On finite type invariants of knots and 3-manifolds

（結び目と 3 次元多様体の有限型不変量の研究）

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1 Introduction

In 1990, Vassiliev defined Vassiliev invariants (finite type invariants) by considering the whole continuous mapping of the circle to R^3 and by calculating a complementary set of a singularity set. Vassiliev invariants (Finite type invariants) are formulated by giving a sort of filtration,

$$\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \cdots$$

to vector space $\mathcal{K}$ spanned by all the isotopy types of knots. Quotient vector space of each degree $\mathcal{K}_d/\mathcal{K}_{d+1}$ of this filtration has a finite dimension.

Remark 1.1. “ Vassiliev invariants (Finite type invariants) ” in Section 2 are expressed as “ Vassiliev invariants ” in Section 3, and as finite type invariants ” in Section 4-6.

Generally, for a Vassiliev invariant v of degree d, when a link K has a crossing number n, it is known that the value of $v(K)$ is controlled by the order of n^d. In short, the value of $v(K)/n^d$ is bounded, and therefore the following set is bounded:

$$\mathcal{S}(K, D) := \left\{ \left(\frac{\mathcal{V}_2(K)}{n^2}, \frac{\mathcal{V}_3(K)}{n^3} \right) \in \mathbb{R} \times \mathbb{R} \right\}.$$

where $\mathcal{V}_2(K)$ and $\mathcal{V}_3(K)$ are the values of the primitive Vassiliev invariants of degrees 2 and 3 of the knot K respectively. It is important to obtain the shape of this set. For example, if we obtain the value of Vassiliev invariants $v_2(K), v_3(K)$ of a knot K, we can estimate the minimum number of crossings of the link through the shape of the set.

Finite type invariants have two properties:

1. The finite type invariants, i.e. invariant linear map of links to an abelian group, become zero mapping when filtration is restricted to $\mathcal{K}_{d+1}$.

2. Quotient vector space of each degree of filtration $\mathcal{K}_d/\mathcal{K}_{d+1}$ has a finite dimension.

By generalizing these two characters, we defined quasi finite type invariants, the characters of which are as follows:

1. They are invariant linear map to an abelian group.

2. They become zero mapping when we freely fix filtration $\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$ in a certain class of knots so that they are not empty set and we restrict it to $\mathcal{K}_{d+1}$.

3. Quotient vector space $\mathcal{K}_d/\mathcal{K}_{d+1}$ has a finite dimension.

Concretely speaking, we calculate quasi finite type invariants for 2-bridge links and torus links, for which quandle shadow cocycle invariants are calculated by using dihedral quandles. Furthermore, in 2012, Hatakenaka, Nosaka showed that quandle shadow cocycle invariants, calculated by using dihedral quandles for space M_L where double branched covering is given to link L, are equal to scalar multiples of Dijkgraaf-Witten invariants. This caused us to find quasi finite type invariants are extended to general finite type invariants of 3-dimensional manifolds. Of course, we also discover how to show that, for each d, the whole finite type invariant space of degree d has a finite dimension, especially by using Heegaard splittings and mapping class groups of the 3-manifold.

For general quandles, it is difficult to obtain finite type invariants from quandle (shadow) cocycle invariants, however, in case of trivial quandles, it is possible. Though it is known that quandle (shadow) cocycle invariants are a part of quantum invariants (an R-matrix $\mathcal{R}$ is obtained.), their relation to finite type invariants is not clear because a Lie ring is not obtained, which means a weight system to obtain quantum invariants from finite type invariants has not yet been found. Though a Lie ring was not able to be obtained this time, we obtained finite type invariants from quandle (shadow) cocycle invariants with trivial quandles derived from a matrix $\mathcal{R}$. When we calculate quandle (shadow) cocycle invariants for trivial quandles, they are calculable only for a link with two or more components. We clearly showed a part of the relation of quandle (shadow) cocycle invariants and finite type invariants to links. By using quandle (shadow) cocycle invariants with trivial quandle, a value of linking number was obtained [CJKLS]. Therefore, the relation between linking numbers and finite type invariants is also clear.

In Section 2.1, we state the fundamentals of knots (or links), and define knot diagrams and Reidemeister moves to deal with the problems of 3-dimensional knots in a combinational way by using 2-dimensional knot diagrams.

In Section 2.2, we define braid groups and explain that link invariants are composed of the representation of braid groups derived from an R-matrix.

In Section 2.3, we define Vassiliev invariants (finite type invariants). Then, we show that quotient vector space $\mathcal{K}_d/\mathcal{K}_{d+1}$ of each degree of filtration has a finite dimension by using the chord diagrams we define.

In Section 2.4, we introduce quandles and quandle cohomology and define quandle (shadow) cocycle invariants of knots or links.

In Section 2.5, we define Heegaard splittings and mapping class groups of 3-manifolds in order to get background knowledge to show that, for each d, the whole finite type invariant space of degree d has a finite dimension.

In Section 2.6, we define Dijkgraaf-Witten invariants, which is necessary to introduce finite type invariants of 3-manifolds used in Section 4.2.

In Section 2.7, we define lens space considered in Section 4.2.

In Section 3, For torus knots, we obtained the shape of $\mathcal{S}(K, D)$, and we prove that the following inequality holds for torus links, which is the Willerton conjecture:

$$|\mathcal{V}_3(K)| \leq [\frac{n(n^2-1)}{24}].$$

In Section 4, first, in Section 4.1, we define quasi finite type invariants and obtain quasi finite type invariants from quandle (shadow) cocycle invariants of

2-bridge links and torus links. Next, in Section 4.2, we define general finite type invariants of 3-manifolds and obtain new finite type invariants of lens space and Brieskorn manifolds where double branched covering is given to 2-bridge links and torus links respectively.

In Section 5, we show that finite type invariants are obtained from cocycle invariants of trivial quandles in case of 2-cocycle invariants (Section 5.1) and 3-cocycle invariants (Section 5.2) respectively.

In Appendix of Section 6, we show that quandle (shadow) cocycle invariants are a part of quantum invariants, which is a well-known fact.

2 Preliminaries

A *braid* is introduced by Artin in the 1920s. By using braids, we can deal with the problems of links. Besides, based on the indication of braids, invariants of the link are formed. It is a great advantage to use the structure of a braid group derived from a set of braids. In order to form link invariants, the representation of braid groups is constituted by using a matrix R which is introduced by Yang and Baxter around 1970. Since Jones defined the Jones polynomial in 1984, a great number of link invariants are formed by using a matrix R. Furthermore, in 1990, Vassiliev introduced *Vassiliev invariants (finite type invariants)* by means of calculating approximately the cohomology group of the space embedded smoothly in $\mathbb{R}^3$ with a circle S^1. A few years later, Birman and Lin defined these invariants by using a diagram of links and pointed out that these invariants include the information of quantum invariants.

A *quandle* is a set with a binary operation satisfying an axiom analogical to the operation taking the conjugation in a group [Joy]. For example, when we define a quandle operation for a residue ring Z/mZ(m is an odd number) as $x*y = 2y-x \pmod{m}$ for any $x, y \in Z/mZ$, $(Z/mZ, *)$ becomes quandle. This quandle is called a *dihedral quandle* R_m. Cohomology of a quandle was defined as an analogy of group cohomology by Carter, Jelsovsky, Kamada, Langford and Saito [CJKLS]. We regard $H^Q_n(X; A)$ as a cohomogy group of degree n of a quandle X with a coefficient group A. Especially when p is a prime number, by R_p, quandle shadow cocycle invariants are calculated in a certain class of links [I].

2.1 Knots and their diagrams

A reference of this subsection is [*chap.I*[Oh2]]. An image embedded smoothly to three-dimensional Euclid space $\mathbb{R}^3$ with circle S^1 is called a *knot*. An image embedded smoothly to $\mathbb{R}^3$ with l S^1's is called a *link* of l components. To give an example of the family of infinite knots, with respect to natural numbers p, q which are prime to each other, when a straight line with the inclination p/q in torus $\mathbb{R}^2/\mathbb{Z}^2$ is given and this torus is embedded in $\mathbb{R}^3$, a knot formed by the image of the line is called a $T(p, q)$ *torus knot*. A diagram of the figure 1 is a 2-bridge link and represented by $C(a_1, a_2, \cdots, a_k)$. This diagram has a different

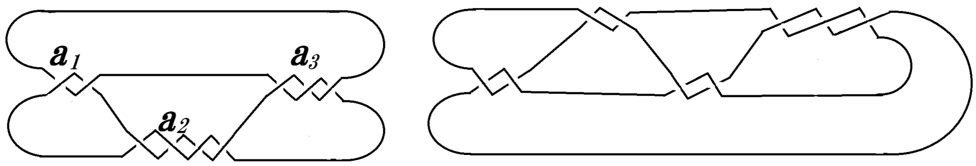


Figure 1: 2-bridge links

way to connect the rightmost string (The figure on the left shows when k is an odd number and that on the right when k is an even number.). When we extend l/n into a connected fraction, it is changed into a $S(l, n)$ 2-bridge link by taking l, n so that

$$\frac{l}{n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

With respect to two knots (or two links) K, K', when the family of homeomorphism $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ $(t \in [0, 1])$ such that h_0 is identity map, $h_1(K) = K', K$ and K' are called *isotopic*, and its transformation process h_t is called *isotopy*.

In order to prove an isotopic knot to be isotopic, we have only to show the process of transformation concretely, while it is not easy to prove a non-isotopic knot to be non-isotopic and this proof needs invariants. Here, when

$$Map \quad I : \{\text{a knot (a link)}\} \to (\text{a certain set})$$

satisfies $I(K) = I(K')$ for isotopic knots(links) K, K', I is called an isotopy invariant of the knot (the link). Mostly " a certain set " where invariants have a value is a well-known set such as a polynomial ring.

A knot (a link) with the top and the bottom at the point where the lines intersect on the plane $\mathbb{R}^2$ given by the projection $\mathbb{R}^3 \to \mathbb{R}^2$ is called a diagram. In a diagram, the intersection of lines with the top and the bottom is called a crossing. Note that, while a knot is a three-dimensional figure, a diagram of a knot is a two-dimensional figure. When two diagrams move by using an isotopy $\mathbb{R}^2$, it is called isotopic.

Theorem 2.1 ([BZ])**.** *When we regard K, K' (or two links, in general) as knots and D, D' as diagrams, a necessary and sufficient condition for K and K' being isotopic is that we obtain D' by giving the following RI, RII, RIII moves and isotopies of the diagram of $\mathbb{R}^2$ to D finite times.*

RI, RII and RIII moves are called the *Reidemeister moves*.

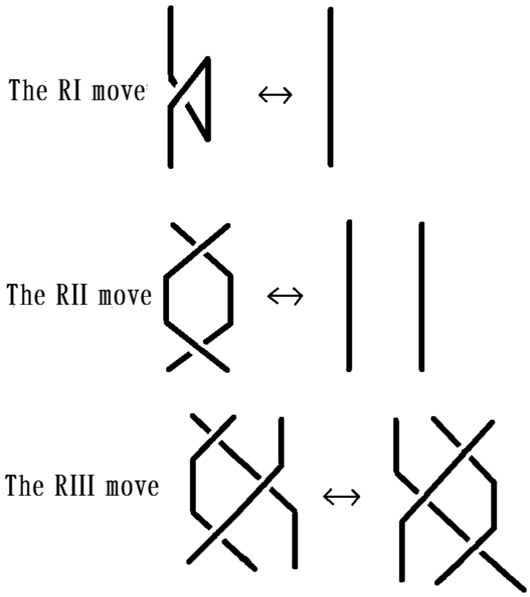


Figure 2: Reidemeister moves

The above theorem 2.1 is important. This theorem changes a three-dimensional problem in knots (or links) into dealing with a combinatorial way of two-dimensional knot (or link) diagrams modulo the Reidemeister moves.

A knot with a string oriented is called an *oriented knot*. A link with each oriented knot is called an *oriented link*. A diagram with a circle in oriented $\mathbb{R}^2$ is called an *oriented diagram*. As an oriented version of Theorem 2.1, the following Corollary holds:

Corollary 2.2. *When we regard K, K' as oriented knots (or two oriented links, in general) and D, D' as diagrams, a necessary and sufficient condition for K and K' being isotopic is that we obtain D' by giving the following $\overrightarrow{RI}, \overrightarrow{RII}, \overrightarrow{RIII}$ moves and isotopies of the diagram of $\mathbb{R}^2$ to D finite times.*

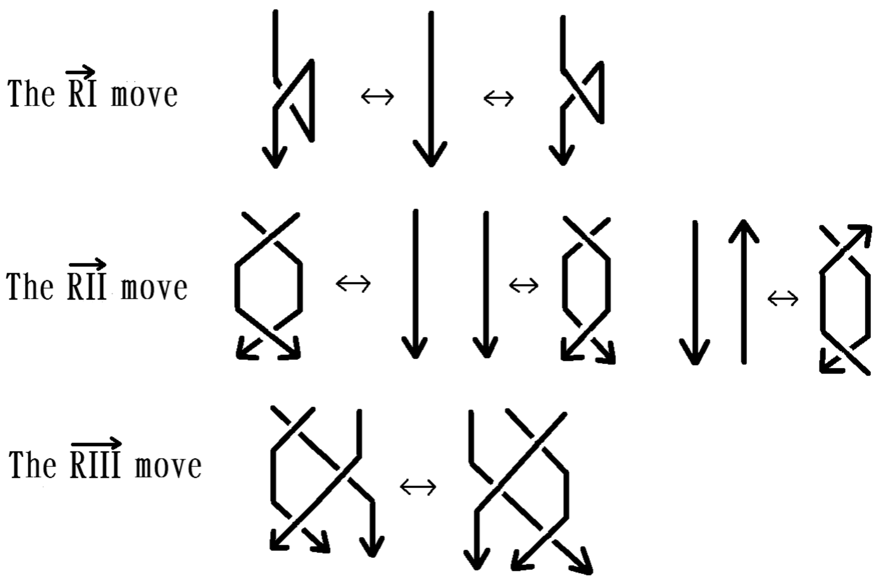


Figure 3: The Reidemeister moves for oriented diagrams

2.2 Quantum invariants

A reference of this subsection is [*chap.II*[Oh2]]. An image embedded in $\mathbb{R}^2 \times [0,1]$ with n strings so that an end point is $\{1, 2 \cdots, n\} \times \{0\} \times \{0, 1\}$ and that it is monotonous with respect to the height of function is called a *braid* of n strings. Each string of the braid is oriented downward. When 2 strings b_1, b_2 move by using an isotopy of $R^2 \times [0,1]$ such that they keep the height of function by fixing the boundary, it is called isotopic, i.e. $h_t : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ for $t \in [0,1]$ such that h_0 is the identity map of $(R^2 \times [0,1])$ and $h_1(b_1) = b_2$.

We obtain any n braids by connecting vertically the copies of σ_i and σ_i^{-1} $(i = 1, 2, \cdots, n-1)$. Here, σ_i and σ_i^{-1} represent a braid with the i-th string and $(i+1)$-th string twisted as shown in the figure below.

