Psi,2p}}^{2p} \leq \liminf_{j \rightarrow \infty} \|\mathbf{g}_{\varepsilon_j}\|_{W^{\Psi,2p}}^{2p} \\
&\leq 2^{2p-1} \left\{ \left(1 + \frac{2^{2p}\mathcal{L}}{\Phi(\frac{\varepsilon\mathcal{L}}{2})} \right) \|\mathbf{g}\|_{L^{2p}}^{2p} + C(p, \varepsilon, \varphi) \tilde{\mathcal{E}}_{(\Phi,p)}(\mathbf{g}) \right\}
\end{aligned}$$

by lower semi-continuity of a weakly convergent sequence. \square

Proof of Theorem 4.5. We use the notation

$$g_a(t) := \frac{1}{\Phi(t)} - \frac{1}{\Phi(a)}$$

for $0 < t \leq a$. For $\varepsilon \in (0, \frac{\mathcal{L}}{2})$, let

$$\mathcal{E}_{(\Phi,p),\varepsilon}(\mathbf{f}) := \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{\varepsilon \leq |s_1| \leq \frac{\mathcal{L}}{2}} (g_{|s_1|}(\|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n}))^p ds_1 ds_2.$$

Then, we have $\mathcal{E}_{(\Phi,p)}(\mathbf{f}) = \lim_{\varepsilon \rightarrow +0} \mathcal{E}_{(\Phi,p),\varepsilon}(\mathbf{f})$. By the mean value theorem, for $s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, $\varepsilon \leq |s_1| \leq \frac{\mathcal{L}}{2}$, there exists $\theta = \theta(s_1, s_2) \in (\|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n}, |s_1|)$ such that

$$g_{|s_1|}(\|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n}) = -g'_{|s_1|}(\theta)(|s_1| - \|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n}). \quad (4.2)$$

By (A1), for all $\eta > 0$, there exists $\delta > 0$ such that if $0 < x < \delta$ then we have

$$K - \eta \leq G(x) \leq K + \eta.$$

First, we assume that \mathbf{f} is bi-Lipschitz continuous and belongs to $W^{1+\Psi, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. By Hölder's inequality and (4.2), we have

$$\begin{aligned} \mathcal{E}_{(\Phi, p), \varepsilon}(\mathbf{f}) &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{\varepsilon \leq |s_1| \leq \frac{\varepsilon}{2}} (g_{|s_1|}(\|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n}))^p ds_1 ds_2 \\ &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{\varepsilon \leq |s_1| \leq \frac{\varepsilon}{2}} \{-g'_{|s_1|}(\theta)(|s_1| - \|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n})\}^p ds_1 ds_2 \\ &\leq \frac{1}{2^p} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{\varepsilon \leq |s_2| \leq \frac{\varepsilon}{2}} \int_0^1 \int_0^1 (-g'_{|s_1|}(\theta)|s_1|)^p \|\Delta_{s_2+s_4s_1}^{s_2+s_3s_1} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p} ds_3 ds_4 ds_1 ds_2 \\ &= (*). \end{aligned}$$

By the bi-Lipschitz continuity of \mathbf{f} and (A2-1), we have

$$-g'_{|s_1|}(\theta)|s_1|\Phi(|\Delta_4^3 s||s_1|) \leq C_b G(\theta) \varphi(C_b |\Delta_4^3 s|)$$

for $s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, $\varepsilon \leq |s_1| \leq \frac{\varepsilon}{2}$, $s_3, s_4 \in [0, 1]$, where $C_b > 0$ is the bi-Lipschitz constant of \mathbf{f} . By (A2-3) and Fubini's theorem, we have

$$\begin{aligned} (*) &\leq C_b^p \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_0^1 \int_0^1 G(\theta)^p \varphi(C_b |\Delta_4^3 s|)^p \frac{\|\Delta_{s_2+s_4s_1}^{s_2+s_3s_1} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Phi(|\Delta_4^3 s||s_1|)^p} ds_3 ds_4 ds_1 ds_2 \\ &= C_b^p \int_0^1 \int_0^1 \varphi(C_b |\Delta_4^3 s|)^p \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} G(\theta)^p \frac{\|\Delta_{s_2+s_4s_1}^{s_2+s_3s_1} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Phi(|\Delta_4^3 s||s_1|)^p} ds_2 ds_1 ds_3 ds_4 \\ &= C_b^p \int_0^1 \int_0^1 \frac{\varphi(C_b |\Delta_4^3 s|)^p}{|\Delta_4^3 s|} \\ &\quad \times \int_{|\Delta_4^3 s|\varepsilon \leq |t_1| \leq |\Delta_4^3 s|\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} G(\tilde{\theta})^p \frac{\|\Delta_{t_2}^{t_1+t_2} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Psi(|t_1|)^p} \frac{1}{|t_1|} dt_2 dt_1 ds_3 ds_4, \end{aligned}$$

where s_1, s_2 are transformed into $t_1 = (\Delta_4^3 s)s_1$, $t_2 = s_2 + s_4s_1$, and we set $\tilde{\theta} = \tilde{\theta}(t_1, t_2) = \theta(s_1, s_2)$ in the last equality. We take $\varepsilon > 0$ satisfying $\varepsilon \leq \delta$. For $s_3, s_4 \in [0, 1]$, we decompose

$$\int_{|\Delta_4^3 s|\varepsilon \leq |t_1| \leq |\Delta_4^3 s|\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} G(\tilde{\theta})^p \frac{\|\Delta_{t_2}^{t_1+t_2} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Psi(|t_1|)^p} \frac{1}{|t_1|} dt_2 dt_1 = I_{\varepsilon, \delta}^1(\boldsymbol{\tau}) + I_{\varepsilon, \delta}^2(\boldsymbol{\tau}),$$

where

$$\begin{aligned} I_{\varepsilon, \delta}^1(\boldsymbol{\tau}) &:= \int_{|\Delta_4^3 s|\varepsilon \leq |t_1| \leq |\Delta_4^3 s|\delta} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} G(\tilde{\theta})^p \frac{\|\Delta_{t_2}^{t_1+t_2} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Psi(|t_1|)^p} \frac{1}{|t_1|} dt_2 dt_1 \\ I_{\varepsilon, \delta}^2(\boldsymbol{\tau}) &:= \int_{|\Delta_4^3 s|\delta \leq |t_1| \leq |\Delta_4^3 s|\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} G(\tilde{\theta})^p \frac{\|\Delta_{t_2}^{t_1+t_2} \boldsymbol{\tau}\|_{\mathbb{R}^n}^{2p}}{\Psi(|t_1|)^p} \frac{1}{|t_1|} dt_2 dt_1. \end{aligned}$$

If $|\Delta_4^3 s|\varepsilon \leq |t_1| \leq |\Delta_4^3 s|\delta$, we have $G(\tilde{\theta}) \leq K + \eta$ because $0 < \tilde{\theta} \leq \delta$. Hence, we get

$$I_{\varepsilon, \delta}^1(\boldsymbol{\tau}) \leq (K + \eta)^p [\boldsymbol{\tau}]_{\Psi, 2p}^{2p}.$$

If $|\Delta_4^3 s| \delta \leq |t_1| \leq |\Delta_4^3 s| \frac{\mathcal{L}}{2}$, then we have $C_b^{-1} \delta \leq \tilde{\theta} \leq \frac{\mathcal{L}}{2}$. Hence, we get

$$I_{\varepsilon, \delta}^2(\boldsymbol{\tau}) \leq G_\delta^p [\boldsymbol{\tau}]_{\Psi, 2p}^{2p},$$

where $G_\delta := \max_{x \in [C_b^{-1} \delta, \frac{\mathcal{L}}{2}]} G(x)$. By (A3), we obtain

$$\mathcal{E}_{(\Phi, p), \varepsilon}(\mathbf{f}) \leq \frac{C_b^p}{2^p} \{(K + \eta)^p + G_\delta^p\} M(C_b) [\boldsymbol{\tau}]_{\Psi, 2p}^{2p} < \infty$$

for all $\varepsilon \leq \delta$. Thus it holds that $\mathcal{E}_{(\Phi, p)}(\mathbf{f}) < \infty$.

Next, we assume $\mathcal{E}^{\Phi, p}(\mathbf{f}) < \infty$. Then, we have

$$\begin{aligned} \mathcal{E}_{(\Phi, p)}(\mathbf{f}) &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} (g_{|s_1|}(\|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n})^p ds_1 ds_2 \\ &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \{-g'_{|s_1|}(\theta)(|s_1| - \|\Delta_{s_2}^{s_1+s_2} \mathbf{f}\|_{\mathbb{R}^n})\}^p ds_1 ds_2 \\ &\geq \frac{1}{4^p} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \left(G(\theta) \int_0^1 \int_0^1 \|\Delta_{s_2+s_4 s_1}^{s_2+s_3 s_1} \boldsymbol{\tau}\|_{\mathbb{R}^n} ds_3 ds_4 \right)^p ds_1 ds_2 \\ &\geq \frac{(K + \tilde{G}_\delta)^p}{4^p} \tilde{\mathcal{E}}_{(\Phi, p)}(\boldsymbol{\tau}), \end{aligned}$$

where $\tilde{G}_\delta := \min_{x \in [C_b^{-1} \delta, \frac{\mathcal{L}}{2}]} G(x)$. Hence, we get inequality (4.1) because it holds that

$$\|\boldsymbol{\tau}\|_{W^{\Psi, 2p}} \leq C_g \left(\|\boldsymbol{\tau}\|_{L^{2p}} + \tilde{\mathcal{E}}_{(\Phi, p)}(\boldsymbol{\tau}) \right) \leq C_g \left(\|\boldsymbol{\tau}\|_{L^{2p}} + \mathcal{E}_{(\Phi, p)}(\mathbf{f}) \right)$$

by Lemma 4.7, where C_g is a positive constant depending only p , \mathcal{L} , and Φ and which may not be the same in each case. Therefore, we obtain $\mathbf{f} \in W^{1+\Psi, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. \square

