線形, 非線形の波動現象と近似方程式の導出

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概 要

この論文では, 線形波動現象の性質をもとに非線形波動現象において KdV 方程式と非線形シュレディンガー方程式を導くことについて考察する. その中で, 主に, 水面波における現象について考察する. 今, 水深 h の浅い水を考える. 水面波の近似方程式の導出には 2 通りの方法がある. 1 つ目の方法は逓減摂動法である. 逓減摂動法とは, 線形近似の近傍でスケール不変な非線形発展方程式を導出する漸近摂動展開のことで, 従属変数の展開とともに独立変数の変換を行うことである. このような手法を用いると, KdV 方程式

$$\frac{\partial \zeta^{(1)}(\xi,\tau)}{\partial \tau}+\frac{3}{2}\frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial \zeta^{(1)}(\xi,\tau)}{\partial \xi}+\frac{1}{6}c_0h^2\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial \xi^3}=0$$

を求めることができる. ここで, x, t に対して新しい独立変数

$$\xi=\epsilon^{\frac{1}{2}}(x-c_0t),\quad \tau=\epsilon^{\frac{3}{2}}t$$

を導入し, $\xi(x,t), \Phi(x,z,t)$ を

$$\zeta(x,t)=\sum_{n=1}^{\infty}\epsilon^n\zeta^{(n)}(\xi,\tau),\quad \Phi(x,z,t)=\epsilon^{-\frac{1}{2}}\sum_{n=1}^{\infty}\epsilon^n\phi^{(n)}(\xi,z,\tau)$$

と展開した. 2 つ目のやり方は, 多重スケールの方法である. 多重スケールの方法は, 微小パラメーター ϵ を導入して, 独立変数 x, t をスケールの異なる多くの変数

$$x_n=\epsilon^n x,\quad t_n=\epsilon^n t\quad (x_0=x, t_0=t)\qquad (n=0,1,2\cdots)$$

に拡張する方法である. このような手法により, 非線形シュレディンガー方程式

$$i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial \tau}+\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}+\nu|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)=0$$

を求めることができる. ここで, A は複素振幅であり, $\xi\equiv x_1-v_gt_1$ の組み合わせで t_1, x_1 に依存する. また, A は $\xi_1\equiv x_2-v_gt_2, \tau\equiv t_2$ の組み合わせで t_2, x_2 に依存する. 以上より, 複雑な非線形偏微分方程式から様々な摂動展開の手法を用いることにより, KdV 方程式や非線形シュレディンガー方程式などの簡単な近似方程式が導かれることがわかった.

1　線形波動

本章では, 和達 [2, p.8-11] に沿って, 線形波動について述べる.

1.1　波動方程式

複雑な非線形偏微分方程式系が与えられた時, まず最初に行うのが線形化である. 非線形項をすべて落とし, 線形化した方程式の解の性質を調べる. 本文では波動を例に挙げ複雑な非線形偏微分方程式を求めていきたいと考えている. そのために, 非線形波動を考える準備としてまず線形波動の性質を調べる. わかりやすくするために 1 次元格子で考える.

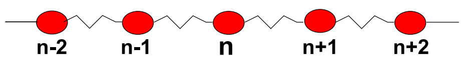


図 1: 1 次元格子

質量 m, 長さ a, バネ定数 κ, ばねの伸び Δ, バネの力 F で表す. バネの力を次のように表す.

$$F_n t = \kappa\Delta$$

このときの運動方程式は

$$m\frac{d^2y_n(t)}{dt^2} = F_n(t) = \kappa(y_{n+1}(t) - 2y_n(t) + y_{n-1}(t))\ (n = 0, \pm1, \pm2, \cdots) \tag{1}$$

となる.

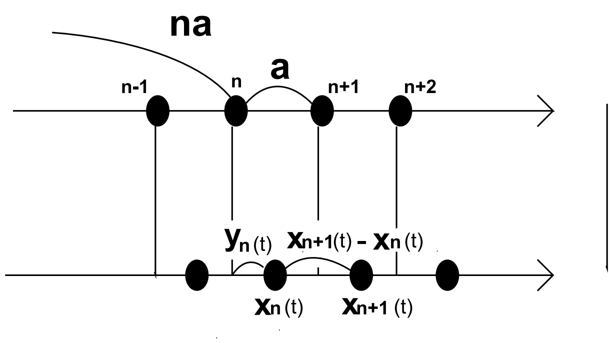


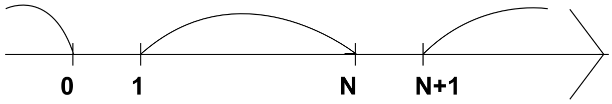
図 2: 平衡状態でのばねのずれ

＜証明＞黒い点を平衡点とする. 上の線を時刻 (0) におけるばねの状態を表し, 下の線を時刻 (t) におけるばねを表す. 時刻 t における n 番目の粒子の位置を $x_n(t)$ とし,

$$x_n(0) = na$$

とする. n 番目の粒子の平衡点からのずれを $y_n(t)$ とし,

$$y_n(t) = x_n(t) - na$$



と表す．このとき n 番目の粒子が受ける力は,

$$\begin{aligned} F_n &= \kappa(x_{n+1}(t) - x_n(t) - a) - \kappa(x_n(t) - x_{n-1}(t) - a) \\ &= \kappa(y_{n-1}(t) - 2y_n(t) + y_{n-1}(t)) \ (n = 0, \pm1, \pm2, \cdots) \end{aligned}$$

証明終了.

以上より運動方程式を満たすことがわかる．運動方程式の解を

$$y_n(t) = e^{i(kan-\omega t)} \tag{2}$$

と仮定して求めてみる．k を波数, ω を角振動数とする．(1) の式に代入すると,

$$me^{i(kan-wt)}(-\omega^2) = \kappa(e^{i(ka(n+1)-wt)} - 2e^{i(kan-wt)} + e^{i(ka(n-1)-wt)})$$

となる. 両辺を $e^{i(kan-wt)}$ で割る.

$$\begin{aligned} -m\omega^2 &= \kappa(e^{ika} - 2 + e^{-ika}) \\ &= 2\kappa(\cos(ka) - 1), \\ \omega^2 &= \frac{2\kappa}{m}(1 - \cos(ka)) \\ &= \frac{2\kappa}{m}(2\sin^2\frac{ka}{2}), \end{aligned}$$

$$\omega^2 = \frac{4\kappa}{m}(\sin^2\frac{ka}{2}) \tag{3}$$

k, ω が (3) を満たすならば, (2) は (1) の解であり, 又 $t \to -t$ としても (1) は変化しないので,

$$y_{n_2}(t) = e^{i(kan+\omega t)} \tag{4}$$

も解である．(3) のように k, ω の関係式を分散関係式という．次に周期的境界条件の時を考える．今,

$$n = N \text{ として } y_{N+1} = y_1$$

とおく．

(2) に代入する．

$$\begin{aligned} e^{i(ka(N+1)-\omega t)} &= e^{i(ka-\omega t)}, \\ e^{i(ka-\omega t)}e^{ikaN} &= e^{i(ka-\omega t)}, \end{aligned}$$

$$e^{ikaN} = 1 \Leftrightarrow kaN = 2\pi l \ (l \in Z),$$

$$k_l = \frac{2\pi l}{aN} \ (l = 1 \cdots N) \tag{5}$$

となる. 周期的境界条件を加えると, グラフから波数に制限ができ, とびとびの波長になる. 又, (3) に (5) を代入すると,

$$\omega_l = 2\sqrt{\frac{\kappa}{m}}\sin\frac{l\pi}{N}\ (l = 1\cdots N) \tag{6}$$

と表示でき, N 個の波が基準振動を与える. これはとり方によって同じ基準振動として表示できる. 例えば, N が偶数の時は,

$$k_l = \frac{2\pi l}{aN},\ \omega_l = 2\sqrt{\frac{\kappa}{m}}\sin\frac{l\pi}{N}\ (l = -\frac{N}{2}, -\frac{N}{2}+1, \cdots, \frac{N}{2}-1)$$

となる. わかりやすいように,ω の $\sin\frac{l\pi}{N}$ だけを考えてみる.

例として $N = 4$ の時を考える.

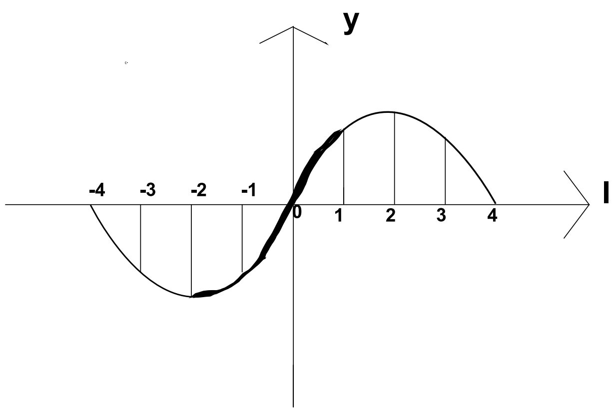


図 3: $y = \sin\frac{l\pi}{4}$

グラフより $N = 4$ の時で考えても同じことがわかる.

次に格子が無限に長いとして, 進行波を考える. 今,

$$\omega(k) = 2\sqrt{\frac{\kappa}{m}}\sin\frac{ka}{2}\quad(-\pi < ka < \pi) \tag{7}$$

としておき, (2) を右, (3) を左に進む波とする. (2), (3) は解の線形結合により (7) の解である. このとき (1) の一般解は, フーリエ積分を使って

$$y_n(t) = \frac{1}{2\pi}[\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dkA(k)e^{i(kan-\omega t)} + \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dkA(k)e^{i(kan+\omega t)}] \tag{8}$$

＜証明＞ (2), (3) を使い, フーリエ積分より,

$$\begin{aligned} y_{n1}(t) &= \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} y_n(w)e^{i(kan-\omega t)}dkd\omega \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}[\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} y_n(w)d\omega]e^{i(kan-\omega t)}dk \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} A(k)e^{i(kan-\omega t)}dk, \\ & \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} y_n(w)d\omega = A(k) \end{aligned}$$

となる. 同様に

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} y_n(w)d\omega = B(k)$$

としてまとめると,

$$y_{n1}(t) + y_{n2}(t) = \frac{1}{2\pi}[\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} A(k)e^{i(kan-\omega t)}dk + \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} B(k)e^{i(kan-\omega t)}dk]$$

となる. 証明終了.

このように, 重ね合わせの原理として波を表すことができるのが, 線形波動の特徴である. 次に位相速度と群速度を求めてみる. (7) を使えば, 位相速度 v_p と群速度 v_g は,

$$v_p \equiv \frac{\omega(k)}{k} = c_0(1 - \frac{1}{24}(ka)^2 + \cdots), \tag{9}$$

$$v_g \equiv \frac{\partial\omega(k)}{\partial k} = c_0(1 - \frac{1}{8}(ka)^2 + \cdots) \tag{10}$$

と表示できる. c_0 は音速であり,

$$c_0 = \sqrt{\frac{\kappa}{m}}a$$

で与えられる.

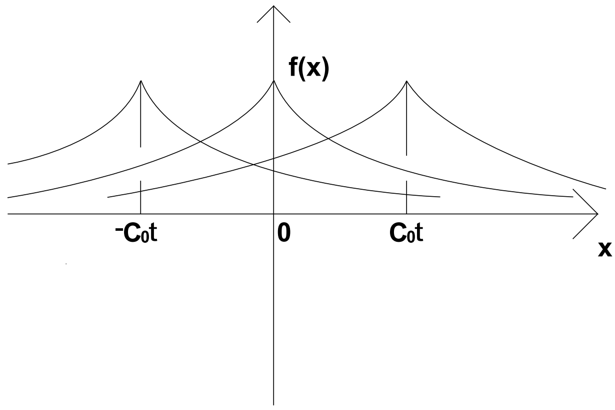
又, v_p が k に依存するとき又は, v_p と v_g が異なる時分散があるという. もし, 分散がない場合は,

$$\omega = c_0 k$$

とおくと

$$y_n(t) = \frac{1}{2\pi}[\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dkA(k)e^{ik(an-c_0t)} + \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dkB(k)e^{ik(an+c_0t)}] \tag{11}$$

となる.



又,

$$f(x)=\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{A(k)}{2\pi}e^{ik\pi}dk$$

とし, グラフから, 左の波形を

$$f(an-c_0t)=\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dk\frac{A(k)}{2\pi}e^{ik(an-c_0t)},$$

右の波形を

$$f(an+c_0t)=\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dk\frac{B(k)}{2\pi}e^{ik(an+c_0t)}$$

と表示すれば, 波形を変えずに平行移動していることがわかる.

実際, 格子波には分散があるので, 各波数成分は異なる位相速度をもち, 伝播につれて波は変形する. その理由として次のことを考えてみる.
格子間隔 a と波長 $\lambda=\frac{2\pi}{k}$ を比べる.

$$a<<\frac{2\pi}{k}$$

の時, $y_n(t),y_{n+1}(t)$ があまり変化しない. よって $x=na$ を連続変数とみなせることができる.

$$\begin{aligned}y_{n+1}(t)&=\frac{1}{2\pi}[\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dkA(k)e^{ik(a(n+1)-c_0t)}+\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dkB(k)e^{ik(a(n+1)+c_0t)}]\\&=\frac{1}{2\pi}[\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dkA(k)e^{ik(an-c_0t)}\cdot e^{ika}+\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dkB(k)e^{ik(an+c_0t)}\cdot e^{ika}]\end{aligned}$$

$a\ll\frac{2\pi}{k}$ より, $e^{ika}<1$ となるので,

$$y_{n+1}(t)\equiv y_n(t)$$

となる.$y_{n-1}(t)$ も同様にできる.
こう考えれば連続体近似が使える.

$$y_{n\pm1}(t)=y((n+1)a,t)=y(na+a,t)$$

となる. 又,

$$y_n(t) = y(na, t) = y(x, t)$$

とおけるので, $y_{n+1}(t) \equiv y_n(t)$ から,

$$y(na + a, t) \equiv y(x, t)$$

となる. 空間変化の差分を微分で近似する手法、連続体近似を使うと,

$$y_{n\pm1}(t) = y(x \pm a, t) = y(x, t) \pm a\frac{\partial y(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} \pm \frac{a^3}{6}\frac{\partial^3 y(x,t)}{\partial x^3} + \frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4} \pm \cdots \tag{12}$$

と展開できる.

これを (1) に代入する.

$$\begin{aligned} m\frac{d^2 y(x,t)}{dt^2} &= \kappa(y(x,t) + a\frac{\partial y(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} + \cdots - 2y(x,t) \\ &+ y(x,t) - a\frac{\partial y(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} + \cdots) \\ &= \kappa(a^2\frac{\partial^2 y(x,t)}{\partial x^2})\ (\frac{\partial^4 y(x,t)}{\partial x^4}\text{以降を無視して考える.}) \end{aligned}$$

まとめると,

$$\frac{d^2 y(x,t)}{dt^2} = {c_0}^2\frac{\partial^2 y(x,t)}{\partial x^2} \quad (c_0 = \sqrt{\frac{\kappa}{m}}a) \tag{13}$$

となり, 波動方程式が得られる.

2　1次元非線形格子

前章では非線形波動現象を理解する上で線形現象の性質を調べてみた. 本章では, 和達 [2, p.12-15] に沿って,1 次元非線形格子について述べる. 線形現象の性質をもとに非線形現象での近似方程式を導く事を目的とする. 簡単な例として 1 次元非線形格子の時を考えてみる.

2.1　KdV 方程式

線形での 1 次元格子の時と同様に

$$F_n(t) = \kappa(\Delta + \alpha\Delta^2)$$

と仮定する. この時の運動方程式は,

$$\begin{aligned} m\frac{d^2y_n(t)}{dt^2} &= F_n(t) = \kappa(y_{n+1}(t) - y_n(t) + \alpha(y_{n+1}(t) - y_n(t))^2) \\ &-\kappa(y_n(t) - y_{n-1}(t) + \alpha(y_n(t) - y_{n-1}(t))^2)\ (n = 0, \pm1, \pm2, \cdots) \end{aligned} \tag{14}$$

となる. この方程式に対して, 連続体近似を行なう. (14) に (12) を代入すると,

$$\begin{aligned} m\frac{d^2y(x,t)}{dt^2} &= \kappa\{(y(x,t) + a\frac{\partial y(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^3}{6}\frac{\partial^3 y(x,t)}{\partial x^3} + \frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) \\ &-y(x,t) \\ &+\alpha(a\frac{\partial y(x,t)}{\partial x} + \cdots)^2\} - \kappa\{y(x,t) - (y_(x,t) - a\frac{\partial y(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} \\ &-\frac{a^3}{6}\frac{\partial^3 y(x,t)}{\partial x^3} \\ &+\frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) + \alpha(a\frac{\partial y(x,t)}{\partial x} - \cdots)^2\} \\ &= \kappa(a^2\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^4}{12}\frac{\partial^4 y(x,t)}{\partial x^4} + \alpha(2a^3\frac{\partial y(x,t)}{\partial x}\frac{\partial^2 y(x,t)}{\partial x^2} \\ &+\frac{1}{3}a^5\frac{\partial^2 y(x,t)}{\partial x^2}\frac{\partial^3 y(x,t)}{\partial x^3} + \frac{1}{6}a^5\frac{\partial y(x,t)}{\partial x}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots)) \end{aligned}$$

$$\begin{aligned} \frac{d^2y(x,t)}{dt^2} &= {c_0}^2(\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^2}{12}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots + 2\alpha a\frac{\partial y(x,t)}{\partial x}\frac{\partial^2 y(x,t)}{\partial x^2} \\ &+\frac{1}{3}\alpha a^3\frac{\partial^2 y(x,t)}{\partial x^2}\frac{\partial^3 y(x,t)}{\partial x^3} + \frac{1}{6}\alpha a^3\frac{\partial y(x,t)}{\partial x}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) \end{aligned} \tag{15}$$

となる. ただし,

$$c_0 = \sqrt{\frac{\kappa}{m}}a$$

とする. この時, (15) を近似した式を

$$\frac{d^2y(x,t)}{dt^2} = {c_0}^2(\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^2}{12}\frac{\partial^4 y(x,t)}{\partial x^4} + 2\alpha a\frac{\partial y(x,t)}{\partial x}\frac{\partial^2 y(x,t)}{\partial x^2}) \tag{16}$$

と表示する. これを, Boussinesq 方程式と言う.

Boussinesq 方程式は, ソリトンを記述する方程式として広く使われているが, よいモデル方程式である事は幸運によるところがある. より複雑な非線形波動を取り扱うには, 統一的な近似方法が

必要になる. 分散の効果だけを高次まで取り入れても実際の現象をよく近似するとは限らないし, 非線形の効果だけを高次まで考えてもよい近似になるとは限らない. そこで, 分散のある波で振幅は小さいが無限小でないとする. 波はだいたい音速 c_0 で進むとする.c_0 で動く座標系を導入すると, この座標系では波の変化はゆっくりしたものになる. この変化と, 非線形項が同じ大きさ程度なら, そのオーダーで閉じた方程式が得ることができる. この事を数式で表してみる.

まず $\epsilon > 0$ を小さい無次元パラメータとして, 独立変数 x, t を新しい独立変数 ξ, τ に変える.

$$\xi = \frac{\epsilon^p (x - c_0 t)}{a}, \tau = \frac{\epsilon^q c_0 (t)}{a}, \tag{17}$$

$$y(x,t) = \epsilon^r y^{(1)}(\xi, \tau) \tag{18}$$

(17), (18) を (15) に代入すると,

$$\begin{aligned} &{c_0}^2 \left(\epsilon^{2p} \frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2} - 2\epsilon^{p+q} \frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi \partial \tau} + \epsilon^{2q} \frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \tau^2}\right) \\ &= {c_0}^2 \left(\epsilon^{2p} \frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2} + \frac{\epsilon^{4p}}{12} \frac{\partial^4 y^{(1)}(\xi,\tau)}{\partial \xi^4} + 2\alpha \epsilon^{3p+r} \frac{\partial y^{(1)}(\xi,\tau)}{\partial \xi} \frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2} + \cdots \right) \end{aligned} \tag{19}$$

となる.ϵ の最低次の項だけに着目して考える.

$$q > p$$

という仮定を使う. 又右辺の第四項以降を無視する.

左辺, 右辺の第一項は同じなので, それ以外を考える. 仮定から左辺の第三項は第二項より高いオーダーの項なので, それ以外の左辺の第二項と右辺の第二項, 第三項の ϵ が同じオーダーであると考える.

$$\begin{gathered} p + q = 4p = 3p + r, \\ p = r, p = \frac{1}{3} q \end{gathered} \tag{20}$$

となり, p, q, r が (20) を満たせば何でもよい.

$$r = 1, p = 1, q = 3$$

として代入すると,

$$2\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi \partial \tau} + \frac{1}{12} \frac{\partial^4 y^{(1)}(\xi,\tau)}{\partial \xi^4} + 2\alpha \frac{\partial y^{(1)}(\xi,\tau)}{\partial \xi} \frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2} = 0 \tag{21}$$

となる.

$$u = \alpha \frac{\partial y^{(1)}(\xi,\tau)}{\partial \xi}$$

とおいて, (21) に代入すると,

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial \xi} + \frac{1}{24} \frac{\partial u^3}{\partial \xi^3} = 0 \tag{22}$$

となる. この方程式を KdV 方程式という.

2.2 Gardner-Morikawa 変換と逓減摂動法

KdV 方程式を求める際に, (17) のように 2 つの独立変数を新しい 2 つの独立変数に変換するやり方を Gardner-Morikawa 変換と言う.Gardner-Morikawa 変換は分散関係式から決められる事がわかる.
$k \ll 1$ の時, (7) から,

$$\begin{aligned}\omega(k) &= 2\sqrt{\frac{k}{m}}\sin\frac{ka}{2} \\ &= 2\sqrt{\frac{k}{m}}(\frac{ka}{2}-\frac{k^3}{6}(\frac{a}{2})^3+\cdots) \\ &= c_0k-\frac{1}{24}c_0k(ka)^2+\cdots \end{aligned} \qquad (23)$$

である. 又, (23), $k=\epsilon^p\tilde{k}$ を使うと,

$$\begin{aligned}kx-\omega(k)t &= \epsilon^p\tilde{k}x-c_0\epsilon^p\tilde{k}t+\frac{1}{24}c_0\epsilon^{3p}\tilde{k}ta^2+\cdots \\ &= \tilde{k}\epsilon^p(x-c_0t)+\frac{1}{24}c_0\epsilon^{3p}t\tilde{k}a^2+\cdots \end{aligned} \qquad (24)$$

となる. ここで,

$$\xi=\frac{\epsilon^p(x-c_0t)}{a},\tau'=\frac{\epsilon^{3p}c_0(t)}{a}$$

とおく. 又, 第三項目以降を無視すると,

$$kx-\omega(k)t=\tilde{k}a\xi+\frac{1}{24}a\tau\tilde{k}a^2$$

と表示できる. 位相速度で動く座標系でゆっくりと変化する波を記述するには, Gardner-Morikawa 変換を使えばよく, x,t が大きく変わっても, ξ,τ があまり変化しないので, (17) は, おそい変数とも呼ばれている.

又,Gardner-Morikawa 変換を使って非線形発展方程式を導く方法を逓減摂動法と呼ぶ. 逓減摂動法とは, 近似方程式を導く方法の一つであり, 問題とする現象の変化のスケールが他と異なる事に着目し, スケールの異なる変化を引き出して記述するように摂動展開を行うという考え方である.