A set of all isotopy types with n braids becomes a group with respect to the product of attaching braids vertically. This group is shown as B_n and called a *braid group.* It is known that a braid group B_n is shown as a group:

$$B_n = \langle \sigma_1, \cdots, \sigma_{n-1} \mid \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i \text{ if } |i-j| \geq 2,\ \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} \rangle.$$

An oriented braid given by connecting the top and the bottom of the braid is called a *closure* of the braid. With respect to this, the following theorem holds. By the following two theorems, we pick quarrel and can consider the geometric

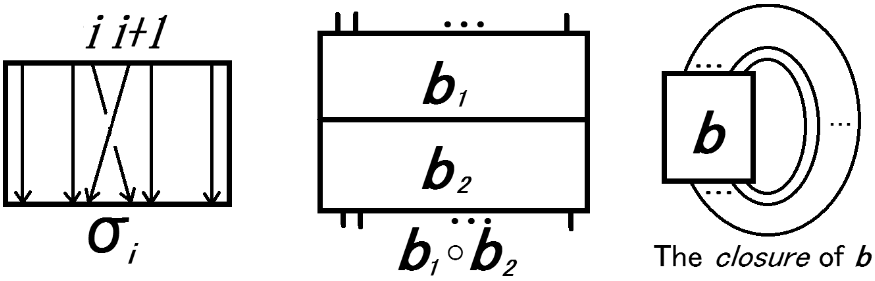


Figure 4: Braid.

object of links as an algebraic object.

Theorem 2.3 ([A])**.** *Any oriented link is shown as a closure of a certain braid.*

Theorem 2.4 ([Bi])**.** *With respect to two braids b_1, b_2, a necessary and sufficient condition for their closure being an isotopic link is that we obtain b_2 by giving the following MI, MII moves to b_1 finite times.*

$$MI : ab \longleftrightarrow ba \ (a, b \in B_n), \qquad MII : b\sigma_n \longleftrightarrow b \longleftrightarrow b\sigma_n^{-1} \ (b \in B_n),$$

where, for MII moves, we regard $b\sigma_n^{\pm 1}$ as an element of B_{n+1}.

From now on, vector space is on $\mathbb{C}$. For finite dimensional vector space V, we let the dual space be V^* and the whole space of linear map $V \to V$ be $End(V)$. By the correspondence of $f \otimes x \in V^* \otimes V$ and $(y \to f(y)x) \in End(V)$, we make an equation such as $V^* \otimes V = End(V)$. The *trace* of $End(V)$ is shown as

$$\text{trace} : \text{End}(V) \to V^* \otimes V \to \mathbb{C},$$

where, the contraction is linear map $V^* \otimes V \to \mathbb{C}$ fixed by $f \otimes x \to f(x)$. Further, let V_1 and V_2 be vector space. For an endomorphism in $\text{End}(V_1 \otimes V_2)$

we define the trace$_2$ of the endomorphism to be the trace with respect to the 2-th entry as

$$\begin{aligned}\mathrm{trace}_2 : \mathrm{End}(V_1 \otimes V_2) &= (V_1 \otimes V_2)^* \otimes (V_1 \otimes V_2) \\ &= V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \xrightarrow{\text{contraction}} V_1^* \otimes V_1 = \mathrm{End}(V_1),\end{aligned}$$

where the contraction is the contraction $V_2^* \otimes V_2 \to \mathbb{C}$ respectively.

Definition 2.5 ([Ji]). The construction of the general quantum invariable is explained as below. Let V be a vector space over $\mathbb{C}$. We obtain a representation $\psi_n : B_n \to End(V^{\otimes n})$ defined by

$$\psi_n(\sigma_i) = (id_V)^{\otimes(i-1)} \otimes R \otimes (id_V)^{\otimes(n-i-1)}. \tag{1}$$

Such a map ψ_n given in (1) always satisfies $\psi_n(\sigma_i\sigma_j) = \psi_n(\sigma_j\sigma_i)$ ($|i-j| \geq 2$). To obtain $\psi_n(\sigma_i\sigma_{i+1}\sigma_i) = \psi_n(\sigma_{i+1}\sigma_i\sigma_{i+1})$, the matrix R is required to satisfy the relation

$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R).$$

We call this equation the *Yang-Baxter equation*, and call a solution of it an *R-matrix*.

Theorem 2.6 (chap.I [Tu] and chap.X [K]). *Let L be an oriented link and $b \in B_n$ be a braid such that the closure is isotopic to L. We regard $\mathcal{R}$ as an R-matrix and $h \in End(V)$ as a linear map which satisfies*

- $\mathrm{trace}_2\big((id_V \otimes h) \cdot R^{\pm}\big) = id_V$,
- $R \cdot (h \otimes h) = (h \otimes h) \cdot R$.

Then, a $\mathrm{trace}\big(h^{\otimes n} \cdot \psi_n(b)\big)$ is unchanged by MI, MII moves. Therefore, by Theorem 2.4, this is an isotopy invariant of L.

We call these invariants isotopy invariants of L obtained from an R-matrix. But nowadays the invariants of a quantum $(\mathfrak{g}, V')$ (let V' be a vector representation of a Lie algebra $\mathfrak{g}$) are defined via the quantum group and the representation.

2.3 Vassiliev invariants (Finite type invariants).

When a knot invariant v with a value in the complex number is given, it is extended to a singular knot by the following formula:

$$v(K_D) = v(K_+) - v(K_-). \tag{2}$$

where, a singular knot is an immersion in $\mathbb{R}^3$ of the circle whose singularities are only double points. Moreover, a part of K_D, K_+, K_- is shown in the following figure 5 and the other parts are the similar three singular knots. We regard

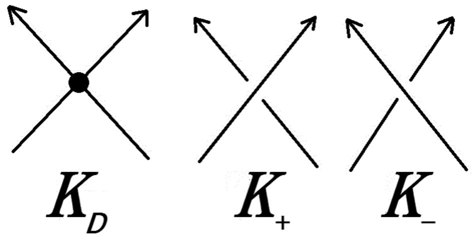


Figure 5: K_D, K_+, K_-

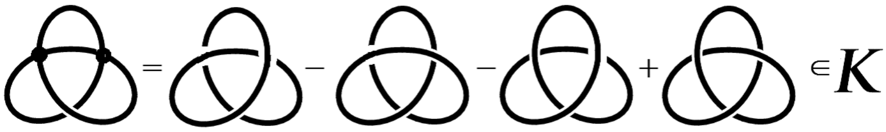


Figure 6: singular knot is an element of $\mathcal{K}$.

vector space on $\mathbb{C}$ spanned by all the isotopy types of oriented knots as $\mathcal{K}$. As a result of linearly resolving its each double point by using the formula (2) for a singular knot, we look on the singular knot as an element of $\mathcal{K}$. For example, it is shown in the following figure 6. We regard vector subspace of $\mathcal{K}$ spanned by singular knots spanned with d double points as $\mathcal{K}_d$, which fixes filtration of $\mathcal{K}$:

$$\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \cdots$$

Linear map $\mathcal{K} \to \mathbb{C}$ which becomes zero mapping when restricted to $\mathcal{K}_{d+1}$ is called *Vassiliev invarianats (finite type invariants)* of *degree* d [G1, G2, V].

We calculate a quantum invariant of singular knots by regarding a matrix $\mathcal{R}$ at a singularity as $\mathcal{R} - \mathcal{R}^{-1}$. The definition shows that especially a Vassiliev invariant (finite type invariant) of degree 0 is an constant function with the same value for all knots. In other words, with respect to a Vassiliev invariant (finite type invariant) v of degree 0, even if the crossings of the knot are replaced as shown in the figure below, the value of v is invariable, and therefore v has the same value for all the knots. Furthermore, the definition of Vassiliev invariants (finite type invariants) shows that, as the degree of finite type invariants gets raised, a set of Vassiliev invariants (finite type invariants) makes a hierarchy structure as follows:

{Vassiliev invariants (finite type invariants) of degree 0} $\subset$ {Vassiliev invariants (finite type invariants) of degree 1} $\subset$ {Vassiliev invariants (finite type invariants) of degree 2} $\subset \cdots$.

With respect to a Vassiliev invariant (finite type invariant) v of degree d, we give consideration to the value for a singular knot K^d with d double points.

Then, when the singular knot given by transforming a singular knot K^d with the self-intersection of edges is regarded as $K^{d'}$, we find $v(K^d) = v(K^{d'})$, for the changing value of v caused by the self-intersection of edges is shown by the value of v for a singular knot with $(d+1)$ double points which appears the moment that self-intersection occurs, which is 0 derived from the definition of Vassiliev invariants of degree d. Therefore, with respect to the given v, the value of $v(K^d)$ is decided by the order of the double points of K^d forming a line on the circumference. The order is shown by a chord diagram.

A *chord diagram* is a diagram connecting pairs of points by using dotted lines (called *chords*), and it is derived from singular knots by connecting marks corresponding the same double points by using chords after marking on the circle the places where the double points appear when a string of the singular knot goes round. When a Vassiliev invariant (finite type invariant) v of degree d is given, let a singular knot corresponding a chord diagram D with d chords be K_D. Note that K_D is an element of K_d. Although a singular knot K_D is not unique, the equivalence class of K_D is unique for any singular knot corresponding D as stated above. In short, as an element of $\mathcal{K}_d/\mathcal{K}_{d+1}$, the equivalence class $[K_D]$ is unique.

Furthermore, with respect to a singular knot K with d double points, when we let a chord diagram which represents a type of K be D, we regard a value of $v(K)$ as $W_v(D)$. As shown above, note that a value of $W_v(D)$ from a chord diagram D of degree d on S^1 is decided only by D, (which is independent of K). In short, a Vassiliev invariant (finite type invariant) v of degree d induces the following linear map:

$$W_v : \mathrm{span}_{\mathbb{C}}\{\text{chord diagrams on } S^1 \text{ of degree } d\} \to \mathbb{C}.$$

This linear map has two restrictions. One is that it becomes 0 for a chord diagram with an isolated chord. The other is that it has the relations obtained according to four ways to resolve a triple point. Based on the conditions mentioned above, we consider the following relations in vector space spanned by a chord diagram:

When

$$A(S^1) = \mathrm{span}_{\mathbb{C}}\{\text{ chord diagrams on } S^1\}/ \text{ the 4T relation}$$

is given, let vector subspace of $A(S^1)$ spanned by the chord diagram of degree d be $A(S^1)^{(d)}$. Let map where D corresponds $[K_D]$ be

$$\varphi : A(S^1)^{(d)}/FI \to \mathcal{K}_d/\mathcal{K}_{d+1}.$$

By using this map, the weight system W_v of a Vassiliev invariant (finite type invariant) v of degree d is shown as the following composed mapping:

$$W_v : A(S^1)^{(d)}/FI \xrightarrow{\varphi} \mathcal{K}_d/\mathcal{K}_{d+1} \subset \mathcal{K}/\mathcal{K}_{d+1} \xrightarrow{[V]} \mathbb{C}$$

Here, $[v]$ indicates the linear map which v naturally induces on the quotient vector space given by $\mathcal{K}_{d+1}$. Linear map φ is a surjection, which is given by the constitution method and the number of chord diagrams of degree d is finite. Therefore $\mathcal{K}_d/\mathcal{K}_{d+1}$ has a finite dimension.

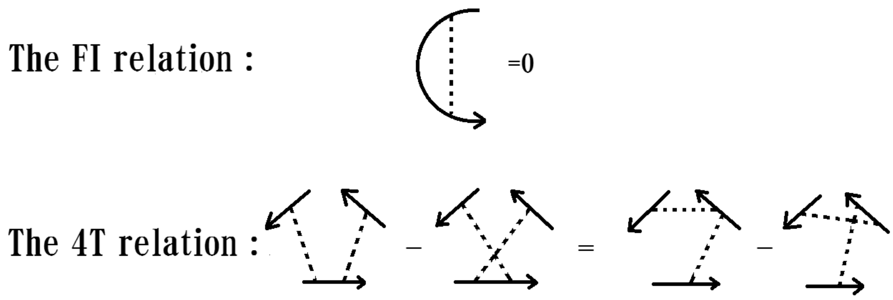


Figure 7: The FI and 4T relations.

2.4 Quandle (shadow) cocycle invariants

A *quandle* is a set X with a binary operation $* : X \times X \to X$ such that the following three are satisfied:

(i) For any $a \in X$, $a * a = a$.

(ii) For any $a, b \in X$, there exists a unique $c \in X$ such that $a = c * b$.

(iii) For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

For example, any module over $\mathbb{Z}[\omega^{\pm}]$ is a quandle with operations

$$x * y = \omega x + (1 - \omega) y.$$

Such a quandle is called an *Alexander quandle*. When $\omega = 1$ i.e. $x * y = x$, we call this type of quandle a *trivial quandle*. For an odd $p \in \mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$ is a quandle under the operation

$$x * y = 2y - x$$

for $x, y \in \mathbb{Z}/p\mathbb{Z}$. We call this type of quandle a *dihedral quandle* R_p.

For quandle X, we give consideration to natural map,

$$X \to \mathrm{Aut}(X), \ \ x \mapsto (\bullet * x).$$

The *inner automorphism group* $\mathrm{Inn}(X)$ denotes the subgroup of $\mathrm{Aut}(X)$ which the image of the above map generates. A connected component of X represents an orbit of the action of $\mathrm{Inn}(X)$ on X. When the action of $\mathrm{Inn}(X)$ is transitive, quandle X is called *connected*,

A link quandle of L is a set $Q(L) = \{[(D, a)]\}$ composed of a homotopy class of a pair of path a in complementary space of L from a boundary between meridian disc D of L and D to a certain basic point, and it becomes a quandle

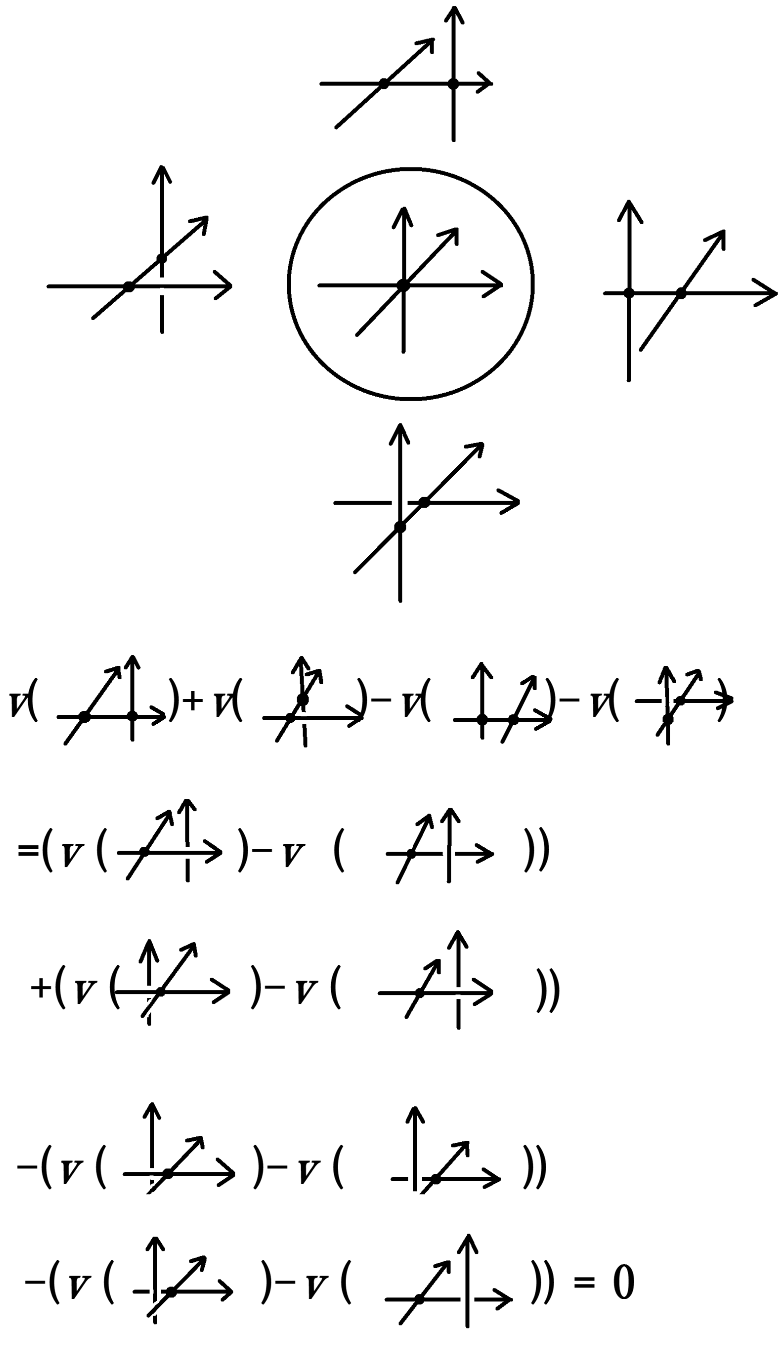


Figure 8: triple point.