5 Variational formulae and estimates of O'Hara's energy

The explicit expression of the first variational formula of the (α, p) -energy in the sense of Cauchy's principle value was given in [9]. However, the absolute integrability seems not to be shown. In this section, the first and second variational formulae of (α, p) -energies will be calculated, and several estimates will be shown: absolute integrability, uniform boundedness, and continuity. We, however, do not have a decomposition like the $(\alpha, 1)$ case, and therefore the technique in [17, 19] cannot be used. Instead, we pay attention to the function

$$\mathcal{N}(\mathbf{u}, \mathbf{v}) = \frac{1}{2 \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_2} \int_{s_1}^{s_2} \Delta_{s_4}^{s_3} \mathbf{u} \cdot \Delta_{s_4}^{s_3} \mathbf{v} ds_3 ds_4.$$

By use of \mathcal{N} , the (α, p) -energy may be written as

$$\mathcal{E}_{(\alpha, p)}(\mathbf{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{\varphi_\alpha(\mathcal{N}(\boldsymbol{\tau}, \boldsymbol{\tau}))}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right)^p ds_1 ds_2,$$

where

$$\varphi_\alpha(t) = 1 - \frac{1}{(1+t)^{\frac{\alpha}{2}}}.$$

Since $\mathcal{N}(\boldsymbol{\tau}, \boldsymbol{\tau})$ is non-negative, and since φ_α is smooth for $t \geq 0$, the derivation and estimation of variational formulae mainly reduce to the study of \mathcal{N} .

We are now in a position to describe our main result of this section. The first and second variational formulae, $\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}]$ and $\mathcal{H}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]$, are given by

$$\begin{aligned}\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}] ds_1 ds_2 &= \delta(\mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2)[\boldsymbol{\phi}], \\ \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] ds_1 ds_2 &= \delta^2(\mathcal{M}_{(\alpha,p)}(\mathbf{f}) ds_1 ds_2)[\boldsymbol{\phi}, \boldsymbol{\psi}].\end{aligned}$$

The purpose of this section is to give certain new expressions and estimates for these variational formulae. The expression will be given in § 4.1. In § 4.2, we prove the L^1 , L^∞ and C^0 -estimates for them on the appropriate function spaces. We will show the L^1 -estimate for the variational formulae on the *Sobolev-Slobodeckij space* $W^{k+\sigma,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$.

By use of an appropriate weight, the L^∞ -estimate holds on some Hölder or Lipschitz space $C^{k,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ with $k \in \mathbb{N} \cup \{0\}$ and $\beta \in (0, 1]$. Here,

$$C^{k,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) = \{\mathbf{f} \in C^k(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \mid [\mathbf{f}^{(k)}]_{C^{0,\beta}} < \infty\}$$

and this space is equipped with the norm

$$\|\mathbf{f}\|_{C^{k,\beta}} = \|\mathbf{f}\|_{C^k} + [\mathbf{f}^{(k)}]_{C^{0,\beta}},$$

where

$$[\mathbf{f}^{(k)}]_{C^{0,\beta}} = \sup_{s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\|\Delta \mathbf{f}^{(k)}\|_{\mathbb{R}^n}}{|\Delta s|^\beta}.$$

Note that we will see later that the weight is necessary.

The completion of $C^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ in the Hölder space is known as the *little Hölder space* $h^{k,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ when $\beta < 1$. The completion of $C^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ in $C^{k,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ is $C^{k+1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. We have the continuity of the energy density, and the first and second variational formulae in the completion space. The precise statement is as follows.

Theorem 5.1. *Let $\alpha \in (0, \infty)$, $p \in [1, \infty)$ satisfy $2 \leq \alpha p < 2p + 1$, and set $\sigma = \frac{\alpha p - 1}{2p}$. Assume that \mathbf{f} is bi-Lipschitz, i.e., there exists a positive constant $C_b > 0$ such that $\mathcal{D}(\mathbf{f}) \leq C_b \|\Delta \mathbf{f}\|_{\mathbb{R}^n}$.*

1. *If $\mathbf{f}, \boldsymbol{\phi}, \boldsymbol{\psi} \in W^{1+\sigma, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then $\mathcal{M}_{(\alpha,p)}(\mathbf{f}), \mathcal{G}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}], \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Moreover, there exists a positive constant C depending on $\|\boldsymbol{\tau}\|_{W^{\sigma, 2p} \cap L^\infty}$, C_b , α , and p such that*

$$\begin{aligned}\|\mathcal{M}_{(\alpha,p)}(\mathbf{f})\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C, \\ \|\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}]\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C \|\boldsymbol{\phi}'\|_{W^{\sigma, 2p} \cap L^\infty}, \\ \|\mathcal{H}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C \|\boldsymbol{\phi}'\|_{W^{\sigma, 2p} \cap L^\infty} \|\boldsymbol{\psi}'\|_{W^{\sigma, 2p} \cap L^\infty}.\end{aligned}$$

2. *Let $\beta \in (0, 1]$. If $\mathbf{f}, \boldsymbol{\phi}, \boldsymbol{\psi} \in C^{1,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then $\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{M}_{(\alpha,p)}(\mathbf{f}), \mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{G}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}], \mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$.*

Moreover, there exists a positive constant C depending on $\|\tau\|_{C^{0,\beta}}$, C_b , α , and p such that

$$\begin{aligned} \|\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{M}_{(\alpha,p)}(\mathbf{f})\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C, \\ \|\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi]\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C \|\phi'\|_{C^{0,\beta}}, \\ \|\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi]\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C \|\phi'\|_{C^{0,\beta}} \|\psi'\|_{C^{0,\beta}}. \end{aligned}$$

3. For $\beta \in (0, 1]$, let

$$X^\beta(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) = \begin{cases} h^{0,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) & \text{for } 0 < \beta < 1, \\ C^1(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) & \text{for } \beta = 1. \end{cases}$$

If $\tau, \phi', \psi' \in X^\beta(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then $\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{M}_{(\alpha,p)}(\mathbf{f})$, $\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi]$, $\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi]$ can be extended to the diagonal set $\{(s, s) \mid s \in \mathbb{R}/\mathcal{L}\mathbb{Z}\}$ such that these functions are continuous everywhere on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$, and they satisfy the estimates

$$\begin{aligned} \|\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{M}_{(\alpha,p)}(\mathbf{f})\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C, \\ \|\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi]\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C \|\phi'\|_{X^\beta}, \\ \|\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi]\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} &\leq C \|\phi'\|_{X^\beta} \|\psi'\|_{X^\beta} \end{aligned}$$

for some positive constant C depending on $\|\tau\|_{X^\beta}$, C_b , α , and p . The limit functions

$$\begin{aligned} \lim_{(s_1, s_2) \rightarrow (s, s)} \mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{M}_{(\alpha,p)}(\mathbf{f}), \\ \lim_{(s_1, s_2) \rightarrow (s, s)} \mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi], \\ \lim_{(s_1, s_2) \rightarrow (s, s)} \mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p} \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi] \end{aligned}$$

exist and are finite. These vanish everywhere on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ when $\beta \in (0, 1)$.

The second assertion gives us the L^∞ -estimates for $\mathcal{M}_{(\alpha,p)}$, $\mathcal{G}_{(\alpha,p)}$, and $\mathcal{H}_{(\alpha,p)}$ without the weight, when $\alpha \leq 2$.

Corollary 5.2. Let $\alpha \in (0, \infty)$, $p \in [1, \infty)$ satisfy $2/p \leq \alpha \leq 2$. If \mathbf{f} , ϕ , $\psi \in C^{1, \frac{\alpha}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and \mathbf{f} is bi-Lipschitz, then $\mathcal{M}_{(\alpha,p)}(\mathbf{f})$, $\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi]$, and $\mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi]$ belong to $L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$.

The corresponding estimates to our main result were shown for the spacial case $\mathcal{E}_{(2,1)}$ in [17], and in this sense our result is an extension to a wider class of O'Hara's energy.

Remark 5.3. When $\alpha > 2$, we need the weight $\mathcal{D}(\mathbf{f})^{(\alpha-2\beta)p}$ to obtain the uniform boundedness even if \mathbf{f} is analytic. For example, let us consider the right circle with the total length 2π , i.e.,

$$\mathbf{f}(s) = (\cos s, \sin s, 0, \dots, 0) \quad \text{for } s \in \mathbb{R}/2\pi\mathbb{Z}.$$

Since $\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2 = 2(1 - \cos \Delta s)$, it follows that

$$\mathcal{M}_{(\alpha,p)}(\mathbf{f}) = \frac{1}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha p}} \left[1 - \left\{ \frac{2(1 - \cos \Delta s)}{(\Delta s)^2} \right\}^{\frac{\alpha}{2}} \right]^p. \quad (5.1)$$

Using Taylor's theorem

$$2(1 - \cos x) = x^2 - \frac{1}{12}x^4 + \mathcal{O}(x^6) \quad \text{as } x \rightarrow 0,$$

and thus we have

$$1 - \left\{ \frac{2(1 - \cos \Delta s)}{(\Delta s)^2} \right\}^{\frac{\alpha}{2}} = \frac{\alpha}{24}(\Delta s)^2 + \mathcal{O}((\Delta s)^4) \quad \text{as } \Delta s \rightarrow 0.$$

Hence, the right-hand side of (5.1) is equal to

$$\left(\frac{|\Delta s|}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}} \right)^{\alpha p} \left(\frac{\alpha}{24} |\Delta s|^{2-\alpha} + \mathcal{O}(|\Delta s|^{4-\alpha}) \right)^p \quad \text{as } \Delta s \rightarrow 0.$$

Therefore, when $\alpha > 2$, $\mathcal{D}(\mathbf{f})^\gamma \mathcal{M}_{(\alpha,p)}(\mathbf{f})$ is uniformly bounded if and only if $\gamma \geq (\alpha - 2)p$.