2.3 変形 KdV 方程式

KdV 方程式を求めた時と同様に逓減摂動法を用いて, 近似方程式の解を求めていく.

$$F_n(t) = \kappa(\Delta + \alpha^2\Delta^2)$$

と仮定する. この時の運動方程式は,

$$\begin{aligned} m\frac{d^2y_n(t)}{dt^2} &= F_n(t) = \kappa(y_{n+1}(t) - y_n(t) + \alpha^2(y_{n+1}(t) - y_n(t))^3) \\ &- \kappa(y_n(t) - y_{n-1}(t) + \alpha^2(y_n(t) - y_{n-1}(t))^3)\ (n = 0, \pm1, \pm2, \cdots) \end{aligned} \quad (25)$$

である. この方程式に対して, 連続体近似を行なう. (12) に (25) を代入すると,

$$\begin{aligned} m\frac{d^2y(x,t)}{dt^2} &= \kappa\{a\frac{\partial y(x,t)}{\partial x} + \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^3}{6}\frac{\partial^3 y(x,t)}{\partial x^3} + \frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) \\ &+ \alpha^2(a^3(\frac{\partial y(x,t)}{\partial x})^3 + \frac{a^6}{8}(\frac{\partial^2 y(x,t)}{\partial x^2})^3 + \cdots + \frac{3}{2}a^4(\frac{\partial y(x,t)}{\partial x})^2\frac{\partial^2 y(x,t)}{\partial x^2} + \cdots)\} \\ &- \kappa\{(a\frac{\partial y(x,t)}{\partial x} - \frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^3}{6}\frac{\partial^3 y(x,t)}{\partial x^3} - \frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) \\ &+ \alpha^2(a^3(\frac{\partial y(x,t)}{\partial x})^3 - \frac{a^6}{8}(\frac{\partial^2 y(x,t)}{\partial x^2})^3 + \cdots - \frac{3}{2}a^4(\frac{\partial y(x,t)}{\partial x})^2\frac{\partial^2 y(x,t)}{\partial x^2} + \cdots)\}, \end{aligned}$$

$$\begin{aligned} &m\frac{d^2y(x,t)}{dt^2} = \kappa\{(a^2\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) + \alpha^2(\frac{a^6}{4}(\frac{\partial^2 y(x,t)}{\partial x^2})^3 \\ &+3a^4(\frac{\partial y(x,t)}{\partial x})^2\frac{\partial^2 y(x,t)}{\partial x^2} + \cdots)\} \end{aligned}$$

となる. ここで

$$c_0 = \sqrt{\frac{\kappa}{m}}a$$

を使うと,

$$\begin{aligned} &\frac{d^2y(x,t)}{dt^2} = {c_0}^2\{\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{a^2}{24}\frac{\partial^4 y(x,t)}{\partial x^4} + \cdots) + \alpha^2(\frac{a^4}{4}(\frac{\partial^2 y(x,t)}{\partial x^2})^3 \\ &+3a^2(\frac{\partial y(x,t)}{\partial x})^2\frac{\partial^2 y(x,t)}{\partial x^2} + \cdots)\} \end{aligned}$$

となる. ここで, 左辺と右辺の第一項を見ると, 波動方程式になっており, 又, (18) を使えば,

$$\begin{aligned} &\frac{\epsilon^r {c_0}^2}{a^2}(\epsilon^{2p}\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2} - 2\epsilon^{p+q}\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi\partial \tau} + \epsilon^{2q}\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \tau^2}) \\ =\ &\frac{\epsilon^r {c_0}^2}{a^2}\{(\epsilon^{2p}\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2} + \frac{\epsilon^{4p}}{12}\frac{\partial^4 y^{(1)}(\xi,\tau)}{\partial \xi^4} + \cdots) \\ +\ &\alpha^2(\frac{\epsilon^{6p+2r}}{4}(\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2})^3 + 3\epsilon^{4p+2r}(\frac{\partial y^{(1)}(\xi,\tau)}{\partial \xi})^2(\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial \xi^2}) + \cdots)\} \end{aligned}$$

となる.ϵ の最低次の項だけに着目して考える.

$$q > p$$

という仮定を使う. KdV 方程式と同様に考える.

$$p + q = 4p = 4p + 2r,$$

$$q = 3p, r = 0 \tag{26}$$

となり, p, q, r が (26) を満たせば何でもよい.

$$r = 0, p = \frac{1}{2}, q = \frac{3}{2}$$

として代入すると,

$$\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial\xi\partial\tau} + \frac{1}{24}\frac{\partial^4 y^{(1)}(\xi,\tau)}{\partial\xi^4} + \frac{3}{2}\alpha^2(\frac{\partial y^{(1)}(\xi,\tau)}{\partial\xi})^2\frac{\partial^2 y^{(1)}(\xi,\tau)}{\partial\xi^2} = 0 \tag{27}$$

となる.

$$u = \alpha\frac{\partial y^{(1)}(\xi,\tau)}{\partial\xi}$$

とおいて, (27) に代入すると,

$$\frac{\partial u}{\partial\tau} + \frac{3}{2}u^2\frac{\partial u}{\partial\xi} + \frac{1}{24}\frac{\partial u^3}{\partial\xi^3} = 0 \tag{28}$$

となる. この方程式を変形 KdV 方程式という.

3　水面波の近似方程式

本章では, 和達 [2, p.16-17], 川原 [3, p.28-30] に沿って, 水面波の近似方程式について述べる.

3.1　基礎方程式

物理現象を記述する基礎方程式系から, 近似方程式を導く具体的な例として, 水面波を取り上げる. 水面波の運動は粘性のない非圧縮性の運動方程式として表示できる. 深さを水深 h の浅い水を考える. 今, 垂直方向を z 軸とし, 水平方向に (x,y) 面をとる。水面は曲線 $z=\zeta(x,y,t)$ で表す。静止水面を $z=0$, 底面を $z=-h$ とする. 基礎方程式は水の圧縮性をほとんど無視できるので, 非圧縮性流体に対するナビエ・ストークス方程式で与えられる. まず初めに非圧縮性の式を (29), 運動方程式を (30) として表示する.

$$\nabla \cdot \mathbf{u} = 0, \tag{29}$$

$$(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(\frac{P}{\rho} + gz) + \nu\nabla^2\mathbf{u}. \tag{30}$$

このとき,

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

とする. 又, u は速度ベクトル, P は圧力であり,
$\mathbf{u}(x,y,z,t) = (u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$, $P = P(x,y,z,t)$ と表示する.
速度場の回転を

$$\omega(x,y,z,t) = \nabla \times \mathbf{u}(x,y,z,t) \tag{31}$$

と表示する. 又、$\omega(x,y,z,t) = (\omega_1(x,y,z,t), \omega_2(x,y,z,t), \omega_3(x,y,z,t))$ であり, 渦度と呼ばれている. (31) において, $\omega(x,y,z,t)=0$ を渦なし運動と呼ぶ.
(30) で,

$$\nu = 0, \omega = 0 \text{ の時}, \nabla \times \mathbf{u}(x,y,z,t) = 0$$

を満たす.$\mathbf{u}(x,y,z,t)$ は速度ポテンシャル $\Phi(x,y,z,t)$ を使って,

$$\mathbf{u}(x,y,z,t) = \nabla\Phi(x,y,z,t) = (\frac{\partial\Phi(x,y,z,t)}{\partial x}, \frac{\partial\Phi(x,y,z,t)}{\partial y}, \frac{\partial\Phi(x,y,z,t)}{\partial z}) \tag{32}$$

と表示できる。このとき, (32) を (29) に代入すると $\Phi(x,y,z,t)$ は, ラプラス方程式

$$\triangle\Phi(x,y,z,t) = \frac{\partial^2\Phi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\Phi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\Phi(x,y,z,t)}{\partial z^2} = 0 \qquad (-h < z < \zeta) \tag{33}$$

を満たすことになる. (33) を解く時に, 速度ポテンシャル $\Phi(x,y,z,t)$ が与えられると, (30) から P が次のようにして求められる. ベクトル場の公式

$$(\mathbf{u}(x,y,z,t)\cdot\nabla)\mathbf{u}(x,y,z,t) = \frac{1}{2}\nabla|\mathbf{u}(x,y,z,t)|^2 - \mathbf{u}(x,y,z,t)\times(\nabla\times\mathbf{u}(x,y,z,t))$$

を用いて,(30) を書き換えると,

$$\frac{\partial\mathbf{u}(x,y,z,t)}{\partial t} + \frac{1}{2}\nabla|\mathbf{u}(x,y,z,t)|^2 - \mathbf{u}(x,y,z,t)\times(\nabla\times\mathbf{u}(x,y,z,t)) = -\nabla(\frac{P}{\rho} + gz)$$

又, $\omega = 0$, (32) を使えば,

$$\nabla(\frac{\partial \Phi(x,y,z,t)}{\partial t} + \frac{1}{2}|\nabla\Phi(x,y,z,t)|^2 + \frac{P}{\rho} + gz) = 0$$

となる. 空間変数に関して積分すると,

$$-\frac{P}{\rho} \equiv \frac{\partial \Phi(x,y,z,t)}{\partial t} + \frac{1}{2}|\nabla\Phi(x,y,z,t)|^2 + gz + C(t)$$

これがベルヌイの定理である. $C(t)$ は空間積分で出てくる時間のみに依存する任意関数であるが, 速度を変えることなく, Φ に繰り込むことができるので, $C(t)$ を繰り込んで消去したものと考える.
次に境界条件を考える. 時刻 t においての水の表面での方程式を

$$F(x,y,z,t) \equiv z - \zeta(x,y,t) = 0 \tag{34}$$

とする. ζ は, 静止水面 $z = 0$ から, 測った境界面の高さを表す. このとき, 境界面上と流体の運動が一致する条件式

$$\frac{\partial \zeta(x,y,t)}{\partial t} + \frac{\partial \Phi(x,y,z,t)}{\partial x}\frac{\partial \zeta(x,y,t)}{\partial x} + \frac{\partial \Phi(x,y,z,t)}{\partial y}\frac{\partial \zeta(x,y,t)}{\partial y} = \frac{\partial \Phi(x,y,z,t)}{\partial z} \quad (z = \zeta \quad \text{上}) \tag{35}$$

が成り立つ.
<証明>時刻 $t + \Delta t$ 上での水の粒子の位置は (x,y,z) から $(x + u_1\Delta t, y + u_2\Delta t, z + u_3\Delta t)$ に移行するが、水の表面上に留まるとするならば、

$$F(x + u_1\Delta t, y + u_2\Delta t, z + u_3\Delta t, t + \Delta t) = 0 \tag{36}$$

が成り立つ.

$$A_1 = x + u_1\Delta t, A_2 = y + u_2\Delta t, A_3 = z + u_3\Delta t, A_4 = t + \Delta t \tag{37}$$

とする. この式を $\Delta t = 0$ として展開する.

$$F(x,y,z,t) + \frac{\partial F(x,y,z,t)}{\partial \Delta t}\Delta t + \frac{\partial_2 F(x,y,z,t)}{\partial \Delta t_2}\frac{(\Delta t)^2}{2} + \cdots \tag{38}$$

境界面上と流体の運動が一致する条件式と二回微分以降はすごく小さいので無視して考えれば、

$$\frac{\partial F(x,y,z,t)}{\partial \Delta t} = 0$$

となる. 合成関数の微分より,

$$\frac{\partial F(x,y,z,t)}{\partial x}\frac{\partial x}{\partial \Delta t} + \frac{\partial F(x,y,z,t)}{\partial y}\frac{\partial y}{\partial \Delta t} + \frac{\partial F(x,y,z,t)}{\partial z}\frac{\partial z}{\partial \Delta t} + \frac{\partial F(x,y,z,t)}{\partial t}\frac{\partial t}{\partial \Delta t} = 0$$

(34) と (38) より

$$\frac{\partial \zeta(x,y,t)}{\partial x}u_1 + \frac{\partial \zeta(x,y,t)}{\partial y}u_2 - u_3 + \frac{\partial \zeta(x,y,t)}{\partial t} = 0$$

となる. 最後に (32) を使うと,

$$\frac{\partial \zeta(x,y,t)}{\partial t} + \frac{\partial \Phi(x,y,z,t)}{\partial x}\frac{\partial \zeta(x,y,t)}{\partial x} + \frac{\partial \Phi(x,y,z,t)}{\partial y}\frac{\partial \zeta(x,y,t)}{\partial y} = \frac{\partial \Phi(x,y,z,t)}{\partial z} \quad (z = \zeta \quad \text{上})$$

となり, 主張が成り立つことがわかる.
証明終了

又, 水の底面での速度の鉛直成分が 0 となる条件は

$$\frac{\partial \Phi(x,y,z,t)}{\partial z} = 0 \qquad (z = -h \text{ 上}) \tag{39}$$

となる. ここまでに出た式をまとめたいが, 水面上での境界条件が必要であり, 表面張力があるかないかで条件が変わるので分けて考える.

3.2 表面張力がある場合

$z=\zeta$ 上では, 流体と大気の圧力差が表面張力と釣り合う関係から,

$$P_w - P_a = -\frac{T}{R}, \tag{40}$$

$$\frac{1}{R} \equiv [\{1+(\frac{\partial\zeta(x,y,t)}{\partial x})^2\}\frac{\partial^2\zeta(x,y,t)}{\partial y^2} + \{1+(\frac{\partial\zeta(x,y,t)}{\partial y})^2\}\frac{\partial^2\zeta(x,y,t)}{\partial x^2}$$
$$-2(\frac{\partial\zeta(x,y,t)}{\partial x})(\frac{\partial\zeta(x,y,t)}{\partial y})\frac{\partial^2\zeta(x,y,t)}{\partial x\partial y}]\{1+(\frac{\partial\zeta(x,y,t)}{\partial x})^2+(\frac{\partial\zeta(x,y,t)}{\partial y})^2\}^{-\frac{3}{2}} \tag{41}$$

となる. ここで,P_a は大気の圧力, P_w は流体の圧力, T は表面張力, R は表面張力定数である. 又,

$$P = P_w, z = \zeta$$

の時,　ベルヌーイの定理より

$$\frac{P_w}{\rho} + \frac{\partial\Phi(x,y,\zeta,t)}{\partial t} + \frac{1}{2}|\nabla\Phi(x,y,\zeta,t)|^2 + g\zeta \equiv \mathbf{const}$$

となる.(40) より,

$$\frac{P_a}{\rho} - \frac{T}{\rho R} + \frac{\partial\Phi(x,y,\zeta,t)}{\partial t} + \frac{1}{2}|\nabla\Phi(x,y,\zeta,t)|^2 + g\zeta(x,y,t) \equiv \mathbf{const}$$

となる. ここで, 右辺 $=\frac{P_a}{\rho}$ と置いて, 書き直す.

$$-\frac{T}{\rho R} + \frac{\partial\Phi(x,y,z,t)}{\partial t} + \frac{1}{2}|\nabla\Phi(x,y,z,t)|^2 + g\zeta(x,y,t) = 0 \qquad (z=\zeta \quad \text{上}) \tag{42}$$

とベルヌーイの定理が修正される. (33), (35), (39), (42) をまとめて書くと,

$$\triangle\Phi(x,y,z,t) = \frac{\partial^2\Phi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\Phi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\Phi(x,y,z,t)}{\partial z^2} = 0 \qquad (-h<z<\zeta),$$

$$\frac{\partial\zeta(x,y,t)}{\partial t} + \frac{\partial\Phi(x,y,z,t)}{\partial x}\frac{\partial\zeta(x,y,t)}{\partial x} + \frac{\partial\Phi(x,y,z,t)}{\partial y}\frac{\partial\zeta(x,y,t)}{\partial y} = \frac{\partial\Phi(x,y,z,t)}{\partial z} \qquad (z=\zeta \quad \text{上}),$$

$$-\frac{T}{\rho R} + \frac{\partial\Phi(x,y,z,t)}{\partial t} + \frac{1}{2}|\nabla\Phi(x,y,z,t)|^2 + g\zeta(x,y,t) = 0 \qquad (z=\zeta \quad \text{上}),$$

$$\frac{\partial\Phi(x,y,z,t)}{\partial z} = 0 \qquad (z=-h\ \text{上})$$

となる.

次に方程式を線形化して分散関係を調べる. 波の振幅が微小で ζ が小さい場合、表面の境界条件を $z=0$ のまわりに展開することができる. 最低次の近似で境界条件は静止水面 $z=0$ 上で線形化されたものになる. 上の 4 つの式はつぎのように表示できる.

$$\triangle\Phi(x,y,z,t) = \frac{\partial^2\Phi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\Phi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\Phi(x,y,z,t)}{\partial z^2} = 0 \qquad (-h<z<\zeta),$$

$$\frac{\partial\zeta(x,y,t)}{\partial t} - \frac{\partial\Phi(x,y,z,t)}{\partial z} = 0 \quad (z=0 \quad \text{上}),$$

$$-\frac{T}{\rho}(\frac{\partial^2\zeta(x,y,t)}{\partial x^2}+\frac{\partial^2\zeta(x,y,t)}{\partial y^2})+\frac{\partial\Phi(x,y,z,t)}{\partial t}+g\zeta(x,y,t)=0 \qquad (z=0 \quad 上),$$

$$\frac{\partial\Phi(x,y,z,t)}{\partial z}=0 \qquad (z=-h\ 上).$$

上の 4 つの式を見ると, $\Phi(x,y,z,t)$ と $\zeta(x,y,t)$ を決める問題であり,z の依存性を除けば, x,y に依存する 2 次元問題である．簡単のため 1 次元の平面波にかぎって議論する．上の 4 つの式での $\frac{\partial}{\partial y}\equiv 0$ で考える．

$$\triangle\Phi(x,z,t)=\frac{\partial^2\Phi(x,z,t)}{\partial x^2}+\frac{\partial^2\Phi(x,z,t)}{\partial z^2}=0 \qquad (-h<z<\zeta), \tag{43}$$

$$\frac{\partial\zeta(x,t)}{\partial t}-\frac{\partial\Phi(x,z,t)}{\partial z}=0 \qquad (z=0 \quad 上), \tag{44}$$

$$-\frac{T}{\rho}(\frac{\partial^2\zeta(x,t)}{\partial x^2})+\frac{\partial\Phi(x,z,t)}{\partial t}+g\zeta(x,t)=0 \qquad (z=0 \quad 上), \tag{45}$$

$$\frac{\partial\Phi(x,z,t)}{\partial z}=0 \qquad (z=-h\ 上), \tag{46}$$

となる. 今,

$$\Phi(x,z,t)=f(z)\cos(kx-wt) \tag{47}$$

と表示する．(43) に代入すると,

$$(-k^2)f(z)\cos(kx-wt)+\frac{d^2f(z)}{dz^2}\cos(kx-wt)=0$$

が得られる.$\cos(kx-wt)$ で両辺を割ると,

$$\frac{d^2f(z)}{dz^2}-k^2f(z)=0$$

となる．$\frac{\partial}{\partial z}=D$ とおき, 計算する.

$$(D^2-k^2)f(z)=0$$

となる. ここで $D=\lambda$ として解く.

$$\lambda^2-k^2=0,$$
$$\lambda=\pm k$$

となり,

$$f(z)=C_1e^{kz}+C_2e^{-kz} \quad (C_1,C_2: 定数) \tag{48}$$

が得られる．又, (46) を用いると

$$(C_1ke^{kh}-C_2ke^{-kh})\cos(kx-\omega t)=0, \tag{49}$$

$$C_1k-C_2k=0,$$
$$C_1=C_2$$

となる.(48) に戻すと

$$\begin{aligned} f(z) &= C_1(e^{kz}+e^{-kz}) \\ &= 2C_1\cosh(kz) \end{aligned}$$

となる.(47) に戻すと,

$$\begin{aligned}\Phi(x,z,t) &= 2C_1\cosh(kz)\cos(kx-\omega t)\\ &\quad 2C_1 = A \text{ とおく}\\ &= A\cosh(kz)\cos(kx-\omega t)\end{aligned}$$

(49) を用いると,

$$\Phi(x,z,t) = A\cosh\{k(z+h)\}\cos(kx-\omega t)$$

と表示できる. 又 (44), (45) を使って ζ を消去する. (44) を計算すると

$$\frac{\partial\zeta(x,t)}{\partial t} - kA\sinh(kh)\cos(kx-\omega t) = 0 \tag{50}$$

となる.(45) を計算すると,

$$-\frac{T}{\rho}(\frac{\partial^2\zeta(x,t)}{\partial x^2}) - A\cosh(kh)\sin(kx-\omega t)(-\omega) + g\zeta(x,t) = 0 \tag{51}$$

となる.(50) を積分すると,

$$\zeta(x,t) = -\frac{1}{\omega}kA\sinh(kh)\sin(kx-\omega t)$$

となり, これを (51) に代入する.

$$\begin{aligned}&-\frac{T}{\rho}\frac{1}{\omega}k^3A\sinh(kh)\sin(kx-\omega t) + A\cosh(kh)\sin(kx-\omega t)(\omega)\\ &-\frac{g}{\omega}kA\sinh(kh)\sin(kx-\omega t) = 0\end{aligned}$$

となる. $-\frac{A}{\omega}\sinh(kh)\sin(kx-\omega t)$ で割ると,

$$-\frac{\omega^2}{\tanh(kh)} + gk + \frac{T}{\rho}k^3 = 0,$$

$$\omega^2 = (gk + \frac{Tk^3}{\rho})\tanh kh$$

となり, 分散関係式を得る. 位相速度 v_p と群速度 v_g を求めるために, まず ω を求める.

$$\begin{aligned}\omega^2 &= (gk + \frac{Tk^3}{\rho})\tanh kh \ (k=0 \text{ でテイラー展開})\\ &= (gk + \frac{Tk^3}{\rho})kh(1 - \frac{1}{3}(kh)^2 + \cdots)\end{aligned}$$

$$\omega^2 = gk^2h(1 + \frac{Tk^2}{g\rho})(1 - \frac{1}{3}(kh)^2)$$

となる. ここで, $(1 - \frac{1}{3}(kh)^2 + \cdots)$ での 3 項目を無視した.

$$\omega \quad = k\sqrt{gh}\sqrt{1 + \frac{Tk^2}{\rho g}}\sqrt{1 - \frac{1}{3}(kh)^2} \tag{52}$$

において, 下の 2 つの式を $k=0$ でテイラー展開する.

$$\sqrt{1+\frac{Tk^2}{\rho g}} = (1+\frac{Tk^2}{2\rho g}+\cdots)$$

$$\sqrt{1-\frac{1}{3}(kh)^2} = (1-\frac{h^2k^2}{6}+\cdots)$$

これらを (52) に代入すると,

$$\omega = kc_0\{1+\frac{1}{2}(\frac{T}{\rho gh^2}-\frac{1}{3})k^2h^2+\cdots\}$$

となる. ここで,

$$c_0 = \sqrt{gh}$$

とする.

$$kh \ll 1$$

として考える. 位相速度 v_p と群速度 v_g は,

$$v_p \equiv \frac{\omega(k)}{k} = c_0(1+\frac{1}{2}(\frac{T}{\rho gh^2}-\frac{1}{3})(kh)^2+\cdots)$$

$$v_p = (gh)^{\frac{1}{2}} \tag{53}$$

$$v_g \equiv \frac{\partial\omega(k)}{\partial k} = v_p + (gh)^{\frac{1}{2}}k((\frac{T}{\rho gh^2}-\frac{1}{3})kh^2+\cdots)$$

$$v_g = (gh)^{\frac{1}{2}} \tag{54}$$

と表示できる.

$$v_p = v_g = (gh)^{\frac{1}{2}}$$

となり, 分散性がないことがわかる. 又, 分散性を表す係数は,

$$T > \frac{1}{3}\rho gh^2$$

の時「正」であり, 逆の時「負」と表される. この場合は「正」となる.

