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Stability of Antisymmetric Tensor Fields of Chern-Simons Type in AdS Spacetime

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Abstract

Stability of massive antisymmetric tensor fields with the Chern-Simons type action in anti de Sitter spacetime is studied. It is found that there exists a complete set of solutions whose energy is conserved and positive definite if the mass is positive. Scalar products of the solutions are shown to be well-defined and conserved. In contrast to the previously studied scalar field case there is no other set of stable solutions with a different kind of boundary condition.

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1. Introduction

Anti de Sitter (AdS) spacetime is a maximally symmetric spacetime with a negative constant curvature. It naturally appears as a solution of Einstein equation with a negative cosmological constant. It also appears in compactifications of higher dimensional supergravities in the Kaluza-Klein theory [1]. Recently interest in field theories in AdS spacetime has been much increased due to their relevance to the AdS/CFT correspondence in the string/M theory [2, 3, 4] (For a review see ref. [5].). An important issue of field theories in AdS spacetime is their stability, which was previously discussed in refs. [6, 7, 8]. (For recent studies on the stability see ref. [9].) Another important issue is a choice of boundary conditions of fields at spatial infinity. In ref. [10] boundary conditions are chosen such that the Cauchy problem for field equations is well-defined by requiring the conservation of the scalar product of fields.

The purpose of this paper is to study the stability of free massive antisymmetric tensor fields of arbitrary rank n in AdS spacetime. The string/M theory and supergravities contain antisymmetric tensor fields, which play an important role. Therefore, their stability is an important issue. There are two types of theories of antisymmetric tensor fields. One type of theories have an action with the second order kinetic term of the Maxwell type and the other type of theories have an action with the first order kinetic term of the Chern-Simons type [11]. (Equivalent theories to the latter were studied in refs. [12, 13].) Both types of theories appear in supergravities [14]. In this paper we only consider the Chern-Simons type theories, which are theories of n-th rank antisymmetric tensor fields in $d = 2n + 1$ dimensions. As a preparation for study of the stability we first obtain the general solution of the field equation. Then we obtain conditions for the stability by studying the conservation and the positivity of the energy. We also study scalar products of the solutions.

The conditions for the stability of scalar fields in AdS spacetime were obtained in refs. [7, 8]. A free massive scalar field theory in d-dimensional AdS spacetime is stable if the mass m satisfies

$$\left(\frac{m}{a}\right)^2 > -\left(\frac{d-1}{2}\right)^2, \tag{1.1}$$

where a^{-1} is the radius of AdS spacetime. More precisely, there exists a complete set of solutions of the field equation whose energy is conserved and positive definite.

Furthermore, when the mass satisfies

$$-\left(\frac{d-1}{2}\right)^2 < \left(\frac{m}{a}\right)^2 < 1 - \left(\frac{d-1}{2}\right)^2, \tag{1.2}$$

there exists another set of stable solutions satisfying a different kind of boundary condition at spatial infinity. In the latter case the coefficient of the improvement term in the energy-momentum tensor must take a particular value. Scalar fields can be stable even if the mass squared is negative due to a positive contribution from the kinetic term to the energy.

As in the scalar field case we find that there exists a complete set of solutions for antisymmetric tensor fields whose energy is conserved and positive definite if the mass is positive $m > 0$. There are three kinds of improvement terms in the energy-momentum tensor for antisymmetric tensor fields of rank $n \geq 2$. The coefficients of these terms can take arbitrary values although they do not contribute to the energy. The scalar products of the solutions are shown to be well-defined and conserved.

In contrast to the scalar field case, however, there is no other set of stable solutions. The conservation of the energy allows another set of solutions satisfying a different kind of boundary condition at spatial infinity as in the scalar field case but their energy turns out to be divergent. Therefore, only one kind of boundary condition is possible for antisymmetric tensor fields with the Chern-Simons type action. A Chern-Simons type theory of the second rank atisymmetric tensor fields in five-dimensional AdS spacetime was previously studied in ref. [15] in the context of the AdS/CFT correspondence. By using a different approach it was found there that only one kind of boundary condition is possible, which is consistent with our result.

In the next section we introduce the first order action of antisymmetric tensor fields in AdS spacetime and obtain the energy-momentum tensor. In sect. 3 the general solution of the field equation is obtained in terms of the hypergeometric functions. The conservation and the positivity of the energy are discussed in sects. 4 and 5 respectively. In sect. 6 we show that there exists a well-define and conserved scalar product for the solutions. In Appendix we give a construction and useful identities of spherical harmonics for antisymmetric tensors on the $(d-2)$-dimensional sphere S^{d-2}.

2. Antisymmetric tensor fields in AdS spacetime

We consider antisymmetric tensor fields in d-dimensional AdS spacetime. AdS spacetime is a maximally symmetric spacetime and has the metric

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{a^2\cos^2\rho}\left[-dt^2 + d\rho^2 + \sin^2\rho\ h_{ab}d\theta^a d\theta^b\right], \tag{2.1}$$

where $\mu, \nu = 0, 1, \cdots, d-1$ are d-dimensional world indices and the constant a^{-1} is the radius of AdS spacetime. The time coordinate t has a range $-\infty < t < \infty$, which corresponds to considering the universal covering of AdS spacetime. The radial coordinate ρ has a range $0 \leq \rho < \frac{\pi}{2}$ with the spatial infinity at $\rho = \frac{\pi}{2}$. θ^a and h_{ab} $(a, b = 1, 2, \cdots, d-2)$ are coordinates and the metric of the $(d-2)$-dimensional unit sphere S^{d-2}. Non-vanishing components of the Christoffel connection are

$$\begin{aligned}
&\Gamma^{\ 0}_{0\rho} = \Gamma^{\ \rho}_{00} = \Gamma^{\ \rho}_{\rho\rho} = \tan\rho, \\
&\Gamma^{\ \rho}_{ab} = -\tan\rho h_{ab}, \qquad \Gamma^{\ a}_{\rho b} = \frac{1}{\sin\rho\cos\rho}\delta^a_b, \\
&\Gamma^{\ a}_{bc} = \frac{1}{2}h^{ad}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right) = \gamma^{\ a}_{bc},
\end{aligned} \tag{2.2}$$

where $\gamma^{\ a}_{bc}$ is the Christoffel connection of S^{d-2}. The Riemann tensor is given by

$$R_{\mu\nu}{}^\tau{}_\sigma = -a^2\left(\delta^\tau_\mu g_{\nu\sigma} - \delta^\tau_\nu g_{\mu\sigma}\right). \tag{2.3}$$

Our conventions for the curvature tensors are $R_{\mu\nu}{}^\tau{}_\sigma = \partial_\mu\Gamma^{\ \tau}_{\nu\sigma} + \Gamma^{\ \tau}_{\mu\lambda}\Gamma^{\ \lambda}_{\nu\sigma} - (\mu \leftrightarrow \nu)$, $R_{\mu\nu} = R_{\tau\mu}{}^\tau{}_\nu$, $R = g^{\mu\nu}R_{\mu\nu}$. For other properties of AdS spacetime see, e.g. ref. [5].