5.1 Variational formulae

We begin by recalling the definition of the function \mathcal{N} . For functions $\mathbf{u}, \mathbf{v} : \mathbb{R}/\mathbb{L}\mathbb{Z} \rightarrow \mathbb{R}^d$, where $d = 1$ or n , we set

$$\mathcal{N}(\mathbf{u}, \mathbf{v}) = \frac{1}{2\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \Delta_{s_4}^{s_3} \mathbf{u} \cdot \Delta_{s_4}^{s_3} \mathbf{v} ds_3 ds_4.$$

We use $\mathcal{N}(\mathbf{u})$ instead of $\mathcal{N}(\mathbf{u}, \mathbf{u})$ for simplicity. Then, we can write $\mathcal{M}_\alpha(\mathbf{f})$ as

$$\mathcal{M}_\alpha(\mathbf{f}) = \frac{\varphi_\alpha(\mathcal{N}(\boldsymbol{\tau}))}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}, \quad (5.2)$$

The first variational formula $\mathcal{G}_{(\alpha,p)}$ can be derived as

$$\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi] ds_1 ds_2 = \delta\{(\mathcal{M}_\alpha(\mathbf{f}))^p\}[\phi] ds_1 ds_2 + \mathcal{M}_\alpha(\mathbf{f}) \delta(ds_1 ds_2)[\phi],$$

and

$$\delta\{(\mathcal{M}_\alpha(\mathbf{f}))^p\}[\phi] = p(\mathcal{M}_\alpha(\mathbf{f}))^{p-1} \delta \mathcal{M}_\alpha(\mathbf{f})[\phi].$$

The second variational formula $\mathcal{H}_{(\alpha,p)}$ is calculated similarly. Since $\delta(ds_j)[\phi] = \boldsymbol{\tau}(s_j) \cdot \boldsymbol{\phi}'(s_j) ds_j$ holds (see, for example, [17]), we obtain the following.

Theorem 5.4. $\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi]$ and $\mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi]$ can be written as

$$\mathcal{G}_{(\alpha,p)}(\mathbf{f})[\phi] = \sum_{i=1}^2 \mathcal{G}_i(\mathbf{f})[\phi], \quad \mathcal{H}_{(\alpha,p)}(\mathbf{f})[\phi, \psi] = \sum_{i=1}^6 \mathcal{H}_i(\mathbf{f})[\phi, \psi],$$

where

$$\begin{aligned} \mathcal{G}_1(\mathbf{f})[\phi] &= p(\mathcal{M}_\alpha(\mathbf{f}))^{p-1} \delta \mathcal{M}_\alpha(\mathbf{f})[\phi], \\ \mathcal{G}_2(\mathbf{f})[\phi] &= (\mathcal{M}_\alpha(\mathbf{f}))^p (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}'(s_1) + \boldsymbol{\tau}(s_2) \cdot \boldsymbol{\phi}'(s_2)), \\ \mathcal{H}_1(\mathbf{f})[\phi, \psi] &= p(\mathcal{M}_\alpha(\mathbf{f}))^{p-1} \delta^2 \mathcal{M}_\alpha(\mathbf{f})[\phi, \psi], \\ \mathcal{H}_2(\mathbf{f})[\phi, \psi] &= p(p-1)(\mathcal{M}_\alpha(\mathbf{f}))^{p-2} \delta \mathcal{M}_\alpha(\mathbf{f})[\phi] \delta \mathcal{M}_\alpha(\mathbf{f})[\psi], \\ \mathcal{H}_3(\mathbf{f})[\phi, \psi] &= \mathcal{G}_1(\mathbf{f})[\phi] (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\psi}'(s_1) + \boldsymbol{\tau}(s_2) \cdot \boldsymbol{\psi}'(s_2)), \\ \mathcal{H}_4(\mathbf{f})[\phi, \psi] &= \mathcal{G}_1(\mathbf{f})[\psi] (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}'(s_1) + \boldsymbol{\tau}(s_2) \cdot \boldsymbol{\phi}'(s_2)), \\ \mathcal{H}_5(\mathbf{f})[\phi, \psi] &= (\mathcal{M}_\alpha(\mathbf{f}))^p \{ \boldsymbol{\phi}'(s_1) \cdot \boldsymbol{\psi}'(s_1) + \boldsymbol{\phi}'(s_2) \cdot \boldsymbol{\psi}'(s_2) \\ &\quad - 2(\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}'(s_1))(\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\psi}'(s_1)) - 2(\boldsymbol{\tau}(s_2) \cdot \boldsymbol{\phi}'(s_2))(\boldsymbol{\tau}(s_2) \cdot \boldsymbol{\psi}'(s_2)) \}, \\ \mathcal{H}_6(\mathbf{f})[\phi, \psi] &= (\mathcal{M}_\alpha(\mathbf{f}))^p (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}'(s_1) + \boldsymbol{\tau}(s_2) \cdot \boldsymbol{\phi}'(s_2)) (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\psi}'(s_1) + \boldsymbol{\tau}(s_2) \cdot \boldsymbol{\psi}'(s_2)). \end{aligned}$$

In the remainder of this subsection, we will give the exact expression of $\delta\mathcal{M}_\alpha(\mathbf{f})[\phi]$ and $\delta^2\mathcal{M}_\alpha(\mathbf{f})[\phi, \psi]$. First we note that, by (5.2), we have

$$\delta\mathcal{M}_\alpha(\mathbf{f})[\phi] = \varphi'_\alpha(\mathcal{N}(\boldsymbol{\tau})) \frac{\delta\mathcal{N}(\boldsymbol{\tau})[\phi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^\alpha} - \frac{\alpha}{2} \frac{\varphi_\alpha(\mathcal{N}(\boldsymbol{\tau}))\delta\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2[\phi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^{\alpha+2}}, \quad (5.3)$$

and therefore we need variational formulae for $\mathcal{N}(\boldsymbol{\tau})$, and $\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2$. It follows from the definition of $\mathcal{N}(\boldsymbol{\tau})$ that $\delta\mathcal{N}(\boldsymbol{\tau})$ can be written in terms of $\delta\boldsymbol{\tau}$ as well as $\delta\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2$. We can derive the second variational formula from the corresponding ingredients.

Firstly, we show the ingredients as Lemma 5.5. The variational formulae $\delta\mathcal{N}(\boldsymbol{\tau})$ and $\delta^2\mathcal{N}(\boldsymbol{\tau})$ will be given in the forthcoming Lemma 5.7. Also, we give representations of $\delta\mathcal{M}_\alpha(\mathbf{f})$ and $\delta^2\mathcal{M}_\alpha(\mathbf{f})$ in Proposition 5.8.

For $\mathbf{u}, \mathbf{v} : \mathbb{R}/\mathbb{L}\mathbb{Z} \rightarrow \mathbb{R}^n$, let

$$\mathcal{K}(\mathbf{u}, \mathbf{v}) = \frac{\Delta\mathbf{u} \cdot \Delta\mathbf{v}}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2}.$$

Lemma 5.5. *The following variational formulae hold.*

1. $\delta\boldsymbol{\tau}(s_j)[\phi] = \phi'(s_j) - (\boldsymbol{\tau}(s_j) \cdot \phi'(s_j))\boldsymbol{\tau}(s_j).$
2. $\delta\|\Delta_j^i\boldsymbol{\tau}\|_{\mathbb{R}^n}^2[\phi] = 2\Delta_j^i\boldsymbol{\tau} \cdot \Delta_j^i\phi' - \|\Delta_j^i\boldsymbol{\tau}\|_{\mathbb{R}^n}^2(\boldsymbol{\tau}(s_i) \cdot \phi'(s_i) + \boldsymbol{\tau}(s_j) \cdot \phi'(s_j)).$
3. $\delta\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2[\phi] = 2\mathcal{K}(\mathbf{f}, \phi)\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2.$
4. $\delta\left(\frac{1}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2}\right)[\phi] = -2\frac{\mathcal{K}(\mathbf{f}, \phi)}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2}.$
5. $\delta^2\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2[\phi, \psi] = 2\mathcal{K}(\phi, \psi)\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2.$

Proof. See [17, Lemma 1]. □

Before proving Lemma 5.7, we establish the following sublemma.

Lemma 5.6. 1. *Assume that $\mathbf{u} : \mathbb{R}/\mathbb{L}\mathbb{Z} \rightarrow \mathbb{R}^n$ is $\boldsymbol{\tau}$ or ϕ' . Then, $\delta\mathcal{N}(\boldsymbol{\tau}, \mathbf{u})[\psi]$ can be written as*

$$\begin{aligned} \delta\mathcal{N}(\boldsymbol{\tau}, \mathbf{u})[\psi] &= -2\mathcal{K}(\mathbf{f}, \psi)\mathcal{N}(\boldsymbol{\tau}, \mathbf{u}) \\ &\quad + \frac{1}{2\|\Delta\mathbf{f}\|_{\mathbb{R}^n}} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \{\delta(\Delta_{s_4}^{s_3}\boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3}\mathbf{u})[\psi] \\ &\quad + (\Delta_{s_4}^{s_3}\boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3}\mathbf{u})(\boldsymbol{\tau}(s_3) \cdot \psi'(s_3) + \boldsymbol{\tau}(s_4) \cdot \psi'(s_4))\} ds_3 ds_4, \end{aligned}$$

where

$$\begin{aligned} &\delta(\Delta_{s_4}^{s_3}\boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3}\mathbf{u})[\psi] \\ &= \begin{cases} 2\Delta_{s_4}^{s_3}\boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3}\phi' - \|\Delta_{s_4}^{s_3}\boldsymbol{\tau}\|_{\mathbb{R}^n}^2(\boldsymbol{\tau}(s_3) \cdot \phi'(s_3) + \boldsymbol{\tau}(s_4) \cdot \phi'(s_4)), & \mathbf{u} = \boldsymbol{\tau}, \\ \Delta_{s_4}^{s_3}\phi' \cdot \Delta_{s_4}^{s_3}\psi' - (\Delta_{s_4}^{s_3}\boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3}\phi')(\boldsymbol{\tau}(s_3) \cdot \psi'(s_3) + \boldsymbol{\tau}(s_4) \cdot \psi'(s_4)) \\ \quad - \{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \phi')\}\{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \psi')\}, & \mathbf{u} = \phi'. \end{cases} \end{aligned}$$