3.3　表面張力がない場合

(40) より, $T=0$ であり, P_w において $z=\zeta$ 上では,

$$P_w(x,y,\zeta,t)=P_a(x,y,\zeta,t)=P(x,y,\zeta,t)$$

が成り立つので, ベルヌーイの定理は

$$-\frac{P_a}{\rho}+\frac{\partial\Phi(x,y,\zeta,t)}{\partial t}+\frac{1}{2}|\nabla\Phi(x,y,\zeta,t)|^2+g\zeta(x,y,t)=0$$

と修正される. 方程式を線形化して分散関係を調べる. 表面張力がある場合と同様に考える. (47) より

$$\Phi(x,z,t)=f(z)\cos(kx-wt)$$

と表示する. (43) に代入すると,

$$(-k^2)f(z)\cos(kx-wt)+\frac{d^2f(z)}{dz^2}\cos(kx-wt)=0$$

となる.$\cos(kx-wt)$ で両辺を割ると,

$$\frac{d^2f(z)}{dz^2}-k^2f(z)=0$$

となる. $\frac{\partial}{\partial z}=D$ とおき, 計算する

$$(D^2-k^2)f(z)=0$$

となる. ここで $D=\lambda$ として解く.

$$\lambda^2-k^2=0$$
$$\lambda=\pm k$$

よって (48) から,

$$f(z)=C_1e^{kz}+C_2e^{-kz}\quad(C_1,C_2:\text{定数}) \tag{55}$$

となる. 又, (46) を用いる.

$$(C_1ke^{kh}-C_2ke^{kh})\cos(kx-\omega t)=0$$
$$C_1k-C_2k=0$$
$$C_1=C_2$$

となる.(48) に戻すと,

$$\begin{aligned}f(z)&=C_1(e^{kz}+e^{-kz})\\&=2C_1\cosh(kz)\end{aligned}$$

(47) に戻すと,

$$\begin{aligned}\Phi(x,z,t)&=2C_1\cosh(kz)\cos(kx-\omega t)\\&\quad 2C_1=A\text{ とおく}\\&=A\cosh(kz)\cos(kx-\omega t)\end{aligned}$$

(49) を用いると,

$$\Phi(x,z,t) = A\cosh\{k(z+h)\}\cos(kx-\omega t)$$

と表示できる.$T=0$ であるので,

$$\frac{\partial\Phi(x,z,t)}{\partial t} + g\zeta(x,t) = 0 \qquad (z=0 \quad 上) \tag{56}$$

である.(44), (56) を使って ζ を消去する. (44) を計算すると,

$$\frac{\partial\zeta(x,t)}{\partial t} - kA\sinh(kh)\cos(kx-\omega t) = 0$$

となる.(56) を計算すると,

$$-A\cosh(kh)\sin(kx-\omega t)(-\omega) + g\zeta(x,t) = 0 \tag{57}$$

となる.(50) を積分すると,

$$\zeta(x,t) = -\frac{1}{\omega}kA\sinh(kh)\sin(kx-\omega t)$$

となりこれを (57) に代入する.

$$A\cosh(kh)\sin(kx-\omega t)(\omega) - \frac{g}{\omega}kA\sinh(kh)\sin(kx-\omega t) = 0$$

となる. $-\frac{A}{\omega}\sinh(kh)\sin(kx-\omega t)$ で割ると,

$$-\frac{\omega^2}{\tanh(kh)} + gk = 0$$

$$\omega^2 = gk\tanh kh \tag{58}$$

となり, 分散関係式を得る. 位相速度 v_p と群速度 v_g を求めるために, まず ω を求める.

$$\begin{aligned}\omega^2 &= gk\tanh kh \ (k=0 \text{ でテイラー展開}) \\ &= gk^2h(1-\frac{1}{3}(kh)^2+\cdots)\end{aligned}$$

$$\omega^2 = gk^2h(1-\frac{1}{3}(kh)^2)$$

ここで,$(1-\frac{1}{3}(kh^2)+\cdots)$ での 3 項目を無視した.

$$\omega \quad = k\sqrt{gh}\sqrt{1-\frac{1}{3}(kh)^2} \tag{59}$$

$c_0 = \sqrt{gh}$ より,

$$\omega = kc_0\{1-\frac{1}{6}k^2h^2+\cdots\} \tag{60}$$

又,

$$kh = \infty \tag{61}$$

とする. (58) を書き直せば

$$\omega = \sqrt{gk}$$

となる. 位相速度 v_p と群速度 v_g は,(61) より, 次のように表示する.

$$v_p \equiv \frac{\omega(k)}{k} = c_0(1 - \frac{1}{6}(kh)^2 + \cdots)$$

$$v_p = (gh)^{\frac{1}{2}} \tag{62}$$

$$v_g \equiv \frac{\partial\omega(k)}{\partial k} = v_p + (gh)^{\frac{1}{2}} k(-\frac{1}{3}kh^2 + \cdots)$$

$$v_g = \frac{1}{2}(\frac{g}{k})^{\frac{1}{2}} \tag{63}$$

と表示できる. $v_p = (\frac{g}{k})^{\frac{1}{2}}, v_g = \frac{1}{2}(\frac{g}{k})^{\frac{1}{2}}$ となり, 分散性がある. 又, 分散性を表す係数は,「負」となる.

4 水面波の近似方程式の導出

本章では, 和達 [2, p.18-19] に沿って逓減摂動法を, 川原 [3, p.31-35],T.Kawahara J. Phys.[4, p.265-270] に沿って多重スケールについて述べる. 水面波の近似方程式を導く手法として 2 通りの手法を用いる. 置き方が異なっても, 摂動展開の仕方は変わらないので, 逓減摂動法と多重スケールの方法を用いて, 近似方程式を導出する. 水面波での近似方程式を導出する際に, 短波長, 長波長によって導かれる方程式が異なるので分けて考える. ここでは, 逓減摂動法では長波長, 多重スケールでは, 両方を求める.

4.1 逓減摂動法

1 つ目は, KdV 方程式, 変形 KdV 方程式を求める際に使用した逓減摂動法を用いて近似方程式を導出する. 簡単のため表面張力がない場合で考える.

$z = h$ が静止水面で $z = 0$ に底がある流体層における 1 次元平面波に対する方程式と境界条件を考える.

$$\frac{\partial}{\partial y} \equiv 0, c_0 = \sqrt{gh}$$

とする.

$$\triangle\Phi(x,z,t) = \frac{\partial^2\Phi(x,z,t)}{\partial^2 x} + \frac{\partial^2\Phi(x,z,t)}{\partial^2 z} = 0 \qquad (0 < z < \zeta(x,t)), \tag{64}$$

$$\frac{\partial\zeta(x,t)}{\partial t} + \frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x} - \frac{\partial\Phi(x,z,t)}{\partial z} = 0 \qquad (z = \zeta(x,t) \text{ 上}), \tag{65}$$

$$\frac{\partial\Phi(x,z,t)}{\partial t} + \frac{1}{2}\{(\frac{\partial\Phi(x,z,t)}{\partial x})^2 + (\frac{\partial\Phi(x,z,t)}{\partial z})^2\} + g(\zeta(x,t) - h) = 0 \qquad (z = \zeta(x,t) \text{ 上}), \tag{66}$$

$$\frac{\partial\Phi(x,z,t)}{\partial z} = 0 \qquad (z = 0 \text{ 上}) \tag{67}$$

に逓減摂動法を適用する. $\epsilon > 0$ を小さいパラメーターとする. 浅い水 ($kh \sim \epsilon^{\frac{1}{2}}$, すなわち, 波長 $\lambda \gg h$) の時, 線形波の位相 θ は,

$$\begin{aligned}
\theta = kx - \omega t &= k(x - c_0 t) + \frac{1}{6}c_1 h^2 k^3 t \qquad \text{((60) の右辺の第三項以降を無視する)} \\
&= kh\frac{x - c_0 t}{h} + \frac{1}{6}(kh)^3\frac{c_0 t}{h} \\
&= \epsilon^{\frac{1}{2}}\frac{x - c_0 t}{h} + \frac{1}{6}\epsilon^{\frac{3}{2}}\frac{c_0 t}{h} \qquad (kh \sim \epsilon^{\frac{1}{2}})
\end{aligned}$$

と表示される. x, t に対して, 新しい独立変数,

$$\xi = \epsilon^{\frac{1}{2}}(x - c_0 t), \tau = \epsilon^{\frac{3}{2}}t \tag{68}$$

を導入し, $\zeta(x,t), \Phi(x,z,t)$

$$\zeta(x,t) = h + \sum_{n=1}^{\infty}\epsilon^n\zeta^{(n)}(\xi,\tau), \tag{69}$$

$$\Phi(x,z,t) = \epsilon^{-\frac{1}{2}}\sum_{n=1}^{\infty}\epsilon^n\Phi^{(n)}(\xi,z,\tau) \tag{70}$$

と展開する.

(64), (65), (66), (67) で独立変数 x, t を ξ, τ に置き換え, ζ, Φ の展開式を代入し, ϵ のべきについて整理する.

(64) は,

$$\begin{aligned}
\triangle\Phi(x,z,t) &= \frac{\partial^2\Phi(x,z,t)}{\partial x^2}+\frac{\partial^2\Phi(x,z,t)}{\partial z^2} \\
&= \epsilon\frac{\partial^2\Phi(x,z,t)}{\partial \xi^2}+\frac{\partial^2\Phi(x,z,t)}{\partial z^2} \\
&= \epsilon(\epsilon^{\frac{1}{2}}\frac{\partial^2\Phi^{(1)}(\xi,z,\tau)}{\partial \xi^2}+\epsilon^{\frac{3}{2}}\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial \xi^2}+\cdots) \\
&+ (\epsilon^{\frac{1}{2}}\frac{\partial^2\Phi^{(1)}(\xi,z,\tau)}{\partial z^2}+\epsilon^{\frac{3}{2}}\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial z^2}+\cdots) \\
&= 0
\end{aligned}$$

となるので,

$$\begin{aligned}
&\epsilon^{\frac{1}{2}}:\frac{\partial^2\Phi^{(1)}(\xi,z,\tau)}{\partial z^2}=0, \\
&\epsilon^{\frac{3}{2}}:\frac{\partial^1\Phi^{(1)}(\xi,z,\tau)}{\partial \xi^2}+\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial z^2}=0, \\
&\epsilon^{\frac{5}{2}}:\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial \xi^2}+\frac{\partial^2\Phi^{(3)}(\xi,z,\tau)}{\partial z^2}=0
\end{aligned}$$

を得る. これより, 同様に,

$$\frac{\partial^2\Phi^{(1)}(\xi,z,\tau)}{\partial z^2}=0\ \ (0\le z\le h), \tag{71}$$

$$\frac{\partial^2\Phi^{(n-1)}(\xi,z,\tau)}{\partial \xi^2}+\frac{\partial^2\Phi^{(n)}(\xi,z,\tau)}{\partial z^2}=0\ \ (n\ge 2, 0\le z\le h) \tag{72}$$

となる.

又, (67) より,

$$\frac{\partial\Phi^{(n)}(\xi,z,\tau)}{\partial z}=0\quad (n\ge 1, z=0) \tag{73}$$

となる. (71), (72), (73) から, $\Phi^{(1)}(\xi,z,\tau), \Phi^{(2)}(\xi,z,\tau), \Phi^{(3)}(\xi,z,\tau)$ を求めることができる.

まず最初に (71) を使って $\Phi^{(1)}(\xi,z,\tau)$ を求める.

$$\begin{aligned}
&\frac{\partial^2\Phi^{(1)}(\xi,z,\tau)}{\partial z^2}=0, \\
&\frac{\partial\Phi^{(1)}(\xi,z,\tau)}{\partial z}=0 \qquad (z\text{ で積分し}, (73)\text{ を使う}), \\
&\Phi^{(1)}(\xi,z,\tau)=\Phi^{(1)}(\xi,\tau) \qquad (\text{右辺は}, z\text{ に依存しない関数とする}).
\end{aligned} \tag{74}$$

次に (72) を使って $\Phi^{(2)}(\xi,z,\tau)$ を求める.

$$\frac{\partial^2\Phi^{(1)}(\xi,z,\tau)}{\partial\xi^2}+\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial z^2}=0 \qquad (n=2),$$
$$\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial z^2}=-\frac{\partial^2}{\partial\xi^2}(\Phi^{(1)}(\xi,\tau)) \qquad ((74)\text{ を使う}),$$
$$\frac{\partial\Phi^{(2)}(\xi,z,\tau)}{\partial z}=-z\frac{\partial^2}{\partial\xi^2}(\Phi^{(1)}(\xi,\tau)) \qquad (z\text{ で積分し}, (73)\text{ を使う}),$$
$$\Phi^{(2)}(\xi,z,\tau)=-\frac{z^2}{2}\frac{\partial^2\Phi^{(1)}(\xi,\tau)}{\partial\xi^2}+a(\xi,\tau) \qquad (a\text{ は }z\text{ に依存しない任意関数}). \tag{75}$$

最後に (72) を使って $\Phi^{(3)}(\xi,z,\tau)$ を求める.

$$\frac{\partial^2\Phi^{(2)}(\xi,z,\tau)}{\partial\xi^2}+\frac{\partial^2\Phi^{(3)}(\xi,z,\tau)}{\partial z^2}=0 \qquad (n=3),$$
$$\frac{\partial^2\Phi^{(3)}(\xi,z,\tau)}{\partial z^2}=-\frac{\partial^2}{\partial\xi^2}(-\frac{z^2}{2}\frac{\partial^2\Phi^{(1)}(\xi,\tau)}{\partial\xi^2}+a(\xi,\tau)) \qquad ((75)\text{ を使う}),$$
$$\frac{\partial\Phi^{(3)}(\xi,z,\tau)}{\partial z}=\frac{z^3}{6}\frac{\partial^4\Phi^{(1)}(\xi,\tau)}{\partial\xi^4}-z\frac{\partial^2 a}{\partial\xi^2} \qquad (z\text{ で積分し}, (73)\text{ を使う}),$$
$$\Phi^{(3)}(\xi,z,\tau)=\frac{z^4}{24}\frac{\partial^4\Phi^{(1)}(\xi,\tau)}{\partial\xi^4}-\frac{z^2}{2}\frac{\partial^2 a}{\partial\xi^2}+b(\xi,\tau)$$
$$(b\text{ は }z\text{ に依存しない任意関数}) \tag{76}$$

となる. 又, $z=\zeta$ についての境界条件は, $z=h$ からの展開として考えることができる. (69) を使うと,

$$\begin{aligned}\Phi(\xi,\zeta,\tau) &= \Phi(\xi,h,\tau)+\frac{\partial\Phi(\xi,h,\tau)}{\partial z}(\zeta-h)+\frac{\partial^2\Phi(\xi,h,\tau)}{\partial z^2}\frac{(\zeta-h)^2}{2}+\cdots\\ &= \Phi(\xi,h,\tau)+\frac{\partial\Phi(\xi,h,\tau)}{\partial z}(\epsilon\zeta^{(1)}+\epsilon^2\zeta^{(2)}+\cdots)\\ &+\frac{\partial^2\Phi(\xi,h,\tau)}{\partial z^2}\frac{1}{2}(\epsilon\zeta^{(1)}+\epsilon^2\zeta^{(2)}+\cdots)^2+\cdots\end{aligned} \tag{77}$$

となる.(77) の両辺を (n) 次式で書き換えると,

$$\begin{aligned}\Phi^{(n)}(\xi,\zeta,\tau) &= \Phi^{(n)}(\xi,h,\tau)+\epsilon\frac{\partial\Phi^{(n)}(\xi,h,\tau)}{\partial z}\zeta^{(1)}\\ &+\epsilon^2(\frac{\partial\Phi^{(n)}(\xi,h,\tau)}{\partial z}\zeta^{(2)}+\frac{\partial\Phi^{(n)}(\xi,h,\tau)}{\partial z^2}\frac{1}{2}{\zeta^{(1)}}^2+\cdots)+\cdots\end{aligned} \tag{78}$$

となる. 次に (65) を導いていく. この時, (68) ～ (70), (74) ～ (76), (78) を使うと,

$$\frac{\partial\zeta(x,t)}{\partial x}=\sum_{n=1}^{\infty}\epsilon^{n+\frac{1}{2}}\frac{\partial\zeta^{(n)}(\xi,\tau)}{\partial\xi}=\epsilon^{\frac{3}{2}}\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi}+\cdots,$$
$$\frac{\partial\Phi(x,z,t)}{\partial z}|_{z=\zeta(x,t)} = \epsilon^{\frac{3}{2}}\frac{\partial\Phi^{(2)}(\xi,h,\tau)}{\partial z}+\epsilon^{\frac{5}{2}}(\frac{\partial^2\Phi^{(2)}(\xi,h,\tau)}{\partial z^2}\zeta^{(1)}(\xi,\tau)+\frac{\partial\Phi^{(3)}(\xi,h,\tau)}{\partial z}) +\cdots,$$
$$\frac{\partial\zeta(x,t)}{\partial t} = \epsilon^{\frac{3}{2}}(-c_0)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi}+\epsilon^{\frac{5}{2}}((-c_0)\frac{\partial\zeta^{(2)}(\xi,\tau)}{\partial\xi}+\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\tau}) +\cdots,$$
$$\frac{\partial\Phi(x,z,t)}{\partial x}|_{z=\zeta(x,t)}=\epsilon\frac{\partial\Phi^{(1)}(\xi,\tau)}{\partial\xi}+\cdots$$

となる.(65) に代入して, 最低次の項だけでまとめると

$$\epsilon^{\frac{3}{2}} : (-c_0)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} - \frac{\partial\Phi^{(2)}(\xi,h,\tau)}{\partial z} = 0 \quad (z=\zeta(x,t)) \tag{79}$$

$$\epsilon^{\frac{5}{2}} : (-c_0)\frac{\partial\zeta^{(2)}(\xi,\tau)}{\partial\xi} + \frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\tau} + \frac{\partial\Phi^{(1)}(\xi,\tau)}{\partial\xi}\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi}$$

$$-\frac{\partial^2\Phi^{(2)}(\xi,h,\tau)}{\partial z^2}\zeta^{(1)}(\xi,\tau) - \frac{\partial\Phi^{(3)}(\xi,h,\tau)}{\partial z} = 0 \quad (z=\zeta(x,t)) \tag{80}$$

を得る. さらに (66) 式を導いていく. この時, (68) ∼ (70), (74) ∼ (76), (78) を使う.

$$\frac{\partial\Phi(x,z,t)}{\partial t} = (-c_0)\epsilon^{\frac{1}{2}}\frac{\partial\Phi(x,z,t)}{\partial\xi} + \epsilon^{\frac{3}{2}}\frac{\partial\Phi(x,z,t)}{\partial\tau}$$

$$= (-c_0)\sum_{n=1}^{\infty}\epsilon^n\frac{\partial\Phi^{(n)}(\xi,z,\tau)}{\partial\xi} + \sum_{n=1}^{\infty}\epsilon^{n+1}\frac{\partial\Phi^{(n)}(\xi,z,\tau)}{\partial\tau}$$

$$= \epsilon(-c_0)\frac{\partial\Phi^{(1)}(\xi,z,\tau)}{\partial\xi} + \epsilon^2((-c_0)\frac{\partial\Phi^{(2)}(\xi,z,\tau)}{\partial\xi} + \frac{\partial\Phi^{(1)}(\xi,z,\tau)}{\partial\tau}) + \cdots$$

$$= \epsilon(-c_0)\frac{\partial\Phi^{(1)}(\xi,\zeta,\tau)}{\partial\xi} + \epsilon^2((-c_0)\frac{\partial\Phi^{(2)}(\xi,\zeta,\tau)}{\partial\xi} + \frac{\partial\Phi^{(1)}(\xi,\zeta,\tau)}{\partial\tau}) + \cdots \quad (z=\zeta(x,t))$$

$$= \epsilon(-c_0)\frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi} + \epsilon^2((-c_0)\frac{\partial\Phi^{(2)}(\xi,h,\tau)}{\partial\xi}$$

$$+\frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\tau}) + \cdots ((74),(78) \text{ を使う}, z=h),$$

$$\frac{\partial\Phi(x,z,t)}{\partial x} = \epsilon^{\frac{1}{2}}\frac{\partial\Phi(x,z,t)}{\partial\xi}$$

$$= \sum_{n=1}^{\infty}\epsilon^n\frac{\partial\Phi^{(n)}(\xi,z,\tau)}{\partial\xi}$$

$$= \epsilon\frac{\partial\Phi^{(1)}(\xi,z,\tau)}{\partial\xi} + \epsilon^2\frac{\partial\Phi^{(2)}(\xi,z,\tau)}{\partial\xi} + \cdots,$$

$$(\frac{\partial\Phi(x,z,t)}{\partial x})^2 = \epsilon^2(\frac{\partial\Phi^{(1)}(\xi,z,\tau)}{\partial\xi})^2 + \cdots$$

$$= \epsilon^2(\frac{\partial\Phi^{(1)}(\xi,\zeta,\tau)}{\partial\xi})^2 + \cdots \quad (z=\zeta(x,t))$$

$$= \epsilon^2(\frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi})^2 + \cdots \quad ((74),(78) \text{ を使う}) \quad (z=h),$$

$$\frac{\partial\Phi(x,z,t)}{\partial z} = \epsilon^{-\frac{1}{2}}\sum_{n=1}^{\infty}\epsilon^n\frac{\partial\Phi^{(n)}(\xi,z,\tau)}{\partial z}$$

$$= \epsilon^{\frac{3}{2}}\frac{\partial\Phi^{(2)}(\xi,z,\tau)}{\partial z} + \cdots$$

$$= \epsilon^{\frac{3}{2}}\frac{\partial\Phi^{(2)}(\xi,\zeta,\tau)}{\partial z} + \cdots \quad (z=\zeta(x,t)),$$

$$(\frac{\partial\Phi(x,z,t)}{\partial z})^2 = \epsilon^3(\frac{\partial\Phi^{(2)}(\xi,h,\tau)}{\partial z})^2 + \cdots \quad ((74),(78) \text{ を使う}) \quad (z=h),$$

$$g(\zeta(x,t)-h) = g(\epsilon\zeta^{(1)}(\xi,\tau)+\epsilon^2\zeta^{(2)}(\xi,\tau)+\cdots) \quad (z=h)$$

となる. 以上から,ϵ の各次数で表示する.

$$\epsilon : -c_0\frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi}+g\zeta^{(1)}(\xi,\tau)=0 \quad (z=h) \tag{81}$$

$$\begin{aligned}\epsilon^2 : &-c_0\frac{\partial\Phi^{(2)}(\xi,h,\tau)}{\partial\xi}+\frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\tau}+\frac{1}{2}(\frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi})^2+g\zeta^{(2)}(\xi,\tau)\\ &=0 \quad (z=h)\end{aligned} \tag{82}$$

(80) を見ると, 各項が $\zeta^{(1)}(\xi,\tau)$ で表されている事がわかるので書き直す.

$$\begin{aligned}\frac{\partial^2\Phi^{(2)}(\xi,h,\tau)}{\partial z^2}\zeta^{(1)}(\xi,\tau) &= \zeta^{(1)}(\xi,\tau)\frac{\partial^2}{\partial z^2}\{-\frac{z^2}{2}\frac{\partial^2\Phi^{(1)}(\xi,h,\tau)}{\partial\xi^2}+a(\xi,\tau)\} \quad ((74),(75))\\ &= \zeta^{(1)}(\xi,\tau)(-\frac{\partial^2\Phi^{(1)}(\xi,\tau)}{\partial\xi^2})\\ &= -\zeta^{(1)}(\xi,\tau)\frac{g}{c_0}\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} \ (81)\\ &= -\frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi},\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi^{(1)}(\xi,\tau)}{\partial\xi}\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} &= \frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi}\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} \quad ((74)\text{ を使う})\\ &= \frac{g}{c_0}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi}(81)\\ &= \frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi},\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi^{(3)}(\xi,h,\tau)}{\partial z} &= \frac{\partial}{\partial z}\frac{z^4}{24}\frac{\partial^4\Phi^{(1)}(\xi,\tau)}{\partial\xi^4}-\frac{z^2}{2}\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2}+b(\xi,\tau) \quad ((76)\text{ を使う})\\ &= \frac{z^3}{6}\frac{\partial^4\Phi^{(1)}(\xi,\tau)}{\partial\xi^4}-z\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2}\\ &= \frac{h^3}{6}\frac{\partial^4\Phi^{(1)}(\xi,h,\tau)}{\partial\xi^4}-h\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2} \quad ((74), z=h\text{ を使う})\\ &= \frac{h^3}{6}\frac{g}{c_0}\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial\xi^3}-h\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2} \quad ((81)\text{ を使う})\\ &= \frac{1}{6}h^2c_0\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial\xi^3}-h\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2},\end{aligned}$$

$$
\begin{aligned}
(-c_0)\frac{\partial\zeta^{(2)}(\xi,\tau)}{\partial\xi} &= \frac{c_0}{g}\{\frac{\partial^2\Phi^{(1)}(\xi,h,\tau)}{\partial\tau\partial\xi} - c_0\frac{\partial^2\Phi^{(2)}(\xi,h,\tau)}{\partial\xi^2} \\
&+ \frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi}\frac{\partial^2\Phi^{(1)}(\xi,h,\tau)}{\partial\xi^2}\} \quad ((82)\text{ を使う}) \\
&= \frac{c_0}{g}\{\frac{\partial^2\Phi^{(1)}(\xi,h,\tau)}{\partial\tau\partial\xi} + \frac{1}{2}c_0h^2\frac{\partial^4\Phi^{(1)}(\xi,h,\tau)}{\partial\xi^4} - c_0\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2} \\
&+ \frac{\partial\Phi^{(1)}(\xi,h,\tau)}{\partial\xi}\frac{\partial^2\Phi^{(1)}(\xi,h,\tau)}{\partial\xi^2}\} \quad ((75)\text{ を使う}) \\
&= \frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\tau} + \frac{1}{2}c_0h^2\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial\xi^3} - h\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2} \\
&+ \frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} \quad ((81)\text{ を使う})
\end{aligned}
$$

となる. 以上をまとめると,(80) は,

$$
\begin{aligned}
&\frac{1}{6}h^2c_0\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial\xi^3} - h\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2} - \frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} = \frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\tau} + \frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\tau} \\
&+ \frac{1}{2}c_0h^2\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial\xi^3} - h\frac{\partial^2 a(\xi,\tau)}{\partial\xi^2} + \frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} + \frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi}
\end{aligned}
$$

となる. 左辺を右辺に移項して整理すると,

$$
\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\tau} + \frac{3}{2}\frac{c_0}{h}\zeta^{(1)}(\xi,\tau)\frac{\partial\zeta^{(1)}(\xi,\tau)}{\partial\xi} + \frac{1}{6}c_0h^2\frac{\partial^3\zeta^{(1)}(\xi,\tau)}{\partial\xi^3} = 0 \tag{83}
$$

となり, KdV 方程式が導かれる事がわかった.