We consider a free theory of a complex antisymmetric tensor field $B_{\mu_1\cdots\mu_n}$ of rank n in d-dimensional AdS spacetime for $d = 2n+1$. The Chern-Simons type action [11] is

$$\begin{aligned}
S = \int d^dx\Big[&(-1)^{\frac{1}{2}(n+1)}\frac{1}{(n!)^2}\epsilon^{\mu_1\cdots\mu_{2n+1}}B^*_{\mu_1\cdots\mu_n}\partial_{\mu_{n+1}}B_{\mu_{n+2}\cdots\mu_{2n+1}} \\
&-\frac{m}{n!}\sqrt{-g}B^*_{\mu_1\cdots\mu_n}B^{\mu_1\cdots\mu_n}\Big],
\end{aligned} \tag{2.4}$$

where $\epsilon^{\mu_1\cdots\mu_{2n+1}}$ is the totally antisymmetric tensor and $*$ denotes the complex conjugation. The reality of the action requires that the mass m is real. When $m = 0$, the action consists of only the kinetic term, which do not depends on the metric but is invariant under general coordinate transformations. This is an action of a topological field theory. We do not discuss the stability of the $m = 0$ case since there is no local degrees of freedom and the energy is zero. The field equation derived from this action is

$$(-1)^{\frac{1}{2}(n+1)} \frac{1}{n!} \epsilon^{\mu_1 \cdots \mu_{2n+1}} \partial_{\mu_{n+1}} B_{\mu_{n+2} \cdots \mu_{2n+1}} - m\sqrt{-g} B^{\mu_1 \cdots \mu_n} = 0. \tag{2.5}$$

We define the energy-momentum tensor of the theory as a variation of the action with respect to the metric. To do this we need an action for general metric, which reduces to the original action (2.4) for the AdS metric (2.1). We use the action

$$\begin{aligned} S' = \int d^d x \ \Big[& (-1)^{\frac{1}{2}(n+1)} \frac{1}{(n!)^2} \epsilon^{\mu_1 \cdots \mu_{2n+1}} B^*_{\mu_1 \cdots \mu_n} \partial_{\mu_{n+1}} B_{\mu_{n+2} \cdots \mu_{2n+1}} \\ & - \frac{\mu}{n!} \sqrt{-g} B^*_{\mu_1 \cdots \mu_n} B^{\mu_1 \cdots \mu_n} + \frac{\alpha}{n! a} R B^*_{\mu_1 \cdots \mu_n} B^{\mu_1 \cdots \mu_n} \\ & + \frac{\beta}{n! a} R_{\mu\nu} B^{*\mu}{}_{\mu_2 \cdots \mu_n} B^{\nu \mu_2 \cdots \mu_n} + \frac{\gamma}{n! a} R_{\mu\nu\tau\sigma} B^{*\mu\nu}{}_{\mu_3 \cdots \mu_n} B^{\tau\sigma\mu_3 \cdots \mu_n} \Big], \end{aligned} \tag{2.6}$$

where α, β and γ are arbitrary constant parameters. The parameter μ is chosen as

$$\mu = m - [d(d-1)\alpha + (d-1)\beta + 2\gamma]\, a \tag{2.7}$$

so that this action coincides with eq. (2.4) for the AdS metric. The last three terms in eq. (2.6) containing the curvature tensors are generalizations of the well-known $R\phi^2$ term in the scalar field theory. These terms give improvement terms in the energy-momentum tensor. Note that we do not need to introduce the term $R_{\mu\tau\nu\sigma} B^{*\mu\nu}{}_{\mu_3 \cdots \mu_n} B^{\tau\sigma\mu_3 \cdots \mu_n}$ since it is related to the last term in eq. (2.6) by the Bianchi identity of the Riemann tensor. From eq. (2.6) we obtain the energy-momentum tensor as

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S'}{\delta g^{\mu\nu}} \\ &= \frac{2}{n!} [nm + (d-1)\beta a + 2\gamma a] \left(B^*_{(\mu} B_{\nu)} \right) \\ &\quad - \frac{1}{n!} [m - 2(d-1)\alpha a]\, g_{\mu\nu} (B^* B) \\ &\quad + \frac{2\alpha}{n! a} \left(D_\mu D_\nu - g_{\mu\nu} D^2 \right) (B^* B) \\ &\quad + \frac{\beta}{n! a} \Big[2 D_\sigma D_{(\mu} (B^*_{\nu)} B^\sigma) - D^2 (B^*_{(\mu} B_{\nu)}) - g_{\mu\nu} D_{(\tau} D_{\sigma)} (B^{*\tau} B^\sigma) \Big] \\ &\quad - \frac{4\gamma}{n! a} D_\tau D_\sigma (B^*_{(\mu}{}^\tau B_{\nu)}{}^\sigma), \end{aligned} \tag{2.8}$$

where $(B^*B) = B^*_{\mu_1\mu_2\cdots\mu_n}B^{\mu_1\mu_2\cdots\mu_n}$, $(B^*_\mu B_\nu) = B^*_{\mu\rho_2\cdots\rho_n}B_\nu{}^{\rho_2\cdots\rho_n}$, etc. There is no contribution from the kinetic term.

The energy is defined as follows [6]. The energy-momentum tensor (2.8) satisfies $D_\mu T^{\mu\nu} = 0$ for arbitrary α, β and γ when the field equation is used. We can construct a conserved current

$$\partial_\mu\left(\sqrt{-g}T^\mu{}_\nu\xi^\nu\right) = 0 \tag{2.9}$$

for each Killing vector ξ^μ satisfying $D_\mu\xi_\nu + D_\nu\xi_\mu = 0$. The energy is the charge of this current for a timelike Killing vector $\xi^\mu = (1, 0, \cdots, 0)$

$$E = -\int d^{d-1}x\sqrt{-g}T^t{}_t, \tag{2.10}$$

where the integral is over $(d-1)$-dimensional space.

3. Solutions of the field equation

We shall obtain the general solution of the field equation (2.5). It is more convenient to rewrite the field equation in another form. Applying ∂_{μ_1} to eq. (2.5) we obtain a constraint

$$D_{\mu_1}B^{\mu_1\cdots\mu_n} = 0, \tag{3.1}$$

where D_μ is the covariant derivative in AdS spacetime. Applying the differential operation in the first term of eq. (2.5) to eq. (2.5) again and using eq. (2.5) in the second term we obtain the second order equation

$$\left[D_\mu D^\mu + n(n+1)a^2 - m^2\right]B^{\mu_1\cdots\mu_n} = 0. \tag{3.2}$$

We first solve these equations and then substitute the solutions into the first order equation (2.5) to obtain further conditions.