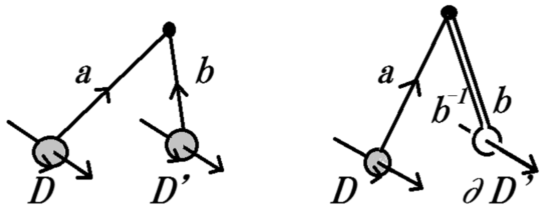


Figure 9: link quandle.

by operation $(D, a) * (D', b) = (D, a \cdot b^{-1} \cdot \partial D' \cdot b)$ [Joy]. Furthermore, quandle homomorphism $Q_L \to X$ is called an *X-coloring* of L.

A quandle cocycle invariant of a link associated with a 2-cocycle (as an analogy of group cohomology) f of a finite quandle X. Let D be a diagram of link L. A map $C : D \to X$ is an *X-coloring* of D [Definition.4.1 [CKS]], the following figures 10 of an X-coloring of crossing points. We need the relation

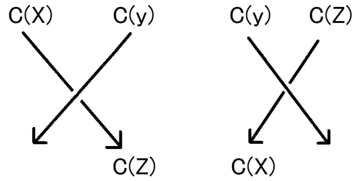


Figure 10: $W_f(x; C)$

$C(z) = C(x) * C(y)$. The bijection $\mathrm{Hom}_{\mathrm{Qnd}}(Q_L, X) \to \{X - \text{coloring of } D\}$ exists. Especially, in case of quandles, the right side of the equation is finite as shown by the definition and the left side depends only on the link L, and thus the X-coloring number is an invariant of the link L.

Using a quandle 2-cocycle f, we define the weight $W_f(x; C)$ at a crossing x of a diagram D, with a coloring C in X as stated above for the two types of crossings. We fix $t \in \mathbb{C} \setminus \{0\}$. The *quandle 2-cocycle invariant* of knots and links [Definition.4.3 and Theorem.4.4 [CJKLS]] is the state-sum

$$\sum_C \prod_x W_f(x; C) \cong \sum_C \prod_x t^{W_f(x;C)} \in \mathbb{Z}[t^i, t^{-i}] \ (i \in A)$$

which is an invariant of the link L. We put

$$\Phi_f(L) = \sum_C \prod_x t^{W_\phi(x;C)}$$

A quandle cocycle invariant of a link associated with a 3-cocycle (as an analogy of group cohomology) ϕ of a finite quandle X. Let D be a diagram of link L, and $\sum(D)$ the set of arcs of D and the set of regions separated by the underlying immersed curve of D. A map $C : \sum(D) \to X$ is an X-*shadow coloring* of D [Definition.4.3 [CKS]], as shown in the following figure 11 of an X-shadow coloring of crossing points. We need the relation $C(w) =$

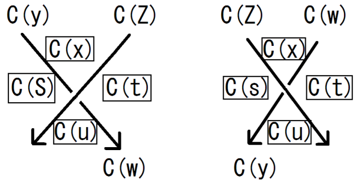


Figure 11: $W_\phi(x;C)$

$C(y) * C(z), C(x) = C(s) * C(y), C(u) = C(s) * C(z)$ and $C(t) = C(x) * C(z) = C(u) * C(w)$. Using a quandle 3-cocycle ϕ, we define the weight $W_\phi(x;C)$ at a crossing x of a diagram D, with a shadow coloring C in X as stated above for the two types of crossings. We fix $t \in \mathbb{C} \setminus \{0\}$. The *shadow cocycle invariant* of knots and links [Definition.5.5 and Theorem.5.6 [CJKLS]] is the state-sum

$$\sum_C \prod_x W_\phi(x;C) \cong \sum_C \prod_x t^{W_\phi(x;C)} \in \mathbb{Z}[t^i, t^{-i}]\ (i \in A)$$

which is an invariant of the link L. We put

$$\Phi_\phi(L) = \sum_C \prod_x t^{W_\phi(x;C)}$$

2.5 Heegaard splitting

Definition 2.7. Let $\sum_g$ be a closed surface of the genus g and a set of all the isotopy types of automorphisms keeping the orientation of $\sum_g$ makes a group by the product of composed mapping. This group is called a *mapping class group* and represented by $\mathfrak{M}_g$.

Definition 2.8. An orientable 3-manifold that is a 3-ball with g 1-handles added is called a *handlebody of genus g H_g*. For autohomeomorphic mapping ϕ of $\sum_g$, we consider 3-manifolds $H_g \cup_\phi H_g$ obtained by gluing a boundary of two copies of H_g with ϕ. When the given 3-manifold M is homeomorphic to $H_g \cup_\phi H_g$ for a certain ϕ, the $H_g \cup_\phi H_g$ is called a *Heegaard splitting of M*.

Proposition 2.9. *Any closed connected orientable 3-manifold has a Heegaard splitting.*

Definition 2.10. Minimum genus of a Heegaard splitting of M:

$g(M) = \min\{g|M$ has a Heegaard splitting of genus g$\}$

is called the Heegaard genus of M represented by $g(M)$.

Definition 2.11. Two Heegaard splittings of $H_g \cup_{\varphi_1} H_g$ and $H_g \cup_{\varphi_2} H_g$are *equivalent* if an ambient isotopy carries φ_1 to φ_2, preserving orientation.

Definition 2.12. A *stabilization* of a genus g Heegaard splitting is a new splitting of genus $g+1$ obtained by adding a trivial 1-handle in the following figure 12 to the Heegaard surface.

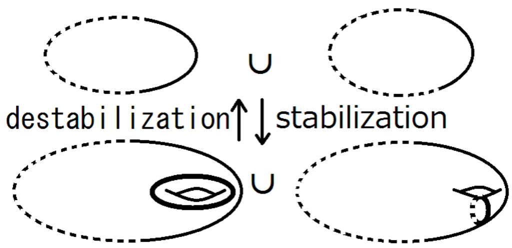


Figure 12: stabilization

Remark 2.13. τ is a homeomorphism of the surface sending the bold line of a closed curve in the boundary of the left handlebody to that of the right handlebody respectively.

Proposition 2.14 ([L1, L2])**.** *A mapping class group* $\mathfrak{M}_g$*consists of the twist about the set of* $(3g-1)$ *simple closed curves of* l_i, m_j, n_k $(1 \leq i, j \leq g,\ 1 \leq k \leq g-1)$ *shown in the following figure 13. This generators are Lickorish generators*

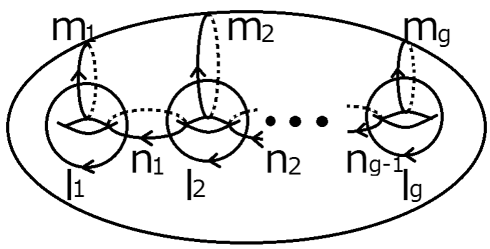


Figure 13: Lickorish generators

2.6 Dijkgraaf-Witten invariants

Let A be an abelian group. A *Dijkgraaf-Witten invariant*[DW] of oriented 3-manifolds is a topological invariant fixed every time 3-cocycle $\alpha : G^3 \to A$ with values in a finite group G and in a one-dimensional unitary group $U(1)$ of the classifying space BG is given, and, for a connected closed 3-manifold M, it is defined as the following formula:

$$DW_\alpha(M) = \frac{1}{|G|} \sum_{f \in \mathrm{Hom_{gr}}(\pi_1(M), G)} \langle f^*(\alpha), [M] \rangle \in \mathbb{Z}[A],$$

where $[M] \in H_3(M; A)$ is the fundamental class of M.

2.7 Lens space

Let p be a positive integer two or more and a positive integer q which is prime to p is given. First, we consider a three-dimensional sphere B^3 in three-dimensional Euclid space R^3, divide the equator $e = \{(x, y, z) \in \partial B^3 \mid x^2 + y^2 + z^2 = 1,\ z = 0\}$ of this boundary ∂B^3 into p equal parts, and regard the points divided equally as $a_0 = (1, 0, 0), a_1 = (\cos \frac{2\pi}{p}, \sin \frac{2\pi}{p}, 0), \cdots, a_i = (\cos \frac{2\pi i}{p}, \sin \frac{2\pi i}{p}, 0), \cdots, (\cos \frac{2\pi(p-1)}{p}, \sin \frac{2\pi(p-1)}{p}, 0)$. Let l_i be the longitude going through a_i, and the domain surrounded by l_i and l_{i+1} is divided by e into two parts; the upper one is named U_i and the lower one is D_i. Then, we put equivalence relation $\sim$ in ∂B^3 as follows:

We glue D_i and U_{i+q} so that l_i becomes l_{i+p}, l_{i+1} becomes l_{i+q+1} and $N = (0, 0, 1)$ becomes $S = (0, 0, -1)$. Then, when $s_1 = (x_1, y_1, z_1)$ and $s_2 = (x_2, y_2, z_2)$, and when $s_1 \sim s_2$ is shown as $x_1 = \sqrt{1-h^2}\cos\varphi$, $y_1 = \sqrt{1-h^2}\sin\varphi$, $z_1 = h$ by using $h \geq 0, 0 \leq \varphi \leq 2\pi$, we decide this equivalence relation when:

$$(\text{Inside}) x_2 = x_1,\ y_2 = y_1,\ z_2 = z_1$$

$$(\text{On the sphere}) x_2 = \sqrt{1-h^2}\cos(\varphi + \frac{2\pi q}{p}),\ y_2 = \sqrt{1-h^2}\sin(\varphi + \frac{2\pi q}{p}),\ z_2 = -h.$$

Quotient space $B^3/\sim$ where B^3 is divided by this equivalence relation is called *lens space* and represented by $L(p, q)$.

3 On Vassiliev invariants of degrees 2 and 3 for torus knots

The present work is motivated by [Oh3, §2.4 Vassiliev invarians and cossing numbers]. As to the Vassiliev invariants of degrees 2 and 3, N. Okuda [Ok] posed the following problem:

Problem 3.1 (cf.[Oh3, Problem 2.10])**.** Let K be a knot and n be the crossing number of a diagram D of K, and $\mathcal{V}_2(K)$ and $\mathcal{V}_3(K)$ be the primitive Vassiliev invariants of degree 2 and 3 of K, respectively. Then, describe the following set

$$\mathcal{S}(K, D) := \left\{ \left(\frac{\mathcal{V}_2(K)}{n^2}, \frac{\mathcal{V}_3(K)}{n^3} \right) \in \mathbb{R} \times \mathbb{R} \right\}.$$

The most ideal solution for this problem would be to give a precise function $f(x,y)$ such that

$$f\left(\frac{\mathcal{V}_2(K)}{n^2}, \frac{\mathcal{V}_3(K)}{n^3}\right) = 0.$$

However so far a reasonable solution would be to give a domain as sharp as possible containing the set $\mathcal{S}(K,D)$. As to each of $\mathcal{V}_2(K)$ and $\mathcal{V}_3(K)$, N. Okuda [Ok] showed the following inequalities:

$$-\frac{n^2}{16} \leq \mathcal{V}_2(K) \leq \frac{n^2}{8},$$

$$|\mathcal{V}_3(K)| \leq \frac{n(n-1)(n-2)}{15}.$$

Here the right-hand-side inequality of the first one is due to Polyak-Viro [PV]. It follows from these two inequalities that the set $\mathcal{S}(K,D)$ is contained in the rectangle

$$\left[-\frac{1}{16}, \frac{1}{8}\right] \times \left[-\frac{1}{15}, \frac{1}{15}\right].$$

Then, as pointed out in the second Remark right after Problem 2.10 in [Oh3, pp.404-405], it is a problem to describe the smallest domain containing the set $\mathcal{S}(K,D)$. In this paper we give a non-trivial domain (i.e., non-rectangle domain) containing the set $\mathcal{S}(K,D)$ in the case of torus knots.

Theorem 3.2. *Let K be a torus knot and let n be the crossing number of a diagram D of K. Then we have*

$$\mathcal{S}(K,D) \subset \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{8}{3}x^2 < |y| \leq \frac{1}{3}x\right\} \bigcup \{(0,0) \in \mathbb{R}^2\} \tag{3}$$

As to $\mathcal{V}_3(L)$, S. Willerton [W2] made the following conjecture:

Conjecture 3.3 ([W2] and cf. [Oh3, Conjecture 2.11])**.** Let $\mathcal{V}_3$ be as above. If a knot K has a diagram with n crossings, then

$$|\mathcal{V}_3(K)| \leq [\frac{n(n^2-1)}{24}],$$

where $[x]$ denotes the Gauss symbol of x.

In this section we show that the above conjecture is correct in the case of torus knots.

3.1 Primitive Vassiliev invariants and torus knots

In [V] V. A. Vassiliev introduced what is now called the *Vassiliev invariant of a knot*, using the cohomology of the complement of discriminants in space of maps. In [G1] M.N. Goussarov redefined or independently defined the Vassiliev invariant more axiomatically.

A Vassiliev invariant $\mathcal{V}$ is called *primitive* if it is additive under the connected sum of knots K_1, K_2, that is, $\mathcal{V}(K_1 \sharp K_2) = \mathcal{V}(K_1) + \mathcal{V}(K_2)$. Let $\mathcal{V}_2$ and $\mathcal{V}_3$ be the $\mathbb{R}$-valued Vassiliev invariants of degree 2 and 3 of K, respectively, normalized by the conditions that $\mathcal{V}_2(K) = \mathcal{V}_2(\overline{K})$ and $\mathcal{V}_3(K) = -\mathcal{V}_3(\overline{K})$ for any K and its mirror image $\overline{K}$ and that they take 0 on the unknot and 1 on the trefoil.