4.2　多重スケール

2つ目の手法として多重スケールという手法で求めてみる．多重スケールとは微小パラメーターを導入して，独立変数を多くの異なるスケールに拡張する手法である．この手法を用いて近似方程式を導出する．又，水面波の近似方程式は多重スケールの手法を用いると長波長と短波長で方程式が変わってくる．よって別々に考えてみる．表面張力がある場合で考える．

4.2.1　短波方程式

短波長の波のみが存在するとき，非線形シュレディンガー方程式が求められる．実際，非線形シュレディンガー方程式が導いてみる．$z=0$ が静止水面で $z=-h$ に底がある流体層における 1 次元平面波に対する方程式と境界条件を考える．

$$\frac{\partial}{\partial y} \equiv 0$$

とすると，

$$\triangle\Phi(x,z,t) = \frac{\partial^2\Phi(x,z,t)}{\partial x^2} + \frac{\partial^2\Phi(x,z,t)}{\partial z^2} = 0 \qquad (-h < z < \zeta(x,t)), \tag{84}$$

$$\frac{\partial\zeta(x,t)}{\partial t} + \frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x} - \frac{\partial\Phi(x,z,t)}{\partial z} = 0 \quad (z=\zeta(x,t)\text{ 上}), \tag{85}$$

$$\begin{aligned}&\frac{\partial\Phi(x,z,t)}{\partial t} + \frac{1}{2}\{(\frac{\partial\Phi(x,z,t)}{\partial x})^2 + (\frac{\partial\Phi(x,z,t)}{\partial z})^2\} + g\zeta(x,t)\\ &-\frac{T}{\rho}(\frac{\partial^2\zeta(x,t)}{\partial x^2})\{1+(\frac{\partial\zeta(x,t)}{\partial x})^2\}^{-\frac{3}{2}} = 0 \qquad (z=\zeta(x,t)\text{ 上}),\end{aligned} \tag{86}$$

$$\frac{\partial\Phi(x,z,t)}{\partial z} = 0 \qquad (z=-h\text{ 上}) \tag{87}$$

となる．空間・時間スケールを多重スケールに拡張する。

$$x_n = \epsilon^n x, t_n = \epsilon^n t, \quad (x_0 = x, t_0 = t) \quad (n = 0,1,2\cdots) \tag{88}$$

今，ζ と Φ を

$$\zeta(x,t) = \sum_{n=1}^{\infty}\epsilon^n\zeta_n(x_0,x_1,x_2,\cdots,t_0,t_1,t_2,\cdots), \tag{89}$$

$$\Phi(x,z,t) = \sum_{n=1}^{\infty}\epsilon^n\Phi_n(x_0,x_1,x_2,\cdots,z,t_0,t_1,t_2,\cdots) \tag{90}$$

を使って (84) ∼ (87) に代入して $\epsilon,\epsilon^2,\epsilon^3$ の項としてまとめる．

境界条件を $z=0$ でテイラー展開する. (90) より,

$$
\begin{aligned}
&\frac{\partial^2\Phi(x,z,t)}{\partial x^2}\\
&=\frac{\partial}{\partial x}(\epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\epsilon+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_2}\epsilon^2+\cdots)\\
&+\epsilon^2(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\epsilon+\cdots)\\
&+\epsilon^3(\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\cdots)+\cdots)\\
&=\epsilon(\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0^2}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}\epsilon+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_2}\epsilon^2+\cdots)\\
&+\epsilon^2(\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1^2}\epsilon+\cdots)\\
&+\epsilon^3(\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_2}+\cdots)\\
&+\epsilon^2(\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0^2}+\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}\epsilon+\cdots)\\
&+\epsilon^3(\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}+\cdots)+\epsilon^3(\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_0^2}+\cdots)+\cdots\\
&=\epsilon(\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0^2})+\epsilon^2(\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0^2}+2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1})\\
&+\epsilon^3(2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_2}+2\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1^2}\\
&+\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_0^2})\\
&+\cdots,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\Phi(x,z,t)}{\partial z^2}&=\epsilon\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}+\epsilon^2\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&+\epsilon^3\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}+\cdots
\end{aligned}
$$

となる. よって,(84) より,

$$
\begin{aligned}
&O(\varepsilon):\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial {x_0}^2}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}=0,\\
&O(\varepsilon^2):\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial {x_0}^2}+\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}=-2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1},\\
&O(\varepsilon^3):\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial {x_0}^2}+\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&=-2\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}-2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_2}-\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial {x_1}^2}
\end{aligned}
$$

となる．(89), (90) より，

$$
\begin{aligned}
&\frac{\partial\zeta(x,t)}{\partial t} = \frac{\partial}{\partial t}(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\epsilon^2\zeta_2(x_0,\cdots,t_0,\cdots)+\epsilon^3\zeta_3(x_0,\cdots,t_0,\cdots)+\cdots) \\
&= \epsilon(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_0}+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_1}\epsilon+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_2}\epsilon^2+\cdots) \\
&+\epsilon^2(\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_0}+\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_1}\epsilon+\cdots) \\
&+\epsilon^3(\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0}+\cdots)+\cdots \\
&= \epsilon(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_0})+\epsilon^2(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_1}+\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_0}) \\
&+\epsilon^3(\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0}+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_2}+\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_1})+\cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\Phi(x,z,t)}{\partial x} \\
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})+\epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}) \\
&+\epsilon^3(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_2}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_1} \\
&+\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})+\cdots
\end{aligned}
$$

となる．ここで $z=0$ で展開して, (34), (89) を使うと,

$$
\begin{aligned}
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0}+\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z}(\epsilon\zeta_1+\epsilon^2\zeta_2+\cdots) \\
&+\frac{\partial^3\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z^2}\frac{(\epsilon^2\zeta_1^2+\cdots)}{2}+\cdots) \\
&+\epsilon^2\{(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_1}+\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_1\partial z}(\epsilon\zeta_1+\cdots)+\cdots) \\
&+(\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial x_0}+\frac{\partial^2\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z}(\epsilon\zeta_1+\cdots)+\cdots)\} \\
&+\epsilon^3(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial x_1}+\frac{\partial\Phi_3(x_0,\cdots,0,t_0,\cdots)}{\partial x_0}+\cdots) \\
&+\cdots \\
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0})+\epsilon^2(\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z} \\
&+\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_1} \\
&+\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial x_0})+\epsilon^3(\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z} \\
&+\frac{\zeta_1(x,t)^2}{2}\frac{\partial^3\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z^2} \\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_1\partial z}+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial x_0\partial z} \\
&+\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial x_2}+\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial x_1}+\frac{\partial\Phi_3(x_0,\cdots,0,t_0,\cdots)}{\partial x_0})+\cdots,
\end{aligned}
$$

となる. 同様に

$$\frac{\partial\zeta(x,t)}{\partial x}=\frac{\partial}{\partial x}(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\epsilon^2\zeta_2(x_0,\cdots,t_0,\cdots)+\epsilon^3\zeta_3(x_0,\cdots,t_0,\cdots)+\cdots)$$
$$=\epsilon(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1}\epsilon+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_2}\epsilon^2+\cdots)$$
$$+\epsilon^2(\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1}\epsilon+\cdots)$$
$$+\epsilon^3(\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\cdots)+\cdots$$
$$=\epsilon(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})+\epsilon^2(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1}+\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0})$$
$$+\epsilon^3(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_2}+\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1}+\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0})+\cdots,$$

$$\frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x}=\epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})$$
$$+\epsilon^3(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}(\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1})$$
$$+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0}(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}$$
$$+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}))+\cdots,$$

$$\frac{\partial\Phi(x,z,t)}{\partial z}$$
$$=\epsilon\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z}+\epsilon^2\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z}$$
$$+\epsilon^3\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z}+\cdots$$

となる. $z=0$ で展開して, (34), (89) を使うと

$$=\epsilon(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z}+\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z^2}(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)$$
$$+\epsilon^2\zeta_2(x_0,\cdots,t_0,\cdots)+\cdots)$$
$$+\frac{\partial^3\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z^3}(\frac{\epsilon^2\zeta_1^2(x_0,\cdots,t_0,\cdots)}{2}+\cdots)+\cdots)+\epsilon^2(\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial z}$$
$$+\frac{\partial^2\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial z^2}(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\cdots)+\cdots)$$
$$+\epsilon^3(\frac{\partial\Phi_3(x_0,\cdots,0,t_0,\cdots)}{\partial z}+\cdots)+\cdots$$
$$=\epsilon(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z})+\epsilon^2(\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z^2}$$
$$+\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial z})$$
$$+\epsilon^3(\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z^2}+\frac{\zeta_1^2(x_0,\cdots,t_0,\cdots)}{2}\frac{\partial^3\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial z^3}$$
$$+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial z^2}+\frac{\partial\Phi_3(x_0,\cdots,0,t_0,\cdots)}{\partial z})+\cdots$$

を得る. よって,(85) より

$$
\begin{aligned}
O(\varepsilon) \quad &: \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_0} - \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial z} = 0 \ (z=\zeta(x,t) \text{ 上}),\\
O(\varepsilon^2) \quad &: \frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_0} - \frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} = -\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_1}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&-\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0} \ (z=\zeta(x,t) \text{ 上}),\\
O(\varepsilon^3) \quad &: \frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0} - \frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z} = -\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_1}\\
&-\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_2} + \zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&+\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&+\frac{1}{2}\zeta_1(x_0,\cdots,t_0,\cdots)^2\frac{\partial^3\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^3}\\
&-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\{\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0} + \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1}\}\\
&-(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}\} \ (z=\zeta(x,t) \text{ 上})
\end{aligned}
$$

が得られる.

$$
\begin{aligned}
\frac{\partial\Phi(x,z,t)}{\partial t} \quad &= \frac{\partial}{\partial t}(\epsilon\Phi_1(x_0,\cdots,z,t_0,\cdots) + \epsilon^2\Phi_2(x_0,\cdots,z,t_0,\cdots)\\
&+\epsilon^3\Phi_3(x_0,\cdots,z,t_0,\cdots)+\cdots)\\
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0} + \epsilon\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\\
&+\epsilon^2\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2} + \cdots) + \epsilon^2(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}\\
&+\epsilon\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1} + \cdots)\\
&+\epsilon^3(\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0} + \cdots) + \cdots\\
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}) + \epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\\
&+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0})\\
&+\epsilon^3(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2} + \frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\\
&+\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}) + \cdots
\end{aligned}
$$

$z=0$ で展開して, (34), (89) を使うと,

$$
\begin{aligned}
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}\\
&(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\epsilon^2\zeta_2(x_0,\cdots,t_0,\cdots)+\cdots)\\
&+\frac{\partial^3\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z^2}(\frac{\epsilon^2\zeta_1^2(x_0,\cdots,t_0,\cdots)}{2}+\cdots)+\cdots)\\
&+\epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\\
&+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1\partial z}(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\cdots)+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}\\
&+\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\cdots)+\cdots)\\
&+\epsilon^3(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}\\
&+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}+\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0})+\cdots\\
&= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_0})+\epsilon^2(\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_0\partial z}\\
&+\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial t_0}\\
&+\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_1})+\epsilon^3(\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_0\partial z}\\
&+\frac{1}{2}\zeta_1(x_0,\cdots,t_0,\cdots)^2\frac{\partial^3\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_0\partial z^2}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_1\partial z}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial t_0\partial z}\\
&+\frac{\partial\Phi_1(x_0,\cdots,0,t_0,\cdots)}{\partial t_2}+\frac{\partial\Phi_2(x_0,\cdots,0,t_0,\cdots)}{\partial t_1}\\
&+\frac{\partial\Phi_3(x_0,\cdots,0,t_0,\cdots)}{\partial t_0})+\cdots
\end{aligned}
$$

となる. また,

$$
\begin{aligned}
g\zeta(x,t) &= g(\epsilon\zeta_1(x_0,\cdots,t_0,\cdots)+\epsilon^2\zeta_2(x_0,\cdots,t_0,\cdots)+\epsilon^3\zeta_3(x_0,\cdots,t_0,\cdots)+\cdots)\\
&= \epsilon(g\zeta_1(x_0,\cdots,t_0,\cdots))+\epsilon^2(g\zeta_2(x_0,\cdots,t_0,\cdots))+\epsilon^3(g\zeta_3(x_0,\cdots,t_0,\cdots))+\cdots
\end{aligned}
$$

$$
\begin{aligned}
(\frac{\partial\Phi(x,z,t)}{\partial x})^2 &= \epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})^2+\epsilon^3(2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\\
&(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}))+\cdots
\end{aligned}
$$

$$
\begin{aligned}
(\frac{\partial\Phi(x,z,t)}{\partial z})^2 \quad &= \epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})^2 \\
&+\epsilon^3(2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} \\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2})+)\cdots,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\zeta(x,t)}{\partial x^2} \quad &= \frac{\partial}{\partial x}(\epsilon(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0}) + \epsilon^2(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1} + \frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0}) \\
&+\epsilon^3(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_2} + \frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1} + \frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0}) + \cdots \\
&= \epsilon(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2} + \frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1}\epsilon \\
&+\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_2}\epsilon^2 + \cdots) \\
&+\epsilon^2(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1} + \frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1^2}\epsilon + \cdots \\
&+\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0^2} + \frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1}\epsilon + \cdots) \\
&+\epsilon^3(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_2\partial x_0} + \frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1} \\
&+\frac{\partial^2\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0^2} + \cdots) + \cdots \\
&= \epsilon(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2}) + \epsilon^2(2\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1} + \frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0^2}) \\
&+\epsilon^3(2\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_2} + \frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1^2} \\
&+2\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1} + \frac{\partial^2\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0^2}) + \cdots
\end{aligned}
$$

$$
(\frac{\partial\zeta(x,t)}{\partial x})^2 = \epsilon^2(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})^2 + \cdots
$$

$$
\{1+(\frac{\partial\zeta(x,t)}{\partial x})^2\}^{-\frac{3}{2}} = 1 + (-\frac{3}{2}\epsilon^2(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})^2) + \cdots,
$$

$$
\begin{aligned}
(\frac{\partial^2\zeta(x,t)}{\partial x^2})\{1+(\frac{\partial\zeta(x,t)}{\partial x})^2\}^{-\frac{3}{2}} \quad &= \epsilon(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2}) \\
&+\epsilon^2(2\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1} + \frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0^2}) \\
&+\epsilon^3(2\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_2} + \frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1^2} \\
&+2\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1} + \frac{\partial^2\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0^2} \\
&-\frac{3}{2}\epsilon^2(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})^2(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2}) + \cdots
\end{aligned}
$$

となる. 以上より,(86) より,

$$
\begin{aligned}
O(\varepsilon) \quad & : \frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0} + g\zeta_1(x_0,\cdots,t_0,\cdots) \\
& -(\frac{T}{\rho})(\frac{\partial^2 \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2}) = 0 \qquad (z=\zeta(x,t)\ \text{上}), \\
O(\varepsilon^2) \quad & : \frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0} + g\zeta_2(x_0,\cdots,t_0,\cdots) - (\frac{T}{\rho})(\frac{\partial^2 \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2}) \\
& = -\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1} - \zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0 \partial z} \\
& -\frac{1}{2}\{(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})^2 \\
& +(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})^2\} + 2(\frac{T}{\rho})(\frac{\partial^2 \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0 \partial x_1}) \qquad (z=\zeta(x,t)\ \text{上}), \\
O(\varepsilon^3) \quad & : \frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0} + g\zeta_3(x_0,\cdots,t_0,\cdots) - (\frac{T}{\rho})(\frac{\partial^2 \zeta_3(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2}) \\
& = -\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1} - \frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2} \\
& -\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_2}{\partial t_0 \partial z} - \zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1 \partial z} \\
& +\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0 \partial z} \\
& -\frac{1}{2}\zeta_1(x_0,\cdots,t_0,\cdots)^2\frac{\partial^3 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0 \partial z^2} \\
& -(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\{\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1} \\
& +\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0 \partial z}\} \\
& -(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})\{\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} \\
& +\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\} + (\frac{T}{\rho})\{2\frac{\partial^2 \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0 \partial x_1} \\
& +2\frac{\partial^2 \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0 \partial x_2} \\
& +\frac{\partial^2 \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1^2} \\
& -(\frac{3}{2})(\frac{\partial^2 \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2})(\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})^2\} \qquad (z=\zeta(x,t)\ \text{上}),
\end{aligned}
$$

が得られる.

$$
\frac{\partial \Phi(x,z,t)}{\partial z} = \epsilon\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z} + \epsilon^2\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} + \epsilon^3\frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z} + \cdots
$$

であるので,(87) より

$$
\begin{aligned}
O(\varepsilon) : \frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z} &= 0 \qquad (z=-h\ \text{上}), \\
O(\varepsilon^2) : \frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} &= 0 \qquad (z=-h\ \text{上}), \\
O(\varepsilon^3) : \frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z} &= 0 \qquad (z=-h\ \text{上})
\end{aligned}
$$

が得られる. 以上 4 つの式から得られる関係式を $\epsilon, \epsilon^2, \epsilon^3$ に分けて, まとめると,
$O(\varepsilon)$:

$$\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial {x_0}^2}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}=0, \tag{91}$$

$$\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_0}-\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial z}=0 \qquad (z=\zeta(x,t)\ \text{上}), \tag{92}$$

$$\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+g\zeta_1(x_0,\cdots,t_0,\cdots)-(\frac{T}{\rho})(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2})=0 \qquad (z=\zeta(x,t)\ \text{上}), \tag{93}$$

$$\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z}=0 \qquad (z=-h\ \text{上}), \tag{94}$$

$O(\varepsilon^2)$:

$$\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial {x_0}^2}+\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}=-2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}, \tag{95}$$

$$\begin{aligned}&\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_0}-\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z}=-\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_1}\\&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\&-\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0}(z=\zeta(x,t)\ \text{上}),\end{aligned} \tag{96}$$

$$\begin{aligned}&\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+g\zeta_2(x_0,\cdots,t_0,\cdots)-(\frac{T}{\rho})(\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2})\\&=-\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}-\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}\\&-\frac{1}{2}\{(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})^2+(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})^2\}\\&+2(\frac{T}{\rho})(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1}) \qquad (z=\zeta(x,t)\ \text{上}),\end{aligned} \tag{97}$$

$$\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z}=0 \qquad (z=-h\ \text{上}), \tag{98}$$

$O(\varepsilon^3)$:

$$\begin{aligned}&\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial {x_0}^2}+\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}=-2\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}\\&-2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_2}-\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial {x_1}^2}\end{aligned} \tag{99}$$

$$\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0}-\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z}=-\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_1}$$
$$-\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_2}+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}$$
$$+\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}$$
$$+\frac{1}{2}\zeta_1^2(x_0,\cdots,t_0,\cdots)\frac{\partial^3\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^3}$$
$$-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\{\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1}\}$$
$$-(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}$$
$$+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}\}\qquad(z=\zeta(x,t)\text{ 上}),\tag{100}$$

$$\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+g\zeta_3(x_0,\cdots,t_0,\cdots)-(\frac{T}{\rho})(\frac{\partial^2\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2})$$
$$=-\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}-\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}$$
$$-\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}$$
$$-\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1\partial z}+\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}$$
$$-\frac{1}{2}\zeta_1^2(x_0,\cdots,t_0,\cdots)\frac{\partial^3\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z^2}$$
$$-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}$$
$$+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}\}$$
$$-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z}$$
$$+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\}$$
$$+(\frac{T}{\rho})\{2\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1}+2\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_2}$$
$$+\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1^2}-(\frac{3}{2})(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2})(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})^2\}$$
$$(z=\zeta(x,t)\text{ 上})\tag{101}$$

$$\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z}=0\qquad(z=-h\text{ 上})\tag{102}$$

となる．次に $O(\varepsilon)$ の関係式の解を求める． $A=A(x_1,\cdots,t_1,\cdots),\Psi_1=\Psi_1(x_1,\cdots,t_1,\cdots)$ とし,$e^{i\theta}$ によらない関数する．又, $\theta\equiv kx_0-\omega t_0$,$B(z)$ を z の関数とする．$c.c.$ は複素共役を表す.

$$\zeta_1(x_0,\cdots,t_0,\cdots)=A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.\tag{103}$$

と仮定する.

$$\Phi_1(x_0,\cdots,z,t_0,\cdots)=B(z)e^{i\theta}+c.c.+\Psi_1(x_1,\cdots,t_1,\cdots)$$

とおける.
まず最初に (91) に代入すると

$$
\begin{aligned}
&-k^2B(z)e^{i\theta}+c.c.+\frac{d^2B(z)}{dz^2}e^{i\theta}+c.c.=0,\\
&-k^2B(z)+\frac{d^2B(z)}{dz^2}=0 \quad (c.c\text{ は省略})
\end{aligned}
$$