By the constraint (3.1) the components $B_{ta_2\cdots a_n}$ and $B_{t\rho a_3\cdots a_n}$ are not independent but can be expressed by $B_{a_1\cdots a_n}$ and $B_{\rho a_2\cdots a_n}$. The field equation (3.2) for $[\mu_1\cdots\mu_n] = [a_1\cdots a_n]$, $[\rho a_2\cdots a_n]$ gives

$$\begin{aligned}
&\mathcal{L}_1 B^{a_1\cdots a_n} - \frac{2(-1)^n n}{\sin^3\rho\cos\rho}\nabla^{[a_1}B^{a_2\cdots a_n]\rho} = 0, \\
&\mathcal{L}_2 B^{\rho a_2\cdots a_n} - \frac{2}{\tan\rho}\nabla_{a_1}B^{a_1a_2\cdots a_n} = 0,
\end{aligned} \tag{3.3}$$

where ∇_a is the covariant derivative on S^{d-2} using the Christoffel connection γ^a_{bc} in eq. (2.2). The differential operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as

$$
\begin{aligned}
\mathcal{L}_1 &= -\partial_t^2 + \partial_\rho^2 + \frac{4n-1}{\sin\rho\cos\rho}\partial_\rho + \frac{\nabla_a\nabla^a + 3n(n-1)}{\sin^2\rho} + \frac{4n^2 - \left(\frac{m}{a}\right)^2}{\cos^2\rho}, \\
\mathcal{L}_2 &= -\partial_t^2 + \partial_\rho^2 + \left(4\tan\rho + \frac{4n-3}{\sin\rho\cos\rho}\right)\partial_\rho + \frac{\nabla_a\nabla^a + 3n^2 - 7n + 3}{\sin^2\rho} \\
&\quad + \frac{(2n+1)^2 - \left(\frac{m}{a}\right)^2}{\cos^2\rho} - 4.
\end{aligned} \tag{3.4}
$$

It can be shown that when eqs. (3.1) and (3.3) are satisfied, then the remaining components of eq. (3.2) are automatically satisfied.

We decompose the antisymmetric tensor field into transverse and longitudinal modes by using spherical harmonics $Y^{(l)}_{a_1\cdots a_n}(\theta)$ for antisymmetric tensor fields on S^{d-2}. The spherical harmonics are transverse $\nabla^{a_1}Y^{(l)}_{a_1\cdots a_n} = 0$ and are eigenfunctions of the Laplacian $\nabla_a\nabla^a$ on S^{d-2} with the eigenvalue $-[l(l+d-3)-n]$. The quantum number l takes values $l = 0, 1, 2, \cdots$ for $n = 0$ and $l = 1, 2, 3, \cdots$ for $n \geq 1$. In the Appendix we sketch how to construct $Y^{(l)}_{a_1\cdots a_n}$ and give some useful identities. Using the spherical harmonics the components of the antisymmetric tensor field are decomposed as

$$
\begin{aligned}
B^{a_1\cdots a_n}(x) &= R_1(t,\rho)Y^{(l)a_1\cdots a_n}(\theta) + R_2(t,\rho)\nabla^{[a_1}Y^{(l)a_2\cdots a_n]}(\theta), \\
B^{\rho a_2\cdots a_n}(x) &= \sin\rho\cos\rho R_3(t,\rho)Y^{(l)a_2\cdots a_n}(\theta) + R_4(t,\rho)\nabla^{[a_2}Y^{(l)a_3\cdots a_n]}(\theta), \\
B^{t a_2\cdots a_n}(x) &= R_5(t,\rho)Y^{(l)a_2\cdots a_n}(\theta) + R_6(t,\rho)\nabla^{[a_2}Y^{(l)a_3\cdots a_n]}(\theta), \\
B^{t\rho a_3\cdots a_n}(x) &= R_7(t,\rho)Y^{(l)a_3\cdots a_n}(\theta) + R_8(t,\rho)\nabla^{[a_3}Y^{(l)a_4\cdots a_n]}(\theta),
\end{aligned} \tag{3.5}
$$

where $Y^{(l)a_1\cdots a_n} = h^{a_1b_1}\cdots h^{a_nb_n}Y^{(l)}_{b_1\cdots b_n}$ and $\nabla^a = h^{ab}\nabla_b$. The factor $\sin\rho\cos\rho$ in front of R_3 is for later convenience. Substituting eq. (3.5) into the constraint (3.1) for $[\mu_2\cdots\mu_n] = [a_2\cdots a_n]$, $[\rho a_3\cdots a_n]$, R_5, $\cdots$, R_8 are expressed in terms of R_1, $\cdots$, R_4 as

$$
\begin{aligned}
\partial_t R_5 &= \frac{1}{n}(l+n-1)^2 R_2 - (\sin\rho\cos\rho\partial_\rho + 2n)R_3, \\
\partial_t R_6 &= -\left(\partial_\rho + 2\tan\rho + \frac{2n-1}{\sin\rho\cos\rho}\right)R_4, \\
\partial_t R_7 &= -\frac{1}{n-1}(l+n)(l+n-2)R_4, \\
\partial_t R_8 &= 0.
\end{aligned} \tag{3.6}
$$

Other components of the constraint (3.1) are then automatically satisfied. R_1, $\cdots$, R_4, in turn, are determined by solving eq. (3.3). To solve eq. (3.3) it is convenient to change the variable as

$$v = \sin^2 \rho. \tag{3.7}$$

Let us first consider R_1. We define the function $f_1(v)$ by

$$R_1(t,\rho) = \bar{N}_1(t) v^{\frac{1}{2}\kappa} (1-v)^{\frac{1}{2}\lambda} f_1(v), \tag{3.8}$$

where

$$\bar{N}_1(t) = N_1 e^{-i\omega_1 t} + \tilde{N}_1 e^{i\omega_1 t} \tag{3.9}$$

and N_1 and $\tilde{N}_1$ are complex constants. The transverse part of the first equation in (3.3) gives

$$\Big[4v(1-v)\partial_v^2 + 2\left(2\kappa + 4n - 2(\kappa+\lambda+1)v\right)\partial_v + \omega_1^2 - (\kappa+\lambda)^2 \\ + \frac{(\kappa - l + n)(\kappa + l + 3n - 2)}{v} + \frac{(\lambda - 2n)^2 - \left(\frac{m}{a}\right)^2}{1-v} \Big] f_1(v) = 0. \tag{3.10}$$