2. *It holds that*

$$\delta\mathcal{K}(\mathbf{f}, \phi)[\psi] = \mathcal{K}(\phi, \psi) - 2\mathcal{K}(\mathbf{f}, \phi)\mathcal{K}(\mathbf{f}, \psi).$$

Proof. 1. By Lemma 5.5, we have

$$\begin{aligned}
\delta \mathcal{N}(\boldsymbol{\tau}, \mathbf{u})[\boldsymbol{\psi}] &= \delta \left(\frac{1}{2\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u} ds_3 ds_4 \right) [\boldsymbol{\psi}] \\
&= \frac{1}{2} \delta \left(\frac{1}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \right) [\boldsymbol{\psi}] \int_{s_2}^{s_1} \int_{s_2}^{s_1} \Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u} ds_3 ds_4 \\
&\quad + \frac{1}{2\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \delta(\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u})[\boldsymbol{\psi}] ds_3 ds_4 \\
&\quad + \frac{1}{2\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} (\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u}) \delta(ds_3 ds_4) [\boldsymbol{\psi}] \\
&= -2\mathcal{K}(\mathbf{f}, \boldsymbol{\psi}) \mathcal{N}(\boldsymbol{\tau}, \mathbf{u}) + \frac{1}{2\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \{ \delta(\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u})[\boldsymbol{\phi}] \\
&\quad + (\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u})(\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\psi}'(s_3) + \boldsymbol{\tau}(s_4) \cdot \boldsymbol{\psi}'(s_4)) \} ds_3 ds_4.
\end{aligned}$$

It remains to calculate $\delta(\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \mathbf{u})[\boldsymbol{\phi}]$. Since the case where $\mathbf{u} = \boldsymbol{\tau}$ may be handled by Lemma 5.5, we deal with the case where $\mathbf{u} = \boldsymbol{\phi}'$. Then, it follows from Lemma 5.5 and $\delta \boldsymbol{\phi}'(s_j)[\boldsymbol{\psi}] = -(\boldsymbol{\tau}(s_j) \cdot \boldsymbol{\psi}'(s_j))\boldsymbol{\phi}'(s_j)$ that

$$\begin{aligned}
&\delta(\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}')[\boldsymbol{\psi}] \\
&= \delta(\Delta_{s_4}^{s_3} \boldsymbol{\tau})[\boldsymbol{\psi}] \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}' + \Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \delta(\Delta_{s_4}^{s_3} \boldsymbol{\phi}')[\boldsymbol{\psi}] \\
&= \Delta_{s_4}^{s_3} \boldsymbol{\psi}' \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}' - \Delta_{s_4}^{s_3} \{(\boldsymbol{\tau} \cdot \boldsymbol{\psi}')\boldsymbol{\tau}\} \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}' - \Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \{(\boldsymbol{\tau} \cdot \boldsymbol{\psi}')\boldsymbol{\phi}'\} \\
&= \Delta_{s_4}^{s_3} \boldsymbol{\phi}' \cdot \Delta_{s_4}^{s_3} \boldsymbol{\psi}' - [(\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\psi}'(s_3))\Delta_{s_4}^{s_3} \boldsymbol{\tau} + \{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \boldsymbol{\psi}')\}\boldsymbol{\tau}(s_4)] \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}' \\
&\quad - \Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot [\{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \boldsymbol{\psi}')\}\boldsymbol{\phi}'(s_3) + (\boldsymbol{\tau}(s_4) \cdot \boldsymbol{\psi}'(s_4))\Delta_{s_4}^{s_3} \boldsymbol{\phi}'] \\
&= \Delta_{s_4}^{s_3} \boldsymbol{\phi}' \cdot \Delta_{s_4}^{s_3} \boldsymbol{\psi}' \\
&\quad - (\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}')(\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\psi}'(s_3) + \boldsymbol{\tau}(s_4) \cdot \boldsymbol{\psi}'(s_4)) \\
&\quad - \{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \boldsymbol{\psi}')\}(\boldsymbol{\tau}(s_4) \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}' + \Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \boldsymbol{\phi}'(s_3)) \\
&= \Delta_{s_4}^{s_3} \boldsymbol{\phi}' \cdot \Delta_{s_4}^{s_3} \boldsymbol{\psi}' \\
&\quad - (\Delta_{s_4}^{s_3} \boldsymbol{\tau} \cdot \Delta_{s_4}^{s_3} \boldsymbol{\phi}')(\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\psi}'(s_3) + \boldsymbol{\tau}(s_4) \cdot \boldsymbol{\psi}'(s_4)) \\
&\quad - \{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')\}\{\Delta_{s_4}^{s_3}(\boldsymbol{\tau} \cdot \boldsymbol{\psi}')\}.
\end{aligned}$$

2. Using Lemma 5.5, we have

$$\begin{aligned}
\delta \mathcal{K}(\mathbf{f}, \boldsymbol{\phi})[\boldsymbol{\psi}] &= \frac{\delta(\Delta \mathbf{f} \cdot \Delta \boldsymbol{\phi})[\boldsymbol{\psi}]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} + (\Delta \mathbf{f} \cdot \Delta \boldsymbol{\phi}) \delta \left(\frac{1}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \right) [\boldsymbol{\psi}] \\
&= \frac{\Delta \boldsymbol{\psi} \cdot \Delta \boldsymbol{\phi}}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} - 2\Delta \mathbf{f} \cdot \Delta \boldsymbol{\phi} \frac{\mathcal{K}(\mathbf{f}, \boldsymbol{\psi})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \\
&= \mathcal{K}(\boldsymbol{\phi}, \boldsymbol{\psi}) - 2\mathcal{K}(\mathbf{f}, \boldsymbol{\phi})\mathcal{K}(\mathbf{f}, \boldsymbol{\psi}).
\end{aligned}$$

□

Lemma 5.7. $\delta \mathcal{N}(\boldsymbol{\tau})[\boldsymbol{\phi}]$ and $\delta^2 \mathcal{N}(\boldsymbol{\tau})[\boldsymbol{\phi}, \boldsymbol{\psi}]$ can be written as

$$\delta \mathcal{N}(\boldsymbol{\tau})[\boldsymbol{\phi}] = \sum_{i=1}^2 \mathcal{R}_i(\mathbf{f})[\boldsymbol{\phi}], \quad \delta^2 \mathcal{N}(\boldsymbol{\tau})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \sum_{i=1}^5 \mathcal{S}_i(\mathbf{f})[\boldsymbol{\phi}, \boldsymbol{\psi}],$$

where

$$\begin{aligned}
\mathcal{R}_1(\mathbf{f})[\phi] &= -2\mathcal{K}(\mathbf{f}, \phi)\mathcal{N}(\tau), \\
\mathcal{R}_2(\mathbf{f})[\phi] &= 2\mathcal{N}(\tau, \phi'), \\
\mathcal{S}_1(\mathbf{f})[\phi, \psi] &= -2(\mathcal{K}(\phi, \psi) - 2\mathcal{K}(\mathbf{f}, \phi)\mathcal{K}(\mathbf{f}, \psi))\mathcal{N}(\tau), \\
\mathcal{S}_2(\mathbf{f})[\phi, \psi] &= -\mathcal{K}(\mathbf{f}, \phi)\delta\mathcal{N}(\tau)[\psi] - \mathcal{K}(\mathbf{f}, \psi)\delta\mathcal{N}(\tau)[\phi], \\
\mathcal{S}_3(\mathbf{f})[\phi, \psi] &= -2\mathcal{K}(\mathbf{f}, \psi)\mathcal{N}(\tau, \phi') - 2\mathcal{K}(\mathbf{f}, \phi)\mathcal{N}(\tau, \psi'), \\
\mathcal{S}_4(\mathbf{f})[\phi, \psi] &= 2\mathcal{N}(\phi', \psi'), \\
\mathcal{S}_5(\mathbf{f})[\phi, \psi] &= -2\mathcal{N}((\tau \cdot \phi'), (\tau \cdot \psi')).
\end{aligned}$$

Proof. By Lemmas 5.5 and 5.6, we have

$$\begin{aligned}
\delta\mathcal{N}(\tau)[\phi] &= -2\mathcal{K}(\mathbf{f}, \phi)\mathcal{N}(\tau) \\
&\quad + \frac{1}{2\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \{\delta\|\Delta_{s_4}^{s_3}\tau\|_{\mathbb{R}^n}^2[\phi] \\
&\quad + \|\Delta_{s_4}^{s_3}\tau\|_{\mathbb{R}^n}^2(\tau(s_3) \cdot \phi'(s_3) + \tau(s_4) \cdot \phi'(s_4))\} ds_3 ds_4 \\
&= -2\mathcal{K}(\mathbf{f}, \phi)\mathcal{N}(\tau) + \frac{1}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \Delta_{s_4}^{s_3}\tau \cdot \Delta_{s_4}^{s_3}\phi' ds_3 ds_4 \\
&= -2\mathcal{K}(\mathbf{f}, \phi)\mathcal{N}(\tau) + 2\mathcal{N}(\tau, \phi') \\
&= \sum_{i=1}^2 \mathcal{R}_i(\mathbf{f})[\phi].
\end{aligned}$$

Next we calculate $\delta^2\mathcal{N}(\tau)[\phi, \psi]$. Firstly, we have

$$\delta^2\mathcal{N}(\tau)[\phi, \psi] = \frac{1}{2}\delta(\delta\mathcal{N}(\tau)[\phi])[\psi] + \frac{1}{2}\delta(\delta\mathcal{N}(\tau)[\psi])[\phi].$$