となる.$\frac{d}{dz}=D$ とおくと

$$
\begin{aligned}
&-k^2-D^2=0,\\
&D=\pm k,\\
B(z)&=\dot{c_1}e^{kz}+\dot{c_2}e^{-kz} \quad (\dot{c_1},\dot{c_2}\text{は定数}) \qquad (104)
\end{aligned}
$$

となる. 次に, (94) に代入する.

$$
\frac{dB(-h)}{dz}e^{i\theta}=0
$$

ここで,$B(-h)=\dot{c_1}e^{-kh}+\dot{c_2}e^{kh}$ を代入する.

$$
\begin{aligned}
&k\dot{c_1}e^{-kh}-k\dot{c_2}e^{kh}=0\\
&\dot{c_1}e^{-kh}=\dot{c_2}e^{kh}\\
&\dot{c_1}=\dot{c_2}e^{2kh}
\end{aligned}
$$

(104) に代入すると

$$
B(z)=\frac{\dot{c_2}}{e^{-kh}}(e^{k(h+z)}+e^{-k(h+z)})
$$

となる. これを $\Phi_1(x_0,\cdots,z,t_0,\cdots)$ の仮定の式に戻す.

$$
\begin{aligned}
\Phi_1(x_0,\cdots,z,t_0,\cdots) &= \frac{\dot{c_2}}{e^{-kh}}(e^{k(h+z)}+e^{-k(h+z)})e^{i\theta}+c.c.+\Psi_1(x_1,\cdots,t_1,\cdots)\\
&=\frac{2\dot{c_2}}{e^{-kh}}\cos hk(z+h)e^{i\theta}+c.c.+\Psi_1(x_1,\cdots,t_1,\cdots)
\end{aligned}
$$

となる.

$$
\frac{2\dot{c_2}}{e^{-kh}}=\dot{c_3}
$$

とおいて書き換えると,

$$
\Phi_1(x_0,\cdots,z,t_0,\cdots)=\dot{c_3}\cos hk(z+h)e^{i\theta}+c.c.+\Psi_1(x_1,\cdots,t_1,\cdots) \qquad (105)
$$

となる. そして,(92) に代入する.

$$
\begin{aligned}
&-A(x_1,\cdots,t_1,\cdots)e^{i\theta}i\omega-\dot{c_3}k\sin hk(h)e^{(i\theta)}=0\\
&-A(x_1,\cdots,t_1,\cdots)i\omega-\dot{c_3}k\sinh k(h)=0\\
&\dot{c_3}=-A(x_1,\cdots,t_1,\cdots)i\omega(k\sinh k(h))^{-1}
\end{aligned}
$$

を得る. これを (105) に代入する.

$$\Phi_1(x_0,\cdots,z,t_0,\cdots) = -i\omega(k\sinh(kh))^{-1}\cosh\{k(z+h)\}A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c. + \Psi_1(x_1,\cdots,t_1,\cdots) \quad (106)$$

又,(93) に代入する.

$$-\omega^2(k\sinh(kh))^{-1}\cosh\{(kh)\}A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c. + gA(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.+\frac{T}{\rho}A(x_1,\cdots,t_1,\cdots)e^{i\theta}k^2 = 0$$

$A(x_1,\cdots,t_1,\cdots)e^{i\theta}$ で割る

$$-\omega^2(k\sinh(kh))^{-1}\cosh\{(kh)\}+g+\frac{T}{\rho}k^2 = 0 \quad (c.c.\text{省略})$$

$$g+\frac{T}{\rho}k^2 = \frac{\omega^2}{k}\frac{1}{\tanh(kh)}$$

より,

$$\omega^2 = (g+\frac{T}{\rho}k^2)k\tanh(kh)$$

となる.$\sigma \equiv \tanh kh$ と置くと,

$$\omega^2 = (g+\frac{T}{\rho}k^2)k\sigma \quad (107)$$

となり, 分散関係式が得られたことがわかる.

次に (103), (106) を $O(\varepsilon^2)$ の関係式に代入する. この時の解を次のように仮定する.

ただし, 条件として, $e^{i\theta}$ に比例する項の係数を 0 とする.

$$\eta_2 = \eta_2(x_1,\cdots,t_1,\cdots),$$
$$\Psi_2 = \Psi_2(x_1,\cdots,t_1,\cdots)$$

とし, $e^(\pm 2i\theta), e^(\pm i\theta)$ によらない関数する.

$$\begin{aligned}&\zeta_2(x_0,\cdots,t_0,\cdots)\\&=[(gk+\frac{Tk^3}{\rho})(3-\sigma^2)\{2g\sigma^3+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)^{-1}]A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}\\&-i(\sigma+kh)(k\sigma)^{-1}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}-i\omega^{-1}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta}\\&+c.c.+\eta_2(x_1,\cdots,t_1,\cdots)\end{aligned} \quad (108)$$

$$\begin{aligned}&\Phi_2(x_0,\cdots,z,t_0,\cdots) = -3i\omega[\{g(1-\sigma^2)+\frac{Tk^2}{\rho}(3-\sigma^2)\}\{2g\sigma^3+2\frac{Tk^2}{\rho}\sigma(\sigma^3-3)\}^{-1}]\\&(\sinh 2kh)^{-1}\cosh\{2k(z+h)\}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}\\&-\omega(k\sinh kh)^{-1}(z+h)\sinh\{k(z+h)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}\\&+c.c.+\Psi_2(x_1,\cdots,t_1,\cdots)\end{aligned} \quad (109)$$

である.(108), (109) が $O(\varepsilon^2)$ の関係式をみたすか確かめる． Φ_1, ζ_2, Φ_2 を簡単にするために定数 $\hat{c_i}(i = 1, \cdots, 6)$ を使って表す.

$$\hat{c_1} = \omega(k\sinh(kh))^{-1},$$
$$\hat{c_2} = [(gk + \frac{Tk^3}{\rho})(3 - \sigma^2)\{2g\sigma^3 + 2(\frac{Tk^2}{\rho})\sigma(\sigma^2 - 3)^{-1}],$$
$$\hat{c_3} = (\sigma + kh)(k\sigma)^{-1},$$
$$\hat{c_4} = \omega^{-1},$$
$$\hat{c_5} = 3\omega[\{g(1 - \sigma^2) + \frac{Tk^2}{\rho}(3 - \sigma^2)\}\{2g\sigma^3 + 2\frac{Tk^2}{\rho}\sigma(\sigma^2 - 3)\}^{-1}](\sinh 2kh)^{-1},$$
$$\hat{c_6} = \omega(k\sinh kh)^{-1}$$

として書き換える.

$$\Phi_1(x_0, \cdots, z, t_0, \cdots) = -i\hat{c_1}\cosh\{k(z + h)\}A(x_1, \cdots, t_1, \cdots)e^{i\theta}$$
$$+c.c. + \Psi_1(x_1, \cdots, t_1, \cdots), \tag{110}$$
$$\zeta_2(x_0, \cdots, t_0, \cdots) = \hat{c_2}A^2(x_1, \cdots, t_1, \cdots)e^{2i\theta} - i\hat{c_3}(\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1})e^{i\theta}$$
$$-i\hat{c_4}(\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial t_1})e^{i\theta} + c.c. + \eta_2(x_1, \cdots, t_1, \cdots), \tag{111}$$
$$\Phi_2(x_0, \cdots, z, t_0, \cdots) = -i\hat{c_5}\cosh\{2k(z + h)\}A^2(x_1, \cdots, t_1, \cdots)e^{2i\theta}$$
$$-\hat{c_6}(z + h)\sinh\{k(z + h)\}(\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1})e^{i\theta}$$
$$+c.c. + \Psi_2(x_1, \cdots, t_1, \cdots) \tag{112}$$

となる. まず最初に,(95) に代入する．左辺は,

$$\frac{\partial^2\Phi_2(x_0, \cdots, z, t_0, \cdots)}{\partial {x_0}^2} + \frac{\partial^2\Phi_2(x_0, \cdots, z, t_0, \cdots)}{\partial z^2}$$
$$= -i\hat{c_5}\cosh\{2k(z + h)\}A^2(x_1, \cdots, t_1, \cdots)e^{2i\theta}(2ik^2)$$
$$-\hat{c_6}(z + h)\sinh\{k(z + h)\}(\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1})e^{i\theta}(ik)^2 + c.c.$$
$$-i\hat{c_5}\cosh\{2k(z + h)\}4k^2A^2(x_1, \cdots, t_1, \cdots)e^{2i\theta}$$
$$-\hat{c_6}\{2k\cosh\{k(z + h)\} + k^2(z + h)\sinh\{k(z + h)\}(\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1})e^{i\theta} + c.c.$$
$$= -2\hat{c_6}\{2k\cosh\{k(z + h)\}\}(\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1})e^{i\theta} + c.c.$$

となり, 右辺は,

$$-2\frac{\partial^2\Phi_1(x_0, \cdots, z, t_0, \cdots)}{\partial x_0\partial x_1}$$
$$= 2(i\hat{c_1}\cosh\{k(z + h)\}\frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1}e^{i\theta}(ik)) + c.c.$$

となる.左辺 = 右辺 より,

$$\hat{c_6} = \hat{c_1} = \omega(k\sinh(kh))^{-1}$$

となり, $\hat{c_6}, \hat{c_1}$ が確かめられた. 次に, (98) に代入する.

$$\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} = -i\hat{c_5}\sinh\{2k(0)\}2kA^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}$$
$$-\hat{c_6}\{\sinh\{k(0)\}+k(0)\cosh\{k(0)\}\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}+c.c.$$
$$=0$$

さらに, (96) に代入する. 左辺は,

$$\frac{\partial \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_0} - \frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z} = \hat{c_2}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(-2i\omega)$$
$$-i\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(-i\omega) - i\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta}(-i\omega)$$
$$+c.c. + 2ki\hat{c_5}\sinh\{2kh\}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}$$
$$+\hat{c_6}\{\sinh(kh)+kh\cosh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}+c.c.$$
$$=(-2i\omega\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)+2ki\hat{c_5}\sinh(2kh)A^2(x_1,\cdots,t_1,\cdots))e^{2i\theta}$$
$$+(-\omega\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) - \omega\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})$$
$$+\hat{c_6}\{\sinh(kh)+kh\cosh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}))e^{i\theta}+c.c.$$

であり, 右辺は,

$$-\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_1} + \zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2 \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}$$
$$-\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0}$$
$$=-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{i\theta}+c.c.$$
$$+(A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)(-ik^2\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)$$
$$-(-i\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)+c.c.)(A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)+c.c.)$$
$$=-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{i\theta}+c.c. - i\hat{c_1}k^2A^2(x_1,\cdots,t_1,\cdots)\cosh(kh)e^{2i\theta}$$
$$+ik^2\hat{c_1}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 - ik^2\hat{c_1}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2+c.c.$$
$$-ik^2\hat{c_1}\cosh(kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}$$
$$+ik^2\hat{c_1}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 - ik^2\hat{c_1}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2+c.c$$
$$=-2i\hat{c_1}k^2A^2(x_1,\cdots,t_1,\cdots)\cosh(kh)e^{2i\theta} - \frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{i\theta}+c.c.$$

となる.左辺 = 右辺 から, 最初に $e^{2i\theta}$ の係数を求めると

$$\begin{aligned} \omega\hat{c_2} - k\hat{c_5}\sinh(2kh) &= \hat{c_1}k^2\cosh(kh) \\ &= \frac{k\omega}{\sigma} \end{aligned} \tag{113}$$

となる. 次に,$e^{i\theta}$ の係数を求めてみる.

$$
\begin{aligned}
&(-\omega\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) - \omega\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}) \\
&+\hat{c_6}\{\sinh(kh) + kh\cosh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})) = -\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1} \\
&-\omega\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) + \hat{c_6}\{\sinh(kh) + kh\cosh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) \\
&= 0 \\
&-\omega\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) + \omega(\frac{1}{k} + \frac{h}{\sigma})(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) \\
&= 0 \\
&-\hat{c_3} + \frac{1}{k} + \frac{h}{\sigma} = 0 \\
&\hat{c_3} = (\sigma + kh)(k\sigma)^{-1}, \hat{c_4} = \frac{1}{\omega}
\end{aligned}
$$

$\hat{c_3}, \hat{c_4}$ が確かめられた. さらに, (97) に代入する. 左辺は,

$$
\begin{aligned}
&\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0} + g\zeta_2(x_0,\cdots,t_0,\cdots) - (\frac{T}{\rho})(\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2}) \\
&= -i\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(-2i\omega) \\
&-\hat{c_6}h\sinh(kh)(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(-i\omega) + c.c. \\
&+g(\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta} - i\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta} \\
&-i\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta} + c.c. + \eta_2) \\
&-(\frac{T}{\rho})(\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(2ik)^2 - i\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(ik)^2 \\
&-i\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta}(ik)^2 + c.c.) \\
&= (-2\omega\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots) + g\hat{c_2}A^2(x_1,\cdots,t_1,\cdots) \\
&+(\frac{T}{\rho})(4k^2\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)))e^{2i\theta} + (i\omega\hat{c_6}h\sinh(kh)(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) \\
&-ig\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) - gi\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}) \\
&-(\frac{T}{\rho})(ik^2\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}) + ik^2\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})))e^{i\theta} + g\eta_2 + c.c.
\end{aligned}
$$

であり, 右辺は,

$$
\begin{aligned}
&-\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}-\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}\\
&-\frac{1}{2}\{(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})^2+(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})^2\}\\
&+2(\frac{T}{\rho})(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1})\\
&=i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{(i\theta)}+c.c.-\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1}\\
&-(A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)(-ik\hat{c_1}\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(-i\omega)+c.c.)\\
&-\frac{1}{2}\{(-i\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)+c.c.)^2\\
&+(-ik\hat{c_1}\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)^2\}\\
&+2(\frac{T}{\rho})(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{i\theta}(ik)+c.c.)\\
&=i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{i\theta}+c.c.-\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1}\\
&+k\omega\hat{c_1}\sinh(kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}\\
&+\omega k\hat{c_1}\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2+\omega k\hat{c_1}\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2)+c.c.\\
&-\frac{1}{2}\{(k\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots))^2e^{2i\theta}+2(k\hat{c_1}\cosh(kh))^2|A(x_1,\cdots,t_1,\cdots)|^2+c.c.\\
&-(k\hat{c_1}\sinh(kh)A(x_1,\cdots,t_1,\cdots))^2e^{2i\theta}+2(k\hat{c_1}\sinh(kh))^2|A(x_1,\cdots,t_1,\cdots)|^2+c.c.\}\\
&+2(\frac{T}{\rho})ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{i\theta}+c.c.\\
&=(k\omega\hat{c_1}\sinh(kh)A^2(x_1,\cdots,t_1,\cdots)-\frac{1}{2}(k\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots))^2\\
&+\frac{1}{2}(k\hat{c_1}\sinh(kh)A(x_1,\cdots,t_1,\cdots))^2)e^{2i\theta}\\
&+(i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}+2(\frac{T}{\rho})ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}\\
&-\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1}+2\omega k\hat{c_1}\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2\\
&-(k\hat{c_1}\cosh(kh))^2|A(x_1,\cdots,t_1,\cdots)|^2-(k\hat{c_1}\sinh(kh))^2|A(x_1,\cdots,t_1,\cdots)|^2+c.c.
\end{aligned}
$$

となる.左辺 = 右辺 より, 最初に $e^{2i\theta}$ の係数を求めてみる.

$$
\begin{aligned}
&-2\omega\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)+g\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)+(\frac{T}{\rho})(4k^2\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)\\
&=k\omega\hat{c_1}\sinh(kh)A^2(x_1,\cdots,t_1,\cdots)-\frac{1}{2}(k\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots))^2\\
&+\frac{1}{2}(k\hat{c_1}\sinh(kh)A(x_1,\cdots,t_1,\cdots))^2\\
&-2\omega\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)+\hat{c_2}(gA^2(x_1,\cdots,t_1,\cdots)\\
&+(\frac{T}{\rho})(4k^2A^2(x_1,\cdots,t_1,\cdots)))\\
&=(\omega^2-\frac{1}{2}(\frac{\omega^2}{\sigma^2}-\omega^2))A^2(x_1,\cdots,t_1,\cdots)\\
&-2\omega\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)+\hat{c_2}(gA^2(x_1,\cdots,t_1,\cdots)\\
&+(\frac{T}{\rho})(4k^2A^2(x_1,\cdots,t_1,\cdots)))\\
&=\omega^2(1-\frac{1}{2}(\frac{1}{\sigma^2}-1))A^2(x_1,\cdots,t_1,\cdots)\\
&-2\omega\hat{c_5}\cosh(2kh)+\hat{c_2}(g+(\frac{T}{\rho})4k^2)\\
&=\omega^2(\frac{3}{2}-\frac{1}{2\sigma^2})
\end{aligned}
\tag{114}
$$

となる.(113) と (114) から $\hat{c_2},\hat{c_5}$ を求める. (113) から,

$$
\begin{aligned}
&\omega c_2-\frac{k\omega}{\sigma}=kc_5\sinh(2kh),\\
&c_5=\frac{1}{k\sinh(2kh)}(\omega c_2-\frac{k\omega}{\sigma})
\end{aligned}
\tag{115}
$$

を得る.(115) を (114) に代入する.

$$
\begin{aligned}
&\frac{-2\omega\cosh(2kh)}{k\sinh(2kh)}(\omega c_2-\frac{k\omega}{\sigma})+c_2(g+4k^2(\frac{T}{\rho}))\\
&=\omega^2(\frac{3}{2}-\frac{1}{2\sigma^2})\\
&\frac{-2\omega}{k}\frac{1+\sigma^2}{2\sigma}(\omega c_2-\frac{k\omega}{\sigma})+c_2(g+4k^2(\frac{T}{\rho}))\\
&=\omega^2(\frac{3}{2}-\frac{1}{2\sigma^2})\quad(\frac{1}{\tanh(2kh)}=\frac{1+\sigma^2}{2\sigma})\\
&c_2(-\frac{1+\sigma^2}{k\sigma}\omega^2+g+4k^2(\frac{T}{\rho}))\\
&=\omega^2(\frac{3}{2}-\frac{1}{2\sigma^2})-\frac{1+\sigma^2}{\sigma^2}\omega^2\\
&c_2(-(1+\sigma^2)(g+\frac{Tk^2}{\rho})+g+4k^2(\frac{T}{\rho}))\\
&=(\frac{3}{2}-\frac{1}{2\sigma^2}-\frac{1+\sigma^2}{\sigma^2})(g+\frac{Tk^2}{\rho})k\sigma
\end{aligned}
$$

両辺に $2\sigma^2$ をかけると

$$c_2(-2\sigma^2(1+\sigma^2)(g+\frac{Tk^2}{\rho})+g2\sigma^2+8k^2\sigma^2(\frac{T}{\rho})$$
$$(3\sigma^2-1-2(1+\sigma^2))(g+\frac{Tk^2}{\rho})k\sigma$$
$$c_2(-2g\sigma^2-2\sigma^2\frac{Tk^2}{\rho}-2\sigma^4 g-2\sigma^4\frac{Tk^2}{\rho}+2g\sigma^2+8k^2\sigma^2(\frac{T}{\rho}))$$
$$=(\sigma^2-3)(g+\frac{Tk^2}{\rho})k\sigma$$
$$c_2(-2\sigma^4 g-2(\frac{Tk^2}{\rho})\sigma^2(\sigma^2-3))$$
$$=-(3-\sigma^2)(g+\frac{Tk^2}{\rho})k\sigma$$

を得る. 両辺を $-\sigma$ で割ると

$$c_2(2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3))$$
$$=(3-\sigma^2)(g+\frac{Tk^2}{\rho})k$$
$$c_2=[(3-\sigma^2)(g+\frac{Tk^2}{\rho})k\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}^{-1}]$$

となる. 又, (113) に $\hat{c_2}$ を代入すると,

$$\omega[(3-\sigma^2)(g+\frac{Tk^2}{\rho})k\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}^{-1}]-\frac{k\omega}{\sigma}=kc_5\sinh(2kh)$$

となる. 両辺を k で割ると

$$\omega[(3-\sigma^2)(g+\frac{Tk^2}{\rho})\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}^{-1}]-\frac{\omega}{\sigma}=c_5\sinh(2kh)$$
$$\omega[\frac{(3-\sigma^2)(g+\frac{Tk^2}{\rho})\sigma-\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}}{\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}\sigma}]=c_5\sinh(2kh)$$
$$\omega[\frac{3g(1-\sigma^2)\sigma+3(\frac{Tk^2}{\rho})\sigma(3-\sigma^2)}{\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}\sigma}]=c_5\sinh(2kh)$$
$$c_5=\frac{1}{\sinh(2kh)}3\omega[\frac{g(1-\sigma^2)+(\frac{Tk^2}{\rho})(3-\sigma^2)}{\{2\sigma^3 g+2(\frac{Tk^2}{\rho})\sigma(\sigma^2-3)\}}]$$

となり,$\hat{c_2},\hat{c_5}$ が確かめられた. 次に $e^{i\theta}$ の係数を求める.

$$i\omega\hat{c_6}h\sinh(kh)(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})-ig\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})-ig\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})$$
$$-(\frac{T}{\rho})(ik^2\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})+ik^2\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}))$$
$$=i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}+2\frac{T}{\rho}(ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})$$

$$
\begin{aligned}
&i\omega^2\frac{h}{k}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})-ig\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})-ig\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})\\
&-(\frac{T}{\rho})(ik^2\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})+ik^2\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}))\\
&=i\omega\frac{1}{k\sigma}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}+2(\frac{T}{\rho})ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}
\end{aligned}
$$

$$
\begin{aligned}
&(i\omega^2\frac{h}{k}-ig\hat{c_3}-(\frac{T}{\rho})ik^2\hat{c_3}-2(\frac{T}{\rho})ik)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}\\
&+(-ig\hat{c_4}-(\frac{T}{\rho})ik^2\hat{c_4}-i\omega)\frac{1}{k\sigma}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}=0
\end{aligned}
$$

$$
\begin{aligned}
&(-\omega^2\frac{h}{k}+g\hat{c_3}+(\frac{T}{\rho})(k^2\hat{c_3}+2(\frac{T}{\rho})k(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})\\
&+(g\hat{c_4}+(\frac{T}{\rho})k^2\hat{c_4}+\omega\frac{1}{k\sigma})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}=0\quad(-\frac{1}{i}\text{をかける})
\end{aligned}
$$

$$
\begin{aligned}
&(-gh\sigma-\frac{Tk^2}{\rho}\sigma h+\frac{g}{k}+\frac{gh}{\sigma}+\frac{Tk}{\rho}+\frac{Tk^2}{\rho}\frac{h}{\sigma}+2\frac{Tk}{\rho})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}\\
&+(\frac{g}{\omega}+\frac{Tk^2}{\rho\omega}+\frac{\omega}{k\sigma})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}=0\quad((107)\text{ を使う})
\end{aligned}
$$

$$
\begin{aligned}
&(-ghk\sigma^2-\frac{Tk^3}{\rho}\sigma^2 h+g\sigma+ghk+\frac{Tk^2}{\rho}\sigma+\frac{Tk^3}{\rho}h+2\frac{Tk^2}{\rho}\sigma)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}\\
&+(\frac{gk\sigma}{\omega}+\frac{Tk^3\sigma}{\rho\omega}+\omega)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}=0\quad(\text{両辺に } k\sigma\text{をかける})
\end{aligned}
$$

$$
\begin{aligned}
&\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1-\sigma^2)\}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}\\
&+(2\omega)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}=0\quad((107)\text{ を使う})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\omega}\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1-\sigma^2)\}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}\\
&+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}=0
\end{aligned}
\tag{116}
$$

又, (107) と,

$$
\frac{\partial\omega^2}{\partial k}=2\omega\frac{\partial\omega}{\partial k}
$$

を使うと.

$$
\begin{aligned}
&(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1-\sigma^2)=2\omega\frac{\partial\omega}{\partial k}\\
&\frac{\partial\omega}{\partial k}=\frac{1}{2\omega}\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1-\sigma^2)\}
\end{aligned}
$$

となる.$v_g = \frac{\partial \omega}{\partial k}$ より

$$v_g = \frac{1}{2\omega}\{(g + \frac{3Tk^2}{\rho})\sigma + (g + \frac{Tk^2}{\rho})kh(1-\sigma^2)\} \tag{117}$$

を得る.(117) を使って (116) を書き直すと,

$$v_g \frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial x_1} + \frac{\partial A(x_1, \cdots, t_1, \cdots)}{\partial t_1} = 0 \tag{118}$$

となり,$\Phi_2(x_0, \cdots, z, t_0, \cdots), \zeta_2(x_0, \cdots, t_0, \cdots)$ を得る際の条件として与えられる.
最後に e^0 は,

$$\begin{aligned}
&g\eta_2 = -\frac{\partial \Psi_1(x_1, \cdots, t_1, \cdots)}{\partial t_1} + 2\omega k\hat{c_1}\sinh(kh)|A(x_1, \cdots, t_1, \cdots)|^2 \\
&-(k\hat{c_1}\cosh(kh))^2|A(x_1, \cdots, t_1, \cdots)|^2 - (k\hat{c_1}\sinh(kh))^2|A(x_1, \cdots, t_1, \cdots)|^2 \\
&\Rightarrow g\eta_2 = -\frac{\partial \Psi_1(x_1, \cdots, t_1, \cdots)}{\partial t_1} + 2\omega^2|A(x_1, \cdots, t_1, \cdots)|^2 - (\frac{\omega}{\rho})^2|A(x_1, \cdots, t_1, \cdots)|^2 \\
&-\omega^2|A(x_1, \cdots, t_1, \cdots)|^2 \\
&\Rightarrow g\eta_2 + \frac{\partial \Psi_1(x_1, \cdots, t_1, \cdots)}{\partial t_1} + (-2\omega^2 + (\frac{\omega}{\sigma})^2 + \omega^2)|A(x_1, \cdots, t_1, \cdots)|^2 = 0 \\
&g\eta_2(x_1, \cdots, t_1, \cdots) + \frac{\partial \Psi_1(x_1, \cdots, t_1, \cdots)}{\partial t_1} + \omega^2(1-\sigma^2)\sigma^{-2}|A(x_1, \cdots, t_1, \cdots)|^2 \\
&= 0
\end{aligned} \tag{119}$$

となり, これも,$\Phi_2(x_0, \cdots, z, t_0, \cdots), \zeta_2(x_0, \cdots, t_0, \cdots)$ を得る際の条件として与えられる.
$O(\varepsilon^3)$ でも同様に関係式の解を求める.