We choose the parameters κ and λ as

$$\kappa = l - n, \qquad (\lambda - 2n)^2 = \left(\frac{m}{a}\right)^2 \tag{3.11}$$

so that the v^{-1} and $(1-v)^{-1}$ terms in eq. (3.10) vanish. There are two possible values of λ

$$\lambda = \lambda_{\pm} \equiv 2n \pm \frac{|m|}{a}. \tag{3.12}$$

Then, eq. (3.10) becomes a hypergeometric equation

$$\left[v(1-v)\partial_v^2 + \left(c_1 - (a_1 + b_1 + 1)v\right)\partial_v - a_1 b_1 \right] f_1(v) = 0, \tag{3.13}$$

where

$$\begin{aligned} a_1 &= \frac{1}{2}(\lambda + l - n - \omega_1), \\ b_1 &= \frac{1}{2}(\lambda + l - n + \omega_1), \\ c_1 &= l + n. \end{aligned} \tag{3.14}$$

The solution which gives $B_{a_1 \cdots a_n}$ regular at $\rho = 0$ is a hypergeometric function

$$\begin{aligned} f_1(v) &= {}_2F_1(a_1, b_1, c_1; v) \\ &= \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(b_1)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\Gamma(b_1+n)}{\Gamma(c_1+n)} \frac{v^n}{n!}. \end{aligned} \tag{3.15}$$

The equation for R_4 can be solved similarly. We put

$$R_4(t,\rho) = \bar{N}_4(t) v^{\frac{1}{2}(l-n+1)}(1-v)^{\frac{1}{2}(\lambda+1)} f_4(v), \tag{3.16}$$

where

$$\bar{N}_4(t) = N_4 e^{-i\omega_4 t} + \tilde{N}_4 e^{i\omega_4 t} \tag{3.17}$$

and λ satisfies the second equation in (3.11). The longitudinal part of the second equation in (3.3) gives a hypergeometric equation on f_4. The solution which gives $B_{\rho a_2 \cdots a_n}$ regular at $\rho = 0$ is

$$f_4(v) = {}_2F_1(a_4, b_4, c_4; v), \tag{3.18}$$

where

$$\begin{aligned} a_4 &= \frac{1}{2}(\lambda + l - n - \omega_4), \\ b_4 &= \frac{1}{2}(\lambda + l - n + \omega_4), \\ c_4 &= l + n. \end{aligned} \tag{3.19}$$

The equations for R_2 and R_3 are slightly more complicated since they are coupled equations. The longitudinal part of the first equation and the transverse part of the second equation in (3.3) become

$$\begin{aligned} \mathcal{L}R_2 + \frac{2n}{v}R_3 &= 0, \\ \mathcal{L}R_3 + \frac{2(l+n-1)^2}{nv}R_2 &= 0, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \mathcal{L} = &-\partial_t^2 + 4v(1-v)\partial_v^2 + 4(2n-v)\partial_v \\ &- \frac{l(l+2n-2)-(n-1)(3n+1)}{v} + \frac{4n^2 - \left(\frac{m}{a}\right)^2}{1-v}. \end{aligned} \tag{3.21}$$

These equations can be diagonalized by defining new functions $\hat{R}_2$ and $\hat{R}_3$ as

$$\begin{aligned} \hat{R}_2 &= -(l+n-1)R_2 + nR_3, \\ \hat{R}_3 &= (l+n-1)R_2 + nR_3. \end{aligned} \tag{3.22}$$

Eq. (3.20) then becomes

$$
\begin{aligned}
\left[\mathcal{L} - \frac{2}{v}(l+n-1)\right] \hat{R}_2 &= 0, \\
\left[\mathcal{L} + \frac{2}{v}(l+n-1)\right] \hat{R}_3 &= 0.
\end{aligned}
\tag{3.23}
$$

Putting

$$
\begin{aligned}
\hat{R}_2(t,\rho) &= \bar{N}_2(t) v^{\frac{1}{2}(l-n+1)} (1-v)^{\frac{1}{2}\lambda} f_2(v), \\
\hat{R}_3(t,\rho) &= \bar{N}_3(t) v^{\frac{1}{2}(l-n-1)} (1-v)^{\frac{1}{2}\lambda} f_3(v),
\end{aligned}
\tag{3.24}
$$

where

$$
\begin{aligned}
\bar{N}_2(t) &= N_2 e^{-i\omega_2 t} + \tilde{N}_2 e^{i\omega_2 t}, \\
\bar{N}_3(t) &= N_3 e^{-i\omega_3 t} + \tilde{N}_3 e^{i\omega_3 t},
\end{aligned}
\tag{3.25}
$$

these equations become hypergeometric equations on f_2 and f_3. The solutions which give $B_{a_1\cdots a_n}$ and $B_{\rho a_2\cdots a_n}$ regular at $\rho = 0$ are

$$
\begin{aligned}
f_2(v) &= {}_2F_1(a_2, b_2, c_2; v), \\
f_3(v) &= {}_2F_1(a_3, b_3, c_3; v),
\end{aligned}
\tag{3.26}
$$

where

$$
\begin{aligned}
a_2 &= \frac{1}{2}(\lambda + l - n + 1 - \omega_2), \\
b_2 &= \frac{1}{2}(\lambda + l - n + 1 + \omega_2), \\
c_2 &= l + n + 1, \\
a_3 &= \frac{1}{2}(\lambda + l - n - 1 - \omega_3), \\
b_3 &= \frac{1}{2}(\lambda + l - n - 1 + \omega_3), \\
c_3 &= l + n - 1.
\end{aligned}
\tag{3.27}
$$

Thus we have obtained the general solution of the second order equation (3.2). We now consider the first order equation (2.5). Substituting eq. (3.5) into eq. (2.5) and using eq. (A.11) and the second order equation (3.2) we find that eq. (2.5) is satisfied if the functions R's satisfy

$$
\begin{aligned}
\frac{m}{a} R_4 &= (-1)^{\frac{1}{2}n(n-1)} \frac{n-1}{l+n} \sin\rho \cos\rho \, \partial_t R_1, \\
\frac{m}{a} R_3 &= (-1)^{\frac{1}{2}n(n+1)+\frac{1}{2}} \frac{l+n-1}{n} \left(\partial_t R_2 + \frac{n}{\sin^2\rho} R_5 \right).
\end{aligned}
\tag{3.28}
$$

These equations are satisfied if $\omega_1=\omega_4$, $\omega_2=\omega_3$ and $\bar{N}_i$ satisfy

$$\begin{aligned}\frac{d}{dt}\bar{N}_1 &= (-1)^{\frac{1}{2}n(n-1)}\frac{l+n}{n-1}\frac{m}{a}\bar{N}_4,\\ \frac{d}{dt}\bar{N}_3 &= (-1)^{\frac{1}{2}n(n+1)+\frac{1}{2}}\frac{a}{m}\left[\frac{1}{2}\Big((l+n-1)^2-\Big(\frac{m}{a}\Big)^2-\omega_2^2\Big)\bar{N}_3\right.\\ &\qquad \left.-2(l+n)(l+n-1)\bar{N}_2\right].\end{aligned} \tag{3.29}$$

By these relations $\bar{N}_4$ and $\bar{N}_3$ are related to $\bar{N}_1$ and $\bar{N}_2$ respectively and independent degrees of freedom are reduced from four to two.