By the symmetry with respect to ϕ and ψ , it is sufficient to calculate $\delta(\delta\mathcal{N}(\tau)[\phi])[\psi]$, for which we first note that

$$\delta(\delta\mathcal{N}(\tau)[\phi])[\psi] = \sum_{i=1}^2 \delta(\mathcal{R}_i(\mathbf{f})[\phi])[\psi].$$

Using Lemma 5.6 again, we can show that

$$\begin{aligned}
\delta(\mathcal{R}_1(\mathbf{f})[\phi])[\psi] &= -2\delta\mathcal{K}(\mathbf{f}, \phi)[\psi]\mathcal{N}(\tau) - 2\mathcal{K}(\mathbf{f}, \phi)\delta\mathcal{N}(\tau)[\psi] \\
&= -2\mathcal{K}(\phi, \psi)\mathcal{N}(\tau) + 4\mathcal{K}(\mathbf{f}, \phi)\mathcal{K}(\mathbf{f}, \psi)\mathcal{N}(\tau) \\
&\quad - 2\mathcal{K}(\mathbf{f}, \phi)\delta\mathcal{N}(\tau)[\psi],
\end{aligned}$$

and

$$\begin{aligned}
\delta(\mathcal{R}_2(\mathbf{f})[\phi])[\psi] &= 2\delta(\mathcal{N}(\tau, \phi'))[\psi] \\
&= -4\mathcal{K}(\mathbf{f}, \psi)\mathcal{N}(\tau, \phi') \\
&\quad + \frac{1}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} [\Delta_{s_4}^{s_3}\phi' \cdot \Delta_{s_4}^{s_3}\psi' \\
&\quad - (\Delta_{s_4}^{s_3}\tau \cdot \Delta_{s_4}^{s_3}\phi')(\tau(s_3) \cdot \psi'(s_3) + \tau(s_4) \cdot \psi'(s_4)) \\
&\quad - \{\Delta_{s_4}^{s_3}(\tau \cdot \phi')\}\{\Delta_{s_4}^{s_3}(\tau \cdot \psi')\} \\
&\quad + (\Delta_{s_4}^{s_3}\tau \cdot \Delta_{s_4}^{s_3}\phi')(\tau(s_3) \cdot \psi'(s_3) + \tau(s_4) \cdot \psi'(s_4))] ds_3 ds_4 \\
&= -4\mathcal{K}(\mathbf{f}, \psi)\mathcal{N}(\tau, \phi') + 2\mathcal{N}(\phi', \psi') - 2\mathcal{N}((\tau \cdot \phi'), (\tau \cdot \psi')),
\end{aligned}$$

from which the claim follows. \square

Next, we give expressions of $\delta \mathcal{M}_\alpha(\mathbf{f})[\phi]$ and $\delta^2 \mathcal{M}_\alpha(\mathbf{f})[\phi, \psi]$ in terms of \mathcal{N} , $\delta \mathcal{N}$, and $\delta^2 \mathcal{N}$.

Proposition 5.8. $\delta \mathcal{M}_\alpha(\mathbf{f})[\phi]$ and $\delta^2 \mathcal{M}_\alpha(\mathbf{f})[\phi, \psi]$ can be written as

$$\delta \mathcal{M}_\alpha(\mathbf{f})[\phi] = \sum_{i=1}^2 \mathcal{P}_i(\mathbf{f})[\phi], \quad \delta^2 \mathcal{M}_\alpha(\mathbf{f})[\phi, \psi] = \sum_{i=1}^6 \mathcal{Q}_i(\mathbf{f})[\phi, \psi],$$

where

$$\begin{aligned} \mathcal{P}_1(\mathbf{f})[\phi] &= \varphi'_\alpha(\mathcal{N}(\tau)) \frac{\delta \mathcal{N}(\tau)[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}, \\ \mathcal{P}_2(\mathbf{f})[\phi] &= -\frac{\alpha}{2} \mathcal{M}_\alpha(\mathbf{f}) \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}, \\ \mathcal{Q}_1(\mathbf{f})[\phi, \psi] &= \varphi'_\alpha(\mathcal{N}(\tau)) \frac{\delta^2 \mathcal{N}(\tau)[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}, \\ \mathcal{Q}_2(\mathbf{f})[\phi, \psi] &= -\frac{\alpha}{2} \varphi'_\alpha(\mathcal{N}(\tau)) \frac{\delta \mathcal{N}(\tau)[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}, \\ \mathcal{Q}_3(\mathbf{f})[\phi, \psi] &= \varphi''_\alpha(\mathcal{N}(\tau)) \frac{\delta \mathcal{N}(\tau)[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\frac{\alpha}{2}}} \frac{\delta \mathcal{N}(\tau)[\psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\frac{\alpha}{2}}}, \\ \mathcal{Q}_4(\mathbf{f})[\phi, \psi] &= -\frac{\alpha}{2} \delta \mathcal{M}_\alpha(\mathbf{f})[\psi] \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}, \\ \mathcal{Q}_5(\mathbf{f})[\phi, \psi] &= -\frac{\alpha}{2} \mathcal{M}_\alpha(\mathbf{f}) \frac{\delta^2 \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}, \\ \mathcal{Q}_6(\mathbf{f})[\phi, \psi] &= \frac{\alpha}{2} \mathcal{M}_\alpha(\mathbf{f}) \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}. \end{aligned}$$

Proof. The assertion for $\delta \mathcal{M}_\alpha(\mathbf{f})$ follows immediately from (5.2) and (5.3).

Regarding $\delta^2 \mathcal{M}_\alpha(\mathbf{f})$, we have

$$\delta^2 \mathcal{M}_\alpha(\mathbf{f})[\phi, \psi] = \delta(\delta \mathcal{M}_\alpha(\mathbf{f})[\phi])[\psi] = \sum_{i=1}^2 \delta(\mathcal{P}_i(\mathbf{f})[\phi])[\psi],$$

and

$$\begin{aligned} \delta(\mathcal{P}_1(\mathbf{f})[\phi])[\psi] &= \varphi''_\alpha(\mathcal{N}(\tau)) \delta \mathcal{N}(\tau)[\psi] \frac{\delta \mathcal{N}(\tau)[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} + \varphi'_\alpha(\mathcal{N}(\tau)) \frac{\delta^2 \mathcal{N}(\tau)[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \\ &\quad - \frac{\alpha}{2} \varphi'_\alpha(\mathcal{N}(\tau)) \frac{\delta \mathcal{N}(\tau)[\phi] \delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha+2}} \\ &= \mathcal{Q}_3(\mathbf{f})[\phi, \psi] + \mathcal{Q}_1(\mathbf{f})[\phi, \psi] + \mathcal{Q}_2(\mathbf{f})[\phi, \psi], \\ \delta(\mathcal{P}_2(\mathbf{f})[\phi])[\psi] &= -\frac{\alpha}{2} \delta \mathcal{M}_\alpha(\mathbf{f})[\psi] \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} - \frac{\alpha}{2} \mathcal{M}_\alpha(\mathbf{f}) \frac{\delta^2 \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \\ &\quad + \frac{\alpha}{2} \mathcal{M}_\alpha(\mathbf{f}) \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi] \delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^4} \\ &= \mathcal{Q}_4(\mathbf{f})[\phi, \psi] + \mathcal{Q}_5(\mathbf{f})[\phi, \psi] + \mathcal{Q}_6(\mathbf{f})[\phi, \psi]. \end{aligned}$$

\square

Plugging Lemma 5.7 and Proposition 5.8 into Theorem 5.4, we obtain the first and second variational formulae of the (α, p) -O'Hara energies. Since the expressions are rather lengthy, we omit their explicit formulae here.

5.2 Estimates of the first and second variational formulae

5.2.1 Strategy

Let $\mathbf{f}, \phi, \psi \in W^{1+\sigma, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1, \infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, and assume that \mathbf{f} is bi-Lipschitz. By [3], we already know the estimate

$$\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p} \leq C. \quad (5.4)$$

Combining Theorem 5.4 and Hölder's inequality, if $p > 1$, we have

$$\begin{aligned} \|\mathcal{G}_1(\mathbf{f})[\phi]\|_{L^1} &\leq p\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p}^{p-1}\|\delta\mathcal{M}_\alpha(\mathbf{f})[\phi]\|_{L^p}, \\ \|\mathcal{G}_2(\mathbf{f})[\phi]\|_{L^1} &\leq 2\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p}^p\|\phi'\|_{L^\infty}, \\ \|\mathcal{H}_1(\mathbf{f})[\phi, \psi]\|_{L^1} &\leq p\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p}^{p-1}\|\delta^2\mathcal{M}_\alpha(\mathbf{f})[\phi, \psi]\|_{L^p}, \\ \|\mathcal{H}_2(\mathbf{f})[\phi, \psi]\|_{L^1} &\leq p(p-1)\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p}^{p-2}\|\delta\mathcal{M}_\alpha(\mathbf{f})[\phi]\|_{L^p}\|\delta\mathcal{M}_\alpha(\mathbf{f})[\psi]\|_{L^p}, \\ \|\mathcal{H}_3(\mathbf{f})[\phi, \psi]\|_{L^1} &\leq 2\|\mathcal{G}_1(\mathbf{f})[\phi]\|_{L^1}\|\psi'\|_{L^\infty}, \\ \|\mathcal{H}_4(\mathbf{f})[\phi, \psi]\|_{L^1} &\leq 2\|\mathcal{G}_1(\mathbf{f})[\psi]\|_{L^1}\|\phi'\|_{L^\infty}, \\ \|\mathcal{H}_5(\mathbf{f})[\phi, \psi]\|_{L^1} &\leq 6\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p}^p\|\phi'\|_{L^\infty}\|\psi'\|_{L^\infty}, \\ \|\mathcal{H}_6(\mathbf{f})[\phi, \psi]\|_{L^1} &\leq 4\|\mathcal{M}_\alpha(\mathbf{f})\|_{L^p}^p\|\phi'\|_{L^\infty}\|\psi'\|_{L^\infty}. \end{aligned}$$

Hence, if there exists $C = C(\mathbf{f}) > 0$ such that

$$\begin{aligned} \|\delta\mathcal{M}_\alpha(\mathbf{f})[\phi]\|_{L^p} &\leq C\|\phi'\|_{W^{\sigma, 2p} \cap L^\infty}, \\ \|\delta^2\mathcal{M}_\alpha(\mathbf{f})[\phi, \psi]\|_{L^p} &\leq C\|\phi'\|_{W^{\sigma, 2p} \cap L^\infty}\|\psi'\|_{W^{\sigma, 2p} \cap L^\infty}, \end{aligned} \quad (5.5)$$

then the desired L^1 -estimates for $\mathcal{G}_{(\alpha, p)}(\mathbf{f})$ and $\mathcal{H}_{(\alpha, p)}(\mathbf{f})$ follow.