まず初めに, (99) の式を求める. 右辺は,

$$
\begin{aligned}
&-2\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_1}-2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial x_2}-\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial {x_1}^2}\\
&=-2(-i\hat{c_5}\cosh\{2k(z+h)\}2A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{2i\theta}(2ik)\\
&-\hat{c_6}(z+h)\sinh\{k(z+h)\}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}e^{i\theta}(ik)+c.c.)\\
&-2(-i\hat{c_1}\cosh\{k(z+h)\}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}e^{i\theta}(ik)+c.c.)\\
&-(-i\hat{c_1}\cosh\{k(z+h)\}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}e^{i\theta}+c.c.+\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2})\\
&=-8k\hat{c_5}\cosh\{2k(z+h)\}A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{2i\theta}\\
&+(2ik\hat{c_6}(z+h)\sinh\{k(z+h)\}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\\
&-2k\hat{c_1}\cosh\{k(z+h)\}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}\\
&+i\hat{c_1}\cosh\{k(z+h)\}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2})e^{i\theta}-\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+c.c.\\
&=-8k\hat{c_5}\cosh\{2k(z+h)\}A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{2i\theta}\\
&+\hat{c_1}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}i(2k(z+h)\sinh\{k(z+h)\}+\cosh\{k(z+h)\})e^{i\theta}\\
&-2k\hat{c_1}\cosh\{k(z+h)\}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}e^{i\theta}-\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+c.c.
\end{aligned}
$$

又, 左辺から, 次のように予測できる.

$$\frac{\partial^2}{\partial z^2}[4(z+h)\sinh\{2k(z+h)\}]e^{2i\theta}$$
$$=(8k\cosh\{k(z+h)\}+16k^2(z+h)\sinh\{2k(z+h)\})e^{2i\theta}$$
$$\frac{\partial^2}{\partial x_0^2}[4(z+h)\sinh\{2k(z+h)\}]e^{2i\theta}=[-16k^2(z+h)\sinh\{2k(z+h)\}]e^{2i\theta}$$
$$(\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial z^2})[4(z+h)\sinh\{2k(z+h)\}e^{2i\theta}]=8k\cosh\{2k(z+h)\}e^{2i\theta}$$
$$\frac{\partial^2}{\partial z^2}[\frac{1}{2}(z+h)^2\cosh\{k(z+h)\}e^{i\theta}]$$
$$=(\cosh\{(k(z+h)\}+2k(z+h)\sinh\{k(z+h)\}+\frac{1}{2}k^2(z+h)^2\cosh\{k(z+h)\})e^{i\theta}$$
$$\frac{\partial^2}{\partial x_0^2}[\frac{1}{2}(z+h)^2\cosh\{k(z+h)\}e^{i\theta}]=-\frac{1}{2}(z+h)^2\cosh\{k(z+h)\}k^2e^{i\theta}$$
$$(\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial z^2})[\frac{1}{2}(z+h)^2\cosh\{k(z+h)\}e^{i\theta}]$$
$$=[\cosh\{(k(z+h)\}+2k(z+h)\sinh\{k(z+h)\}]e^{i\theta}$$
$$\frac{\partial^2}{\partial z^2}[(z+h)\sinh\{k(z+h)\}e^{i\theta}]=[2k\cosh\{k(z+h)\}+k^2(z+h)\sinh\{k(z+h)\}]e^{i\theta}$$
$$\frac{\partial^2}{\partial x_0^2}[(z+h)\sinh\{k(z+h)\}e^{i\theta}]=[-(z+h)k^2\sinh\{k(z+h)\}]e^{i\theta}$$
$$(\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial z^2})[(z+h)\sinh\{k(z+h)\}e^{i\theta}]=2k\cosh\{k(z+h)\}e^{i\theta}$$

より, $\Phi_3(x_0,\cdots,z,t_0,\cdots)$ は次のようになる.

$$\Phi_3(x_0,\cdots,z,t_0,\cdots)$$
$$=-4\hat{c_5}(z+h)\sinh\{2k(z+h)\}A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}\epsilon^{2i\theta}$$
$$+\frac{1}{2}\hat{c_1}i\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}(z+h)^2\cosh\{k(z+h)\}\exp^{i\theta}$$
$$-\hat{c_1}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(z+h)\sinh\{k(z+h)\}e^{i\theta}-\frac{1}{2}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}(z+h)^2$$
$$+c.c.$$

となる. この先を求めていく中で,

$$e^{ni\theta},(n=2,3)$$

は非線形シュレディンガー方程式,KdV 方程式を求める際に,$e^{ni\theta},(n=0,1)$ 以外は使わないのでここでは省略して考える.

$$\Phi_3(x_0,\cdots,z,t_0,\cdots)=$$
$$+\frac{1}{2}\hat{c_1}i\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}(z+h)^2\cosh\{k(z+h)\}e^{i\theta}$$
$$-\hat{c_1}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(z+h)\sinh\{k(z+h)\}e^{i\theta}-\frac{1}{2}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}(z+h)^2$$
$$+c.c. \tag{120}$$

次に (102) に (120) を代入して, 成り立つ事を示す.

$$\frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z}$$
$$= +\frac{1}{2}\hat{c_1}i\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\{2(z+h)\cosh\{k(z+h)\}+k(z+h)^2\sinh\{k(z+h)\}\}e^{i\theta}$$
$$-\hat{c_1}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}\{\sinh\{k(z+h)\}+k(z+h)\cosh\{k(z+h)\}\}e^{i\theta}+c.c.$$
$$-\frac{1}{2}\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}2(z+h)$$

$(z=-h)$ を使う.

$$=0$$

が成り立つ事がわかる. 次に (100) に代入して,$\zeta_3(x_0,\cdots,t_0,\cdots)$ を求める. 左辺は,

$$\frac{\partial \zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0}-\frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z}=\frac{\partial \zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0}$$
$$-\frac{1}{2}\hat{c_1}i\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\{2h\cosh(kh)+kh^2\sinh(kh)\}e^{i\theta}$$
$$+\hat{c_1}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}\{\sinh(kh)+kh\cosh(kh)\}e^{i\theta}+c.c.$$
$$+h\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \tag{121}$$

となり右辺は,

$$
\begin{aligned}
&-\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_1}-\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_2}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&+\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&+\frac{1}{2}\zeta_1^2(x_0,\cdots,t_0,\cdots)\frac{\partial^3\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^3}\\
&-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\{\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1}\}\\
&-(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}\}\\
&=-(\hat{c_2}2A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{(2i\theta)}-i\hat{c_3}(\frac{\partial^2A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1})e^{i\theta}\\
&-i\hat{c_4}(\frac{\partial^2A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2})e^{(i\theta)}+c.c.+\frac{\partial\eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1})\\
&-(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}e^{(i\theta)}+c.c.)\\
&+(A(x_1,\cdots,t_1,\cdots)e^{(i\theta)}+c.c.)(-i\hat{c_5}4k^2\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}\\
&-\hat{c_6}\{2k\cosh(kh)+k^2h\sinh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}+c.c.)\\
&+(\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)e^{(2i\theta)}-i\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}\\
&-i\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta}+c.c.+\eta_2(x_1,\cdots,t_1,\cdots))\\
&(-i\hat{c_1}k^2\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)\\
&+\frac{1}{2}(A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c)^2(-i\hat{c_1}k^3\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)\\
&-(-i\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)+c.c.)\{(\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(2ik)\\
&-i\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(ik)\\
&-i\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta}(ik)+c.c.)+(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{i\theta}+c.c.)\}\\
&-(A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)+c.c.)\{(-i\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(2ik)\\
&-\hat{c_6}h\sinh(kh)(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(ik)+c.c.)\\
&+(-i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{i\theta}+c.c.+\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})\\
&+(A(x_1,\cdots,t_1,\cdots)e^{i\theta}+c.c.)(-i\hat{c_1}k\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)+c.c.)\}
\end{aligned}
$$

$$= (i\hat{c_3}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1})e^{i\theta} + i\hat{c_4}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2})e^{i\theta} + c.c.$$
$$-\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}) + (-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}e^{i\theta} + c.c.)$$
$$+(-i4k^2\hat{c_5}\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta}$$
$$-\hat{c_6}\{2k\cosh(kh) + k^2 h\sinh(kh)\}\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + c.c.)$$
$$+(ik^2\hat{c_1}\hat{c_2}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta}$$
$$+k^2\hat{c_1}\hat{c_3}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + k^2\hat{c_1}\hat{c_4}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial t_1}$$
$$-i\eta_2(x_1,\cdots,t_1,\cdots)\hat{c_1}k^2\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)$$
$$(-\frac{1}{2}i\hat{c_1}k^3\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)$$
$$-\{(2ik^2\hat{c_1}\hat{c_2}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta}$$
$$+k^2\hat{c_1}\hat{c_3}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + k^2\hat{c_1}\hat{c_4}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial t_1} + c.c.)$$
$$+(k\hat{c_1}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + c.c.)\}$$
$$-\{(-2ik^2\hat{c_5}\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta}$$
$$-k^2\hat{c_6}h\sinh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + c.c.)$$
$$+(ik\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)e^{i\theta}$$
$$-k\hat{c_1}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + c.c.)$$
$$+(-ik^3\hat{c_1}\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta}$$
$$+2ik^3\hat{c_1}\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)\}$$

$$
\begin{aligned}
&= (i\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} + i\hat{c_4}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2} - i4k^2\hat{c_5}\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+ik^2\hat{c_1}\hat{c_2}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-i\eta_2(x_1,\cdots,t_1,\cdots)\hat{c_1}k^2\cosh(kh)A(x_1,\cdots,t_1,\cdots) \\
&-\frac{3}{2}i\hat{c_1}k^3\sinh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-2ik^2\hat{c_1}\hat{c_2}\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+2ik^2\hat{c_5}\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-ik\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots))e^{i\theta} \\
&-\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} - \hat{c_6}\{2k\cosh(kh) + k^2 h\sinh(kh)\}\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} \\
&+k^2\hat{c_1}\hat{c_3}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + k^2\hat{c_1}\hat{c_4}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial t_1} \\
&-k^2\hat{c_1}\hat{c_3}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} - k^2\hat{c_1}\hat{c_4}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial t_1} \\
&-k\hat{c_1}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + k^2\hat{c_6}h\sinh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} \\
&+k\hat{c_1}\cosh(kh)\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} \\
&= (i\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} + i\hat{c_4}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2} - i\eta_2(x_1,\cdots,t_1,\cdots)\frac{\omega k}{\sigma}A(x_1,\cdots,t_1,\cdots) \\
&-ik\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots) \\
&+(-2ik^2\hat{c_5}\cosh(2kh) - ik\hat{c_2}\omega\frac{1}{\sigma} - \frac{3}{2}ik^2\omega)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots))e^{i\theta} \\
&-\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} \\
&+(-\frac{2\omega}{\sigma})\frac{\partial |A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1} + c.c. \qquad (122)
\end{aligned}
$$

である. 左辺=右辺より

$$
\begin{aligned}
&\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0} = \frac{1}{2}i(\frac{2h\omega}{k\sigma}+\omega h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}e^{i\theta} \\
&-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{\omega}{k}+\frac{\omega h}{\sigma})e^{i\theta}+c.c.-h\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \\
&+(i\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}+i\hat{c_4}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}-i\eta_2(x_1,\cdots,t_1,\cdots)\frac{\omega k}{\sigma}A(x_1,\cdots,t_1,\cdots) \\
&-ik\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots) \\
&+(-2ik^2\hat{c_5}\cosh(2kh)-ik\hat{c_2}\omega\frac{1}{\sigma}-\frac{3}{2}ik^2\omega)|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))e^{i\theta} \\
&-\frac{\partial\eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} \\
&+(-\frac{2\omega}{\sigma})\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1}+c.c. \\
&=(\frac{1}{2}i(\frac{2h\omega}{k\sigma}+\omega h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \\
&-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{\omega}{k}+\frac{\omega h}{\sigma})+i\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}+i\hat{c_4}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}-i\eta_2(x_1,\cdots,t_1,\cdots)\frac{\omega k}{\sigma}A(x_1,\cdots,t_1,\cdots) \\
&-ik\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots) \\
&+(-2ik^2\hat{c_5}\cosh(2kh)-ik\hat{c_2}\omega\frac{1}{\sigma}-\frac{3}{2}ik^2\omega)|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))e^{i\theta} \\
&-h\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}-\frac{\partial\eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} \\
&+(-\frac{2\omega}{\sigma})\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1}+c.c.
\end{aligned}
$$

となる. よって, $\zeta_3(x_0,\cdots,t_0,\cdots)$ は,

$$\zeta_3(x_0,\cdots,t_0,\cdots)=(\frac{1}{2}i(\frac{2h\omega}{k\sigma}+\omega h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}$$
$$-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{\omega}{k}+\frac{\omega h}{\sigma})+i\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}+i\hat{c_4}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2}$$
$$-\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}-i\eta_2(x_1,\cdots,t_1,\cdots)\frac{k\omega}{\sigma}A(x_1,\cdots,t_1,\cdots)$$
$$-ik\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)$$
$$+(-2ik^2\hat{c_5}\cosh(2kh)-ik\hat{c_2}\omega\frac{1}{\sigma}-\frac{3}{2}ik^2\omega)$$
$$|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))e^{i\theta}(-\frac{1}{i\omega})+c.c.$$
$$+(-h\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}-\frac{\partial\eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}$$
$$+(-\frac{2\omega}{\sigma})\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1})t_0$$

となる. ここで, $\zeta_3(x_0,\cdots,t_0,\cdots)$ が発散しない条件として,

$$\frac{\partial\eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}+h\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+\frac{2\omega}{\sigma}\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1}=0 \tag{123}$$

となるので, $\zeta_3(x_0,\cdots,t_0,\cdots)$ を書き直すと,

$$\zeta_3(x_0,\cdots,t_0,\cdots)=(-\frac{1}{2}(\frac{2h}{k\sigma}+h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}$$
$$+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{1}{ki}+\frac{h}{\sigma i})-\frac{1}{\omega}(\frac{1}{k}+\frac{h}{\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}-\frac{1}{\omega^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2}$$
$$+\frac{1}{\omega i}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+\frac{k}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots)$$
$$+\frac{k}{\omega}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)$$
$$+(2\frac{k^2\hat{c_5}\cosh(2kh)}{\omega}+k\hat{c_2}\frac{1}{\sigma}+\frac{3}{2}k^2)|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))e^{i\theta}$$
$$+c.c. \tag{124}$$

となる．そして,(101) に代入して成り立つか確かめる．左辺は,

$$\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+g\zeta_3(x_0,\cdots,t_0,\cdots)-(\frac{T}{\rho})(\frac{\partial^2\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial {x_0}^2})$$

$$=\frac{1}{2}\hat{c_1}i\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}h^2\cosh(kh)\epsilon^{i\theta}(-i\omega)$$

$$-\hat{c_1}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}h\sinh(kh)\exp^{i\theta}(-i\omega)+c.c.+g((-\frac{1}{2}(\frac{2h}{k\sigma}+h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}$$

$$+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{1}{ki}+\frac{h}{\sigma i})-\frac{1}{\omega}(\frac{1}{k}+\frac{h}{\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}-\frac{1}{\omega^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2}$$

$$+\frac{1}{\omega i}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+\frac{k}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots)$$

$$+\frac{k}{\omega}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)$$

$$+(2\frac{k^2\hat{c_5}\cosh(2kh)}{\omega}+k\hat{c_2}\frac{1}{\sigma}+\frac{3}{2}k^2)|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))\exp^{i\theta}+c.c.)$$

$$+(\frac{T}{\rho})((-\frac{1}{2}(\frac{2h}{k\sigma}+h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}$$

$$+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{1}{ki}+\frac{h}{\sigma i})-\frac{1}{\omega}(\frac{1}{k}+\frac{h}{\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}-\frac{1}{\omega^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2}$$

$$+\frac{1}{\omega i}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+\frac{k}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots)$$

$$+\frac{k}{\omega}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)$$

$$+(2\frac{k^2\hat{c_5}\cosh(2kh)}{\omega}+k\hat{c_2}\frac{1}{\sigma}+\frac{3}{2}k^2)|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))e^{i\theta}k^2+c.c.)$$

$$
\begin{aligned}
&= (\frac{\omega^2 h^2}{2\sigma k}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} + \frac{\omega^2 h}{k} i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2} \\
&-\frac{g}{2}(\frac{2h}{k\sigma} + h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \\
&+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{g}{ki} + \frac{gh}{\sigma i}) - \frac{g}{\omega}(\frac{1}{k} + \frac{h}{\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} \\
&-\frac{g}{\omega^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&+\frac{g}{\omega i}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2} + \frac{gk}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots) \\
&+\frac{gk}{\omega}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots) \\
&+(2g\frac{k^2 \hat{c_5}\cosh(2kh)}{\omega} + gk\hat{c_2}\frac{1}{\sigma} + \frac{3g}{2}k^2)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+(\frac{T}{\rho})(-\frac{1}{2}(\frac{2kh}{\sigma} + k^2h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \\
&+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{k}{i} + \frac{k^2 h}{\sigma i}) - \frac{1}{\omega}(k + \frac{k^2 h}{\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} \\
&-\frac{k^2}{\omega^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&+\frac{k^2}{\omega i}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2} + \frac{k^3}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots) \\
&+\frac{k^3}{\omega}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots) \\
&+(2\frac{k^4 \hat{c_5}\cosh(2kh)}{\omega} + k^3\hat{c_2}\frac{1}{\sigma} + \frac{3}{2}k^4)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots))e^{i\theta} + c.c.) \\
&= (\frac{g}{\omega i} + \frac{Tk^2}{\rho\omega i})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2} \\
&+(\frac{\omega^2 h}{k}i + \frac{g}{ki} + \frac{gh}{\sigma i} + \frac{Tk}{i\rho} + \frac{Tk^2 h}{\rho\sigma i})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2} \\
&+(-\frac{g}{\omega^2} - \frac{Tk^2}{\omega^2\rho})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} \\
&+(-\frac{g}{k\omega} - \frac{gh}{\omega\sigma} - \frac{Tk}{\omega\rho} - \frac{Tk^2 h}{\omega\sigma\rho})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} \\
&+(\frac{\omega^2 h^2}{2\sigma k} - \frac{hg}{k\sigma} - \frac{gh^2}{2} - \frac{Tkh}{\sigma\rho} - \frac{Tk^2h^2}{2\rho})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \\
&+(\frac{gk}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots) + \frac{gk}{\omega}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1} + \frac{Tk^3}{\sigma\rho}\eta_2(x_1,\cdots,t_1,\cdots) \\
&+\frac{Tk^3}{\omega\rho}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})A(x_1,\cdots,t_1,\cdots) \\
&+(2g\frac{k^2\hat{c_5}\cosh(2kh)}{\omega} + gk\hat{c_2}\frac{1}{\sigma} + \frac{3g}{2}k^2 + 2\frac{Tk^4\hat{c_5}\cosh(2kh)}{\omega\rho} \\
&+\hat{c_2}\frac{Tk^3}{\sigma\rho} + \frac{3Tk^4}{2\rho})|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) + c.c. \qquad (125)
\end{aligned}
$$

