プロジェクト名：Fano 多様体上のアフィン錐への加法群スキームの作用の研究（ *On actions of the additive group scheme on affine cones over Fano varieties*）

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Report on the project entitled "*On actions of the additive group scheme on affine cones over Fano varieties*"

Background:

Fano varieties are simply defined as varieties (over the fields of complex numbers) whose anti-canonical divisors are ample. Thus the definition is quite simple and does not contain any ambiguity, nevertheless this class of varieties are remarkably important from the point of view of Minimal Model Program (MMP, for short), which is one of the main streams of the modern algebraic geometry. Very roughly speaking, MMP plays an essential role in order to investigate the structure of (mainly) projective varieties with certain kinds of singularities eventually with boundary divisors in terms of birational geometry. More precisely to say, starting with a projective variety **V**, the mechanism of MMP tells us how to make **V** simpler via finitely many elementary birational operations, so-called divisorial contractions and flips. Let **V'** be the resulting variety obtained from **V** via MMP. Then **V'** is either a Mori Fiber Space (MFS, for short) or a minimal model according to log Kodaira dimension of **V**. In our application, the case of MFS is more interesting because MFS's are usually equipped with structures of fibrations onto lower dimensional varieties, which implies that we may reduce the investigation to the lower-dimensional birational geometry. Notice that the category of **Fano varieties**, which appears in the title of the project, is one of MFS's. In other words, they are defined to be projective varieties with ample anti-canonical divisors whose Picard number are equal to one. Since Fano varieties are one of outputs of MMP, it is significant to analyze them. In fact, there are so far plenty of studies concerning Fano varieties. In case of dimension two, the projective plane is the only possibility of a smooth Fano variety. Meanwhile, in dimension three, the situation becomes drastically complicated. Indeed, there exist 17 deformation types of smooth Fano threefolds. Note that this does not mean that there are 17 smooth Fano threefolds, in fact, this asserts the existence of 17 *deformation types,* i.e., each deformation type has a moduli. As a consequence, we are obliged to deal with infinitely many varieties even in the case of smooth Fano threefolds.

Purpose of the project:

As explained, Fano varieties are interesting objects from the viewpoint of birational geometry. On the other hand, we are interested in affine algebraic geometry. In some sense, affine algebraic geometry is the contrary geometry with respect to birational geometry. Namely, we have to observe "biregular" properties of affine algebraic varieties, which means that birational peculiarities are not enough for us. Thus it seems that MMP is not so efficient in order to investigate structures of affine algebraic varieties. However, if we consider the actions of algebraic groups on affine algebraic varieties, then the quotient varieties become frequently projective varieties to be able to apply MMP. For instance, affine cone over polarized projective varieties are such objects, where the algebraic torus of dimension one acts.

Meanwhile, one of the main interests in affine algebraic geometry lies in the criterion about the existence of actions of the additive group of dimension one **Ga** since it relates an existence of fibrations by the affine line **C**, which is definitively a geometric concept, to locally nilpotent derivations (LND, for short) on coordinate rings of affine algebraic varieties, which is on the contrary purely algebraic concept. In other words, the existence of a fibration by **C** on a given affine algebraic variety **X** is equal to finding an action by **Ga** or an LND on the coordinate ring of **X**. As the fibrations by **C** plays a very important role in affine algebraic geometry (for instance, we can recognize its importance by taking the affirmative solution of Zariski cancellation problem in dimension two into account), it is desired to obtain criteria about the existence of **Ga** actions on **X**. In general, it is never easy to find an action of **Ga**, but in our previous paper [2], we have succeeded into an interpretation of an existence of a **Ga**-action on an affine cone **X** over a polarized projective variety **V** in terms of a certain kind of a cylinder contained in **V**.

In the present research project, by making use of this criterion, we try to construct systematically actions of **Ga** on affine cones **X** over anti-canonically embedded smooth Fano threefolds **V**. The most difficult case to construct such **Ga**-actions is the situation that the Fano index of **V** equals one. In [2], we construct **Ga**-actions in case where the genus of smooth Fano threefolds V are equal to 12. (Usually, the terminology *genus* is used for algebraic curves, however this is also used for smooth Fano threefolds of the Picard number one. Here we do not mention its definition explicitly. But, notice that the genus is less than or equal to 12, furthermore, the value 11 can not occur as genus.)

<u>Results:</u>

This time, we consider the case where the genus of **V** is equal to either 9 or 10, and succeed in constructions of **Ga**-actions on **X** in an explicit manner, which means that we are able to find desired cylinder on **V** concretely. Notice that, as mentioned above, the family of smooth Fano threefolds of Fano index one with genus 9 or 10 form a moduli, say **Hilb**. The main result say that if **V** corresponds to a smooth point of **Hilb**, then the affine cone **X** over **V** admits an effective action by **Ga**. The proof for this depends on the exhaustive knowledge about Fano threefolds of Picard number one in terms of Sarkisov Program. Roughly speaking, Sarkisov Program admits us to bring **V** to the other MFS via a suitable blowing-up and eventual flops. Namely, by transforming to the other MFS, it is often that the situation becomes more visible. From this viewpoint, the advice from experts on Fano threefolds should be necessary. Hence with the help of Prof. Mikhail Zaidenberg at Grenoble University I (France) and Prof. Yuri Prokhorov at Moscow State University (Russia), we can finally obtain the above mentioned result. This is summarized in our article, which have been already submitted to an international mathematical journal with the high-level quality (Michigan Mathematical Journal).

<u>Papers:</u>

[1] Takashi Kishimoto, Yuri Prokhorov and Mikhail Zaidenberg, *Affine cones over Fano threefolds and additive group actions*, submitted to Michigan Mathematical Journal, available at: arXiv:1106.1312v1 [math.AG].

[2] Takashi Kishimoto, Yuri Prokhorov and Mikhail Zaidenberg, *Group actions on affine cones*, In: Affine Algebraic Geometry, The Russell Festschrift (Eds: Daniel Daigle, Richard Ganong and Mariusz Koras), CRM Proceedings & Lecture Notes, Vol. 54 (2011), 123—163.