Next we observe that since $\|\Delta\mathbf{f}\|_{\mathbb{R}^n} \leq \mathcal{D}(\mathbf{f})$ holds, and using the bi-Lipschitz estimate, we deduce that there exists $\tilde{C} = \tilde{C}(\mathbf{f}) > 0$ such that $0 \leq \mathcal{N}(\tau) \leq \tilde{C}$. Hence, we have

$$|\varphi_\alpha^{(j)}(\mathcal{N}(\tau))| \leq \max_{t \in [0, \tilde{C}]} |\varphi_\alpha^{(j)}(t)| < \infty$$

for $j = 0, 1, 2$. By Proposition 5.8, Hölder's inequality, and (5.4), we can show (5.5) if there exists $C = C(\mathbf{f}) > 0$ such that

$$\begin{aligned} \left\| \frac{\delta\mathcal{N}(\tau)[\phi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C\|\phi'\|_{W^{\sigma, 2p} \cap L^\infty}, \\ \left\| \frac{\delta^2\mathcal{N}(\tau)[\phi, \psi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C\|\phi'\|_{W^{\sigma, 2p} \cap L^\infty}\|\psi'\|_{W^{\sigma, 2p} \cap L^\infty}, \\ \left\| \frac{\delta\|\Delta\mathbf{f}\|_{\mathbb{R}^d}^2[\phi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2} \right\|_{L^\infty} &\leq C\|\phi'\|_{L^\infty}, \\ \left\| \frac{\delta^2\|\Delta\mathbf{f}\|_{\mathbb{R}^d}^2[\phi, \psi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2} \right\|_{L^\infty} &\leq C\|\phi'\|_{L^\infty}\|\psi'\|_{L^\infty}. \end{aligned}$$

Similarly, to obtain the desired L^∞ -estimates and continuity, it suffices to consider the corresponding properties of

$$\frac{\delta\mathcal{N}(\tau)[\phi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^\alpha}, \quad \frac{\delta^2\mathcal{N}(\tau)[\phi, \psi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^\alpha}, \quad \frac{\delta\|\Delta\mathbf{f}\|_{\mathbb{R}^d}^2[\phi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2}, \quad \frac{\delta^2\|\Delta\mathbf{f}\|_{\mathbb{R}^d}^2[\phi, \psi]}{\|\Delta\mathbf{f}\|_{\mathbb{R}^n}^2}. \quad (5.6)$$

5.2.2 Estimates and continuity of the quantities in (5.6)

As we can see from Lemma 5.7, $\delta\mathcal{N}(\tau)[\phi]$ and $\delta\mathcal{N}(\tau)[\phi, \psi]$ may be expressed in terms of $\mathcal{N}(\mathbf{u}, \mathbf{v})$ and $\mathcal{K}(\mathbf{u}, \mathbf{v})$. First, we discuss $\mathcal{N}(\mathbf{u}, \mathbf{v})$.

Lemma 5.9. *Assume that \mathbf{f} is bi-Lipschitz. Then, the following properties hold.*

1. Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ satisfy $2 \leq \alpha p < 2p + 1$. If $\mathbf{u}, \mathbf{v} \in W^{\sigma, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$, then there exists $C = C(\mathbf{f}) > 0$ such that

$$\left\| \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} \leq C[\mathbf{u}]_{W^{\sigma, 2p}}[\mathbf{v}]_{W^{\sigma, 2p}}.$$

2. Let $0 < \beta \leq 1$. If $\mathbf{u}, \mathbf{v} \in C^{0, \beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$, then there exists $C = C(\mathbf{f}) > 0$ such that

$$\left\| \mathcal{D}(\mathbf{f})^{\alpha-2\beta} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} \leq C[\mathbf{u}]_{C^{0, \beta}}[\mathbf{v}]_{C^{0, \beta}}.$$

3. Let $0 < \beta \leq 1$. If $\mathbf{u}, \mathbf{v} \in X^\beta(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$, then for $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$,

$$\lim_{(s_1, s_2) \rightarrow (s, s)} |\Delta s|^{\alpha-2\beta} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} = \begin{cases} 0 & (0 < \beta < 1), \\ \frac{1}{12} \mathbf{u}'(s) \cdot \mathbf{v}'(s) & (\beta = 1) \end{cases}$$

holds.

Proof. 1. By Hölder's inequality and the bi-Lipschitz continuity of \mathbf{f} , we have

$$\begin{aligned} & \left\| \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p}^p \\ &= \frac{1}{2^p} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_2 - \frac{\epsilon}{2}}^{s_2 + \frac{\epsilon}{2}} \frac{1}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha p + 2p}} \left| \int_{s_2}^{s_1} \int_{s_2}^{s_1} \Delta_{s_4}^{s_3} \mathbf{u} \cdot \Delta_{s_4}^{s_3} \mathbf{v} ds_3 ds_4 \right|^p ds_1 ds_2 \\ &\leq C \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_2 - \frac{\epsilon}{2}}^{s_2 + \frac{\epsilon}{2}} \frac{1}{|\Delta s|^{\alpha p + 2}} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\Delta_{s_4}^{s_3} \mathbf{u}\|_{\mathbb{R}^d}^p \|\Delta_{s_4}^{s_3} \mathbf{v}\|_{\mathbb{R}^d}^p ds_3 ds_4 ds_1 ds_2 \\ &= (\dagger). \end{aligned}$$

We change variables

$$t_1 = s_1 - s_2, \quad t_2 = s_2, \quad s_3 = t_2 + t_1 t_3, \quad s_4 = t_2 + t_1 t_4$$

in (\dagger) . Then, we obtain

$$\begin{aligned} (\dagger) &= C \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{1}{|t_1|^{\alpha p}} \\ &\quad \times \int_0^1 \int_0^1 \|\Delta_{t_2+t_1 t_4}^{t_2+t_1 t_3} \mathbf{u}\|_{\mathbb{R}^d}^p \|\Delta_{t_2+t_1 t_4}^{t_2+t_1 t_3} \mathbf{v}\|_{\mathbb{R}^d}^p dt_3 dt_4 dt_1 dt_2 = (\ddagger). \end{aligned}$$

We use Fubini's theorem and change variables

$$w_1 = (t_3 - t_4)t_1, \quad w_2 = t_2 + t_1 t_4$$

in (\dagger) . Then, we obtain

$$\begin{aligned}
(\dagger) &= C \int_0^1 \int_0^1 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{|t_1|^{\alpha p}} \|\Delta_{t_2+t_1 t_3}^{t_2+t_1 t_3} \mathbf{u}\|_{\mathbb{R}^d}^p \|\Delta_{t_2+t_1 t_4}^{t_2+t_1 t_3} \mathbf{v}\|_{\mathbb{R}^d}^p dt_2 dt_1 dt_3 dt_4 \\
&= C \int_0^1 \int_0^1 \int_{-\frac{\varepsilon}{2}|t_3-t_4|}^{\frac{\varepsilon}{2}|t_3-t_4|} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{|w_1|^{\alpha p}} |t_3 - t_4|^{\alpha p - 1} \\
&\quad \times \|\Delta_{w_2}^{w_1+w_2} \mathbf{u}\|_{\mathbb{R}^d}^p \|\Delta_{w_2}^{w_1+w_2} \mathbf{v}\|_{\mathbb{R}^d}^p dw_1 dw_2 dt_3 dt_4 \\
&\leq C \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\|\Delta_{w_2}^{w_1+w_2} \mathbf{u}\|_{\mathbb{R}^d}^p}{|w_1|^{\frac{\alpha p}{2}}} \frac{\|\Delta_{w_2}^{w_1+w_2} \mathbf{v}\|_{\mathbb{R}^d}^p}{|w_1|^{\frac{\alpha p}{2}}} dw_1 dw_2 \\
&\leq C [\mathbf{u}]_{W^{\sigma, 2p}}^p [\mathbf{v}]_{W^{\sigma, 2p}}^p,
\end{aligned}$$

by Hölder's inequality, and the claim holds.