であり, 右辺は

$$
\begin{aligned}
&-\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\\
&-\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}-\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}\\
&-\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1\partial z}\\
&+\zeta_2(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z}-\frac{1}{2}\zeta_1^2(x_0,\cdots,t_0,\cdots)\frac{\partial^3\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0\partial z^2}\\
&-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0})\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0\partial z}\}-(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z})\\
&\{\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z}\\
&+\zeta_1(x_0,\cdots,t_0,\cdots)\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\}\\
&+(\frac{T}{\rho})\{2\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_1}+2\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0\partial x_2}\\
&+\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1^2}\\
&-(\frac{3}{2})(\frac{\partial^2\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0^2})(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0})^2\}
\end{aligned}
$$

$$= -(-i\hat{c_5}\cosh(2kh)2A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{2i\theta}$$
$$-\hat{c_6}h\sinh(kh)(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1})e^{i\theta}$$
$$+c.c. + \frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1})$$
$$-(-i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}e^{i\theta}$$
$$+c.c + \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2})$$
$$-(A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)(-i\hat{c_5}(2k)\sinh(2kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(-2i\omega)$$
$$-\hat{c_6}\{\sinh(kh) + kh\cosh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(-i\omega) + c.c.)$$
$$-(A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)(-i\hat{c_1}k\sinh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}e^{i\theta} + c.c.)$$
$$-(\hat{c_2}A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta} - i\hat{c_3}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}$$
$$-i\hat{c_4}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1})e^{i\theta} + c.c. + \eta_2(x_1,\cdots,t_1,\cdots))$$
$$(-i\hat{c_1}k\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(-i\omega) + c.c.)$$
$$-\frac{1}{2}(A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)^2$$
$$(-i\hat{c_1}k^2\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(-i\omega) + c.c.)$$
$$-(-i\hat{c_1}\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik) + c.c.)$$
$$\{(-i\hat{c_5}\cosh(2kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}(2ik)$$
$$-\hat{c_6}h\sinh(kh)(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta}(ik) + c.c.)$$
$$+(-i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{i\theta} + c.c. + \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})$$
$$+(A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)(-i\hat{c_1}k\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik) + c.c.)\}$$
$$-(-i\hat{c_1}k\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)\{(-i\hat{c_5}2k\sinh(2kh)A^2(x_1,\cdots,t_1,\cdots)e^{2i\theta}$$
$$-\hat{c_6}\{\sinh(kh) + kh\cosh(kh)\}(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1})e^{i\theta} + c.c.)$$
$$+(A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)(-i\hat{c_1}k^2\cosh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)\}$$
$$+\frac{T}{\rho}\{2(\hat{c_2}2A(x_1,\cdots,t_1,\cdots)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}e^{2i\theta}(2ik)$$
$$-i\hat{c_3}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2})e^{i\theta}(ik)$$
$$-i\hat{c_4}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1 \partial x_1})e^{i\theta}(ik) + c.c.)$$
$$+2(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}e^{i\theta}(ik) + c.c.) + (\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}e^{i\theta} + c.c.)$$
$$-(\frac{3}{2})(A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik)^2 + c.c.)(A(x_1,\cdots,t_1,\cdots)e^{i\theta}(ik) + c.c.)^2\}$$

$$
\begin{aligned}
&= +\hat{c_6} h \sinh(kh)(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1})e^{i\theta} \\
&+c.c. - \frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} \\
&+i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}e^{i\theta} \\
&+c.c. - \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2} \\
&+4k\omega\hat{c_5}\sinh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c. \\
&+\omega\hat{c_1}k\sinh(kh)\hat{c_2}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} \\
&+\omega\eta_2(x_1,\cdots,t_1,\cdots)\hat{c_1}k\sinh(kh)A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c. \\
&-\frac{1}{2}(-\omega\hat{c_1}k^2\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} \\
&-2\omega\hat{c_1}k^2\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.) \\
&-\{(2k^2\hat{c_5}\hat{c_1}\cosh(kh)\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.) \\
&+(k\hat{c_1}\cosh(kh)\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.) \\
&+(k^3\hat{c_1}^2\sinh(kh)\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} \\
&+2k^3\hat{c_1}^2\sinh(kh)\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta})\} \\
&-\{(2k^2\hat{c_1}\hat{c_5}\sinh(kh)\sinh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.) \\
&+(\hat{c_1}^2k^3\sinh(kh)\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)\} \\
&+\frac{T}{\rho}\{2k\hat{c_3}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2})e^{i\theta} \\
&+2k\hat{c_4}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1 \partial x_1})e^{i\theta} + c.c. \\
&+2ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}e^{i\theta} + c.c. + \frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}e^{i\theta} + c.c. \\
&-(\frac{3}{2})(-2k^4|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} \\
&+k^4|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)e^{i\theta} + c.c.)\}
\end{aligned}
$$

$$
\begin{aligned}
&= (\hat{c_6} h \sinh(kh) \frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} + i\hat{c_1}\cosh(kh)\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2} \\
&+4k\omega\hat{c_5}\sinh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+\omega\hat{c_1}k\sinh(kh)\hat{c_2}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+\omega\eta_2(x_1,\cdots,t_1,\cdots)\hat{c_1}k\sinh(kh)A(x_1,\cdots,t_1,\cdots) \\
&+\frac{1}{2}\omega\hat{c_1}k^2\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+\omega\hat{c_1}k^2\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-2k^2\hat{c_5}\hat{c_1}\cosh(kh)\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-k\hat{c_1}\cosh(kh)\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots) \\
&-3k^3\hat{c_1}^2\sinh(kh)\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-2k^2\hat{c_1}\hat{c_5}\sinh(kh)\sinh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&-\hat{c_1}^2k^3\sinh(kh)\cosh(kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots) \\
&+\frac{T}{\rho}2k\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} \\
&+2\frac{T}{\rho}k\hat{c_4}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1 \partial x_1} + 2\frac{T}{\rho}ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2} \\
&+\frac{T}{\rho}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} + \frac{3T}{2\rho}k^4|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots))e^{i\theta} \\
&-\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} - \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2} + c.c.
\end{aligned}
$$

$$
\begin{aligned}
&=(\frac{\omega h}{k}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1}+\frac{i\omega}{k\sigma}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}\\
&+4k\omega\hat{c_5}\sinh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&+\omega^2\hat{c_2}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&+\omega^2\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots)\\
&+\frac{k\omega^2}{2\sigma}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&+\frac{k\omega^2}{\sigma}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&-2\frac{k\omega}{\sigma}\hat{c_5}\cosh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&-\frac{\omega}{\sigma}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)\\
&-3\frac{k\omega^2}{\sigma}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&-2k\omega\hat{c_5}\sinh(2kh)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&-\frac{k\omega^2}{\sigma}|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)+\frac{T}{\rho}2k\hat{c_3}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\\
&+2\frac{T}{\rho}\frac{k}{\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1 \partial x_1}+2\frac{T}{\rho}ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}\\
&+\frac{T}{\rho}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+\frac{3T}{2\rho}k^4|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots))e^{i\theta}\\
&-\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}-\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2}+c.c.\\
&=(\frac{i\omega}{k\sigma}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+2\frac{T}{\rho}ik\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}\\
&+(\frac{\omega h}{k}+2\frac{T}{\rho}\frac{k}{\omega})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1}\\
&+(\frac{T}{\rho}2k\hat{c_3}+\frac{T}{\rho})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\\
&+(4k\omega\hat{c_5}\sinh(2kh)+\omega^2\hat{c_2}+\frac{3k\omega^2}{2\sigma}-2\frac{k\omega}{\sigma}\hat{c_5}\cosh(2kh)-4\frac{k\omega^2}{\sigma}\\
&-2k\omega\hat{c_5}\sinh(2kh)+\frac{3T}{2\rho}k^4)|A(x_1,\cdots,t_1,\cdots)|^2 A(x_1,\cdots,t_1,\cdots)\\
&+(\omega^2\eta_2(x_1,\cdots,t_1,\cdots)-\frac{\omega}{\sigma}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})A(x_1,\cdots,t_1,\cdots))e^{i\theta}\\
&-\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}-\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2}+c.c. \qquad (126)
\end{aligned}
$$

である．ここで $e^{i\theta}$ だけ考える．左辺=右辺より，

$$
\begin{aligned}
&((-i2\frac{\omega}{k\sigma})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}\\
&+(-i2\frac{\omega v_g}{k\sigma})\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}\\
&+(-\frac{1}{k\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\\
&+(-\frac{1}{k\omega\sigma}\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}\\
&+(-\frac{1}{k^2\sigma}\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\\
&+(\frac{k^2}{2\sigma^2}\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)\\
&+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&[\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}]^{-1})|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)\\
&+\{(\omega^2(\frac{1}{\sigma^2}-1))\eta_2(x_1,\cdots,t_1,\cdots)+\frac{2\omega}{\sigma}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\}A(x_1,\cdots,t_1,\cdots))e^{i\theta}\\
&=0
\end{aligned}
$$

となる.$-2\frac{\omega}{k\sigma}$ を両辺で割ると，

$$
\begin{aligned}
&(i(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2})+\frac{1}{2\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {t_1}^2}\\
&+\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{1}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1})\\
&+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {x_1}^2}\\
&-k^3\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&[4\omega\sigma\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}]^{-1}|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)\\
&+[\{\frac{1}{2}k\omega(\sigma-\sigma^{-1})\}\eta_2(x_1,\cdots,t_1,\cdots)-k\frac{\partial\Psi(x_1,\cdots,t_1,\cdots)}{\partial x_1}]A(x_1,\cdots,t_1,\cdots))e^{i\theta}=0
\end{aligned}
$$

となる．又，$e^{i\theta}$ の係数は，$e^{i\theta}$ に比例する項の係数を 0 とする条件として与えられる．よって

$$
\begin{aligned}
&i(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2})+\frac{1}{2\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {t_1}^2}+\{(g+\frac{3Tk^2}{\rho})\sigma\\
&+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{1}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1})\\
&+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {x_1}^2}\\
&-k^3\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&[4\omega\sigma\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}]^{-1}|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)\\
&+[\{\frac{1}{2}k\omega(\sigma-\sigma^{-1})\}\eta_2(x_1,\cdots,t_1,\cdots)-k\frac{\partial\Psi(x_1,\cdots,t_1,\cdots)}{\partial x_1}]A(x_1,\cdots,t_1,\cdots)=0 \qquad (127)
\end{aligned}
$$

となる. 又, さきほどの $\epsilon^{(0)}$ の係数は, 左辺=右辺の関係から,

$$-\frac{1}{g}\frac{\partial\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}-\frac{1}{g}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2}=\Psi_3(x_1,\cdots,t_1,\cdots)$$

とおける. よって,

$$\begin{aligned}
&\zeta_3(x_0,\cdots,t_0,\cdots)=(-\frac{1}{2}(\frac{2h}{k\sigma}+h^2)\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\\
&+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2}(\frac{1}{ki}+\frac{h}{\sigma i})-\frac{1}{\omega}(\frac{1}{k}+\frac{h}{\sigma})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}-\frac{1}{\omega^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1^2}\\
&+\frac{1}{\omega i}\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+\frac{k}{\sigma}\eta_2(x_1,\cdots,t_1,\cdots)A(x_1,\cdots,t_1,\cdots)\\
&+\frac{k}{\omega}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}A(x_1,\cdots,t_1,\cdots)\\
&+(2\frac{k^2\hat{c}_5\cosh(2kh)}{\omega}+k\hat{c}_2\frac{1}{\sigma}+\frac{3}{2}k^2)|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots))e^{i\theta}\\
&+c.c.+\Psi_3(x_1,\cdots,t_1,\cdots)
\end{aligned}\tag{128}$$

と考えられることがわかる. $\epsilon^{(i\theta)}$ に比例する項の係数を 0 とする条件が求められたことがわかる. ここでこれまでの摂動展開で得られた式を見てみる. (119), (123) より Ψ_1 を消去する.
まず, (119) の両辺を x_1 で 2 階微分し, h をかけると

$$hg\frac{\partial^2\eta_2(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+h\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1^2}+h\omega^2(1-\sigma^2)\sigma^{-2}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2}=0 \tag{129}$$

となる. 次に (123) の両辺を t_1 で微分すると

$$\frac{\partial^2\eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1^2}+h\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_1}+\frac{2\omega}{\sigma}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1\partial t_1}=0 \tag{130}$$

を得る.(130) から (129) を引くと,

$$\begin{aligned}
&(\frac{\partial^2}{\partial t_1^2}-gh\frac{\partial^2}{\partial x_1^2})\eta_2(x_1,\cdots,t_1,\cdots)+\frac{2\omega}{\sigma}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1\partial t_1}\\
&-h\omega^2(1-\sigma^2)\sigma^{-2}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2}=0
\end{aligned}\tag{131}$$

となる. 次に η_2 を消去する

$$\begin{aligned}
&(\frac{\partial^2}{\partial t_1^2}-gh\frac{\partial^2}{\partial x_1^2})\Phi_1(x_1,\cdots,t_1,\cdots)-\frac{2\omega g}{\sigma}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1\partial t_1}\\
&+\omega^2(1-\sigma^2)\sigma^{-2}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2}=0
\end{aligned}\tag{132}$$

となり Φ_1 に対する同様な式が得られる.

ここで A が (116) を満たしていれば, 長波-短波相互作用の式と同じ形になる. (127) の第 2,3,4 項を,

$$\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}$$

の形に直す.

$$\begin{aligned}&\frac{1}{2\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {t_1}^2}\\&=\frac{1}{2\omega}(-v_g\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1})\\&=\frac{1}{2\omega}(-v_g\frac{\partial}{\partial x_1}[-v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}])\quad(116)\\&=\frac{1}{2\omega}v_g^2\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2},\end{aligned}\tag{133}$$

$$\begin{aligned}&\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{1}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1})\\&=-\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{v_g}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2})\end{aligned}\tag{134}$$

から,

$$\begin{aligned}&\frac{1}{2\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {t_1}^2}+\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\\&\frac{1}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1})\\&+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {x_1}^2}\\&=(\frac{v_g^2}{2\omega}-\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{v_g}{2\omega^2}\\&+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega})\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {x_1}^2}\quad(117)\\&=\frac{1}{8\omega^3}\{-(g+\frac{3Tk^2}{\rho})^2\sigma^2+(g+\frac{Tk^2}{\rho})^2(kh)^2(-1-2\sigma^2+3\sigma^4)\\&+(g+\frac{3Tk^2}{\rho})(g+\frac{Tk^2}{\rho})kh\sigma(2-2\sigma^2)\\&+12(g+\frac{Tk^2}{\rho})\frac{Tk\sigma}{\rho}k\sigma\}\end{aligned}\tag{135}$$

となる. 又,

$$\begin{aligned}&\frac{1}{2}\frac{dv_g}{dk}=\frac{1}{8\omega^3}\{-(g+\frac{3Tk^2}{\rho})^2\sigma^2+(g+\frac{Tk^2}{\rho})^2(kh)^2(-1-2\sigma^2+3\sigma^4)\\&+(g+\frac{Tk^2}{\rho})(g+\frac{3Tk^2}{\rho})kh\sigma(2-2\sigma^2)\\&+12(g+\frac{Tk^2}{\rho})\frac{Tk\sigma}{\rho}k\sigma\}\end{aligned}\tag{136}$$

と書き換えられることができる. 又,

$$|A|^2|A| \text{ の係数を}\nu_1, \eta_2 A \text{ の係数を}\nu_2, \frac{\partial\Psi(x_1,\cdots,t_1,\cdots)}{\partial x_1}A \text{ の係数を}\nu_3$$

として (127) を書き直すと,

$$i(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2})+\frac{1}{2}\frac{dv_g}{dk}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {x_1}^2}$$
$$+\nu_1|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)+(\nu_2\eta_2(x_1,\cdots,t_1,\cdots)$$
$$-\nu_3\frac{\partial\Psi(x_1,\cdots,t_1,\cdots)}{\partial x_1})A(x_1,\cdots,t_1,\cdots)=0 \tag{137}$$

となる. 同様に, (131) の左辺第 2 項を,

$$\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}$$

の形に直す.

$$\frac{2\omega}{\sigma}\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1\partial t_1}$$
$$=\frac{2\omega}{\sigma}(-v_g\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2}) \tag{138}$$

として (131) を書き直すと,

$$(\frac{\partial^2}{\partial t_1^2}-gh\frac{\partial^2}{\partial x_1^2})\eta_2(x_1,\cdots,t_1,\cdots)=(v_g\frac{2\omega}{\sigma}+h\omega^2(1-\sigma^2)\sigma^{-2})\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2} \tag{139}$$

となる. 又,

$$\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2}\text{の係数を}\nu_4$$

とおく.

$$(\frac{\partial^2}{\partial t_1^2}-gh\frac{\partial^2}{\partial x_1^2})\eta_2(x_1,\cdots,t_1,\cdots)=\nu_4\frac{\partial^2|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1^2} \tag{140}$$

となる. これは, (132) も同様にできる. (117) は,A が

$$\xi\equiv x_1-v_gt_1$$

の組み合わせで t_1,x_1 に依存する時,

$$\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_1}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_1}=0$$
$$\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\xi}\frac{\partial\xi}{\partial t_1}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\xi}\frac{\partial\xi}{\partial x_1}=0$$
$$-v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\xi}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\xi}=0 \tag{141}$$

となり満たしていることがわかる. さらに η_2,Ψ_1 が A と同様に ξ のみを通じて t_1,x_1 に依存する時, (119) より,

$$g\eta_2(x_1,\cdots,t_1,\cdots)+\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1}+\omega^2(1-\sigma^2)\sigma^{-2}|A(x_1,\cdots,t_1,\cdots)|^2=0$$
$$g\eta_2(x_1,\cdots,t_1,\cdots)+\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial\xi}\frac{\partial\xi}{\partial t_1}+\omega^2(1-\sigma^2)\sigma^{-2}|A(x_1,\cdots,t_1,\cdots)|^2=0$$
$$g\eta_2(x_1,\cdots,t_1,\cdots)-v_g\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial\xi}+\omega^2(1-\sigma^2)\sigma^{-2}|A(x_1,\cdots,t_1,\cdots)|^2$$
$$=0 \tag{142}$$

(123) より

$$
\begin{aligned}
&\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1}+h\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+\frac{2\omega}{\sigma}\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial x_1}=0\\
&\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi}\frac{\partial \xi}{\partial t_1}+h\frac{\partial}{\partial x}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi}+\frac{2\omega}{\sigma}\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial \xi}\frac{\partial \xi}{\partial x_1}=0\\
&-v_g\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi}+h\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}+\frac{2\omega}{\sigma}\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial \xi}=0 \qquad (143)
\end{aligned}
$$

(143) に (142) を代入する.

$$
\begin{aligned}
&(\frac{-v_g^2+gh}{g})\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}+(\frac{v_g\omega^2(1-\sigma^2)\sigma^{-1}+2\omega g}{g\sigma})\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial \xi}=0\\
&\sigma(-v_g^2+gh)\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}+(v_g\omega^2(1-\sigma^2)\sigma^{-1}+2\omega g)\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial \xi}=0\\
&(v_g^2-gh)\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}-(v_g\omega^2(1-\sigma^2)\sigma^{-2}+2\omega g\sigma^{-1})\frac{\partial|A(x_1,\cdots,t_1,\cdots)|^2}{\partial \xi}=0
\end{aligned}
$$

両辺積分して, 整理すると,

$$
(v_g^2-gh)\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi}=(v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1})|A(x_1,\cdots,t_1,\cdots)|^2+D
$$

となる.D は ξ に依存しない関数とする.

$$
\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi}=\frac{(v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1})|A(x_1,\cdots,t_1,\cdots)|^2}{v_g^2-gh}+\frac{D}{v_g^2-gh}
$$

である. ここで,

$$
\frac{D}{v_g^2-gh}=r(x_2,\cdots,t_2,\cdots)
$$

とおいて書き直すと,

$$
\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi}=\frac{(v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1})|A(x_1,\cdots,t_1,\cdots)|^2}{v_g^2-gh}+r(x_2,\cdots,t_2,\cdots) \qquad (144)
$$

となる.(127) において ξ 依存性を仮定し, さらに A が t_2, x_2 に関して,

$$
\xi_2\equiv x_2-v_gt_2, \tau\equiv t_2
$$

に依存するとする. まず (127) の第 1 項を書き直す. この時, (127) は,

$$
\begin{aligned}
&i(\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial t_2}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial x_2})\\
&=i(-v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial \xi_2}+\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial \tau}+v_g\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial \xi_2})\\
&=i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial \tau} \qquad (145)
\end{aligned}
$$

となる.(127) の第 2 項以降を書き直す.

$$
\begin{aligned}
&\frac{1}{2\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {t_1}^2}+\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\\
&\frac{1}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial t_1 \partial x_1})\\
&+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial {x_1}^2}-k^3\{g^2(9\sigma^4-10\sigma^2+9)\\
&+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)\\
&+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}[4\omega\sigma\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}]^{-1}\\
&|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)\\
&+[\{\frac{1}{2}k\omega(\sigma-\sigma^{-1})\}\eta_2(x_1,\cdots,t_1,\cdots)-k\frac{\partial\Psi(x_1,\cdots,t_1,\cdots)}{\partial x_1}]A(x_1,\cdots,t_1,\cdots)=0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{v_g^2}{2\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}-\{(g+\frac{3Tk^2}{\rho})\sigma\\
&+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{v_g}{2\omega^2}(\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial \xi^2})\\
&+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial \xi^2}-k^3\{g^2(9\sigma^4-10\sigma^2+9)\\
&+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)\\
&+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}[4\omega\sigma\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}]^{-1}\\
&|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)\\
&+[\{\frac{1}{2}k\omega(\sigma-\sigma^{-1})\}\{(\frac{v_g}{g}\frac{(v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1})|A(x_1,\cdots,t_1,\cdots)|^2}{v_g^2-gh}\\
&+r(x_2,\cdots,t_2,\cdots))\\
&-\frac{\omega^2}{g}(1-\sigma^2)\sigma^{-2}|A(x_1,\cdots,t_1,\cdots)|^2\}\\
&-k\{(\frac{v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1}}{v_g^2-gh})|A(x_1,\cdots,t_1,\cdots)|^2\\
&+r(x_2,\cdots,t_2,\cdots)\}]A(x_1,\cdots,t_1,\cdots)=0,
\end{aligned}
\tag{146}
$$

$\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}$ の係数を計算すると,

$$
\begin{aligned}
&\frac{v_g^2}{2\omega}-\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1+\sigma^2)\}\frac{v_g}{2\omega^2}+\{(\frac{3Tk^2\sigma}{\rho})+(g+\frac{3Tk^2}{\rho})kh\}\frac{1}{2k\omega}\\
&=-\frac{1}{8\omega^3}((g+\frac{3Tk^2}{\rho})^2\sigma^2+(g+\frac{Tk^2}{\rho})^2k^2h^2(1+2\sigma^2-3\sigma^4)\\
&+2(g+\frac{3Tk^2}{\rho})(g+\frac{Tk^2}{\rho})kh(\sigma+\sigma^3))\\
&+\frac{1}{2\omega}(\frac{3Tk}{\rho}\sigma+(g+\frac{3Tk^2}{\rho})h)\\
&=-\frac{1}{8\omega^3}((g+\frac{3Tk^2}{\rho})^2\sigma^2+(g+\frac{Tk^2}{\rho})^2k^2h^2(1+2\sigma^2-3\sigma^4)\\
&+2(g+\frac{3Tk^2}{\rho})(g+\frac{Tk^2}{\rho})kh(-\sigma+\sigma^3)\\
&-12\frac{Tk}{\rho}(g+\frac{Tk^2}{\rho})k\sigma^2)
\end{aligned}
$$

となる. 又,

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2\omega}{dk^2}=-\frac{1}{8\omega^3}((g+\frac{Tk^2}{\rho})kh(1-\sigma^2)+(g+\frac{3Tk^2}{\rho})\sigma)^2\\
&+\frac{1}{4\omega}(\frac{6Tk}{\rho}\sigma+2(g+\frac{3Tk^2}{\rho})h(1-\sigma^2)+(g+\frac{Tk^2}{\rho})kh^2(-2\sigma+2\sigma^3))\\
&=-\frac{1}{8\omega^3}((g+\frac{3Tk^2}{\rho})^2\sigma^2+(g+\frac{Tk^2}{\rho})^2k^2h^2(1-2\sigma^2+\sigma^4)\\
&+2(g+\frac{3Tk^2}{\rho})(g+\frac{Tk^2}{\rho})kh(\sigma-\sigma^3)\\
&-2((g+\frac{Tk^2}{\rho})\frac{6Tk}{\rho}k\sigma^2+2(g+\frac{3Tk^2}{\rho})(g+\frac{Tk^2}{\rho})kh(\sigma-\sigma^3)\\
&+(g+\frac{Tk^2}{\rho})^2k^2h^2(-2\sigma^2+2\sigma^4))\\
&=-\frac{1}{8\omega^3}((g+\frac{3Tk^2}{\rho})^2\sigma^2+(g+\frac{Tk^2}{\rho})^2k^2h^2(1+2\sigma^2-3\sigma^4)\\
&+2(g+\frac{3Tk^2}{\rho})(g+\frac{Tk^2}{\rho})kh(-\sigma+\sigma^3)\\
&-12\frac{Tk}{\rho}(g+\frac{Tk^2}{\rho})k\sigma^2)
\end{aligned}
$$

より

$$\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}\text{の係数を}\frac{1}{2}\frac{d^2\omega}{dk^2}$$

とおけることができる.