Proposition 3.4 ([W2])**.** *Let $J_K(t)$ be the Jones polynomial of K and let $J_K^{(m)}(t)$ denote its m-th derivative with respect to t. Then $\mathcal{V}_2(K)$ and $\mathcal{V}_3(K)$ are described using the derivatives of the the Jones polynomial as follows:*

$$\mathcal{V}_2(K) = -\frac{1}{6} J_K^{(2)}(1),$$

$$\mathcal{V}_3(K) = -\frac{1}{36}\big(J_K^{(3)}(1) + 3J_K^{(2)}(1)\big).$$

Let K be a (p, q)-torus knot and let n be the crossing number of a diagram of K. Then it is known that $n \geq min\{|p(q-1)|, |q(p-1)|\}$ (see [Mu]). A (p, q)-torus knot is trivial if and only if either p or q is equal to 1 or -1. The Vassiliev invariant of a trivial knot is 0. Therefore, since we deal with non-trivial knots, from now on we assume that $|p| \geq 2$ and $|q| \geq 2$. Moreover, we know that the Jones polynomial $J_K(t)$ of the (p, q)-torus knot K is expressed by:

$$J_K(t) = \frac{t^{\frac{(p-1)(q-1)}{2}}(1 - t^{p+1} - t^{q+1} + t^{p+q})}{1 - t^2}.$$

Remark 3.5 (cf.[W2, §4. Torus Knots, p292])**.** M. Alvarez and J. M. F. Labastida [AL] obtained the two formulae for $\mathcal{V}_2(K)$ and $\mathcal{V}_3(K)$ in a different way.

Hence the Vassiliev invariants of degree 2 and 3 for a $(p,\ q)$-torus knot K are respectively given by:

$$\mathcal{V}_2(K) = -\frac{1}{6} J_K^{(2)}(1) = \frac{(p^2-1)(q^2-1)}{24},$$

$$\mathcal{V}_3(K) = -\frac{1}{36}\big(J_K^{(3)}(1) + 3J_K^{(2)}(1)\big) = \frac{pq(p^2-1)(q^2-1)}{144}.$$

3.2 Results of this section

Theorem 3.6. *Let K be a non-trivial torus knot and let n be the crossing number of a diagram D of K. Then we have*

$$\frac{8}{3}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^2 < \Big|\frac{\mathcal{V}_3(K)}{n^3}\Big| \leq \frac{1}{3}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big). \tag{4}$$

Proof. We prove the above inequalities in the case of $q \leq p$, which implies that $n \geq |q(p-1)|$.

$$\Big|\frac{\mathcal{V}_3(K)}{n^3}\Big| = \Big|\frac{pq(p^2-1)(q^2-1)}{144n^3}\Big|$$

$$
\begin{aligned}
&= \frac{1}{6}\frac{(p^2-1)(q^2-1)}{24n^2}\frac{|pq|}{n^2}n \\
&\geq \frac{1}{6}\frac{v_2(K)}{n^2}\frac{(p^2-1)(q^2-1)}{24n^2}\frac{24|pq|}{(p^2-1)(q^2-1)}|q(p-1)| \\
&\geq 4\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^2\frac{|p(p-1)|}{p^2-1}\frac{q^2}{q^2-1}.
\end{aligned}
$$

Since $|p| \geq 2$, we have that the minimum of $\frac{|p(p-1)|}{(p^2-1)}$ is $\frac{2}{3}$ when $p = 2$. Moreover note that $|q| \geq 2$, therefore $\frac{q^2}{q^2-1} > 1$.

$$
\begin{aligned}
\Big|\frac{\mathcal{V}_3(K)}{n^3}\Big| &= \Big|\frac{pq(p^2-1)(q^2-1)}{144n^3}\Big| \\
&= \frac{1}{6}\frac{(p^2-1)(q^2-1)}{24n^2}\Big|\frac{pq}{n}\Big| \\
&\leq \frac{1}{6}\frac{\mathcal{V}_2(K)}{n^2}\Big|\frac{pq}{|q(p-1)|}\Big| \\
&\leq \frac{1}{6}\frac{\mathcal{V}_2(K)}{n^2}\Big|\frac{p}{p-1}\Big|.
\end{aligned}
$$

Note that $|p| \geq 2$, therefore the maximum of $|\frac{p}{p-1}|$ is 2 when $p = 2$. Hence we obtain the following relation:

$$\frac{8}{3}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^2 < \Big|\frac{\mathcal{V}_3(K)}{n^3}\Big| \leq \frac{1}{3}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big).$$

In the case of $q \leq p$, we exchange p and q in the above proof and we obtain the same result. □

Let K be a torus knot. The above inequalities (4) imply that the set $\mathcal{S}(K, D)$ is contained in the domain

$$\Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \frac{8}{3}x^2 < |y| \leq \frac{1}{3}x\Big\} \bigcup \{(0, 0) \in \mathbb{R}^2\}.$$

In particular, we get the following

Corollary 3.7. *Let the situation be as above.*

$$0 \leq \frac{\mathcal{V}_2(K)}{n^2} \leq \frac{1}{8}, \qquad -\frac{1}{24} \leq \frac{\mathcal{V}_3(K)}{n^3} \leq \frac{1}{24}.$$

Remark 3.8. In [W2, Proposition 4.1, p.292] Willerton showed the following inequalities for a torus knot K (he uses T instead of K):

$$\frac{2}{3}\mathcal{V}_2(K)^3 + \frac{1}{3}\mathcal{V}_2(K)^2 \leq \mathcal{V}_3(K)^2 \leq \frac{8}{9}\mathcal{V}_2(T)^3 + \frac{1}{9}\mathcal{V}_2(K)^2.$$

From which we can obtain the following inequalities, by dividing them throughout by n^6:

$$\frac{2}{3}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^3 + \frac{1}{3n^2}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^2 \leq \Big(\frac{\mathcal{V}_3(K)}{n^3}\Big)^2 \leq \frac{8}{9}\Big(\frac{\mathcal{V}_2(T)}{n^2}\Big)^3 + \frac{1}{9n^2}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^2.$$

Hence, we get the following inequalities:

$$\frac{2}{3}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^3 < \Big(\frac{\mathcal{V}_3(K)}{n^3}\Big)^2 \leq \frac{8}{9}\Big(\frac{\mathcal{V}_2(T)}{n^2}\Big)^3 + \frac{1}{9}\Big(\frac{\mathcal{V}_2(K)}{n^2}\Big)^2.$$

Therefore we can see that the set $\mathcal{S}(K, D)$ is contained in the following domain:

$$\Big\{(x, y) \in \mathbb{R}^2 \Big| \frac{2}{3}x^3 < y^2 < \frac{8}{9}x^3 + \frac{1}{9}x^2\Big\}.$$

It follows from the above inequalities $\frac{8}{3}x^2 < |y| < \frac{1}{3}x$ (just before Corollary 3.2) and $\frac{2}{3}x^3 < y^2 < \frac{8}{9}x^3 + \frac{1}{9}x^2$ (at the end of Remark 3.3) that we can get the following corollary:

Corollary 3.9. *Let the situation be as above.*

$$\begin{aligned}\mathcal{S}(K, D) \subset &\Big\{(x, y) \in \mathbb{R}^2 \Big| \frac{2}{3}x^3 < y^2 \leq \frac{1}{9}x^2, \quad 0 < x \leq \frac{3}{32}\Big\} \\ &\bigcup \Big\{(x, y) \in \mathbb{R}^2 \Big| \frac{8}{3}x^2 < |y| \leq \frac{1}{3}x, \quad \frac{3}{32} \leq x \leq \frac{1}{8}\Big\} \bigcup \{(0, 0) \in \mathbb{R}^2\}.\end{aligned}$$

As to the Vassiliev invariant $\mathcal{V}_3$, we obtain the following inequality, i.e. the aforementioned Willerton ' s conjecture [W2](c.f. [Oh3, Conjecture 2.11]).

Theorem 3.10. *Let K be a torus knot and let n be the crossing number of a diagram of K. Then we have*

$$\big|\mathcal{V}_3(K)\big| \leq \Big[\frac{n(n^2-1)}{24}\Big],$$

Proof. We prove the above inequality in the case of $q \leq p$. First we show the following inequality:

$$\Big|\frac{p(p+1)(q^2-1)}{6}\Big| \leq \big|q(p-1)\big|^2 - 1. \tag{5}$$

To show this we observe the following:

$$\begin{aligned}&6\big(\big|q(p-1)\big|^2 - 1\big) - p(p+1)(q^2-1) \\ &= 5p^2q^2 - 13pq^2 + 6q^2 + p^2 + p - 6 \\ &= q^2(p-2)(5p-2) + (p-2)(p+3) \\ &= (p-2)\big((q^2(5p-2) + p + 3\big).\end{aligned}$$

Note that $|p| \geq 2$, therefore

$$\frac{p(p+1)(q^2-1)}{6} \leq \big|q(p-1)\big|^2 - 1.$$

Moreover note that $|q| \geq 2$, therefore $p(p+1)(q^2-1) > 0$. Thus we obtain the above inequality (5).

Next, we observe the following:

$$\begin{aligned} \big|\mathcal{V}_3(K)\big| &= \Big|\frac{pq(p^2-1)(q^2-1)}{144}\Big| && (6)\\ &\le \frac{|q(p-1)|}{24}\Big|\frac{p(p+1)(q^2-1)}{6}\Big|. && (7) \end{aligned}$$

By the inequality (5), we obtain

$$\big|\mathcal{V}_3(K)\big| \le \frac{|q(p-1)|}{24}\big(\big|q(p-1)\big|^2-1\big).$$

Hence

$$\big|\mathcal{V}_3(K)\big| \le \frac{n\big(n^2-1\big)}{24}.$$

In the case of $q \le p$, we exchange p and q in the above proof and we obtain the same result. $\square$

Remark 3.11. As remarked in [W2, §2], the degree-3 Vassiliev invariant $\mathcal{V}_3$ satisfies that for any knot K with the crossing number n

$$\big|\mathcal{V}_3(K)\big| \le \frac{n(n-1)(n-2)}{4},$$

which was obtained in [W1] using Domergue and Donato's integration [DD]. For any n we do have that $\frac{n(n-1)(n-2)}{15} < \frac{n(n-1)(n-2)}{4}$. Thus, Okuda's inequality is sharper than Willerton's inequality. However, if $n > 8$, we have that

$$\frac{n(n^2-1)}{24} < \frac{n(n-1)(n-2)}{15}.$$

Here, we note that the equality holds for $n = 7$. Thus our inequality, i.e. the inequality conjectured by Willerton, is sharper for $n \ge 7$, although the knots considered in the present paper are torus knots.

4 Definition of quasi finite type invariants and finite type invariants of some 3-manifolds

This section examines quandle shadow cocycle invariants of links from the viewpoint of finite type invariants. Dijkgraaf-Witten invariants of 3-manifolds are also studied from the same point of view.

For an odd prime integer p, a quandle shadow cocycle invariant [CJKLS] of the link L is defined by using Mochizuki's 3-cocycle

$$\theta_p(x,y,z) = (x-y)\frac{(2z-y)^p + y^p - 2z^p}{p} \bmod p$$

of the dihedral quandle.

Proposition 4.1. *We write $\Phi_p(L)$ or $\Phi_{\theta_p}(L)$ for the quandle shadow cocycle invariant. Then the values are in the Laurent polynomial ring $\mathbb{Z}[t,t^{-1}]/(t^p-1)$.*

Proof. $\Phi_{\theta_p}(L) \in \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ by the definition of quandle (shadow) cocycle invariants. There exists bijective homomorphism $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{Z}[t,t^{-1}]/(t^p-1)$. This map is $\sum_{i=1}^n c_i \bar{a}_i \to \sum_{i=1}^n c_i t^{a_i}$. □

Thus $\Phi_p(L)$ can be regarded as an element in the ring $\mathbb{Z}[t]/(t^{p-1}+\cdots+t+1)$, and expressed as $\Phi_p(L) = a_{p,0}+a_{p,1}(t-1)+\cdots+a_{p,p-2}(t-1)^{p-2}$ for some integers $a_{p,k}$ ($k=0,1,\cdots,p-2$). As proved in [Oh2, Lemma 9.7.] the coefficients $a_{p,k}$ are not unique, but they are well-defined as elements in $\mathbb{Z}/p\mathbb{Z}$. In this paper properties of $a_{p,k}$ are investigated for two special classes of links, namely, 2-bridge links and torus links by using results of computations due to Iwakiri [I] and Asami and Satoh [AS]. Based on this observation we find the coefficient $a_{p,\frac{p-1}{2}}$ satisfies some kind of "finite type " property, and introduces a definition of quasi finite type invariants as an analogy of finite type invariants introduced by Vassiliev and Goussarov.

Hatakenaka shows that $\Phi_p(L)$ can be essentially viewed as the Dijkgraaf-Witten invariant $DW_{\phi_p}(M_L)$ of 3-manifold M_L obtained as a double branched covering of L, where ϕ_p is the 3-cocycle of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ corresponding to θ_p [H, Theorem 3.2.]. Translating this result, we show that, for Lens space and Brieskorn manifolds $M(2,l,n)$, the coefficient $a_{p,\frac{p-1}{2}}$ of their Dijkgraaf-Witten invariants under the expansion with respect to $(t-1)$ have also some kind of finite type property, and propose a definition of finite type invariants as a generalization of Ohtsuki's finite type invariants.

In this section, to describe the property of $a_{p,\frac{p-1}{2}}$ as a finite type, some descending filtrations of vector spaces spanned by special classes of links or 3-manifolds are introduced. The filtration we consider is new.

The relation between quandle (shadow) cocycle invariants and quantum invariants has not been clarified. In this article, we consider the relation between quandle shadow cocycle invariants and finite type invariants. We transform the calculation result when quandle shadow cocycle invariants are both $q=p$ and $\omega=-1$ into

$$a_{p,0}+a_{p,1}(t-1)+a_{p,2}(t-1)^2+\cdots+a_{p,p-2}(t-1)^{p-2} \in \mathbb{Z}[t]/(t^{p-1}+\cdots+t+1).$$

We define a filtration on the 2-bridge link and the torus link. For the filtration which we have defined, a value of $a_{p,\frac{p-1}{2}}$ looks like a finite type invariant. However, as a result of the consideration, we have found that the filtration at a singular point cannot be defined. Therefore, it is not a finite type invariant. Here, we defined a quasi finite type invariant. This definition is the same as that of a finite type invariant except the links with the singular point. The quasi finite type invariants are naturally defined by the nature of the finite type invariants and they have the similar nature to the finite type invariants. One of the similarities is the linear map $v_d : \mathcal{K} \to \mathbb{C}$ that the value at $\mathcal{K}_{d+1}$ is zero in the link subspace sequence $\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \cdots \supset \mathcal{K}_d \supset \mathcal{K}_{d+1} \supset \cdots$.

