The combination of 3-dimensional Affine Algebraic Geometry and Minimal Model Program
(３次元アフィン代数幾何学と極小モデル理論の融合)

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1 What is our interest in Mathematics ?

Our main interest lies in the investigation concerning algebro-geometric properties on *algebraic varieties* defined over the filed of complex numbers $\mathbb{C}$. Roughly speaking, algebraic varieties are geometrical objects which are locally defined by systems of several algebraic equations. The definition of such varieties is not difficult to state, nevertheless, the analysis of them is, usually, quite difficult. In fact, the local systems of defining algebraic equations are, in general, quite far from being linear and the way to glue together such local informations to obtain global ones is quite complicated. Since any algebraic variety is obtained by gluing together local affine data, it is principally important to look into so-called *affine algebraic varieties*. But, the current research on affine algebraic varieties seems to be motivated by several unsolved problems on polynomial rings. For instance, *Jacobian Conjecture*, *Zariski Cancellation Problem*, *Abhyankar-Sathaye Embedding Problem*, *the structure of groups of algebraic automorphisms on polynomial rings* are the famous ones. Although all such problems are stated in terms of algebra, they can be translated into the terminology in algebraic geometry. Namely, the essential of these problems lies in *how to charactrize complex affine spaces $\mathbb{C}^n$ as algebraic varieties !*

At the present time, there exist good characterizations of $\mathbb{C}^n$ for dimensions $n = 1, 2$ in the sense they are enough to settle out above mentioned problems in such dimensions. Meanwhile, for the higher-dimensional cases $n \geq 3$, we know very little about that. Almost all researchers in affine algebraic geometry deal with affine algebraic varieties directly, more precisely to say, their methods mainly lie in the treatment on polynomial transformations. Certainly, it is sometimes useful to construct some kinds of counterexamples, however, as it depends on the forms of polynomials heavily, we guess that it does not become useful to develop some general theory concerning affine algebraic verieties. That is why it is desirable to establish some techniques which are more algebro-geometric and does not depend on the forms of defining equations. In what follows, we shall state the feature of our approach to the analysis of affine algebraic varieties.

2 The feature of our approach and the appearing obstacles

As mentioned just above, it seems not to be suitable to deal with affine varieties directly. In principal, affine varieties are too far from being linear objects, so that, it is hopeful to make such varieties more linear and accessible via suitable procedures. On the other hand, we bet that the properties on open varieties make influences on boundaries. In consideration of them, we think that it is the best way to embed affine varieties into suitable projective varieties, and make use of the theory of *Birational Geometry.* Birational Geometry is interested in algebro-geometric properties up to birational equivalence. Roughly speaking, *small* difference between varieties are neglected in Birational Geometry. Fortunately, Birational Geometry has been developing drastically in the past two decades, and we have strong enough theory, so-called *Minimal Model Program* (or *Mori Theory*) to investigate higher-dimensional varieties (having certain kinds of singularities). Due to Minimal Model Program, in principal, we are able to, at least, look into projective threefolds (having terminal singularities and boundaries). By making use of Minimal Model Program, our approach to investigate affine algebraic threefolds is stated as in the following fashion:

Our method

Let X be a given smooth affine algebraic threefold defined over $\mathbb{C}$. In order to analyze X, we shall embed X into a smooth projective threefold V in such a way that the boundary part $D := V\backslash X$ is a divisor with only simple normal crossings (SNC). (Note that we are free to take such a variety V as far as D is SNC.) Then the pair (V, D) has a certain kind of singularities, namely *dlt* (=divisorial log terminal) singularities in terms of Minimal Model Program (MMP). Although we do not mention the definition of dlt singularities here, MMP works in this category of singularities. So, let us perform MMP for (V, D) to obtain the following diagram:

$$(*) \qquad \phi : (V, D) \overset{\phi^0}{\cdots\to} (V^1, D^1) \overset{\phi^1}{\cdots\to} \cdots\cdots \to (V^{s-1}, D^{s-1}) \overset{\phi^{s-1}}{\cdots\to} (V^s, D^s),$$

where the final object $(V', D') := (V^s, D^s)$ is either a Log Mori fiber space or a log minimal model, i.e., $K_{V'} + D'$ is nef, according to the value of log Kodaira dimension $\overline{\kappa}(X)$ of X. Set $X' := V'\backslash \mathrm{Supp}\,(D')$. Usually, since (V', D') has a distinguished simpler structure compared with that of the original pair (V, D), we expect that we are able to analyze the structure of X' in detail. Hence, *if we were able to compare X with X' in an explicit manner, then we obtain the data on X from those on X'. But, in general, there occur several unavoidable obstacles in this strategy.* These obstacles are caused by the difference of the points of view between Birational Geometry and Affine Algebraic Geometry. Namely, since the main interest in Birational Geometry lies in birational properties of algebraic varieties, the existence and termination of flips and the Abundance are the most important problems. Once these three have been established, the birational properties on projective varieties can be reduced to those on more simple varieties (i.e., (log) Mori fiber spaces or (log) minimal models). Whereas, since the main interest in Affine Algebraic Geometry lies in *biregular* properties of affine algebraic varieties, we should investigate all divisorial contractions and (log) flips, which do not take place in the boundary parts, appearing in the process of (Log) Minimal Model Program $(*)$, that is, we have to describe how exceptional divisors and flipping/flipped curves intersect the boundary parts explicitly. Thus, the obstacles occuring when we try to compare X with X' are summarized as follows:

OBSTACLE Each of birational maps $\phi^i : (V^i, D^i) \cdots \to (V^{i+1}, D^{i+1})$ appearing in $(*)$ above is either a log-divisorial contraction or a log-flip. Let D^i denote the proper transform of D on V^i, and let $X^i := V^i\backslash \mathrm{Supp}\,(D^i)$ be the complement $(0 \le i \le s)$. Then:

(1) In the case where ϕ^i is of log-divisorial type and the exceptional divisor $E^i := \mathrm{Exc}(\phi^i)$ is NOT contained in the boundary $\mathrm{Supp}\,(D^i)$, then X^{i+1} is strictly smaller than the previous one X^i. If we were to describe the contraction ϕ^i and how E^i meets D^i explicitly, then we can resume the data on X^i from those on X^{i+1} in principle. But this seems to be hopeless in general.

(2) In the case where ϕ^i is of log-flipping type, if we were to know that all the flipping curves (resp. flipped curves) are contained in the boundary $\mathrm{Supp}\,(D^i)$ (resp. $\mathrm{Supp}\,(D^{i+1})$), then there exists no difference between X^i and X^{i+1}. But, this anticipation does not always hold true, namely, some of flipping curves or flipped curves may not be contained in the boundary. As a result, we are not able to compare X^i and X^{i+1} explicitly and, in addition to this, X^i (resp. X^{i+1}) may be no longer affine even if X^{i+1} (resp. X^i) is affine.

These are the most messy issues when applying MMP for affine algebraic geometry.

3 Summarization of our recent works

In this section, we shall summarize our recent results along the approach mentioned in §2. As we said beforehand, the algebro-geometric properties on affine algebraic threefolds make influences on the boundaries. From this point of view, it is indispensable to classify the compactifications of affine threefolds into certain kinds of projective threefolds with distinguished properties in the sense of MMP. For instance, as mentioned in §1, many of problems on affine algebraic varieties are concerned with the characterizations of the affine spaces $\mathbb{C}^n$, it is desirable to classify compactifications of the affine 3-space $\mathbb{C}^3$ into projective threefolds with ample anti-canonical bundles, so-called *Fano threefolds*. In [7], we have succeeded in the classification of compactifications of topologically contractible affine threefolds into smooth Fano threefolds with the second Betti number $B_2 = 2$. (This is a generalization of the famous problem proposed by F. Hirzebruch. In the case that $B_2 = 1$, the classification is completed by the contributions of many researchers.) The article [11] is concerned with compactifications of $\mathbb{C}^3$ into $\mathbb{Q}$-*Fano threefolds*. Different from the case treated in [7], in this case, $\mathbb{Q}$-Fano threefolds are no longer smooth, i.e., they have certain kinds of mild singularities. Due to this, the consideration becomes too complicated. But, by making use of the result due to H. Takagi, we have succeeded in the explicit construction of compactifications of $\mathbb{C}^3$ in terms of 2-ray games.

In [8, 9, 12], we deal with the obstacles mentioned in the end of §2. Although we do not refer here, we can overcome obstacles under a certain mild assumption in [12], which is a refinement of the result in [8]. In [9], we describe the first process of the $(K + D)$-MMP, namely, the rational map $\phi^0 : (V, D) \cdots \to (V^1, D^1)$ with notation in §2. For further development of our approach, this is indispensable.

In 1970's, M. Nagata constructed an automorphism on the affine 3-space $\mathbb{C}^3$, so-called *Nagata automorphism*, which seems to represent the difficulty to investigate $\mathbb{C}^3$. In fact, for almost past three decades, it had remained unknown whether or not the Nagata automorphism is tame. Recently, in 2004, Shestakov & Umirbaev have succeeded in showing that the Nagata automorphism is, in fact, not tame. This is a shocking result because it is well-known that all the automorphisms on the affine plane $\mathbb{C}^2$ are tame. This indicates an aspect of the drastic difference between 2-dimensional and 3-dimensional affine algebraic geometry. But, as thier method to prove the non-tameness is purely algebraic and lack of intrinsic geometry hiding in. In our recent article [6, 13], we have succeeded in giving a new proof on the non-tameness from a point of view of Sarkisov Program. We bet that our method for the proof is more accessible compared with that due to Shestakov & Umirbaev, and useful to investigate the group of algebraic automorphisms on $\mathbb{C}^3$. This is our subject in future.

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