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# On uniform dimensions of uniform spaces

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**Abstract**

We introduce a notion of a uniform dimension, denoted by u-dim, of a uniform space. We construct a uniform space $X$ such that u-dim $X = n$ and $\dim X = m$ for $n, m \in \mathbb{N} \cup \{0\}$. We also prove that the identity mapping of a uniform space $X$ has the pseudo orbit tracing property if and only if u-dim $X \leq 0$.

## 1 Introduction

In this paper we study a uniform dimension u-dim $X$ of a uniform space $X$. All spaces are assumed to be normal unless otherwise stated.

In section 2 we introduce a notion of a uniform dimension of a uniform space. We prove the subspace theorem and the completion theorem for uniform dimension.

In section 3 we consider relationships between uniform dimensions and topological dimensions. We give a uniform space $X$ such that u-dim $X = n$ and $\dim X = m$ for every $n, m \in \mathbb{N} \cup \{0\}$.

Fujii (see [2] Theorem 2.3.2 or [1] Remark 4.30) pointed out that the identity mapping of a compact metric space $X$ has the pseudo orbit tracing property if and only if $X$ is zero-dimensional. In section 4 we generalize this result. Namely, we prove that the identity mapping of a uniform space $X$ has the pseudo orbit tracing property if and only if u-dim $X \leq 0$.

For standard results and notion in General Topology and Dimension Theory we refer to [3] and [4].

## 2 Definition of uniform dimensions

We begin with basic symbols.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a space $X$, $x \in X$ and $Y$ a subset of $X$. We set

$$\begin{aligned} \mathrm{St}(x, \mathcal{A}) &= \bigcup \{A \in \mathcal{A} : x \in A\} \\ \mathrm{St}(Y, \mathcal{A}) &= \bigcup \{A \in \mathcal{A} : Y \cap A \neq \emptyset\} \\ \mathcal{A}^* &= \{\mathrm{St}(A, \mathcal{A}) : A \in \mathcal{A}\} \end{aligned}$$

$$\begin{aligned}\mathcal{A}|Y &= \{A \cap Y : A \in \mathcal{A}\}\\ \mathcal{A} \wedge \mathcal{B} &= \{A \cap B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\} \text{ and}\\ \bigcup \mathcal{A} &= \bigcup \{A : A \in \mathcal{A}\}\end{aligned}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be covers of a space $X$ and $x, y \in X$. We say that $\mathcal{A}$ *refines* $\mathcal{B}$, in symbol $\mathcal{A} < \mathcal{B}$, if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$. If $x \in \mathrm{St}(y, \mathcal{A})$, then we write $d(x, y) < \mathcal{A}$.

By a *uniformity* $\Phi$ on a space $X$, we mean a collection of open covers of $X$ satisfying:
(1) if $\mathcal{U} \in \Phi$ and $\mathcal{V}$ is an open cover of $X$ with $\mathcal{U} < \mathcal{V}$, then $\mathcal{V} \in \Phi$,
(2) if $\mathcal{U}, \mathcal{V} \in \Phi$, then $\mathcal{U} \wedge \mathcal{V} \in \Phi$,
(3) for every $\mathcal{U} \in \Phi$ there exists $\mathcal{V} \in \Phi$ such that $\mathcal{V}^* < \mathcal{U}$,
(4) for every $x \in X$ the collection $\{\mathrm{St}(x, \mathcal{U}) : \mathcal{U} \in \Phi\}$ is a neighborhood base at $x$.

A *uniform space* is a pair $(X, \Phi)$ consisting of a space $X$ and a uniformity $\Phi$ on $X$. An open cover $\mathcal{U}$ of a uniform space $(X, \Phi)$ is a *uniform cover* of $X$ if $\mathcal{U} \in \Phi$. To simplify notions, we often use the symbol $X$ instead of $(X, \Phi)$ for a uniform space.

**2.1. Definition.** To every uniform space $X$ we assign the *uniform dimension*, denoted by u-dim $X$. Let $n$ denote an integer $\geq -1$; we say that:
(1) u-dim $X \leq n$ if every uniform cover of $X$ is refined by some uniform cover of order $\leq n + 1$.
(2) u-dim $X = n$ if u-dim $X \leq n$ and the inequality u-dim $X \leq n - 1$ does not hold.
(3) u-dim $X = \infty$ if the inequality u-dim $X \leq n$ does not hold for any $n \in \mathbb{N}$.