Moreover, we give a general definition of finite type invariants of some 3-manifolds and we define new finite type invariants of 3-manifolds of double branched covering of a link (not finite type invariants of integral homology 3-sphere by T. Ohtsuki [Oh1]). We define new finite type invariants as quasi finite type invariants by the fact that shadow cocycle invariants equal a scalar multiple of Dijkgraaf-Witten invariants [HN] and by using mapping class groups. In particular, it's new that by using mapping class groups we show that the whole finite type invariant pace of 3-manifold of each degree is a finite dimension. We give the proof outline of finite type invariants of 3-manifolds.

The subspace sequence of 3-manifold $\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_d \supset \mathcal{M}_{d+1} \supset \cdots$ is naturally defined by the nature of the quasi finite type invariants. By considering similarly to the quasi finite type invariants, we get the linear map $f_d : \mathcal{M} \to \mathbb{C}$ that the value at $\mathcal{M}_{d+1}$ becomes zero. That the whole finite type invariant space of 3-manifolds of each degree has a finite dimension is proved by the fact that the mapping class group split by the Heegaard genus of the 3-manifolds can be shown by a finite number of products of Lickrish generators [L1, L2].

In the case where X is a dihedral quandle R_p, ϕ is a Mochizuki 3-cocycle θ_p.

4.1 Quasi finite type invariants

We know that finite type invariants have the following conditions.

1. We have a descending filtration of $\mathcal{K}$ as $\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots \supset \mathcal{K}_d \supset \mathcal{K}_{d+1} \supset \cdots$.
2. There exists finite dimension vector space $\mathcal{A}^{(d)}$ over $\mathbb{C}$ such that we have surjective map $\mathcal{A}^{(d)} \twoheadrightarrow \mathcal{K}_d/\mathcal{K}_{d+1}$ for any d.

We weaken the condition of these properties and define quasi finite type invariants as follows.

Definition 4.2. Let A be an abelian group. We give a module space sequence spanned by a certain class of link $\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \cdots \supset \mathcal{K}_d \supset \mathcal{K}_{d+1} \supset \cdots$ ($\mathcal{K}_d$ is not an empty set for all d). We call a homomorphic $v_d : \mathcal{K} \to A$ a *quasi finite type invariant* of *degree* d as follows.

1. v_d are invariants of $\mathcal{K}$ for any d.
2. $v_d|_{\mathcal{K}_{d+1}} = 0$.
3. There exists module of finitely generated spaces $\mathcal{A}^{(d)}$ such that we have surjective maps $\mathcal{A}^{(d)} \twoheadrightarrow \mathcal{K}_d/\mathcal{K}_{d+1}$.

Example 4.3. When, for integers l and n, the 2-bridge link of type $S(l, n)$ is given, we fix $n \in \mathbb{Z}$. Let an odd integer $s < l$ be an integer such that $sn \equiv 1 \pmod{l}$. According to [I, Theorem 1.1.], the quandle shadow cocycle invariant of $S(l,n)$ is:

$$\Phi_{\theta_p}(S(l,n)) = \begin{cases} p^2 \sum_{i=0}^{p-1} t^{\frac{-ls}{p}i^2} \in \mathbb{Z}[t]/(t^p-1), & (\textit{if } l \textit{ is divisible by } p), \\ p^2, & (\textit{otherwise}). \end{cases} \tag{8}$$

Remark 4.4 ([S])**.** For any s, $S(l,n)$ and $S(l,s)$ are links of the same type by ignoring the orientation.

We consider $\Phi_{\theta_p}(S(l,n))/p^2 \in \mathbb{Z}[t]/(t^{p-1}+\cdots+t+1)$. Hence we can put

$$\frac{\Phi_{\theta_p}(S(l,n))}{p^2} = a_{p,0} + a_{p,1}(t-1) + a_{p,2}(t-1)^2 + \cdots + a_{p,p-2}(t-1)^{p-2} \tag{9}$$

for some integers $a_{p,n}$'s. We note that the expansion (9) is not unique. If $a_{p,n} \in \mathbb{Z}/p\mathbb{Z}$ is set to $\pmod p$, it will be uniquely determined by $\Phi_{\theta_p}(S(l,n))/p^2$ [Oh2, Lemma 9.7.].

Let $\mathcal{K}_d$ be vector space over $\mathbb{C}$ freely spanned by the following 2-bridge link sets,

$$\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_1 := \langle S(l,n) \rangle_{\mathbb{C}},$$

$$\mathcal{K}_d := \langle S(l,n) \,|\, p \nmid l \text{ for all (odd) prime numbers } p \text{ such that } 3 \leq p \leq p_{d-2} \rangle_{\mathbb{C}},$$

where p_d is the d-th odd prime number.

We consider map, $v_d : \mathcal{K} \to \mathbb{C}$ which is defined as follows: Let $S(l,n)$ be the 2-bridge link type (l,n). We define $v_d(S(l,n))$ $(2 \leq d)$, as the product of $a_{p_{d-1},\frac{p_{d-1}-1}{2}}$ modulo p_{d-1} (we put $0 \leq a_{p_{d-1},\frac{p_{d-1}-1}{2}} < p_{d-1}$) of (9) and p_{d-1}^2, moreover the mapping $v_0 \equiv v_1 \equiv 0$ is obvious in the definition.

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{v_d} & \mathbb{C} \\ \cup & & \cup \\ S(l,n) & \longmapsto & a_{p_{d-1},\frac{p_{d-1}-1}{2}} \end{array}$$

Theorem 4.5. *v_d is a quasi finite type invariant of degree d.*

Proof. For any $S(l,n) \in \mathcal{K}_{d+1}$, l is not divisible by p_{d-1}, since the value of $\Phi_{\theta_{p_{d-1}}}(S(l,n))$ is trivial by (8). Therefor $a_{p_{d-1},\frac{p_{d-1}-1}{2}} = 0$. Hence $v_d|_{\mathcal{K}_{d+1}} = 0$.

$\mathcal{K}_d/\mathcal{K}_{d+1} = \langle S(p_{d-1},n) \rangle_{\mathbb{C}}$. Recall we fix $n \in \mathbb{Z}$, therefore $\langle S(p_{d-1},n) \rangle_{\mathbb{C}}$ is one dimensional vector space over $\mathbb{C}$. $\square$

Example 4.6. We fix $n = 2$ and $l = 15$.

$$v_2(S(15,2)) = 2,$$

because $a_{3,1} = \frac{15 \cdot 8}{3} \cdot 2 \equiv 2 \pmod 3$.

$$v_3(S(15,2)) = 2,$$

because $a_{5,2} = \frac{15^2 \cdot 8^2}{5^2} \cdot 2 \equiv 2 \pmod 5$.

Example 4.7. For an integer n and an odd number l, we consider a (l, n)-torus link $T(l, n)$. we fix $n \in \mathbb{Z}$. According to [AS, Theorem 6.3.], n is even, and then the quandle shadow cocycle invariant of $T(l, n)$ is:

$$\Phi_{\theta_p}(T(l,n)) = \begin{cases} p\sum_{i=0}^{p-1} t^{\frac{-ln}{2p}i^2} \in \mathbb{Z}[t]/(t^p-1), & (\textit{if } l \textit{ divisible by } p), \\ p, & (otherwise). \end{cases} \tag{10}$$

We consider $\Phi_{\theta_p}(T(l,n))/p \in \mathbb{Z}[t]/(t^{p-1}+\cdots+t+1)$. Hence we can put

$$\frac{\Phi_{\theta_p}(T(l,n))}{p} = a_{p,0} + a_{p,1}(t-1) + a_{p,2}(t-1)^2 + \cdots + a_{p,p-2}(t-1)^{p-2} \tag{11}$$

for some integers $a_{p,n}$'s. We note that the expansion (11) is not unique. If $a_{p,n}$ are set to (mod p), it will be uniquely determined by $\Phi_{\theta_p}(T(l,n))/p$ [Oh2, Lemma 9.7.].

Let $\mathcal{K}_d$ be vector space over $\mathbb{C}$ freely spanned by the following (l, n)-torus link $T(l, n)$ sets,

$$\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_1 := \langle T(l,n)\rangle_{\mathbb{C}},$$

$$\mathcal{K}_d := \langle T(l,n) \,|\, p \nmid l \text{ for all (odd) prime numbers } p \text{ such that } 3 \le p \le p_{d-2}\rangle_{\mathbb{C}},$$

where p_d is the d-th odd prime number.

We consider a map, $v_d : \mathcal{K} \to \mathbb{C}$ which is defined as follows. Let $T(l, n)$ be a torus link type (l, n). We define $v_d(T(l, n))$ $(2 \le d)$, as the product of $a_{p_{d-1},\frac{p_{d-1}-1}{2}}$ modulo p_{d-1} (we put $0 \le a_{p_{d-1},\frac{p_{d-1}-1}{2}} < p_{d-1}$) of (11), moreover the mapping $v_0 \equiv v_1 \equiv 0$ is obvious in the definition.

Theorem 4.8. *v_d is a quasi finite type invariant of degree d.*

Proof. For any $T(l, n) \in \mathcal{K}_{d+1}$, l is not divisible by p_{d-1}, since the value of $\Phi_{\theta_{p_{d-1}}}(S(l, n))$ is trivial by (10). Therefor $a_{p_{d-1},\frac{p_{d-1}-1}{2}} = 0$. Hence $v_d|_{\mathcal{K}_{d+1}} = 0$.

$\mathcal{K}_d/\mathcal{K}_{d+1} = \langle T(p_{d-1}, n)\rangle_{\mathbb{C}}$. Recall we fix $n \in \mathbb{Z}$, therefore $\langle T(p_{d-1}, n)\rangle_{\mathbb{C}}$ is one dimensional vector space over $\mathbb{C}$. □

4.2 Finite type invariants of some 3-manifolds of double branched covering of links.

Definition 4.9. We give any 3-manifolds meeting a certain condition vector space over $\mathbb{C}$ sequence $\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_d \supset \mathcal{M}_{d+1} \supset \cdots$ ($\mathcal{M}_d$ is not an empty set for all d). We define formula $\mathfrak{B}(d)$ as a condition of set $\mathcal{M}_d$. For any 3-manifold $M \in \mathcal{M}$, we define linear map $f_d : \mathcal{M} \to \mathbb{C}$ of finite invariants of degree d as follow.

1. f_d are invariants of $\mathcal{M}$ for any d.
2. $f_d|_{\mathcal{M}_{d+1}} = 0$.

3. For each d, the whole finite type invariant space of degree d has finite dimension.

Example 4.10 ([Oh1])**.** T. Otsuki defined finite type invariants where a component of the link which did Dehn surgery for integral homology 3-spheres became a degree.

The sequences of the subspaces $\{\mathcal{M}_d\}$ have a descending filtration of the connected oriented compact 3-manifold $M \in \mathcal{M}$ such as

$$\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots .$$

As for the finite type invariants of degree d, we ignore the difference that the set $\mathcal{M}_{d+1}$ contains (the difference of $\mathcal{M} - \mathcal{M}' \in \mathcal{M}_{d+1}$), and distinguish the connected oriented compact 3-manifolds.

Definition 4.11. Let M be an oriented compact 3-manifold, G be a finite group and A be an abelian group.

$$DW_{\theta_\phi}(M) = \frac{1}{|G|} \sum_{f \in \mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G)} \langle f^*(\theta_\phi), [M] \rangle \in \mathbb{Z}[A],$$

$$DW^{\mathbb{Z}}_{\theta_\phi}(M) = \sum_{f \in \mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G)} \langle f^*(\theta_\phi), [M] \rangle \in \mathbb{Z}[A],$$

where $\theta_\phi \in H^3_{gr}(G; A)$ and $[M] \in H_3(M; A)$ is the fundamental class of M.

Theorem 4.12 ([HN])**.** *Let $m = p$ be an odd prime of a dihedral quandle R_m, and M_L be the double branched covering of a link $L \subset S^3$. For any quandle 3-cocycle $\phi \in H^3_Q(R_p; \mathbb{Z}/p\mathbb{Z})$, there exists a group 3-cocycle θ_ϕ, then the shadow cocycle invariant $\Phi_\phi(L)$ equals a scalar multiple of the Dijkgraaf-Witten invariant $DW^{\mathbb{Z}}_{\theta_\phi}(M_L)$.*

Remark 4.13. We recall that Mochizuki [Mo1, Mo2] calculated the third quandle cohomology $H^3_Q(R_p; \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ and a presentation of the generator θ_p, called Mochizuki 3-cocycle, is obtained. The cocycle θ_p leads to an invariant $\Phi_{\theta_p}(L) \in \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ of a link L. Therefore, we should consider only $\Phi_{\theta_p}(L)$.

We give the following theorem from Theorem 4.12, Remark 4.13 and [H, Theorem 3.2.].

Theorem 4.14 ([H])**.** *With a notation R_p, $L \subset S^3$ and M_L in Theorem 4.12, let $G = \mathbb{Z}/p\mathbb{Z}$ of an odd prime order p be an Abelian group,*

$$\Phi_{\theta_p}(L) = p^2 \cdot DW_{\theta_\phi}(M_L).$$

Theorem 4.15 ([L1, L2])**.** *A mapping class group $\mathfrak{M}_g$ consists of the twist about the set of $(3g-1)$ simple closed curves of l_i, m_j, n_k $(1 \leq i, j \leq g,\ 1 \leq k \leq g-1)$ shown in the following figure 14. This generators are Lickorish generators.*

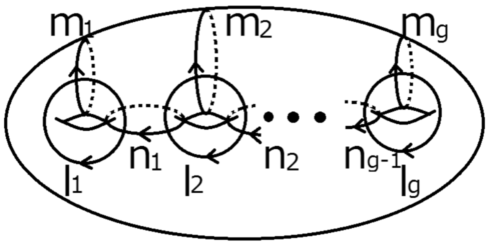


Figure 14: Lickorish generators

Example 4.16 ([DW, MOO, W2])**.** For integers l and n, a 2-bridge link type (l, n) is given. When we fix an odd integer $s < l$ such that $sn \equiv 1 \pmod{l}$, the 2-fold branched covering of S^3 branched along the link is lens space $L(l, s)$. Based on [I, Theorem 1.1.], in case l is divisible by p, the following equations hold:

$$p^2 \cdot DW_{\theta_\phi}(L(l,s)) = \Phi_{\theta_p}(S(l,n)) = p^2 \sum_{i=0}^{p-1} t^{\frac{-ls}{p} i^2} \in \mathbb{Z}[t]/(t^p - 1),$$

if not, the Dijkgraaf-Witten invariant is trivial.

Remark 4.17 ([Br])**.** For any s, $L(l, n)$ and $L(l, s)$ are homeomorphic by preserving the orientation.

We consider $DW_{\theta_\phi}(L(l,s)) = \sum_{i=0}^{p-1} t^{\frac{-ls}{p} i^2} \in \mathbb{Z}[t]/(t^{p-1} + \cdots + t + 1)$. Hence we can put

$$DW_{\theta_\phi}(L(l,s)) = a_{p,0} + a_{p,1}(t-1) + a_{p,2}(t-1)^2 + \cdots + a_{p,p-2}(t-1)^{p-2} \quad (12)$$

for some integers $a_{p,n}$'s. Although the expansion (12) is not unique, $a_{p,n}$, which are set to $\pmod{p}$, will be uniquely fixed by $DW_{\theta_\phi}(L(l,s))$ [Oh2, Lemma 9.7.].