4. Conservation of the energy

We first consider the conservation of the energy as a condition for the stability. From eq. (2.9) the time derivative of the energy (2.10) is given by

$$\frac{d}{dt}E=\int d^{d-2}\theta\,\sqrt{-g}g^{\rho\rho}T_{\rho t}\Big|_{\rho=\frac{\pi}{2}}, \tag{4.1}$$

where the integral is over S^{d-2}. $T_{\rho t}$ can be written as

$$\begin{aligned} n!T_{\rho t} &= 2nm(B^*_{(\rho}B_{t)})+\frac{2\alpha}{a}\left(\partial_\rho-\tan\rho\right)\partial_t(B^*B)\\ &\quad-\frac{\beta}{a\sin\rho\cos\rho}\partial_t\left(B^*_{a_1\cdots a_n}B^{a_1\cdots a_n}\right)\\ &\quad+\frac{\beta}{a}\left(\partial_\rho-\tan\rho-\frac{n-1}{\sin\rho\cos\rho}\right)\partial_t\left(B^*_{ta_2\cdots a_n}B^{ta_2\cdots a_n}\right)\\ &\quad+\frac{\beta}{a}\left(\partial_\rho-\tan\rho+\frac{n}{\sin\rho\cos\rho}\right)\partial_t\left(B^*_{\rho a_2\cdots a_n}B^{\rho a_2\cdots a_n}\right)\\ &\quad+2(n-1)\frac{\beta}{a}\left(\partial_\rho-\tan\rho+\frac{n+1}{2\sin\rho\cos\rho}\right)\partial_t(B^*_{t\rho}B^{t\rho})\\ &\quad-\frac{4\gamma}{a\sin\rho\cos\rho}\partial_t\left(B^*_{ta_2\cdots a_n}B^{ta_2\cdots a_n}\right)+4\gamma a(B^*_{(\rho}B_{t)})\\ &\quad+\frac{4\gamma}{a}\left(\partial_\rho-\tan\rho+\frac{n+1}{\sin\rho\cos\rho}\right)\partial_t(B^*_{t\rho}B^{t\rho})+\nabla_a(\cdots)^a. \end{aligned} \tag{4.2}$$

Here, $\nabla_a(\cdots)^a$ represents total derivative terms on S^{d-2}, which vanish in the integral (4.1). Substituting eq. (3.5) into eq. (4.1) it is divided into three independent parts

$$\frac{d}{dt}E = \frac{1}{n!}\int d\Omega \left[\dot{E}_1 \left|Y^{(l)}_{a_1\cdots a_n}\right|^2 + \dot{E}_2 \left|Y^{(l)}_{a_2\cdots a_n}\right|^2 + \dot{E}_3 \left|Y^{(l)}_{a_3\cdots a_n}\right|^2\right]\Bigg|_{\rho=\frac{\pi}{2}}, \tag{4.3}$$

where $d\Omega = d^{d-2}\theta\sqrt{h}$ is the volume element of S^{d-2} and

$$\left|Y^{(l)}_{a_1\cdots a_n}\right|^2 = h^{a_1b_1}\cdots h^{a_nb_n}Y^{(l)*}_{a_1\cdots a_n}Y^{(l)}_{b_1\cdots b_n}. \tag{4.4}$$

$\dot{E}_1$, $\dot{E}_2$ and $\dot{E}_3$ depend only on R_1, (R_2, R_3) and R_4 respectively.

To evaluate the right hand side of eq. (4.1) we need boundary behaviors of the functions R_i for $\rho \to \frac{\pi}{2}$. They can be obtained from the behavior of the hypergeometric function ${}_2F_1(a,b,c;v)$ for $v \to 1$. When $\lambda - 2n$ is not an integer, we find that near the boundary R's behave as

$$\begin{aligned}
R_1(\rho,t) &\sim \bar{N}_1(t)\Big[A_1(\cos\rho)^{\lambda}\left(1+\mathcal{O}(\cos^2\rho)\right) + B_1(\cos\rho)^{-\lambda+4n}\left(1+\mathcal{O}(\cos^2\rho)\right)\Big],\\
\hat{R}_2(\rho,t) &\sim \bar{N}_2(t)\Big[A_2(\cos\rho)^{\lambda}\left(1+\mathcal{O}(\cos^2\rho)\right) + B_2(\cos\rho)^{-\lambda+4n}\left(1+\mathcal{O}(\cos^2\rho)\right)\Big],\\
\hat{R}_3(\rho,t) &\sim \bar{N}_3(t)\Big[A_3(\cos\rho)^{\lambda}\left(1+\mathcal{O}(\cos^2\rho)\right) + B_3(\cos\rho)^{-\lambda+4n}\left(1+\mathcal{O}(\cos^2\rho)\right)\Big],\\
R_4(\rho,t) &\sim \bar{N}_4(t)\Big[A_4(\cos\rho)^{\lambda+1}\left(1+\mathcal{O}(\cos^2\rho)\right) + B_4(\cos\rho)^{-\lambda+4n+1}\left(1+\mathcal{O}(\cos^2\rho)\right)\Big],
\end{aligned} \tag{4.5}$$

where

$$A_i = \frac{\Gamma(c_i)\Gamma(c_i-a_i-b_i)}{\Gamma(c_i-a_i)\Gamma(c_i-b_i)}, \quad B_i = \frac{\Gamma(c_i)\Gamma(a_i+b_i-c_i)}{\Gamma(a_i)\Gamma(b_i)} \tag{4.6}$$

with a_i, b_i and c_i given in eqs. (3.14), (3.19) and (3.27). We see that the value of λ determines boundary behaviors of the solutions. The case in which $\lambda - 2n$ is an integer is discussed at the end of this section.