The collection of all open covers of a space $X$ refined by a finite open cover of $X$ is a uniformity on $X$. It is obvious that u-dim $X = \dim X$ for this uniform space $X$. Every compact space $X$ has a unique uniformity on $X$, so we have u-dim $X = \dim X$

For a subset $Y$ of a uniform space $(X, \Phi)$ we set

$$\Phi|Y = \{\mathcal{U} : \mathcal{U} \text{ is an open cover of } Y \text{ such that } \mathcal{V}|Y < \mathcal{U} \text{ for some } \mathcal{V} \in \Phi\}.$$

Then the collection $\Phi|Y$ is a uniformity on $Y$. We say that $(Y, \Phi|Y)$ is a *uniform subspace* of $(X, \Phi)$.

**2.2. Theorem.** *For every uniform subspace $Y$ of a uniform space $X$ the inequality* u-dim $Y \leq$ u-dim $X$ *holds.*

**Proof.** The theorem is obvious if u-dim $X = \infty$, so that we can suppose that u-dim $X < \infty$. Suppose that u-dim $X = n$. For every uniform cover $\mathcal{U}$ of $Y$ there exists a uniform cover $\mathcal{V}$ of $X$ such that $\mathcal{V}|Y < \mathcal{U}$. Since u-dim $X = n$, we can take a uniform cover $\mathcal{W}$ of $X$ such that $\mathcal{W} < \mathcal{V}$ and ord $\mathcal{W} \leq n + 1$. Then $\mathcal{W}|Y$ is a uniform cover of $Y$ which refines $\mathcal{U}$ and has order $\leq n + 1$. This implies that u-dim $Y \leq n$, therefore u-dim $Y \leq$ u-dim $X$.

**2.3. Theorem.** *If a uniform space $X$ satisfies the condition* u-dim $X \leq 0$, *then $X$ is zero-dimensional.*

**Proof.** For every $x \in X$ and any neighborhood $U$ of $x$ there exists a uniform cover $\mathcal{U}$ of $X$ such that $\mathrm{St}(x,\mathcal{U}) \subset U$. Take a uniform cover $\mathcal{V}$ of $X$ such that $\mathcal{V} < \mathcal{U}$ and ord $\mathcal{V} \leq 1$. The set $V = \mathrm{St}(x,\mathcal{V})$ is a neighborhood of $x$ and $V = \mathrm{St}(x,\mathcal{V}) \subset \mathrm{St}(x,\mathcal{U}) \subset U$. Since $\mathcal{V}$ is an open cover of order $\leq 1$, $V$ is open-and-closed in $X$. Hence $X$ is zero-dimensional.

For a metric space $(X,d)$ let $\Phi_d$ be the collection of open covers of $X$ which is refined by $\{B(x,\varepsilon) : x \in X\}$ for some $\varepsilon > 0$, where $B(x,\varepsilon)$ is the $\varepsilon$-neighborhood of $x$ in $X$. Then $(X,\Phi_d)$ is a uniform space. We say that $\Phi_d$ is the uniformity which is *induced* by the metric $d$ on $X$.

Let $\Phi_d$ be the uniformity induced by the Euclidean metric on the real line $\mathbb{R}$. It is well-known that $\dim \mathbb{R} = 1$. We consider the uniform dimension of this space.

For every $\varepsilon > 0$ let us set

$$\mathcal{U}_\varepsilon = \{(n\varepsilon,(n+1)\varepsilon) : n \in \mathbb{Z}\} \cup \{((n-\frac{1}{2})\varepsilon,(n+\frac{1}{2})\varepsilon) : n \in \mathbb{Z}\}.$$

Since $\frac{1}{2}\varepsilon$ is a Lebesgue number of $\mathcal{U}_\varepsilon$, $\mathcal{U}_\varepsilon$ is a uniform cover of $\mathbb{R}$. For every uniform cover $\mathcal{U}$ of $\mathbb{R}$ there exists $\varepsilon > 0$ such that $\{B(x,\varepsilon) : x \in X\}$ refines $\mathcal{U}$. On the other hand, since mesh $\mathcal{U}_\varepsilon = \varepsilon$, $\mathcal{U}_\varepsilon$ refines $\{B(x,\varepsilon) : x \in X\}$. Obviously, ord $\mathcal{U}_\varepsilon = 2$. Thus we have u-dim $X \leq 1$. By Theorem 2.3, we have u-dim $X = 1$.