Let $\mathcal{M}_d$ be vector space over $\mathbb{C}$ freely spanned by the following sets of lens space. where, H_1 is a solid torus and $\tau_i \in \mathfrak{M}_1$ indicates Lickorish generators of a mapping class group on the handlebody surface ∂H_1.

$$\begin{gathered} \mathcal{M} = \mathcal{M}_0 = \mathcal{M}_1 = \langle L(l,s) \rangle_{\mathbb{C}}, \\ \mathcal{M}_2 := \langle L(l,s) \,|\, \text{There exists } p \text{ such that } p|l, \text{ where } p \text{ is a prime}, \\ L(l,s) = H_1 \cup_\varphi H_1,\ \varphi = \underbrace{\tau_1 \circ \tau_2 \circ \cdots \circ \tau_d \circ \cdots}_{3 \text{ or more}} \rangle_{\mathbb{C}}, \\ \mathcal{M}_d := \langle L(l,s) \,|\, \text{There exists } p \neq 3, 5, \cdots, p_{d-2} \text{ such that } p|l, \\ L(l,s) = H_1 \cup_\varphi H_1,\ \varphi = \underbrace{\tau_1 \circ \tau_2 \circ \cdots \circ \tau_d \circ \cdots}_{d+1 \text{ or more}} \rangle_{\mathbb{C}}, \end{gathered}$$

where p_d is the d-th odd prime number.

Lemma 4.18. $\mathcal{M}_d \neq \emptyset$.

Proof. (i)Case of $d = 2$, It is the clear that $L(3,2) \in \mathcal{M}_2$.
(ii)Case of $d = k$, we suppose $L(p_{k-1}, 2) \in \mathcal{M}_k$. $p_k > p_{k-1} + 1 \geq k + 2$ by $p_{k-1} \geq k + 1$. Therefore $L(p_k, 2) \in \mathcal{M}_{k+1}$. □

We consider a map, $f_d : \mathcal{M} \to \mathbb{C}$ which is defined as follows. We define $f_d(L(l,s))$ $(2 \leq d)$, as the value of $a_{p_{d-1}, \frac{p_{d-1}-1}{2}}$ modulo p_{d-1} (we put $0 \leq a_{p_{d-1}, \frac{p_{d-1}-1}{2}} < p_{d-1}$) of (12), moreover the mapping $f_0 \equiv f_1 \equiv 0$ is obvious in the definition.

Theorem 4.19. *f_d is a finite type invariant of $L(l,s)$.*

Proof. For any $L(l,s) \in \mathcal{M}_{d+1}$, l is not divisible by p_{d-1}, since the value of $DW^{\mathbb{Z}}_{\theta_\phi}(L(l,s))$ is trivial. Therefor $a_{p_{d-1}, \frac{p_{d-1}-1}{2}} = 0$. Hence $f_d|_{\mathcal{M}_{d+1}} = 0$.

We can define the following map $h : \langle \tau_1, \tau_2 \rangle^{(\leq d)}_{\mathbb{C}} \to \mathcal{M}/\mathcal{M}_{d+1}$,

$$h(\sum \sigma_i \varphi_i) = [\sum \sigma_i H_1 \cup_{\varphi_i} H_1] \ (\sigma \in \mathbb{C}),$$

where $\langle \tau_1, \tau_2 \rangle^{(\leq d)}_{\mathbb{C}}$ is space over $\mathbb{C}$ freely spanned by the product of less than $d+1$ of $\tau_1, \tau_2 \in \mathfrak{M}_1$. Map $h : \langle \tau_1, \tau_2 \rangle^{(\leq d)}_{\mathbb{C}} \to \mathcal{M}/\mathcal{M}_{d+1}$ is surjective, and $\langle \tau_1, \tau_2 \rangle^{(\leq d)}_{\mathbb{C}}$ has a finite dimension. Hence $\mathcal{M}/\mathcal{M}_{d+1}$ are finite dimensional vector space. "The whole finite type invariant space of degree d" has finite dimensional space because this space is the dual vector space of $\mathcal{M}/\mathcal{M}_{d+1}$. □

Example 4.20. For an integer n and an odd number l, a (l,n)-torus link $T(l,n)$ is given. It is clear that the branched covering branched over the link is a Brieskorn manifold $M(2,l,n)$ [R, §10.E. Exercise 6]. Based on [AS, Theorem 6.3.], in case l is divisible by p, the following equations hold:

$$p^2 \cdot DW_{\theta_\phi}(M(2,l,n)) = \Phi_{\theta_p}(T(l,n)) = p \sum_{i=0}^{p-1} t^{\frac{-ls}{2p} i^2} \in \mathbb{Z}[t]/(t^p - 1),$$

if not, the Dijkgraaf-Witten invariant is trivial.

We consider $p \cdot DW_{\theta_\phi}(M(2,l,n)) = \sum_{i=0}^{p-1} t^{\frac{-ln}{2p} i^2} \in \mathbb{Z}[t]/(t^{p-1} + \cdots + t + 1)$. Hence we can put

$$p \cdot DW_{\theta_\phi}(M(2,l,n)) = a_{p,0} + a_{p,1}(t-1) + a_{p,2}(t-1)^2 + \cdots + a_{p,p-2}(t-1)^{p-2} \quad (13)$$

for some integers $a_{p,n}$'s. Although the expansion (13) is not unique, $a_{p,n}$, which is set to (mod p), will be uniquely fixed by $p \cdot DW_{\theta_\phi}(M(2,l,n))$ [Oh2, Lemma 9.7.].

Remark 4.21. For a fundamental group to change into a Brieskorn manifold by the value of l, n, and a Heegaard splitting is not unique.

If $n = 3$, then a Heegarrd splitting of $M(2,l,3)$ of genus 2 [IK]. Let $\mathcal{M}_d$ be vector space over $\mathbb{C}$ freely spanned by the following sets of Brieskorn manifolds $M(2,l,3)$. Here, H_2 is a handlebody of genus 2 and $\tau_i \in \mathfrak{M}_2$ indicates Lickorish generators of a mapping class group on the handlebody surface ∂H_2.

$$\begin{gathered}
\mathcal{M} = \mathcal{M}_0 = \mathcal{M}_1 = \langle M(2,l,3)\rangle_{\mathbb{C}}, \\
\mathcal{M}_2 := \langle M(2,l,3)\,|\,\text{There exists } p \text{ such that } p|l, \text{ where } p \text{ is a prime}, \\
M(2,l,3) = H_2 \cup_\varphi H_2,\ \varphi = \underbrace{\tau_1 \circ \tau_2 \circ \cdots \circ \tau_d \circ \cdots}_{3 \text{ or more}}\rangle_{\mathbb{C}}, \\
\mathcal{M}_d := \langle M(2,l,3)\,|\,\text{There exists } p \neq 3, 5, \cdots, p_{d-2} \text{ such that } p|l, \\
L(l,s) = H_2 \cup_\varphi H_2,\ \varphi = \underbrace{\tau_1 \circ \tau_2 \circ \cdots \circ \tau_d \circ \cdots}_{d+1 \text{ or more}}\rangle_{\mathbb{C}},
\end{gathered}$$

where p_d is the d-th odd prime number.

We consider a map, $f_d : \mathcal{M} \to \mathbb{C}$ which is defined as follows. We define $f_d(M(2,l,3))$ $(2 \leq d)$, as the value of $a_{p_{d-1}, \frac{p_{d-1}-1}{2}}$ modulo p_{d-1} (we put $0 \leq a_{p_{d-1}, \frac{p_{d-1}-1}{2}} < p_{d-1}$) of (13) by p_{d-1}, moreover the mapping $f_0 \equiv f_1 \equiv 0$ is obvious in the definition.

Theorem 4.22. *If $\mathcal{M}_d \neq \emptyset$ for any d, then f_d is a finite type invariant of $M(2,l,3)$.*

Proof. For any $M(2,l,3) \in \mathcal{M}_{d+1}$, l is not divisible by p_{d-1}, since the value of $DW_{\theta_\phi}(M(2,l,3))$ is trivial. Therefor $a_{p_{d-1}, \frac{p_{d-1}-1}{2}} = 0$. Hence $f_d|_{\mathcal{M}_{d+1}} = 0$.

We can define the following map $h : \langle \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\rangle_{\mathbb{C}}^{(\leq d)} \to \mathcal{M}/\mathcal{M}_{d+1}$,

$$h(\sum \sigma_i \varphi_i) = [\sum \sigma_i H_2 \cup_{\varphi_i} H_2]\ (\sigma \in \mathbb{C}),$$

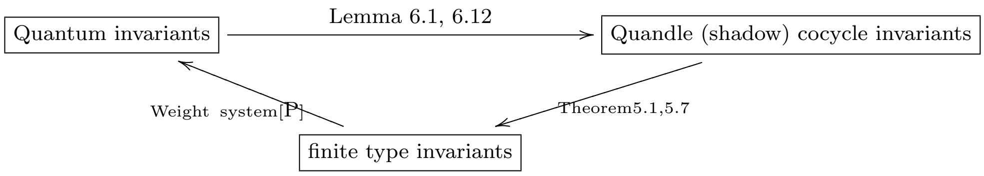
where $\langle \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\rangle_{\mathbb{C}}^{(\leq d)}$ is space over $\mathbb{C}$ freely spanned by the product of less than $d+1$ of $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \in \mathfrak{M}_1$ and the product φ, there exists a Brieskorn manifold $M(2,l,3)$ such that a Heggarrd splitting $H_2 \cup_\varphi H_2$. Map $h : \langle \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\rangle_{\mathbb{C}}^{(\leq d)} \to \mathcal{M}/\mathcal{M}_{d+1}$ is surjective, and $\langle \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\rangle_{\mathbb{C}}^{(\leq d)}$ has a finite dimension. Hence $\mathcal{M}/\mathcal{M}_{d+1}$ is finite dimensional vector space. "The whole finite type invariant space of degree d" has finite dimensional space because this space is the dual vector space of $\mathcal{M}/\mathcal{M}_{d+1}$. $\square$

5 Relations between quandle shadow cocycle invariants and finite type invariants

Quantum invariants are defined via R-matrices, one of which, in general, yields one invariant. These invariants are here defined alternatively via vector space "V" over $\mathbb{C}$ representing a Lie algebra "$\mathfrak{g}$". For each representation "V'" of "$\mathfrak{g}$", a quantum invariant group $U_q(\mathfrak{g})$ is given by transforming "$\mathfrak{g}$" and a representation of $U_q(\mathfrak{g})$ is given by transforming the representation "V'", from which an R-matrix is derived. A knot invariant obtained by using this R-matrix is called a quantum $(\mathfrak{g}, V')$ invariant. By using this construction, many polynomial invariants of knots are classified via quantum invariants.

While the concept of quantum invariants helps generally understand many polynomial invariants, we realize again that there exist a great number of polynomial invariants, in fact, almost as many as Lie algebras and their representations. Since there are too many to study each, we have decided to adopt the concept of 'finite type' invariants which allows us to examine quantum invariants by classifying them according to shared characters.

In this section, as shown in the figure below, we find a relation between the 2 types of invariants described above, which were thought to have no relation to each other at all. In particular, we find that, as regards trivial quandles, finite type invariants can be obtained by using shadow cocycle invariants, and thus this is expected to have applications to surface links and low-dimensional manifolds in the future.



5.1 Main theorems and Proofs of quandle 2-cocycle invariants version

We give the results of Theorems of a quandle 2-cocycle version, and we find that, as regards trivial quandles, finite type invariants can be obtained by using quandle 2-cocycle invariants.

Let $\mathbb{C}\langle X\rangle$ be vector space over $\mathbb{C}$ freely spanned by a finite quandle X and furthermore let n be $|X|$ and $f \in H^2(X;\mathbb{Z})$ be a quandle 2-cocycle and let L be an oriented link and $b \in B_m$ be a braid whose closure $\hat{b}$ is isotopic to L and let D be a diagram of L. We fix $t \in \mathbb{C} \setminus \{0\}$. We assume that $b = \sigma_{s_1}^{(1)} \cdots \sigma_{s_l}^{(l)}$, where the $\sigma_{s_1}^{(1)}, \cdots, \sigma_{s_l}^{(l)}$ are generators of a braid group B_m.

Theorem 5.1. *Let X be a trivial quandle. Then a coefficient of $\hbar^d$ in $\Phi_f(L) \in \mathbb{Z}[t^{\pm}]|_{t=e^h}$ is a finite type invariant of degree d.*

Remark 5.2. For connected and finite quandles, (shadow) cocycle invariants of an integer coefficient $\mathbb{Z}$ are trivial. An Alexander quandle is not a connected quandle.

Theorem 5.3. *The value of a coefficient of degree* 0 *is the coloring number of trivial quandles.*

Theorem 5.4. *The value of a coefficient of degree* 1 *is the cocycle invariant of trivial quandles.*

The following Lemma is a well-known fact, [cf.Remark 2 [E1]] and [cf.Remark 2.2. [E2]].

Lemma 5.5. *Given $R : a_i \otimes a_j \to t^{f(a_i,a_j)} a_j \otimes (a_i * a_j)$ for any $a_i, a_j \in \mathbb{C}\langle X\rangle$, we obtain a representation $\psi_m : B_m \to End(\mathbb{C}\langle X\rangle^{\otimes m})$ defined by map (1). Then,*

$$\Phi_f(L) = \mathrm{trace}\big(\psi_m(b)\big).$$

Remark 5.6. A matrix $\mathcal{R}$ and its inverse $\mathcal{R}^{-1}$ are associated with positive and negative crossings of D respectively. However, we substitute $e^{\hbar}$ for the parameter of $\mathcal{R} - \mathcal{R}^{-1}$. $\mathcal{R} - \mathcal{R}^{-1}$ is not always divisible by $\hbar$.