2. Without loss of generality, we assume that $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ satisfy $0 < |\Delta s| < \mathcal{L}/2$. Then, we have

$$\begin{aligned}
\left| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha}} \right| &\leq C |\Delta s|^{-2\beta-2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\Delta_{s_4}^{s_3} \mathbf{u}\|_{\mathbb{R}^d} \|\Delta_{s_4}^{s_3} \mathbf{v}\|_{\mathbb{R}^d} ds_3 ds_4 \\
&\leq C |\Delta s|^{-2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \frac{\|\Delta_{s_4}^{s_3} \mathbf{u}\|_{\mathbb{R}^d}}{|\Delta_4^3 s|^{\beta}} \frac{\|\Delta_{s_4}^{s_3} \mathbf{v}\|_{\mathbb{R}^d}}{|\Delta_4^3 s|^{\beta}} ds_3 ds_4 \\
&\leq C [\mathbf{u}']_{C^{0,\beta}} [\mathbf{v}']_{C^{0,\beta}}.
\end{aligned}$$

3. First, we consider the case where $0 < \beta < 1$, i.e. $\mathbf{u}, \mathbf{v} \in h^{0,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$. As is well known, the little Hölder space $h^{0,\beta}$ is characterized as

$$h^{0,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) = \left\{ \mathbf{u} \in C^{0,\beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid \lim_{R \rightarrow +0} [\mathbf{u}]_{C^{0,\beta}, R} = 0 \right\},$$

where

$$[\mathbf{u}]_{C^{0,\beta}, R} = \sup_{\substack{s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z} \\ 0 < |\Delta s| < R}} \frac{\|\Delta \mathbf{u}\|_{\mathbb{R}^d}}{|\Delta s|^{\beta}}.$$

For $R > 0$, let $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ satisfy $0 < |\Delta s| < R$. Then, we have

$$\left| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha}} \right| \leq C [\mathbf{u}]_{C^{0,\beta}, R} [\mathbf{v}]_{C^{0,\beta}, R}. \quad (5.7)$$

Taking lim sup on left-hand side in (5.7), we have

$$\limsup_{(s_1, s_2) \rightarrow (s, s)} \left| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha}} \right| \leq C [\mathbf{u}]_{C^{0,\beta}, R} [\mathbf{v}]_{C^{0,\beta}, R}.$$

Taking $R \rightarrow +0$ on right-hand side, we obtain

$$\lim_{(s_1, s_2) \rightarrow (s, s)} |\Delta s|^{\alpha-2\beta} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha}} = 0.$$

Next, we assume $\mathbf{u}, \mathbf{v} \in C^1$. If $s_5, s_6 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ are such that $\|(s_5, s_6) - (s, s)\|_{\mathbb{R}^2}$ is sufficiently small, we can take $\varepsilon > 0$ arbitrarily such that

$$\|\mathbf{u}'(s_5) - \mathbf{u}'(s)\|_{\mathbb{R}^d} \leq \frac{\varepsilon}{2(\|\mathbf{v}'\|_{L^\infty} + 1)}, \quad \|\mathbf{v}'(s_6) - \mathbf{v}'(s)\|_{\mathbb{R}^d} \leq \frac{\varepsilon}{2(\|\mathbf{u}'\|_{L^\infty} + 1)}.$$

Then, using the fact that

$$\int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} ds_5 ds_6 ds_3 ds_4 = \frac{1}{6} |\Delta s|^2,$$

we have

$$\begin{aligned} & \left| \frac{1}{|\Delta s|^4} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} \mathbf{u}'(s_5) \cdot \mathbf{v}'(s_6) ds_5 ds_6 ds_3 ds_4 - \frac{1}{6} \mathbf{u}'(s) \cdot \mathbf{v}'(s) \right| \\ & \leq \frac{1}{|\Delta s|^4} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} |\mathbf{u}'(s_5) \cdot \mathbf{v}'(s_6) - \mathbf{u}'(s) \cdot \mathbf{v}'(s)| ds_5 ds_6 ds_3 ds_4 \\ & \leq \frac{1}{|\Delta s|^4} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} (\|\mathbf{u}'(s_5)\|_{\mathbb{R}^d} \|\mathbf{v}'(s_6) - \mathbf{v}'(s)\|_{\mathbb{R}^d} \\ & \quad + \|\mathbf{v}'(s)\|_{\mathbb{R}^d} \|\mathbf{u}'(s_5) - \mathbf{u}'(s)\|_{\mathbb{R}^d}) ds_5 ds_6 ds_3 ds_4 \\ & \leq \varepsilon. \end{aligned}$$

Hence, we have

$$\begin{aligned} \lim_{(s_1, s_2) \rightarrow (s, s)} \frac{1}{|\Delta s|^4} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} \mathbf{u}'(s_5) \cdot \mathbf{v}'(s_6) ds_5 ds_6 ds_3 ds_4 \\ = \frac{1}{6} \mathbf{u}'(s) \cdot \mathbf{v}'(s). \end{aligned}$$

Using this and

$$\lim_{(s_1, s_2) \rightarrow (s, s)} \frac{|\Delta s|}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}} = \frac{1}{\|\boldsymbol{\tau}(s)\|_{\mathbb{R}^n}} = 1,$$

we obtain

$$\begin{aligned} & |\Delta s|^{\alpha-2} \frac{\mathcal{N}(\mathbf{u}, \mathbf{v})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \\ & = \frac{1}{2} \frac{|\Delta s|^{\alpha+2}}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^{\alpha+2}} \frac{1}{|\Delta s|^4} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} \mathbf{u}'(s_5) \cdot \mathbf{v}'(s_6) ds_5 ds_6 ds_3 ds_4 \\ & \rightarrow \frac{1}{12} \mathbf{u}'(s) \cdot \mathbf{v}'(s) \quad \text{as } (s_1, s_2) \rightarrow (s, s). \end{aligned}$$

□

Since the function $\mathcal{K}(\mathbf{u}, \mathbf{v})$ fulfills

$$|\mathcal{K}(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}'\|_{L^\infty} \|\mathbf{v}'\|_{L^\infty}, \quad \lim_{(s_1, s_2) \rightarrow (s, s)} \mathcal{K}(\mathbf{u}, \mathbf{v}) = \mathbf{u}'(s) \cdot \mathbf{v}'(s),$$

we obtain estimates and continuity of $\frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}$ and $\frac{\delta^2 \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}, \boldsymbol{\psi}]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}$ from Lemma 5.5 as follows.

Lemma 5.10. 1. Assume that $\mathbf{f}, \boldsymbol{\phi}, \boldsymbol{\psi} \in W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and \mathbf{f} is bi-Lipschitz. Then, there exists $C = C(\mathbf{f}) > 0$ such that

$$\left\| \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \right\|_{L^\infty} \leq C \|\boldsymbol{\phi}'\|_{L^\infty}, \quad \left\| \frac{\delta^2 \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}, \boldsymbol{\psi}]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} \right\|_{L^\infty} \leq C \|\boldsymbol{\phi}'\|_{L^\infty} \|\boldsymbol{\psi}'\|_{L^\infty}.$$

2. If $\mathbf{f}, \phi, \psi \in C^1(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then it follows that

$$\lim_{(s_1, s_2) \rightarrow (s, s)} \frac{\delta \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} = \tau(s) \cdot \phi'(s),$$

$$\lim_{(s_1, s_2) \rightarrow (s, s)} \frac{\delta^2 \|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2} = \phi'(s) \cdot \psi'(s)$$

for $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$.

To deal with $\mathcal{S}_5(\mathbf{f})$, we need estimates for $\tau \cdot \phi'$. The proof is easy, therefore we omit the details.

Lemma 5.11. *Assume that \mathbf{f} is bi-Lipschitz. Then, the following properties hold.*

1. Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$ satisfy $2 \leq \alpha p < 2p + 1$. If $\mathbf{f}, \phi \in W^{1+\sigma, 2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \cap W^{1, \infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$, then there exists $C = C(\mathbf{f}) > 0$ such that

$$[(\tau \cdot \phi')]_{W^{\sigma, 2p}} \leq C(\|\tau\|_{L^\infty}[\phi']_{W^{\sigma, 2p}} + [\tau]_{W^{\sigma, 2p}}\|\phi'\|_{L^\infty}).$$

2. Let $0 < \beta \leq 1$. If $\mathbf{f}, \phi \in C^{1, \beta}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then it holds that

$$[(\tau \cdot \phi')]_{C^{0, \beta}} \leq \|\tau\|_{L^\infty}[\phi']_{C^{0, \beta}} + [\tau]_{C^{0, \beta}}\|\phi'\|_{L^\infty}.$$

3. Let $0 < \beta \leq 1$. If $\tau, \phi' \in X^\beta(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then $(\tau \cdot \phi') \in X^\beta(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R})$. Moreover, if $\mathbf{f}, \phi \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, then it holds that

$$\frac{d}{ds}(\tau(s) \cdot \phi'(s)) = \kappa(s) \cdot \phi'(s) + \tau(s) \cdot \phi''(s)$$

for $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$.

5.2.3 Proof of Theorem 5.1

In this subsection we complete the proof of Theorem 5.1 combining the facts in previous subsection. First, we show the L^1 -estimates of the first and second variational formulae for $\mathcal{M}_{(\alpha, p)}(\mathbf{f})$. Using the expression of the first and second variational formulae for $\mathcal{N}(\tau)$ in Lemma 5.7, it follows that

$$\begin{aligned} \left\| \frac{\mathcal{R}_1(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C\|\phi'\|_{L^\infty}, \\ \left\| \frac{\mathcal{R}_2(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C[\phi']_{W^{\sigma, 2p}}, \\ \left\| \frac{\mathcal{S}_1(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C\|\phi'\|_{L^\infty}\|\psi'\|_{L^\infty}, \\ \left\| \frac{\mathcal{S}_2(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C\{\|\phi'\|_{L^\infty}(\|\psi'\|_{L^\infty} + [\psi']_{W^{\sigma, 2p}}) + \|\psi'\|_{L^\infty}(\|\phi'\|_{L^\infty} + [\phi']_{W^{\sigma, 2p}})\}, \\ \left\| \frac{\mathcal{S}_3(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C(\|\psi'\|_{L^\infty}[\phi']_{W^{\sigma, 2p}} + \|\phi'\|_{L^\infty}[\psi']_{W^{\sigma, 2p}}), \\ \left\| \frac{\mathcal{S}_4(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C[\phi']_{W^{\sigma, 2p}}[\psi']_{W^{\sigma, 2p}}, \\ \left\| \frac{\mathcal{S}_5(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C([\phi']_{W^{\sigma, 2p}} + \|\phi'\|_{L^\infty})([\psi']_{W^{\sigma, 2p}} + \|\psi'\|_{L^\infty}) \end{aligned}$$

by Lemmas 5.9 and 5.11. Hence, from these estimates and Lemma 5.10, we obtain

$$\begin{aligned} \left\| \frac{\delta \mathcal{N}(\tau)[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C \|\phi'\|_{W^{\sigma, 2p} \cap L^\infty} \\ \left\| \frac{\delta \mathcal{N}(\tau)[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^p} &\leq C \|\phi'\|_{W^{\sigma, 2p} \cap L^\infty} \|\psi'\|_{W^{\sigma, 2p} \cap L^\infty}, \end{aligned}$$

and L^1 -estimates for $\mathcal{G}_{(\alpha, p)}(\mathbf{f})$ and $\mathcal{H}_{(\alpha, p)}(\mathbf{f})$.