Let us first consider $\dot{E}_1$. We obtain

$$\begin{aligned}
\dot{E}_1 &= \left(\frac{\tan\rho}{a}\right)^{4n-1}\left[\frac{2\alpha}{a}\left(\partial_\rho - \tan\rho\right) + \frac{4n\alpha-\beta}{a\sin\rho\cos\rho}\right]\partial_t|R_1|^2\\
&= \frac{1}{a^{4n}}\partial_t\left|\bar{N}_1\right|^2\Big[-A_1^2\Big(2(2\lambda-2n+1)\alpha+\beta\Big)(\cos\rho)^{2\lambda-4n}\\
&\quad -2A_1B_1\Big(2(2n+1)\alpha+\beta\Big)
\end{aligned}$$

$$
+B_1^2\Big(2(2\lambda-6n-1)\alpha-\beta\Big)(\cos\rho)^{-2\lambda+4n}
$$

$$
+C(\cos\rho)^{2\lambda-4n+2}+D(\cos\rho)^2+E(\cos\rho)^{-2\lambda+4n+2}\Bigg]\Bigg|_{\rho=\frac{\pi}{2}}. \tag{4.7}
$$

The last three terms represent higher order terms than the first three respectively. When we choose $\lambda=\lambda_+$, the first term automatically vanishes. For the second and the third terms to vanish we have to require either (i) $B_1=0$ or (ii) $A_1=0$, $2(2\lambda-6n-1)\alpha-\beta=0$. On the other hand, when we choose $\lambda=\lambda_-$, the third term vanishes automatically and we have to require either (iii) $A_1=0$ or (iv) $B_1=0$, $2(2\lambda-2n+1)\alpha+\beta=0$. It can be shown that the conditions (i) and (ii) are equivalent to (iii) and (iv) respectively. It is enough to consider two cases (i) and (iv) and set $B_1=0$. For $B_1=0$ only C term survives among the higher order terms. In the case (i) it vanishes automatically. In the case (iv) we further need to require $2\lambda_--4n+2>0$, i.e., $|m|<a$.

The conditions for $\dot{E}_2$ and $\dot{E}_3$ to vanish can be obtained in a similar way. It first requires $B_2=B_3=B_4=0$. The remaining terms are

$$
\begin{aligned}
\dot{E}_2 =& -\frac{1}{2na^{4n}}\left[2(2\lambda-2n+1)\alpha+\beta\right](\cos\rho)^{2\lambda-4n}\partial_t|\bar{N}_2A_2-\bar{N}_3A_3|^2\\
&+\frac{1}{8n^2\omega_2^4a^{4n}}\left[2n(2\lambda-2n+1)\alpha+(2\lambda-n)\beta+4\gamma\right](\cos\rho)^{2\lambda-4n}\\
&\times\partial_t|(\lambda-l-3n+1)\dot{\bar{N}}_2A_2+(\lambda+l-n-1)\dot{\bar{N}}_3A_3|^2\\
&+\frac{1}{4n^2\omega_2^2a^{4n}}\left(2\gamma+\frac{nm}{a}\right)\left(\bar{N}_2A_2+\bar{N}_3A_3\right)^*\\
&\times\Big[(\lambda-l-3n+1)\dot{\bar{N}}_2A_2+(\lambda+l-n-1)\dot{\bar{N}}_3A_3\Big]+\text{c.c.},\\
\dot{E}_3 =& -\frac{1}{\omega_1^2a^{4n}}\partial_t\left|\bar{N}_4\right|^2A_4^2(\cos\rho)^{2\lambda-4n}\frac{1}{n-1}(l+n)(l+n-2)(\lambda-2n)\\
&\times\Big[2n(2\lambda-2n+1)(\lambda-2n)\alpha+(2\lambda-n)(\lambda-2n)\beta\\
&+\left(4(\lambda-2n)-2\right)\gamma-\frac{nm}{a}\Big]\Big|_{\rho=\frac{\pi}{2}}.
\end{aligned} \tag{4.8}
$$

For $\lambda=\lambda_+$ all these terms vanish automatically. For $\lambda=\lambda_-$ we have to require the coefficients to vanish, which fixes the parameters α, β and γ.

To summarize, the conservation of the energy leads to one of the two possibilities

$$
\begin{aligned}
&\text{(I)}\quad \lambda=\lambda_+,\quad B_i=0,\\
&\text{(II)}\quad \lambda=\lambda_-,\quad B_i=0,\quad |m|<a.
\end{aligned} \tag{4.9}
$$

In the case (II) the coefficients of the improvement terms must be chosen as

$$\begin{aligned}
\alpha &= -\frac{n}{2(\lambda-n)(2\lambda-2n+1)}\frac{m}{a}, \\
\beta &= \frac{n}{\lambda-n}\frac{m}{a}, \\
\gamma &= -\frac{n}{2}\frac{m}{a}.
\end{aligned} \tag{4.10}$$

The conditions $B_i = 0$ require $a_i = 0$ or $b_i = 0$, which lead to the quantization of ω_i

$$\begin{aligned}
\omega_1 &= \pm(2k_1+\lambda+l-n), \\
\omega_2 &= \pm(2k_2+\lambda+l-n+1), \\
\omega_3 &= \pm(2k_3+\lambda+l-n-1), \\
\omega_4 &= \pm(2k_4+\lambda+l-n),
\end{aligned} \tag{4.11}$$

where k_i are non-negative integers. Then, the hypergeometric functions in eqs. (3.15), (3.18), (3.26) can be expressed by the Jacobi polynomials as in the scalar field theory [8].

The above analysis does not immediately apply to the case $\lambda - 2n = N$ for an integer N. Let us consider the $N > 0$ case. (The $N < 0$ case is equivalent to the $N > 0$ case.) This occurs only when we choose $\lambda = \lambda_+$. In such a case the coefficients A_i in eq. (4.6) are divergent for generic values of ω_i since $c_i - a_i - b_i = -N$. To make A_i finite we have to choose ω_i such that $c_i - a_i = -k'_i$ or $c_i - b_i = -k'_i$ for non-negative integers k'_i. The conservation of the energy requires $B_i = 0$ as above, which restrict the values of k'_i to $k'_i = N, N+1, \cdots$. Redefining $k_i = k'_i - N = 0, 1, \cdots$ we recover the values of ω_i in eq. (4.11). Therefore, the results obtained for non-integer $\lambda - 2n$ is also valid for integer $\lambda - 2n$.

5. Positivity of the energy

We next consider the second condition of stability, i.e., the positivity of the energy. The integrand of the energy (2.10) is

$$\begin{aligned}
-\sqrt{-g}T^t{}_t &= 2m\sqrt{h}\left(\frac{\tan\rho}{a}\right)^{2n-1}\left[n(B_t^*B_t) - \frac{1}{2}g_{tt}(B^*B)\right] \\
&\quad +\partial_\rho\Bigg[\sqrt{h}\left(\frac{\tan\rho}{a}\right)^{2n-1}\frac{1}{a}\Big\{(\partial_\rho - \tan\rho)\left[2\alpha(B^*B) + \beta(B_t^*B^t)\right]
\end{aligned}$$

$$
+\left(\partial_\rho - \tan\rho + \frac{2n-1}{\sin\rho\cos\rho}\right)\left[\beta(B^*_\rho B^\rho) + 4\gamma(B^*_{t\rho}B^{t\rho})\right]
$$

$$
-\frac{1}{\sin\rho\cos\rho}\left[\beta(B^*_a B^a) + 4\gamma(B^*_{ta}B^{ta})\right]\Bigg\}\Bigg] + \sqrt{h}\nabla_a(\cdots)^a. \qquad (5.1)
$$

We see that the contributions from the α, β and γ terms in the action (2.6) are total derivative. Substituting eq. (3.5) into eq. (5.1) it is divided into three independent parts

$$
E = \frac{1}{n!}\int d\Omega\left[E_1\left|Y^{(l)}_{a_1\cdots a_n}\right|^2 + E_2\left|Y^{(l)}_{a_2\cdots a_n}\right|^2 + E_3\left|Y^{(l)}_{a_3\cdots a_n}\right|^2\right]. \qquad (5.2)
$$

E_1, E_2 and E_3 depend only on R_1, (R_2, R_3) and R_4 respectively.