A uniform space $X$ is *totally bounded* if every uniform cover $\mathcal{U}$ has a finite subcover of $X$. Let $\mathcal{F}$ be a collection of subsets of a uniform space $X$. We say that $\mathcal{F}$ *contains arbitrarily small sets* if for every uniform cover $\mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \subset U$ for some $U \in \mathcal{U}$. A uniform space $X$ is *complete* if every collection $\mathcal{F}$ of closed subsets of $X$ which has the finite intersection property and contains arbitrarily small sets has non-empty intersection. It is well-known ([3], 8.3.16) that a uniform space $X$ is compact if and only if $X$ is totally bounded and complete. Every compact space $X$ has a unique uniformity. The unique uniformity on a compact space $X$ is the collection of all open covers of $X$.

For a metric space $(X,d)$ the uniform space $(X,\Phi_d)$ is totally bounded (respectively complete) if and only if the metric space $(X,d)$ is totally bounded (respectively complete) .

We denote by $(\tilde{X},\tilde{\Phi})$ the completion of a uniform space $(X,\Phi)$. The completion $(\tilde{X},\tilde{\Phi})$ is the unique uniform space which is complete and contains $(X,\Phi)$ as a dense uniform subspace.

Let $\Psi$ be a subcollection of a uniformity $\Phi$ on a uniform space $X$. We say that $\Psi$ is a *base* for $\Phi$ if for every $\mathcal{U} \in \Phi$ there exists $\mathcal{V} \in \Psi$ such that $\mathcal{V} < \mathcal{U}$. For an open subset $G$ of a uniform space $X$ we set $\gamma(G) = \tilde{X} - \mathrm{Cl}_{\tilde{X}}(X - G)$. Then it is known that $\{\gamma(\mathcal{U}) : \mathcal{U} \in \Phi\}$ is a base for $\tilde{\Phi}$, where $\gamma(\mathcal{U}) = \{\gamma(U) : U \in \mathcal{U}\}$.

It is easy to show the following Fact.

**Fact.** *Let $\mathcal{U}$ and $\mathcal{V}$ be uniform covers of a uniform space.*
If $\mathcal{U} < \mathcal{V}$, then $\gamma(\mathcal{U}) < \gamma(\mathcal{V})$.
The equality ord $\mathcal{U} =$ ord $\gamma(\mathcal{U})$ holds.

For more detailed information about uniform spaces the reader is referred to [3] and [5].

**2.4. Theorem.** *For every uniform space $X$ the equality* u-dim $X =$ u-dim $\tilde{X}$ *holds.*

**Proof.** Suppose that u-dim $\tilde{X} = n$. For every uniform cover $\mathcal{U}$ of $X$ $\gamma(\mathcal{U})$ is a uniform cover of $\tilde{X}$. Thus we can take a uniform cover $\mathcal{V}$ of $\tilde{X}$ such that $\mathcal{V} < \gamma(\mathcal{U})$ and ord $\mathcal{V} \leq n+1$. Let us set $\mathcal{W} = \mathcal{V}|X$. Then we have

$$\mathcal{W} = \mathcal{V}|X < \gamma(\mathcal{U})|X = \mathcal{U} \text{ and ord } \mathcal{W} = \text{ord } \mathcal{V}|X \leq \text{ord } \mathcal{V} \leq n+1.$$

This implies that u-dim $X \leq n$.

Conversely, suppose that u-dim $X = n$. For every uniform cover $\mathcal{U}$ of $\tilde{X}$ there exists a uniform cover $\mathcal{V}$ of $X$ such that $\gamma(\mathcal{V}) < \mathcal{U}$. Since u-dim $X = n$, $\mathcal{V}$ is refined by some uniform cover $\mathcal{W}$ of order $\leq n+1$. Then we have

$$\gamma(\mathcal{W}) < \gamma(\mathcal{V}) < \mathcal{U} \text{ and ord } \gamma(\mathcal{W}) = \text{ord } \mathcal{W} \leq n+1.$$

This implies that $\dim X \leq n$.

**2.5. Corollary.** *For every totally bounded metric space $(X, d)$ the inequality* u-dim $(X, \Phi_d) \geq \dim X$ *holds.*

**Proof.** Since $X$ is totally bounded, the completion $\tilde{X}$ is compact, therefore u-dim $\tilde{X} = \dim \tilde{X}$. By Theorem 2.4, we have u-dim $X =$ u-dim $\tilde{X}$. Since $X$ is separable and metrizable, we have $\dim X \leq \dim \tilde{X}$. Hence the inequality u-dim $(X, \Phi_d) \geq \dim X$ holds.

# 3 Uniform dimension and topological dimension

This section is concerned with relationships between the uniform dimension and the topological dimension.