Proof of Theorem 5.1. (We show the following proof of [BL] and [Corollary 7.5 [Oh2]].) Let a diagram of L be D. In the constitution of quantum invariants, a matrix $\mathcal{R}$ and its inverse $\mathcal{R}^{-1}$ are associated with positive and negative crossings of D respectively. When $\hbar = 0$, these matrices coincide, and therefore, $\mathcal{R} - \mathcal{R}^{-1}$ is a matrix entry which is divisible by $\hbar$ in $\mathbb{C}[[\hbar]]$. This difference is related to double points of a singular link which occur with the definition of finite type invariants. Recall that we calculate a quantum invariant of singular knots by regarding a matrix $\mathcal{R}$ at a singularity as $\mathcal{R} - \mathcal{R}^{-1}$. Lemma 6.1 shows that a $\mathrm{trace}(\psi_m(b)) = \Phi_f(L)$ and a $\mathrm{trace}(\psi_m(b))$ are composed by the product of the matrix $\mathcal{R}$. Therefore, in case L is a singular link with exactly $d+1$ singular points, $\mathrm{trace}(\psi_m(b)) = \Phi_f(L) \in \mathbb{Z}[t^{\pm}]|_{t=e^{\hbar}}$ is divisible by $\hbar^{d+1}$. Hence, for such singular links, the coefficient of $\hbar^d$ is equal to 0. Here, "be divisible" means "can be treated as power series of $\hbar$ and $t = e^{\hbar} = 1 + \hbar + \frac{1}{2}\hbar^2 + \cdots$." In addition, the singular point of $\mathrm{trace}(\psi_m(b))$ for the singular link L corresponds to $\mathcal{R} - \mathcal{R}^{-1}$. $\square$

Proof of Theorem 5.3. The coefficient of degree 0 of finite invariants is a value obtained when $\hbar = 0$ is substituted. When $\hbar = 0$, linear map R is as follows;

$$R\Big(\sum_{j,k=1}^{n} c_{jk} a_j \otimes a_k\Big) = \sum_{j,k=1}^{n} c_{jk} a_k \otimes a_j$$

A matrix R is a matrix of a value 0 or 1, and through this set-theoretic Yang-Baxter equation, we obtain a trivial quandle-coloring number. $\square$

Proof of Theorem 5.4. The coefficient of degree 1 of finite invariants is calculated by a value obtained when linear map R is once differentiated from $\hbar$ and 0 is substituted for $\hbar$. When $\hbar = 0$ after linear map R is once differentiated by $\hbar$, linear map $R'|_{\hbar=0}$ is as follows;

$$R\Big(\sum_{j,k=1}^{n} c_{jk} a_j \otimes a_k\Big) = \sum_{j,k=1}^{n} c_{jk} t^{f(a_j,a_k)} a_k \otimes (a_j * a_k)$$

$$R'\Big(\sum_{j,k=1}^{n} c_{jk} a_j \otimes a_k\Big) = \sum_{j,k=1}^{n} c_{jk} e^{\hbar f(a_j,a_k)} f(a_j,a_k) a_k \otimes a_j$$

$$R'\Big(\sum_{j,k=1}^{n} c_{jk} a_j \otimes a_k\Big)\Big|_{\hbar=0} = \sum_{j,k=1}^{n} c_{jk} f(a_j,a_k) a_k \otimes a_j$$

In case of trivial quandles, the cocycle conditions are naturally met. $\square$

5.2 Main theorem and Proof of quandle shadow 3-cocycle invariants version

Let $\mathbb{C}\langle X\rangle$ be vector space over $\mathbb{C}$ freely spanned by a finite quandle X and furthermore n be $|X|$ and $\phi \in H^3(X;\mathbb{Z})$ be a quandle 3-cocycle and let L be an oriented link and $b \in B_m$ be a braid whose closure $\hat{b}$ is isotopic to L and let D be an oriented link diagram of L. We fix $t \in \mathbb{C} \setminus \{0\}$.

Theorem 5.7. *Let X be a trivial quandle. Then the coefficient of $\hbar^d$ in $\Phi_\phi(L) \in \mathbb{Z}[t^{\pm}]|_{t=e^\hbar}$ is a finite type invariant of degree d.*

Proof of Theorem 5.7. By Theorem 5.1, we know that the coefficient of $\hbar^d$ in a $\mathrm{trace}(\psi_m(b))|_{t=e^\hbar}$ is a finite type invariant of degree d for any X-coloring C_2 of the set of regions separated by the underlying immersed curve of D. Hence, the coefficient of $\hbar^d$ in $\sum_{C_2} \mathrm{trace}(\psi_m(b))|_{t=e^\hbar}$ is a finite type invariant of degree d. Proposition 6.11 and Lemma 6.20 tell us that the coefficient of $\hbar^d$ in $\Phi_\phi(L) \in \mathbb{Z}[t^{\pm}]|_{t=e^\hbar}$ is a finite type invariant of degree d. $\square$

6 Appendix

Let $\mathbb{C}\langle X\rangle$ be vector space over $\mathbb{C}$ freely spanned by the finite quandle X and furthermore let n be $|X|$ and $f \in H^2(X;\mathbb{Z})$ be a quandle 2-cocycle and let L be an oriented link and $b \in B_m$ be a braid whose closure $\hat{b}$ is isotopic to L and let D be a diagram of L. We fix $t \in \mathbb{C} \setminus \{0\}$. We assume that $b = \sigma_{s_1}^{(1)} \cdots \sigma_{s_l}^{(l)}$, where the $\sigma_{s_1}^{(1)}, \cdots, \sigma_{s_l}^{(l)}$ are generators of a braid group B_m.

Lemma 6.1. *Given $R : a_i \otimes a_j \to t^{f(a_i,a_j)} a_j \otimes (a_i * a_j)$ for any $a_i, a_j \in \mathbb{C}\langle X\rangle$, we obtain a representation $\psi_m : B_m \to End(\mathbb{C}\langle X\rangle^{\otimes m})$ defined by map (1). Then,*

$$\Phi_f(L) = \mathrm{trace}\big(\psi_m(b)\big).$$

Lemma 6.2 (cf.Proposition 2.3.[E2])**.** *We define $R : \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ as $a_j \otimes a_k \to t^{f(a_j,a_k)} a_k \otimes (a_j * a_k)$ for any $a_j, a_k \in X$. Then, R is a Yang-Baxter equation.*

Proof. We obtain the relation of quandle 2-cocycle conditions of $f(a_i, a_j) + f(a_i * a_j, a_k) + f(a_j, a_k) = f(a_j, a_k) + f(a_i, a_k) + f(a_i * a_k, a_j * a_k)$. Hence, R satisfies the solution of the Yang-Baxter equation (Definition 2.5) by the following formulas.

$$(R \otimes id_{\mathbb{C}\langle X\rangle})(id_{\mathbb{C}\langle X\rangle} \otimes R)(R \otimes id_{\mathbb{C}\langle X\rangle})(\sum_{i,j,k=1}^{n} c_{i,j,k} a_i \otimes a_j \otimes a_k)$$

$$= \sum_{i,j,k=1}^{n} c_{i,j,k} t^{f(a_i,a_j)+f(a_i*a_j,a_k)+f(a_j,a_k)} a_k \otimes (a_j * a_k) \otimes ((a_i * a_j) * a_k).$$

$$(id_{\mathbb{C}\langle X\rangle} \otimes R)(R \otimes id_{\mathbb{C}\langle X\rangle})(id_{\mathbb{C}\langle X\rangle} \otimes R)(\sum_{i,j,k=1}^{n} c_{i,j,k} a_i \otimes a_j \otimes a_k)$$

$$= \sum_{i,j,k=1}^{n} c_{i,j,k} t^{f(a_j,a_k)+f(a_i,a_k)+f(a_i*a_k,a_j*a_k)} a_k \otimes (a_j * a_k) \otimes ((a_i * a_k) * (a_j * a_k)).$$

We know that map R is the solution of the Yang-Baxter equation as shown by the following figure 15. $\square$

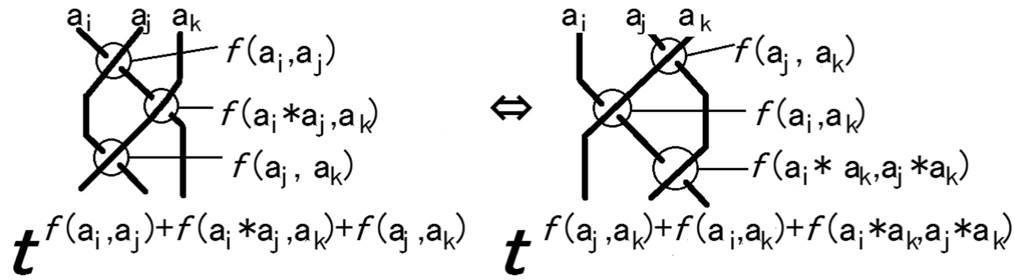


Figure 15: Yang-Baxter equation

Lemma 6.3. *Map R is linear map.*

Proof. For any $r \in \mathbb{C}$, we obtain an equation $rR(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k) = \sum_{j,k=1}^{n} r c_{j,k} t^{f(a_j,a_k)} a_k \otimes (a_j * a_k)$. For any $\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k$, $\sum_{j,k=1}^{n} c'_{j,k} a_j \otimes a_k$, we obtain equations $R(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k + \sum_{j,k=1}^{n} c'_{j,k} a_j \otimes a_k) = \sum_{j,k=1}^{n} (c_{j,k} + c'_{j,k}) t^{f(a_j,a_k)} a_k \otimes (a_j * a_k) = R(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k) + R(\sum_{j,k=1}^{n} c'_{j,k} a_j \otimes a_k)$. $\square$

Lemma 6.4. $R(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle) = \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$.

Proof. $R(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k) = 0$. We know that $a_j \otimes a_k$ is linearly independent for any j, k and $t \neq 0$ therefore $c_{j,k} = 0$ for any j, k. Hence the linear map R is injective. $\mathrm{Ker} R = \{0\}$ if and only if $\dim \mathrm{Ker} R = 0$ if and only if $\dim R(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle) = n^2$ if and only if $R(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle) = \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$. $\square$

Definition 6.5. Let $\mathcal{R}$ be a Yang-Baxter regular matrix representation for the basis $a_1 \otimes a_1, a_1 \otimes a_2, \cdots, a_n \otimes a_n$ of the linear map R by Lemma 6.2, 6.3 and 6.4.

Lemma 6.6. *There is a single non-zero entry in each row and in each column of $\mathcal{R}$.*

Proof. If there were two elements or more in a column, this would be inconsistent with the definition of R: $R(a_j \otimes a_k) = \sum_{l,m,n=1}^{n} t^{f(a_j^m, a_k^n)} a_l \otimes a_m = t^{f(a_j,a_k)} a_k \otimes (a_j * a_k)$.

There can not be two elements or more in a row, as this would be inconsistent with "there is a single element in each column" and "R is a one-to-one mapping." Hence, if $R(a_j \otimes a_k) = t^{f(a_j,a_k)} a_k \otimes (a_j * a_k) = t^{f(a_m,a_n)} a_m \otimes (a_m * a_n) = R(a_m \otimes a_n)$, then $a_j \otimes a_k = a_m \otimes a_n$. □

Lemma 6.7. *Let*

$$\mathcal{R} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{pmatrix} \qquad ,$$

where the R_{jk} *are* $n \times n$ *matrices. We obtain a formula* $\mathrm{trace} R_{jk} = \delta_{jk}$.

Proof. The definition of map R shows that $t^{f(a_j,a_k)} a_k \otimes (a_j * a_k)$ becomes one of the elements of a matrix R_{jk}. Therefore the value on the diagonal of an R_{jj} matrix occurs, which is $t^{\pm f(a_j,a_j)} = t^0 = 1$. When $j = k$, $\mathrm{trace} R_{jk} = 1$. When $j \neq k$, $\mathrm{trace} R_{jk} = 0$. □

Lemma 6.8. $\mathrm{trace}_2\big((\mathrm{id}_{\mathbb{C}\langle X\rangle} \otimes E) \cdot \mathcal{R}^{\pm}\big) = \mathrm{id}_{\mathbb{C}\langle X\rangle}$, *where*

$$\mathrm{id}_{\mathbb{C}\langle X\rangle} = \left.\underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}}_{n \text{ columns}}\right\} n \text{ rows}$$

and $E \in \mathrm{End}(\mathbb{C}\langle X\rangle)$ *is a unit matrix.*

Proof. We calculate a diagonal element of R_{jk}. We obtain an equation $\mathrm{trace} R_{jk} = \delta_{jk}$ by Lemma 6.7.

$$\mathrm{trace}_2\big((\mathrm{id}_{\mathbb{C}\langle X\rangle} \otimes E) \cdot \mathcal{R}^{\pm}\big) = \begin{pmatrix} \mathrm{trace} R_{11} & \mathrm{trace} R_{12} & \cdots & \mathrm{trace} R_{1n} \\ \mathrm{trace} R_{21} & \mathrm{trace} R_{22} & \cdots & \mathrm{trace} R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{trace} R_{n1} & \mathrm{trace} R_{n1} & \cdots & \mathrm{trace} R_{nn} \end{pmatrix}^{\pm} = \mathrm{id}_{\mathbb{C}\langle X\rangle}.$$

□

Lemma 6.9 (cf.Remark 2.2.[E2])**.** $\Phi_f(L) = \mathrm{trace}(\psi_m(b))$.

Proof. We fix one coloring for $\hat{b}$ of a closure of b, and apply the element of a coloring in the upper ends and lower ends of b to $a_{p_1}, \cdots, a_{p_m}$. By this, we obtain one value of $a_{p_1} \otimes \cdots \otimes a_{p_m}$. Since we put the following two maps g,g',

$$g : \{X\text{−coloring of } \hat{b}\} \to \{a_{p_1} \otimes a_{p_2} \otimes \cdots \otimes a_{p_m}\}$$
$$g' : \{X\text{−coloring of } \hat{b}\} \to \{a^*_{p_1} \otimes a^*_{p_2} \otimes \cdots \otimes a^*_{p_m}\}$$

Let $a_{p_1}, a_{p_2}, \cdots, a_{p_m}$ be elements of a quandle and we obtain one coloring in the upper ends and the lower ends of $\hat{b}$. And If two colorings $c \in \mathcal{C}$ and $c' \in \mathcal{C}$ are equal, then $a_{p_1} \otimes a_{p_2} \otimes \cdots \otimes a_{p_m} = a'_{p_1} \otimes a'_{p_2} \otimes \cdots \otimes a'_{p_m}$, and thus $a_{p_1} = a'_{p_1}, a_{p_2} = a'_{p_2}, \cdots, a_{p_m} = a'_{p_m}$. Therefore $c = c'$. Hence the maps g, g' are bijective. Therefore, the sum given by fixing the weight at the crossings and by using a coloring is equal to a value given by calculating a trace($\psi_m(b)$).

$$\begin{aligned}
\mathrm{trace}(\psi_m(b)) &= \sum_{p_1,\cdots,p_m=1}^{n} (a^*_{p_1} \otimes \cdots \otimes a^*_{p_m})(\psi_m(\sigma^{(1)}_{s_1}) \cdots \psi_m(\sigma^{(l)}_{s_l}) a_{p_1} \otimes \cdots \otimes a_{p_m}) \\
&= \sum_{p_1,\cdots,p_m=1}^{n} (a^*_{p_1} \otimes \cdots \otimes a^*_{p_m})(\psi_m(\sigma^{(l)}_{s_l}) \cdots \psi_m(\sigma^{(1)}_{s_1}) a_{p_1} \otimes \cdots \otimes a_{p_m}) \\
&= \sum_{C} (g'(X\text{−coloring of } \hat{b}))(\psi_m(\sigma^{(l)}_{s_l}) \cdots \psi_m(\sigma^{(1)}_{s_1}) g(X\text{−coloring of } \hat{b})) \\
&= \sum_{C} \prod_{x=1}^{l} t^{W_\phi(x;C)} = \Phi_f(\hat{b}) = \Phi_f(L)
\end{aligned}$$

□

Proof of Lemma 6.1. We obtain an equation $\mathrm{trace}_2\big((\mathrm{id}_{\mathbb{C}\langle X\rangle} \otimes E) \cdot \mathcal{R}^{\pm}\big) = \mathrm{id}_{\mathbb{C}\langle X\rangle}$ from Lemma 6.8. Recall that a trace($\psi_m(b)$) is an isotopy invariant of L by Theorem 2.6. This invariant is an isotopy invariant of L by using an R-matrix. Let $\mathcal{C}$ be a coloring of L by a fixed finite quandle X. By Lemma 6.9, we have

$$\Phi_f(L) = \mathrm{trace}(\psi_m(b)).$$

Hence, the 2-cocycle invariants are isotopy invariants of L derived from an R-matrix. □

Let $\mathbb{C}\langle X\rangle$ be vector space over $\mathbb{C}$ freely spanned by a finite quandle X and furthermore n be $|X|$ and $\phi \in H^3(X;\mathbb{Z})$ be a quandle 3-cocycle and let L be an oriented link and $b \in B_m$ be a braid whose closure $\hat{b}$ is isotopic to L and let D be an oriented link diagram of L. We fix $t \in \mathbb{C} \setminus \{0\}$. We assume that $b = \sigma^{(1)}_{s_1} \cdots \sigma^{(l)}_{s_l}$, where the $\sigma^{(1)}_{s_1}, \cdots, \sigma^{(l)}_{s_l}$ are generators of a braid group B_m.