E_1 is evaluated as

$$
E_1 = m\int d\rho\left(\frac{\tan\rho}{a}\right)^{4n+1}\frac{1}{\sin^2\rho}\left|R_1\right|^2 + \Delta E_1, \qquad (5.3)
$$

where ΔE_1 is total derivative terms

$$
\begin{aligned}
\Delta E_1 &= \int d\rho\partial_\rho\left[\left(\frac{\tan\rho}{a}\right)^{4n-1}\frac{1}{a}\left\{2\alpha\left(\partial_\rho - \tan\rho + \frac{2n}{\sin\rho\cos\rho}\right) - \frac{\beta}{\sin\rho\cos\rho}\right\}\left|R_1\right|^2\right] \\
&= \left[-\frac{1}{a^{4n}}\left[2(2\lambda-2n+1)\alpha+\beta\right]\left|\bar{N}_1\right|^2 A_1^2(\cos\rho)^{2\lambda-4n} + \mathcal{O}((\cos\rho)^{2\lambda-4n+2})\right]\Bigg|_{\rho=\frac{\pi}{2}}.
\end{aligned} \qquad (5.4)
$$

The integral of the bulk term in eq. (5.3) is convergent for $\lambda = \lambda_+$ as seen from the boundary behavior of R_1 in eq. (4.5). However, it is divergent for $\lambda = \lambda_-$. Therefore, the choice $\lambda = \lambda_-$ is not allowed and only $\lambda = \lambda_+$ is possible. In the case of scalar fields discussed in refs. [7, 8] the mass term in the energy is also divergent but it is canceled by a divergent contribution from the kinetic term. Both of $\lambda = \lambda_+$ and $\lambda = \lambda_-$ are possible in the scalar field theories. In the present theory there is no kinetic term in the energy since the kinetic term of the action is a topological term. For $\lambda = \lambda_+$ the bulk term in eq. (5.3) is obviously positive definite when $m > 0$. The boundary term ΔE_1 vanishes since it has a positive power of $\cos\rho$.

Similarly, E_2 and E_3 are given by

$$
\begin{aligned}
E_2 = m\int d\rho\left(\frac{\tan\rho}{a}\right)^{4n+1}\frac{1}{\sin^2\rho}\Bigg[\frac{1}{n}(l+n-1)^2\left|R_2\right|^2 \\
+n\left(\cos^2\rho\left|R_3\right|^2 + \frac{1}{\sin^2\rho}\left|R_5\right|^2\right)\Bigg] + \Delta E_2,
\end{aligned}
$$

$$
\begin{aligned}
E_3 = m \int d\rho \left(\frac{\tan\rho}{a}\right)^{4n+1} \frac{1}{\sin^4\rho} \Big[& n(n-1)\frac{1}{\sin^2\rho} \left|R_7\right|^2 \\
& + \frac{n}{n-1}(l+n)(l+n-2)\left(\left|R_4\right|^2 + \left|R_6\right|^2\right)\Big] + \Delta E_3,
\end{aligned} \tag{5.5}
$$

where the boundary terms are

$$
\begin{aligned}
\Delta E_2 = & -\frac{1}{4na^{4n}}\Big[[2(2\lambda-2n+1)\alpha+\beta]\left|\bar{N}_2 A_2 - \bar{N}_3 A_3\right|^2 (\cos\rho)^{2\lambda-4n} \\
& - \frac{1}{n\omega_2^4}\left[2n(2\lambda-2n+1)\alpha + (2\lambda-n)\beta + 4\gamma\right] \\
& \times \left|(\lambda-l-3n+1)\dot{\bar{N}}_2 A_2 + (\lambda+l-n-1)\dot{\bar{N}}_3 A_3\right|^2 (\cos\rho)^{2\lambda-4n} \\
& + \mathcal{O}((\cos\rho)^{2\lambda-4n+2})\Big]\Big|_{\rho=\frac{\pi}{2}}, \\
\Delta E_3 = & \Big[\frac{(l+n)(l+n-2)(\lambda-2n)^2}{(n-1)\omega_4^4 a^{4n}}\left[2n(2\lambda-2n+1)\alpha + (2\lambda-n)\beta + 4\gamma\right] \\
& \times \left|\dot{\bar{N}}_4\right|^2 A_4^2 (\cos\rho)^{2\lambda-4n} + \mathcal{O}((\cos\rho)^{2\lambda-4n+2})\Big]\Big|_{\rho=\frac{\pi}{2}}.
\end{aligned} \tag{5.6}
$$

For $\lambda = \lambda_+$ the bulk integrals in eq. (5.5) are convergent and are positive definite when $m > 0$. The boundary terms ΔE_2 and ΔE_3 vanish in the same way as ΔE_1.

It is thus proved that the energy is well-defined and positive definite for the case (I) in eq. (4.9) if the additional condition $m > 0$ is satisfied. The case (II) in eq. (4.9) is not allowed since the energy is divergent. Therefore, there exists only one complete set of solutions corresponding to the case (I) for antisymmetric tensor fields with the Chern-Simons type action.

6. Scalar product

Finally we consider a scalar product of the fields. The scalar product of two solutions $B_{1\mu_1\cdots\mu_n}$ and $B_{2\mu_1\cdots\mu_n}$ is defined as

$$
(B_1, B_2) = \int d^{d-1}x\sqrt{-g}J^t \tag{6.1}
$$

by using the time component of the conserved current

$$
J^\mu = -i\left(B^*_{1\nu_1\cdots\nu_n}F_2^{\mu\nu_1\cdots\nu_n} - F_1^{*\mu\nu_1\cdots\nu_n}B_{2\nu_1\cdots\nu_n}\right). \tag{6.2}
$$

The integral in eq. (6.1) is convergent since $\sqrt{-g}J^t = \mathcal{O}\left((\cos\rho)^{2\lambda-4n+1}\right)$ for $\rho \to \frac{\pi}{2}$ and $2\lambda - 4n + 1 > -1$ for $\lambda = \lambda_+$ and $m > 0$. The scalar product (6.1) is also conserved

$$\frac{d}{dt}(B_1, B_2) = -\int d^{d-2}\theta \sqrt{-g}J^\rho\Big|_{\rho=\frac{\pi}{2}} = 0 \tag{6.3}$$

since $\sqrt{-g}J^\rho|_{\rho=\frac{\pi}{2}} = \mathcal{O}((\cos\rho)^{2\lambda-4n+2})|_{\rho=\frac{\pi}{2}} = 0$ for $\lambda = \lambda_+$ and $m > 0$. Therefore we have a well-defined conserved scalar product.