**3.1. Example.** *For every $n \in \mathbb{N}$ there exists a uniform space $Y_n$ such that* u-dim $Y_n = 0$ *and* $\dim Y_n = n$.

E. Pol and R. Pol [6] constructed a hereditarily normal space $Y$ with $\dim Y = 0$ containing for every $n \in \mathbb{N}$ a perfectly normal subspace $Y_n$ such that $\dim Y_n = n$. Let $Z_n$ be the Stone-Čech compactification of $\mathrm{Cl}_Y Y_n$. Since $Z_n$ is compact, $Z_n$ has a unique uniformity. We regard $Y_n$ as a uniform subspace of $Z_n$. As the completion of $Y_n$ is uniformly isomorphic to $Z_n$, by Theorem 2.4, u-dim $Y_n =$ u-dim $Z_n = \dim Z_n = \dim \mathrm{Cl}_Y Y_n \leq \dim Y = 0$.

**3.2. Example.** *For every $n \in \mathbb{N}$ there exists a uniform space $Z_n$ such that* u-dim $Z_n = n$ *and* $\dim Z_n = 0$.

First, by using a method of A. K. Steiner and E. F. Steiner [7], we construct a compactification of $\mathbb{N}$ with n-dimensional remainder. Represent the space $\mathbb{N}$ of all positive integers as the union $\bigcup\{N_i : i \in \mathbb{N}\}$ of infinite sets, where $N_i \cap N_j = \emptyset$ whenever $i \neq j$. Take a countable dense subset $D$ of $I^n$, where $I = [0,1]$. Let $\varphi_i$ be a bijection from $N_i$ onto $D$ and let $\varphi : \mathbb{N} \to D$ be the mapping defined by $\varphi(x) = \varphi_i(x)$ for each $x \in N_i$. The space $G = \{(x, \varphi(x)) : x \in \mathbb{N}\}$ is homeomorphic to $\mathbb{N}$. Let us set $Y_n = \mathrm{Cl}_{\omega\mathbb{N} \times I^n} G$, where $\omega\mathbb{N}$ is the Alexandroff compactification of $\mathbb{N}$. Obviously, the space $Y_n$ is a compactification of $\mathbb{N}$, and the remainder

$Y_n - \mathbb{N}$ is homeomorphic to $I^n$. Thus we have $\dim Y_n = n$. We regard $\mathbb{N}$ as a uniform subspace of $Y_n$, and we denote by $Z_n$ this uniform space. By Theorem 2.4, we have u-dim $Z_n$ = u-dim $Y_n = \dim Y_n = n$.

If a uniform space $X$ is the topological sum of two spaces $Y$ and $Z$, then the equality

$$\text{u-dim } X = \max\{\text{u-dim } Y, \text{u-dim } Z\} \text{ and } \dim X = \max\{\dim Y, \dim Z\}$$

hold. Hence we can construct a uniform space $X$ such that u-dim $X = n$ and $\dim X = m$ for $n, m \in \mathbb{N} \cup \{0\}$.

## 4 Pseudo orbit tracing property

A self-mapping $f$ on a uniform space $X$ is *uniformly continuous* if $f^{-1}(\mathcal{U})$ is a uniform cover of $X$ for every uniform cover $\mathcal{U}$ of $X$, where $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$. A self-bijection $f$ on a uniform space $X$ is a *uniform isomorphism* if both $f$ and $f^{-1}$ is uniformly continuous.

Let $f$ be a uniform isomorphism on a uniform space $X$ and let $\mathcal{U}$ be a uniform cover of $X$. A sequence $\{x_n : n \in \mathbb{Z}\}$ of points of $X$ is a *$\mathcal{U}$-pseudo orbit* for $f$ if $d(f(x_n), x_{n+1}) < \mathcal{U}$ for every $n \in \mathbb{Z}$. A sequence $\{x_n : n \in \mathbb{Z}\}$ of points of $X$ is called to *be $\mathcal{U}$-traced by $x \in X$* if $d(f^n(x), x_n) < \mathcal{U}$ for every $n \in \mathbb{Z}$. A uniform isomorphism $f$ on a uniform space $X$ has *the pseudo orbit tracing property* if for every uniform cover $\mathcal{U}$ of $X$ there exists a uniform cover $\mathcal{V}$ of $X$ such that every $\mathcal{V}$-pseudo orbit for $f$ can be $\mathcal{U}$-traced by some point $x \in X$.

Fujii [2] pointed out that the identity mapping of a compact metric space $X$ has the pseudo orbit tracing property if and only if $X$ is zero-dimensional.