Definition 6.10. Let the coloring C_1 be an arc coloring, and every time we fix one arc coloring, the coloring of the set of regions separated by the underlying immersed curve of D becomes one fixed map. Let the coloring C_2 be a coloring of the set of regions, and every time we fix one arc coloring, we get some colorings of the set of regions, which are given by map C_2.

Proposition 6.11.

$$\Phi_\phi(L) = \sum_{C_2}\sum_{C_1}\prod_x t^{W_\phi(x;C_1,C_2)}$$

Proof. An X-shadow coloring C is a combination of an arc coloring C_1 and a coloring C_2 of the set of regions separated by the underlying immersed curve of D. Therefore, we have the following table 16. When we give a coloring to

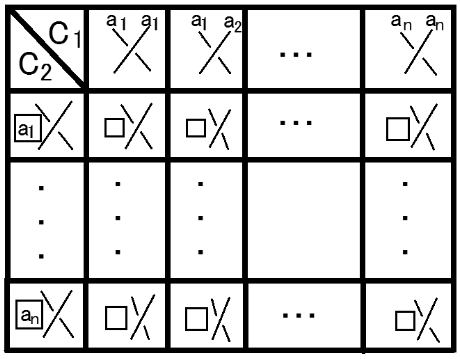


Figure 16: table of X-shadow coloring

the outermost area of a link diagram D, one X-shadow coloring is decided. The coloring C_2 is considered to be the whole coloring (of the outermost area). Hence, $\Phi_\phi(L) = \sum_C \prod_x t^{W_\phi(x;C)} = \sum_{C_2}\sum_{C_1}\prod_x t^{W_\phi(x;C_1,C_2)}$. $\square$

The following Lemma is a well-known fact, but the proof is in a paper for the first time [cf. Remark 2 [E1]] and [cf. Remark 2.2. [E2]].

Lemma 6.12. *Given $R : a_j \otimes a_k \to t^{\phi(a_i,a_j,a_k)} a_k \otimes (a_j * a_k)$ for any $a_j, a_k \in \mathbb{C}\langle X\rangle$, we obtain a representation $\psi_m : B_m \to End(\mathbb{C}\langle X\rangle^{\otimes m})$ defined by map (1). Then,*

$$\Phi_\phi(L) = \sum_{C_2} \mathrm{trace}\big(\psi_m(b)\big).$$

Lemma 6.13 (cf.Proposition 2.3.[E2])**.** *We define $R : \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ as $a_j \otimes a_k \to t^{\phi(a_i,a_j,a_k)} a_k \otimes (a_j * a_k)$ for any $a_j, a_k \in X$, and $a_i \in X$ is fixed when we fix one arc coloring C_1 of D. Then, R satisfies the Yang-Baxter equation.*

Proof. We obtain the relation of quandle 3-cocycle conditions of $\phi(a_i, a_j, a_k) + \phi(a_i * a_k, a_i * a_k, a_l) + \phi(a_i, a_k, a_l) = \phi(a_i * a_j, a_k, a_l) + \phi(a_i, a_j, a_l) + \phi(a_i *$

$a_l, a_j * a_l, a_k * a_l$).Hence, R satisfies the solution of the Yang-Baxter equation (Definition 2.5) by the following formulas.

$$(R \otimes id_{\mathbb{C}\langle X\rangle})(id_{\mathbb{C}\langle X\rangle} \otimes R)(R \otimes id_{\mathbb{C}\langle X\rangle})(\sum_{j,k,l=1}^{n} c_{j,k,l} a_j \otimes a_k \otimes a_l)$$

$$= \sum_{j,k,l=1}^{n} c_{j,k,l} t^{\phi(a_i,a_j,a_k)+\phi(a_i*a_k,a_i*a_k,a_l)+\phi(a_i,a_k,a_l)} a_l \otimes (a_k * a_l) \otimes ((a_j * a_k) * a_l).$$

$$(id_{\mathbb{C}\langle X\rangle} \otimes R)(R \otimes id_{\mathbb{C}\langle X\rangle})(id_{\mathbb{C}\langle X\rangle} \otimes R)(\sum_{j,k,l=1}^{n} c_{j,k,l} a_j \otimes a_k \otimes a_l)$$

$$= \sum_{j,k,l=1}^{n} c_{j,k,l} t^{\phi(a_i*a_j,a_k,a_l)+\phi(a_i,a_j,a_l)+\phi(a_i*a_l,a_j*a_l,a_k*a_l)} a_l \otimes (a_k * a_l) \otimes ((a_j * a_l) * (a_k * a_l)).$$

We see that map R is the solution of the Yang-Baxter equation from the following figure 17. $\square$

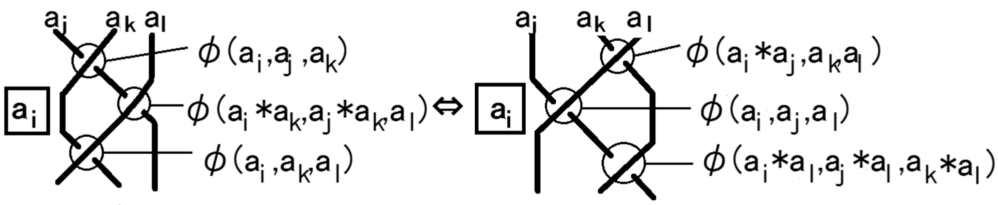


Figure 17: Yang-Baxter equation

Lemma 6.14. *Map R is linear map.*

Proof. For any $r \in \mathbb{C}$, we obtain an equation $rR(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k) = \sum_{j,k=1}^{n} r c_{j,k} t^{\phi(a_i,a_j,a_k)} a_k \otimes (a_j * a_k)$. For any $\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k$, $\sum_{j,k=1}^{n} c'_{j,k} a_j \otimes a_k$, we obtain equations $R(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k + \sum_{j,k=1}^{n} c'_{j,k} a_j \otimes a_k) = \sum_{j,k=1}^{n} (c_{j,k} + c'_{j,k}) t^{\phi(a_i,a_j,a_k)} a_k \otimes (a_j * a_k) = R(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k) + R(\sum_{j,k=1}^{n} c'_{j,k} a_j \otimes a_k)$. $\square$

Lemma 6.15. $R(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle) = \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$.

Proof. $R(\sum_{j,k=1}^{n} c_{j,k} a_j \otimes a_k) = 0$. We know that $a_j \otimes a_k$ are linearly independent for any j, k and $t \neq 0$, therefore $c_{j,k} = 0$ for any j, k. Hence the linear map R is injective. $\mathrm{Ker} R = \{0\}$ if and only if $\dim \mathrm{Ker} R = 0$ if and only if $\dim R(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle) = n^2$ if and only if $R(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle) = \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$. $\square$

Definition 6.16. Let $\mathcal{R}$ be a Yang-Baxter regular matrix representation for the basis $a_1 \otimes a_1, a_1 \otimes a_2, \cdots, a_n \otimes a_n$ of the linear map R by Lemma 6.13, 6.14 and 6.15.

Lemma 6.17. *There is a single non-zero entry in each row and in each column of $\mathcal{R}$.*

Proof. If there were two elements or more in a column, this would be inconsistent with the definition of R: $R(a_j \otimes a_k) = \sum_{l,m,n=1}^{n} t^{\phi(a_i^l, a_j^m, a_k^n)} a_l \otimes a_m = t^{\phi(a_i, a_j, a_k)} a_k \otimes (a_j * a_k)$.

There can not be two elements or more in a row, as this would be inconsistent with "there is a single element in each column" and "R is a one-to-one mapping." Hence, if $R(a_j \otimes a_k) = t^{\phi(a_i, a_j, a_k)} a_k \otimes (a_j * a_k) = t^{\phi(a_l, a_m, a_n)} a_m \otimes (a_m * a_n) = R(a_m \otimes a_n)$, then $a_j \otimes a_k = a_m \otimes a_n$. □

Lemma 6.18. *Let*

$$\mathcal{R} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{pmatrix} \qquad ,$$

where the R_{jk} are $n \times n$ matrices. We obtain a formula $\mathrm{trace} R_{jk} = \delta_{jk}$.

Proof. The definition of map R shows that $t^{\phi(a_i, a_j, a_k)} a_k \otimes (a_j * a_k)$ becomes one of the elements of a matrix R_{jk}. Therefore the value on the diagonal of an R_{jj} matrix occurs, which is $t^{\pm\phi(a_i, a_j, a_j)} = t^0 = 1$. When $j = k$, $\mathrm{trace} R_{jk} = 1$. When $j \neq k$, $\mathrm{trace} R_{jk} = 0$. □

Lemma 6.19. $\mathrm{trace}_2\big((\mathrm{id}_{\mathbb{C}\langle X \rangle} \otimes E) \cdot \mathcal{R}^{\pm}\big) = \mathrm{id}_{\mathbb{C}\langle X \rangle}$, *where*

$$\mathrm{id}_{\mathbb{C}\langle X \rangle} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}}_{n \ columns} \left.\vphantom{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}\right\} n \ rows$$

and $E \in \mathrm{End}(\mathbb{C}\langle X \rangle)$ is a unit matrix.

Proof. We calculate a diagonal element of R_{jk}. We obtain an equation $\mathrm{trace} R_{jk} = \delta_{jk}$ by Lemma 6.18.

$$\mathrm{trace}_2\big((\mathrm{id}_{\mathbb{C}\langle X \rangle} \otimes E) \cdot \mathcal{R}^{\pm}\big) = \begin{pmatrix} \mathrm{trace} R_{11} & \mathrm{trace} R_{12} & \cdots & \mathrm{trace} R_{1n} \\ \mathrm{trace} R_{21} & \mathrm{trace} R_{22} & \cdots & \mathrm{trace} R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{trace} R_{n1} & \mathrm{trace} R_{n1} & \cdots & \mathrm{trace} R_{nn} \end{pmatrix} = \mathrm{id}_{\mathbb{C}\langle X \rangle}.$$

□

Lemma 6.20 (cf.Remark 2.2.[E2])**.**

$$\sum_{C_1} \prod_{x} t^{W_\phi(x;C_1;C_2)} = \mathrm{trace}(\psi_m(b)).$$

Proof. We fix one arc coloring C_1, for $\hat{b}$ of a closure of b, and apply the coloring of the upper and the lower ends of $\hat{b}$ to $a_{p_1}, \cdots, a_{p_m}$. By doing this, we obtain one value of $a_{p_1} \otimes \cdots \otimes a_{p_m}$. Hence, we define map g and map g' by using the coloring C_1.

$$g : \{X\text{−arc coloring of } \hat{b}\} \to \{a_{p_1} \otimes a_{p_2} \otimes \cdots \otimes a_{p_m}\},$$
$$g' : \{X\text{−arc coloring of } \hat{b}\} \to \{a^*_{p_1} \otimes a^*_{p_2} \otimes \cdots \otimes a^*_{p_m}\}.$$

Let $a_{p_1}, a_{p_2}, \cdots, a_{p_m}$ be elements of a quandle and we have one arc coloring in the upper ends and the lower ends of $\hat{b}$. And If two colorings $c \in \mathcal{C}$ and $c' \in \mathcal{C}$ are equal, then $a_{p_1} \otimes a_{p_2} \otimes \cdots \otimes a_{p_m} = a'_{p_1} \otimes a'_{p_2} \otimes \cdots \otimes a'_{p_m}$, and thus $a_{p_1} = a'_{p_1}, a_{p_2} = a'_{p_2}, \cdots, a_{p_m} = a'_{p_m}$. Therefore $c = c'$. Hence the maps g, g' are bijective. Therefore, the sum given by fixing the weight at the crossings and by using all possible arc colorings C_1 is equal to a value given by calculating a trace($\psi_m(b)$).

$$\begin{aligned}
\mathrm{trace}(\psi_m(b)) &= \sum_{p_1,\cdots,p_m=1}^{n} (a^*_{p_1} \otimes \cdots \otimes a^*_{p_m})(\psi_m(\sigma^{(1)}_{s_1}) \cdots \psi_m(\sigma^{(l)}_{s_l}) a_{p_1} \otimes \cdots \otimes a_{p_m}) \\
&= \sum_{p_1,\cdots,p_m=1}^{n} (a^*_{p_1} \otimes \cdots \otimes a^*_{p_m})(\psi_m(\sigma^{(l)}_{s_l}) \cdots \psi_m(\sigma^{(1)}_{s_1}) a_{p_1} \otimes \cdots \otimes a_{p_m}) \\
&= \sum_{C_1} (g'(X\text{−arc coloring of } \hat{b}))(\psi_m(\sigma^{(l)}_{s_l}) \cdots \psi_m(\sigma^{(1)}_{s_1}) g(X\text{−arc coloring of } \hat{b})) \\
&= \sum_{C_1} \prod_{x=1}^{l} t^{W_\phi(x;C_1,C_2)} = \sum_{C_1} \prod_{x} t^{W_\phi(x;C_1;C_2)}.
\end{aligned}$$

□

Proof of Lemma 6.12. We obtain an equation $\mathrm{trace}_2\big((\mathrm{id}_{\mathbb{C}\langle X\rangle} \otimes E) \cdot \mathcal{R}^{\pm}\big) = \mathrm{id}_{\mathbb{C}\langle X\rangle}$ from Lemma 6.19. Recall that a trace($\psi_m(b)$) is an isotopy invariant of L by Theorem 2.6. This invariant is an isotopy invariant of L by using an R-matrix. By Proposition 6.11 and Lemma 6.20, we give

$$\Phi_\phi(L) = \sum_{C_2} \sum_{C_1} \prod_{x} t^{W_\phi(x;C_1;C_2)} = \sum_{C_2} \mathrm{trace}(\psi_m(b)).$$

Hence, the shadow 3-cocycle invariants from R-matrices are isotopy invariants of L. □

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