Next, we consider L^∞ -estimates of $\mathcal{M}_{(\alpha, p)}(\mathbf{f})$, $\mathcal{G}_{(\alpha, p)}(\mathbf{f})$, and $\mathcal{H}_{(\alpha, p)}(\mathbf{f})$ with the weight $|\Delta s|^{\alpha-2\beta}$. We assume that $\mathbf{f}, \phi, \psi \in C^{1, \beta}$ and $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ satisfy $0 < |\Delta s| < \mathcal{L}/2$. Because we have

$$1 - x^\alpha \leq \left(\frac{\alpha}{2} + 1\right) (1 - x^2)$$

for all $x \in [0, 1]$, it follows that

$$\begin{aligned} |\Delta s|^{\alpha-2\beta} \mathcal{M}_\alpha(\mathbf{f}) &\leq C \frac{1}{|\Delta s|^{2\beta}} \left(1 - \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha}{|\Delta s|^\alpha}\right) \\ &\leq C \frac{1}{|\Delta s|^{2\beta}} \left(1 - \frac{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^2}{|\Delta s|^2}\right) \\ &\leq C \frac{1}{|\Delta s|^{2\beta+2}} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\Delta_{s_4}^{s_3} \tau\|_{\mathbb{R}^n}^2 ds_3 ds_4 \\ &\leq C. \end{aligned} \tag{5.8}$$

Moreover, by Lemmas 5.9–5.11, we have

$$\begin{aligned} \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{R}_1(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C \|\phi'\|_{L^\infty}, \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{R}_2(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C [\phi']_{C^{0, \beta}}, \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{S}_1(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C \|\phi'\|_{L^\infty} \|\psi'\|_{L^\infty}, \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{S}_2(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C \{ \|\phi'\|_{L^\infty} (\|\psi'\|_{L^\infty} + [\psi']_{C^{0, \beta}}) + \|\psi'\|_{L^\infty} (\|\phi'\|_{L^\infty} + [\phi']_{C^{0, \beta}}) \}, \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{S}_3(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C (\|\psi'\|_{L^\infty} [\phi']_{C^{0, \beta}} + \|\phi'\|_{L^\infty} [\psi']_{C^{0, \beta}}), \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{S}_4(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C [\phi']_{C^{0, \beta}} [\psi']_{C^{0, \beta}}, \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{S}_5(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C ([\phi']_{C^{0, \beta}} + \|\phi'\|_{L^\infty}) ([\psi']_{C^{0, \beta}} + \|\psi'\|_{L^\infty}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \left\| |\Delta s|^{\alpha-2\beta} \frac{\delta \mathcal{N}(\tau)[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C \|\phi'\|_{C^{0, \beta}} \\ \left\| |\Delta s|^{\alpha-2\beta} \frac{\delta^2 \mathcal{N}(\tau)[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right\|_{L^\infty} &\leq C \|\phi'\|_{C^{0, \beta}} \|\psi'\|_{C^{0, \beta}}, \end{aligned}$$

and the L^∞ -estimates for $\mathcal{M}_{(\alpha,p)}(\mathbf{f})$, $\mathcal{G}_{(\alpha,p)}(\mathbf{f})$, and $\mathcal{H}_{(\alpha,p)}(\mathbf{f})$ with the weight $|\Delta s|^{\alpha-2\beta}$.

Lastly, we consider the continuity. Because continuity on the outside of the diagonal set is clear, we consider the property on the diagonal set. First, we assume $\mathbf{f}, \phi, \psi \in h^{1,\beta}$ with $0 < \beta < 1$, and \mathbf{f} is bi-Lipschitz. Let $R > 0$ be sufficiently small. In a similar manner to the proof of (5.8), it holds that

$$|\Delta s|^{\alpha-2\beta} \mathcal{M}_\alpha(\mathbf{f}) \leq \tilde{C}[\boldsymbol{\tau}]_{C^{0,\beta},R}$$

for $0 < |\Delta s| < R$, and therefore

$$\lim_{(s_1, s_2) \rightarrow (s, s)} |\Delta s|^{\alpha-2\beta} \mathcal{M}_\alpha(\mathbf{f}) = 0,$$

where \tilde{C} is a positive constant depending only on α, p , and C_b . Moreover, by Lemma 5.9, we have

$$\begin{aligned} \lim_{(s_1, s_2) \rightarrow (s, s)} \left| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{R}_i(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right| &= 0, \quad (i = 1, 2), \\ \lim_{(s_1, s_2) \rightarrow (s, s)} \left| |\Delta s|^{\alpha-2\beta} \frac{\mathcal{S}_i(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \right| &= 0, \quad (i = 1, \dots, 5). \end{aligned}$$

Considering Lemma 5.10, we obtain the desired continuity when $\mathbf{f}, \phi, \psi \in h^{1,\beta}$ with $0 < \beta < 1$.

Next, we consider the case where $\beta = 1$. We denote the curvature vector of \mathbf{f} by $\boldsymbol{\kappa}$, i.e. $\boldsymbol{\kappa} = \mathbf{f}''$. Using Lemma 5.9 and L'Hospital's theorem, we have

$$\begin{aligned} |\Delta s|^{\alpha-2} \mathcal{M}_\alpha(\mathbf{f}) &= |\Delta s|^{\alpha-2} \frac{\varphi_\alpha(\mathcal{N}(\boldsymbol{\tau}))}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \\ &= |\Delta s|^{\alpha-2} \frac{\varphi_2(\mathcal{N}(\boldsymbol{\tau}))}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \frac{\varphi_\alpha(\mathcal{N}(\boldsymbol{\tau}))}{\varphi_2(\mathcal{N}(\boldsymbol{\tau}))} \\ &= |\Delta s|^{\alpha-2} \frac{\mathcal{N}(\boldsymbol{\tau})}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} \frac{1}{1 + \mathcal{N}(\boldsymbol{\tau})} \frac{\varphi_\alpha(\mathcal{N}(\boldsymbol{\tau}))}{\varphi_2(\mathcal{N}(\boldsymbol{\tau}))} \\ &\rightarrow \frac{\alpha}{24} \|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2 \quad \text{as } (s_1, s_2) \rightarrow (s, s). \end{aligned}$$

Similarly, it follows that

$$\begin{aligned}
|\Delta s|^{\alpha-2} \frac{\mathcal{R}_1(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow -\frac{1}{6} \boldsymbol{\tau}(s) \cdot \boldsymbol{\phi}'(s) \|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2, \\
|\Delta s|^{\alpha-2} \frac{\mathcal{R}_2(\mathbf{f})[\phi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow \frac{1}{6} \boldsymbol{\kappa}(s) \cdot \boldsymbol{\phi}''(s), \\
|\Delta s|^{\alpha-2} \frac{\mathcal{S}_1(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow -\frac{1}{6} \{ \boldsymbol{\phi}'(s) \cdot \boldsymbol{\psi}'(s) - (\boldsymbol{\tau}(s) \cdot \boldsymbol{\phi}'(s))(\boldsymbol{\tau}(s) \cdot \boldsymbol{\psi}'(s)) \} \|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2, \\
|\Delta s|^{\alpha-2} \frac{\mathcal{S}_2(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow \frac{1}{6} (\boldsymbol{\tau}(s) \cdot \boldsymbol{\phi}'(s))(\boldsymbol{\kappa}(s) \cdot \boldsymbol{\psi}''(s) - \boldsymbol{\tau}(s) \cdot \boldsymbol{\psi}'(s) \|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2) \\
&\quad + \frac{1}{6} (\boldsymbol{\tau}(s) \cdot \boldsymbol{\psi}'(s))(\boldsymbol{\kappa}(s) \cdot \boldsymbol{\phi}''(s) - \boldsymbol{\tau}(s) \cdot \boldsymbol{\phi}'(s) \|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2), \\
|\Delta s|^{\alpha-2} \frac{\mathcal{S}_3(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow -\frac{1}{6} (\boldsymbol{\tau}(s) \cdot \boldsymbol{\psi}'(s))(\boldsymbol{\kappa}(s) \cdot \boldsymbol{\phi}''(s)) - \frac{1}{6} (\boldsymbol{\tau}(s) \cdot \boldsymbol{\phi}'(s))(\boldsymbol{\kappa}(s) \cdot \boldsymbol{\psi}''(s)), \\
|\Delta s|^{\alpha-2} \frac{\mathcal{S}_4(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow \frac{1}{6} \boldsymbol{\phi}''(s) \cdot \boldsymbol{\psi}''(s), \\
|\Delta s|^{\alpha-2} \frac{\mathcal{S}_5(\mathbf{f})[\phi, \psi]}{\|\Delta \mathbf{f}\|_{\mathbb{R}^n}^\alpha} &\rightarrow -\frac{1}{6} \{ (\boldsymbol{\tau}(s) \cdot \boldsymbol{\phi}''(s)) + (\boldsymbol{\phi}'(s) \cdot \boldsymbol{\kappa}(s)) \} \{ (\boldsymbol{\tau}(s) \cdot \boldsymbol{\psi}''(s)) + (\boldsymbol{\psi}'(s) \cdot \boldsymbol{\kappa}(s)) \}
\end{aligned}$$

as $(s_1, s_2) \rightarrow (s, s)$. Hence, the desired continuity holds when $\mathbf{f}, \phi, \psi \in C^2$.

This completes the proof of Theorem 5.1. \square

Acknowledgements The author is grateful to Professor Takeyuki Nagasawa for his direction and many useful advices and remarks. Moreover, the author would like to thank Professor Neal Bez for English language editing and mathematical comments.

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