**4.1. Theorem.** *The identity mapping id on a uniform space $X$ has the pseudo orbit tracing property if and only if* u-dim $X \leq 0$.

**Proof.** Suppose that u-dim $X \leq 0$. For every uniform cover $\mathcal{U}$ of $X$ we take a uniform cover $\mathcal{V}$ of $X$ such that $\mathcal{V} < \mathcal{U}$ and ord $\mathcal{V} \leq 1$. It suffices to show that every $\mathcal{V}$-pseudo orbit for $id$ can be $\mathcal{U}$-traced by some point $x \in X$. To this end let $\{x_n : n \in \mathbb{Z}\}$ be a $\mathcal{V}$-pseudo orbit for $id$. Since ord $\mathcal{V} \leq 1$, it is easy to see that $\{x_n : n \in \mathbb{Z}\} \subset V$ for some $V \in \mathcal{V}$. Take $x \in X$ and $U \in \mathcal{U}$ such that $V \subset U$ and $x \in U$. Obviously, $d(id^n(x), x_n) < \mathcal{U}$. Hence $id$ has the pseudo orbit tracing property.

Conversely, assume that u-dim $X \geq 1$ . Then there exists a uniform cover $\mathcal{U}$ of $X$ such that every uniform cover $\mathcal{V}$ of $X$ which refines $\mathcal{U}$ has order $\geq 2$.

Take uniform covers $\mathcal{W}$ and $\mathcal{W}'$ of $X$ such that $\mathcal{W}^* < \mathcal{W}'$ and $\mathcal{W}'^* < \mathcal{U}$. For every uniform cover $\mathcal{V}$ of $X$ we shall construct a $\mathcal{V}$-pseudo orbit for $id$ which can not be $\mathcal{W}$-traced by any point of $X$.

Let $\mathcal{V}$ be a uniform cover of $X$. For $V, V' \in \mathcal{V}$ we write $V \sim V'$ if there exist $V_0, V_1, \cdots, V_n \in \mathcal{V}$ such that $V = V_0$, $V' = V_n$ and $V_i \cap V_{i+1} \neq \emptyset$ for each $i = 0, 1, \cdots, n-1$. Obviously, the above relation $\sim$ is an equivalence relation on $\mathcal{V}$. Let $\{\mathcal{V}_\lambda : \lambda \in \Lambda\}$ be the equivalence class with respect to $\sim$. We set $V_\lambda = \bigcup \mathcal{V}_\lambda$ for each $\lambda \in \Lambda$ and $\mathcal{V}' = \{V_\lambda : \lambda \in \Lambda\}$. Then ord$\mathcal{V}' \leq 1$. Since $(\mathcal{W}^*)^* < \mathcal{W}'^* < \mathcal{U}$, $\mathcal{V}'$ does not refine $(\mathcal{W}^*)^*$. This implies that there exists $V_\lambda \in \mathcal{V}'$ such that $V_\lambda \not\subset \mathrm{St}(\mathrm{St}(W, \mathcal{W}), \mathcal{W}^*)$ for every $W \in \mathcal{W}$. Take $x \in V_\lambda$ and $W \in \mathcal{W}$ such that $x \in W$. Since $V_\lambda \not\subset \mathrm{St}(\mathrm{St}(W, \mathcal{W}), \mathcal{W}^*)$, we can take $y \in V_\lambda - \mathrm{St}(\mathrm{St}(W, \mathcal{W}), \mathcal{W}^*)$. Since $x \sim y$, there exist $V_0, V_1, \cdots, V_n \in \mathcal{V}$ such that $x \in V_0$, $y \in V_n$ and $V_i \cap V_{i+1} \neq \emptyset$ for each $i = 0, 1, \cdots, n-1$. Take a point $x_{i+1} \in V_i \cap V_{i+1}$ for each $i = 0, 1, \cdots, n-1$. We also set $x_{-i} = x$ for every $i = 0, 1, 2, \cdots$ and $x_i = y$ for every $i = n+1, n+2, \cdots$. The sequence $\{x_i : i \in \mathbb{Z}\}$ is a $\mathcal{V}$-pseudo orbit for $id$. Assume that the sequence $\{x_i : i \in \mathbb{Z}\}$ can be $\mathcal{W}$-traced by some point $z \in X$. Since $z \in \mathrm{St}(W, \mathcal{W})$, $y \in \mathrm{St}(\mathrm{St}(W, \mathcal{W}), \mathcal{W}) \subset \mathrm{St}(\mathrm{St}(W, \mathcal{W}), \mathcal{W}^*)$. This is a contradiction.

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