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Appendix: Spherical harmonics on S^{d-2}

In this Appendix we sketch how to construct spherical harmonics for antisymmetric tensor fields on S^{d-2} following the approach in refs. [16, 17, 18]. See also ref. [19]. We embed a $(d-2)$-dimensional unit sphere S^{d-2} in $(d-1)$-dimensional Euclidean space $\mathbf{R}^{d-1}$ with the Cartesian coordinates x^i $(i = 1, 2, \cdots, d-1)$. The metric of $\mathbf{R}^{d-1}$ is given by

$$\begin{aligned} ds^2 &= \delta_{ij}dx^i dx^j \\ &= dr^2 + r^2 h_{ab}(\theta)d\theta^a d\theta^b, \end{aligned} \tag{A.1}$$

where $r = \sqrt{\delta_{ij}x^i x^j}$ is a radial coordinate and θ^a $(a = 1, 2, \cdots, d-2)$ are angular coordinates parametrizing the unit sphere S^{d-2} with the metric h_{ab}.

Let us consider antisymmetric tensors $T_{i_1\cdots i_n}$ in $\mathbf{R}^{d-1}$ which satisfy

$$n^{i_1}T_{i_1\cdots i_n} = 0, \qquad \partial^{i_1}T_{i_1\cdots i_n} = 0, \tag{A.2}$$

where $n^i(\theta) = r^{-1}x^i$ is a unit vector normal to the sphere. In the polar coordinates $(r,\, \theta^a)$ these conditions become

$$T_{ra_2\cdots a_n} = 0, \qquad \nabla^{a_1}T_{a_1\cdots a_n} = 0, \tag{A.3}$$

where ∇_a is the covariant derivative on S^{d-2}. Therefore, restricting to the unit sphere they represent transverse tensors on the sphere. The relation between $T_{i_1\cdots i_n}$ and $T_{a_1\cdots a_n}$ is

$$T_{a_1\cdots a_n} = r^n \frac{\partial n^{i_1}}{\partial \theta^{a_1}} \cdots \frac{\partial n^{i_n}}{\partial \theta^{a_n}} T_{i_1\cdots i_n}. \tag{A.4}$$

In the Cartesian coordinates spherical harmonics for such transverse antisymmetric tensors for $n \geq 1$ are given by

$$Y^{(l)}_{i_1\cdots i_n}(\theta) = r^{-l} C_{[i_1\cdots i_n j_1](j_2\cdots j_l)} x^{j_1} x^{j_2} \cdots x^{j_l}. \tag{A.5}$$

Here, $C_{[i_1\cdots i_n j_1](j_2\cdots j_l)}$ is a constant coefficient, which is antisymmetric in $i_1, \cdots, i_n, j_1$ and symmetric in $j_2, \cdots, j_l$, and is traceless with respect to any pair of the indices. Spherical harmonics for $n = 0$ are given by

$$Y^{(l)}(\theta) = r^{-l} C_{(j_1\cdots j_l)} x^{j_1} \cdots x^{j_l}, \tag{A.6}$$

where $C_{(j_1\cdots j_l)}$ is symmetric in $j_1, \cdots, j_l$ and is traceless with respect to any pair of the indices. Note that l takes values $l = 0, 1, 2, \cdots$ for $n = 0$ and $l = 1, 2, 3, \cdots$ for $n \geq 1$. One can easily check that these tensors indeed satisfy the conditions (A.2). Applying the Laplacian $\Delta_{d-1} = \delta^{ij}\partial_i\partial_j$ in $\mathbf{R}^{d-1}$ we find

$$\Delta_{d-1} Y^{(l)}_{i_1\cdots i_n} = -\frac{l(l+d-3)}{r^2} Y^{(l)}_{i_1\cdots i_n}. \tag{A.7}$$

On the other hand, in the polar coordinates we have

$$\begin{aligned}
\Delta_{d-1} Y^{(l)}_{a_1\cdots a_n} &= \left[\frac{1}{r^2} \nabla^a \nabla_a + \left(\partial_r + \frac{d-n-2}{r} \right) \left(\partial_r - \frac{n}{r} \right) - \frac{n}{r^2} \right] Y^{(l)}_{a_1\cdots a_n} \\
&= \frac{1}{r^2} \left(\nabla^a \nabla_a - n \right) Y^{(l)}_{a_1\cdots a_n}.
\end{aligned} \tag{A.8}$$

In the last line we have used the fact that $Y^{(l)}_{a_1\cdots a_n}$, which is related to eqs. (A.5), (A.6) by the relation (A.4), has r dependence r^n. Comparing eqs. (A.7) and (A.8) we obtain eigenvalues of the Laplacian on S^{d-2}

$$\nabla^a \nabla_a Y^{(l)}_{a_1\cdots a_n} = -[l(l+d-3) - n] Y^{(l)}_{a_1\cdots a_n}. \tag{A.9}$$

Useful identities, which can be easily derived from eq. (A.9) are

$$\begin{aligned}
\nabla^a \nabla_a \partial_{[a_1} Y^{(l)}_{a_2\cdots a_n]} &= -[l(l+d-3) - d + n + 2] \partial_{[a_1} Y^{(l)}_{a_1\cdots a_n]}, \\
\nabla^{a_1} \nabla_{[a_1} Y^{(l)}_{a_2\cdots a_n]} &= -\frac{1}{n}(l+n-1)(l+d-n-2) Y^{(l)}_{a_2\cdots a_n}.
\end{aligned} \tag{A.10}$$

There is a duality relation between $Y^{(l)}_{a_1\cdots a_m}$ and $Y^{(l)}_{a_1\cdots a_{d-m-3}}$, which we use in sect. 3. By appropriately choosing the coefficients C's in eqs. (A.5) and (A.6) the relation can be written as

$$Y^{(l)a_1\cdots a_m} = \frac{(-1)^{\frac{1}{2}(m+1)(d-m-2)}}{(l+m)(d-m-3)!}\frac{1}{\sqrt{h}}\epsilon^{a_1\cdots a_{2n+1}}\partial_{a_{m+1}}Y^{(l)}_{a_{m+2}\cdots a_{d-2}}. \tag{A.11}$$

One can easily check that both hand sides of this equation are transverse and have the same eigenvalue of $\nabla_a\nabla^a$. The normalization factor on the right hand side is determined by repeated applications of this relation.

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