次に, $|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)$ の係数も同様に計算する.

$$\frac{1}{2}k\omega(\sigma-\sigma^{-1})\frac{v_g}{g}\frac{(v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1})}{v_g^2-gh}$$
$$=\frac{k\omega}{2g(v_g^2-gh)}(-v_g^2\omega^2(\sigma-2\sigma^{-1}+\sigma^{-3})+2v_g\omega g(1-\sigma^{-2})) \quad (147)$$

$$-\frac{1}{2}k\omega(\sigma-\sigma^{-1})\frac{\omega^2}{g}(1-\sigma^2)\sigma^{-2}$$
$$=\frac{k\omega}{2g(v_g^2-gh)}((v_g^2-gh)\omega^2(\sigma-2\sigma^{-1}+\sigma^{-3})) \quad (148)$$

$$-k\{\frac{v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1}}{v_g^2-gh}\}$$
$$=\frac{k\omega}{2g(v_g^2-gh)}(-2g\omega v_g(\sigma^{-2}-1)-4g^2\sigma^{-1}) \quad (149)$$

となる.(147),(148),(149) をまとめれば,

$$\frac{k\omega}{2g(v_g^2-gh)}\{-gh\omega^2(\sigma-2\sigma^{-1}+\sigma^{-3})-4g^2\sigma^{-1}+4\omega gv_g(1-\sigma^{-2})\}$$
$$=-(\frac{k^2}{4\omega\sigma})\frac{1}{(v_g^2-gh)}\{\frac{-2\omega^2\sigma}{kg}(-gh\omega^2(\sigma+\sigma^{-3}-2\sigma^{-1})-4g^2\sigma^{-1}+4\omega gv_g(1-\sigma^{-2})\}$$

が得られる.

以上から

$$
\begin{aligned}
&-k^3\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&[4\omega\sigma\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}]^{-1}\\
&+\{\frac{1}{2}k\omega(\sigma-\sigma^{-1})\}\{\frac{v_g}{g}\frac{(v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1})}{v_g^2-gh}-\frac{\omega^2}{g}(1-\sigma^2)\sigma^{-2}\}\\
&-k(\frac{v_g\omega^2(\sigma^{-2}-1)+2\omega g\sigma^{-1}}{v_g^2-gh})\\
&=-(\frac{k^2}{4\omega\sigma})[k\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}^{-1}\\
&+\frac{1}{(v_g^2-gh)}\{\frac{-2\omega^2\sigma}{kg}(-gh\omega^2(\sigma+\sigma^{-3}-2\sigma^{-1})-4g^2\sigma^{-1}+4\omega gv_g(1-\sigma^{-2})\}\\
&=-(\frac{k^2}{4\omega\sigma})[k\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}^{-1}\\
&+\frac{1}{(v_g^2-gh)}\{2kh(g+\frac{Tk^2}{\rho})^2(\sigma^4+1-2\sigma^2)+8(g+\frac{Tk^2}{\rho})g\sigma-4(\sigma^2-1)(g+\frac{Tk^2}{\rho})\\
&\{(g+\frac{3Tk^2}{\rho})\sigma+(g+\frac{Tk^2}{\rho})kh(1-\sigma^2)\}\}]\\
&=-(\frac{k^2}{4\omega\sigma})[k\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}^{-1}\\
&+\frac{1}{(v_g^2-gh)}2(g+\frac{Tk^2}{\rho})\{(3kh(g+\frac{Tk^2}{\rho})(\sigma^4+1-2\sigma^2)-2(g+\frac{3Tk^2}{\rho})(\sigma^3-\sigma)+4g\sigma)\}\\
&=-(\frac{k^2}{4\omega\sigma})[k\{g^2(9\sigma^4-10\sigma^2+9)+(\frac{gTk^2}{\rho})(15\sigma^4-44\sigma^2+30)+(\frac{T^2k^4}{\rho^2})(6\sigma^4-25\sigma^2+21)\}\\
&\{g\sigma^2+(\frac{Tk^2}{\rho})(\sigma^2-3)\}^{-1}\\
&+\frac{1}{(v_g^2-gh)}2(g+\frac{Tk^2}{\rho})\{3kh(g+\frac{Tk^2}{\rho})(1-\sigma^2)^2-2(g+\frac{3Tk^2}{\rho})\sigma^3+6(g+\frac{Tk^2}{\rho})\sigma\}\\
&=\nu \qquad (150)
\end{aligned}
$$

よりすべてをまとめて表すと,

$$
\begin{aligned}
&i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\tau}+\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}+\nu|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)\\
&+\{\frac{1}{2}k\omega(\sigma-\sigma^{-1})-k\}r(x_2,t_2)A(x_1,\cdots,t_1,\cdots)=0 \qquad (151)
\end{aligned}
$$

となるが, 関数 r は適当な変換により消去できる.

＜証明＞

$$i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\tau}+c_2\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}+c_3|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)$$
$$+r(\tau)A(x_1,\cdots,t_1,\cdots)=0 \tag{152}$$

c_2,c_3 を定数,$r_{(}\tau)$ を実数とする. 又,

$$\begin{aligned} A &= \epsilon^{i\int_0^\tau r(s)ds}\tilde{A} \\ &= E\tilde{A} \end{aligned}$$

を使って書き直す.

$$\begin{aligned}
&E(i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\tau}-r(\tau)\tilde{A}+c_2\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}\\
&+c_3|\tilde{A}(x_1,\cdots,t_1,\cdots)|^2\tilde{A}(x_1,\cdots,t_1,\cdots)\\
&+r(\tau)\tilde{A}(x_1,\cdots,t_1,\cdots))=0\\
&\Rightarrow E(i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\tau}+c_2\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}\\
&+c_3|\tilde{A}(x_1,\cdots,t_1,\cdots)|^2\tilde{A}(x_1,\cdots,t_1,\cdots))=0
\end{aligned}$$

証明終了.

よって

$$i\frac{\partial A(x_1,\cdots,t_1,\cdots)}{\partial\tau}+\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A(x_1,\cdots,t_1,\cdots)}{\partial\xi^2}+\nu|A(x_1,\cdots,t_1,\cdots)|^2A(x_1,\cdots,t_1,\cdots)=0 \tag{153}$$

となり非線形シュレディンガー方程式が導けたことがわかる.

4.2.2 長波方程式

長波長の波のみが存在する時, KdV 方程式が求められる. 実際 KdV 方程式が求められるか解いて見る. 短波長成分が存在しないと考える. 条件として,

$$A = 0, x_0, t_0 \text{が依存性がない} \tag{154}$$

とする. (119) より,

$$g\eta_2(x_1,\cdots,t_1,\cdots) + \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1} = 0 \tag{155}$$

となる.(123) より

$$\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} + h\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} = 0 \tag{156}$$

となる.(155) を t_1 で微分すると,

$$g\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} + \frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1^2} = 0 \tag{157}$$

が得られる.(156) に g をかけると,

$$g\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} + gh\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} = 0 \tag{158}$$

となる.(157) から (158) をひけば

$$(\frac{\partial^2}{\partial t_1^2} - gh\frac{\partial^2}{\partial x_1^2})\Psi_1(x_1,\cdots,t_1,\cdots) = 0 \tag{159}$$

を得る.$\eta_2(x_1,\cdots,t_1,\cdots)$ も (154) を使って表示する.

$$(\frac{\partial^2}{\partial t_1^2} - gh\frac{\partial^2}{\partial x_1^2})\eta_2(x_1,\cdots,t_1,\cdots) = 0 \tag{160}$$

高次近似に進むと,$\Psi_1(x_1,\cdots,t_1,\cdots)$ のみを含む式が表示される.
準備として, ここで (84), (85), (86) の $e^{ni\theta}(n = 4, 5)$ を求めてみる.
又, (154) を使って表す.

$$\zeta_1(x_0,\cdots,t_0,\cdots) = 0, \Phi_1(x_0,\cdots,t_0,\cdots) = \Psi_1(x_1,\cdots,t_1,\cdots), \tag{161}$$

$$\zeta_2(x_0,\cdots,t_0,\cdots) = \eta_2(x_1,\cdots,t_1,\cdots), \tag{162}$$

$$\Phi_2(x_0,\cdots,t_0,\cdots) = \Psi_2(x_1,\cdots,t_1,\cdots), \tag{163}$$

$$\begin{aligned}\zeta_3(x_0,\cdots,t_0,\cdots) &= \Psi_3(x_1,\cdots,t_1,\cdots)\\ &= -\frac{1}{g}\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} - \frac{1}{g}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2},\end{aligned} \tag{164}$$

$$\Phi_3(x_0,\cdots,t_0,\cdots) = -\frac{1}{2}\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}(z+h)^2 \tag{165}$$

と表示される. さらに, (119), (123) から,

$$g\eta_2(x_1,\cdots,t_1,\cdots) + \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1} = 0, \tag{166}$$

$$\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_1} + h\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} = 0 \tag{167}$$

となる. 最初に,(84) を求める. 又, (154) を使って表すと

$$
\begin{aligned}
&\frac{\partial^2\Phi(x,z,t)}{\partial x^2}\\
&=\epsilon^4(2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1\partial x_2}+\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_1^2})\\
&+\epsilon^5(2\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1\partial x_3}+\frac{\partial^2\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_2^2}\\
&+2\frac{\partial^2\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_1\partial x_2}\\
&+\frac{\partial^2\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_1^2})+\cdots
\end{aligned}
$$

となる. 更に,(162) ∼ (167) を使って表わすと

$$
\begin{aligned}
&\frac{\partial^2\Phi(x,z,t)}{\partial x^2}\\
&=+\epsilon^4(2\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial x_2}+\frac{\partial^2\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1^2})\\
&+\epsilon^5(2\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial x_3})+\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_2^2}\\
&+2\frac{\partial^2\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial x_2}\\
&-\frac{1}{2}\frac{\partial^4\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^4}(z+h)^2)+\cdots,
\end{aligned}
\tag{168}
$$

$$
\frac{\partial^2\Phi(x,z,t)}{\partial z^2}=\epsilon^4\frac{\partial^2\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}+\epsilon^5\frac{\partial^2\Phi_5(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}+\cdots \tag{169}
$$

となる. 以上から, (168), (169) を使い,ϵ^4,ϵ^5 でまとめると

$$
\begin{aligned}
O(\varepsilon^4)\quad:\quad&2\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial x_2}+\frac{\partial^2\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}+\frac{\partial^2\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}\\
&=0,
\end{aligned}
\tag{170}
$$

$$
\begin{aligned}
O(\varepsilon^5)\quad:\quad&2\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial x_3})+\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_2^2}\\
&+2\frac{\partial^2\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial x_2}\\
&-\frac{1}{2}\frac{\partial^4\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^4}(z+h)^2+\frac{\partial^2\Phi_5(x_0,\cdots,z,t_0,\cdots)}{\partial z^2}=0
\end{aligned}
\tag{171}
$$

となる.

次に (85) を求める.

$$\frac{\partial\zeta(x_0,\cdots,t_0,\cdots)}{\partial t} = \frac{\partial}{\partial t}(\sum_{n=1}^{\infty}\epsilon^n\zeta_n(x_0,\cdots,t_0,\cdots))$$

$$\begin{aligned}
= \quad & \epsilon(\frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_0} + \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_1}\epsilon + \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_2}\epsilon^2 \\
& + \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_3}\epsilon^3 + \frac{\partial\zeta_1(x_0,\cdots,t_0,\cdots)}{\partial t_4}\epsilon^4 + \cdots) \\
& \epsilon^2(\frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_0} + \frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_1}\epsilon + \frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_2}\epsilon^2 \\
& + \frac{\partial\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial t_3}\epsilon^3 + \cdots) \\
& \epsilon^3(\frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_0} + \frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_1}\epsilon + \frac{\partial\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial t_2}\epsilon^2 + \cdots) \\
& \epsilon^4(\frac{\partial\zeta_4(x_0,\cdots,t_0,\cdots)}{\partial t_0} + \frac{\partial\zeta_4(x_0,\cdots,t_0,\cdots)}{\partial t_1}\epsilon + \cdots) \\
& \epsilon^5(\frac{\partial\zeta_5(x_0,\cdots,t_0,\cdots)}{\partial t_0} + \cdots) + \cdots
\end{aligned}$$

ここで, (154) と, (162) ∼ (167) を使って表わすと

$$\begin{aligned}
& \frac{\partial\zeta(x_0,\cdots,t_0,\cdots)}{\partial t} = \\
& +\epsilon^4(-\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial t_2} + \frac{\partial\Psi_3(x_1,\cdots,t_1,\cdots)}{\partial t_1}) \\
& +\epsilon^5(-\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_2\partial t_3} + \frac{\partial\Psi_3(x_1,\cdots,t_1,\cdots)}{\partial t_2} \\
& +\frac{\partial\zeta_4(x_0,\cdots,t_0,\cdots)}{\partial t_1}) + \cdots
\end{aligned} \tag{172}$$

となる. 又,

$$\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial x} = \frac{\partial}{\partial x}(\sum_{n=1}^{\infty}\epsilon^n\Phi_n(x_0,\cdots,z,t_0,\cdots))$$

$$\begin{aligned}
= \quad & \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\epsilon + \frac{\partial\Phi_1(x_0,z,\cdots,t_0,\cdots)}{\partial x_2}\epsilon^2 \\
& + \frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_3}\epsilon^3 + \frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_4}\epsilon^4 + \cdots) \\
& \epsilon^2(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\epsilon + \frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_2}\epsilon^2 \\
& + \frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_3}\epsilon^3 + \cdots) \\
& \epsilon^3(\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\epsilon + \frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_2}\epsilon^2 + \cdots) \\
& \epsilon^4(\frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}\epsilon + \cdots) \\
& \epsilon^5(\frac{\partial\Phi_5(x_0,\cdots,z,t_0,\cdots)}{\partial x_0} + \cdots)\cdots
\end{aligned}$$

を,(154) を使って表わすと,

$$
\begin{aligned}
\frac{\partial \Phi(x_0,\cdots,z,t_0,\cdots)}{\partial x} &= \epsilon^2(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}) \\
&+\epsilon^3(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_2}+\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}) \\
&+\epsilon^4(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_3}+\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_2} \\
&+\frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}) \\
&+\epsilon^5(\frac{\partial \Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_4}+\frac{\partial \Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial x_3} \\
&+\frac{\partial \Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial x_2}+\frac{\partial \Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}) \\
&+\cdots
\end{aligned}
$$

となる.$z=0$ で展開し, (162) $\sim$ (167) で書き直すと,

$$
\begin{aligned}
\frac{\partial \Phi(x_0,\cdots,z,t_0,\cdots)}{\partial x} &= \epsilon^2(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}) \\
&+\epsilon^3(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_2}+\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1}) \\
&+\epsilon^4(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_3}+\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_2} \\
&-\frac{1}{2}\frac{\partial^3 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^3}h^2) \\
&+\epsilon^5(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_4}+\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_3} \\
&-\frac{1}{2}\frac{\partial^3 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2 \partial x_2}h^2+\frac{\partial \Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}) \\
&+\cdots, \qquad (173)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial \zeta(x_0,\cdots,t_0,\cdots)}{\partial x} = \frac{\partial}{\partial x}(\sum_{n=1}^{\infty}\epsilon^n \zeta_n(x_0,\cdots,t_0,\cdots)) \\
= \ &\epsilon(\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_1}\epsilon+\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_2}\epsilon^2 \\
&+\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_3}\epsilon^3+\frac{\partial \zeta_1(x_0,\cdots,t_0,\cdots)}{\partial x_4}\epsilon^4+\cdots) \\
&\epsilon^2(\frac{\partial \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1}\epsilon+\frac{\partial \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_2}\epsilon^2 \\
&+\frac{\partial \zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_3}\epsilon^3+\cdots) \\
&\epsilon^3(\frac{\partial \zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial \zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_1}\epsilon+\frac{\partial \zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_2}\epsilon^2+\cdots) \\
&\epsilon^4(\frac{\partial \zeta_4(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\frac{\partial \zeta_4(x_0,\cdots,t_0,\cdots)}{\partial x_1}\epsilon+\cdots) \\
&\epsilon^5(\frac{\partial \zeta_5(x_0,\cdots,t_0,\cdots)}{\partial x_0}+\cdots)+\cdots
\end{aligned}
$$

となる. ここで, (154) と, (162) ～ (167) を使って表わす.

$$\frac{\partial\zeta(x_0,\cdots,t_0,\cdots)}{\partial x} = \epsilon^3(-\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1})$$
$$+\epsilon^4(-\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_2} + \frac{\partial\Psi_3}{\partial x_1})$$
$$+\epsilon^5(-\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_3} + \frac{\partial\Psi_3}{\partial x_2}$$
$$+\frac{\partial\zeta_4(x_0,\cdots,t_0,\cdots)}{\partial x_1}) + \cdots \tag{174}$$

となる.(173) と (174) をかけると,

$$\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial x}\frac{\partial\zeta(x_0,\cdots,t_0,\cdots)}{\partial x}$$
$$= \epsilon^5(-\frac{1}{g}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1}) + \cdots \tag{175}$$

$$\begin{aligned}\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial z} &= \frac{\partial}{\partial x}(\sum_{n=1}^{\infty}\epsilon^n\Phi_n(x_0,\cdots,z,t_0,\cdots)) \\ &= \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial z}) \\ &+ \epsilon^2(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial z}) \\ &+ \epsilon^3(\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial z}) \\ &+ \epsilon^4(\frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial z}) \\ &+ \epsilon^5(\frac{\partial\Phi_5(x_0,\cdots,z,t_0,\cdots)}{\partial z}) + \cdots\end{aligned}$$

が得られる.$z = 0$ で展開し, (162) ～ (167) で書き直すと

$$\begin{aligned}\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial z} &= +\epsilon^4(\frac{\partial\Phi_4(x_0,\cdots,0,t_0,\cdots)}{\partial z}) \\ &+\epsilon^5(\frac{\partial\Phi_5(x_1,\cdots,t_1,\cdots)}{\partial z} + \frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\frac{\partial\Psi_1}{\partial t_1}) \\ &+\cdots\end{aligned} \tag{176}$$

となる. 以上から, (172) ～ (176) を使い, ϵ^4, ϵ^5 でまとめる.

$$\begin{aligned}O(\varepsilon^4) \quad : \quad & -\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial t_2} + \frac{\partial\Psi_3(x_1,\cdots,t_1,\cdots)}{\partial t_1} \\ & -\frac{\partial\Phi_4(x_0,\cdots,0,t_0,\cdots)}{\partial z} = 0,\end{aligned} \tag{177}$$

$$\begin{aligned}O(\varepsilon^5) \quad : \quad & -\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial t_3} + \frac{\partial\Psi_3(x_1,\cdots,t_1,\cdots)}{\partial t_2} + \frac{\partial\zeta_4(x_0,\cdots,t_0,\cdots)}{\partial t_1} \\ & -\frac{1}{g}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1\partial t_1} - \frac{\partial\Phi_5(x_1,\cdots,t_1,\cdots)}{\partial z} \\ & -\frac{1}{g}\frac{\partial^2\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2}\frac{\partial\Psi_1}{\partial t_1} = 0\end{aligned} \tag{178}$$

が得られる.

次に (86) を出してみる.

$$
\begin{aligned}
\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial t} &= \frac{\partial}{\partial x}(\sum_{n=1}^{\infty}\epsilon^n\Phi_n(x_0,\cdots,z,t_0,\cdots)) \\
= \quad & \epsilon(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\epsilon+\frac{\partial\Phi_1(x_0,z,\cdots,t_0,\cdots)}{\partial t_2}\epsilon^2 \\
& +\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_3}\epsilon^3+\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_4}\epsilon^4+\cdots) \\
& \epsilon^2(\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\epsilon+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}\epsilon^2 \\
& +\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_3}\epsilon^3+\cdots) \\
& \epsilon^3(\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\epsilon+\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}\epsilon^2+\cdots) \\
& \epsilon^4(\frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+\frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}\epsilon+\cdots) \\
& \epsilon^5(\frac{\partial\Phi_5(x_0,\cdots,z,t_0,\cdots)}{\partial t_0}+\cdots)+\cdots
\end{aligned}
$$

を, (154) を使って表わすと

$$
\begin{aligned}
\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial t} = \quad & \epsilon^2(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial x_1}) \\
& +\epsilon^3(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}) \\
& +\epsilon^4(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_3}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_2} \\
& +\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}) \\
& +\epsilon^5(\frac{\partial\Phi_1(x_0,\cdots,z,t_0,\cdots)}{\partial t_4}+\frac{\partial\Phi_2(x_0,\cdots,z,t_0,\cdots)}{\partial t_3} \\
& +\frac{\partial\Phi_3(x_0,\cdots,z,t_0,\cdots)}{\partial t_2}+\frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial t_1})+\cdots
\end{aligned}
$$

となる.$z=0$ で展開し, (162) $\sim$ (167) で書き直すと,

$$
\begin{aligned}
\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial t} = \quad & +\epsilon^4(\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_3}+\frac{\partial\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_2} \\
& -\frac{1}{2}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_1}h^2) \\
& +\epsilon^5(\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_4}+\frac{\partial\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_3} \\
& -\frac{1}{2}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_2}h^2+\frac{\partial\Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial t_1}) \\
& +\cdots
\end{aligned} \tag{179}
$$

となる. 又,

$$
\begin{aligned}
&(\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial x})^2=\epsilon^4((\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})^2)\\
&+\epsilon^5(\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_2}\\
&+\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\frac{\partial\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1})+\cdots, \qquad (180)
\end{aligned}
$$

$$
(\frac{\partial\Phi(x_0,\cdots,z,t_0,\cdots)}{\partial z})^2=+\cdots, \qquad (181)
$$

である. さらに

$$
g\zeta=\epsilon^4 g\zeta_4(x_0,\cdots,t_0,\cdots)+\epsilon^5 g\zeta_5(x_0,\cdots,t_0,\cdots)+\cdots, \qquad (182)
$$

を (154) を使って表わすと

$$
\begin{aligned}
&\frac{\partial^2\zeta(x_0,\cdots,t_0,\cdots)}{\partial x^2}=\\
&\epsilon^4(\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1^2})\\
&+\epsilon^5(\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1\partial x_2}+\frac{\partial^2\zeta_2(x_0,\cdots,t_0,\cdots)}{\partial x_1\partial x_2}\\
&+\frac{\partial^2\zeta_3(x_0,\cdots,t_0,\cdots)}{\partial x_1^2})+\cdots
\end{aligned}
$$

となる.(162) ∼ (167) で書き直すと,

$$
\begin{aligned}
&\frac{\partial^2\zeta(x_0,\cdots,t_0,\cdots)}{\partial x^2}=\epsilon^4(-\frac{1}{g}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_1})\\
&+\epsilon^5(-2\frac{1}{g}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1\partial x_2}+\frac{\partial^2\Psi_3}{\partial x_1^2})+\cdots, \qquad (183)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{T}{\rho}\frac{\partial^2\zeta(x_0,\cdots,t_0,\cdots)}{\partial x^2}\{1+(\frac{\partial\zeta(x_0,\cdots,t_0,\cdots)}{\partial x})^2\}^{(}-\frac{3}{2})\\
&=\epsilon^4(-\frac{1}{g}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_1})\\
&+\epsilon^5(-2\frac{1}{g}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1\partial x_1\partial x_2}+\frac{\partial^2\Psi_3}{\partial x_1^2})+\cdots \qquad (184)
\end{aligned}
$$

となる. 以上から, (179) ∼ (184) を使い, ϵ^4,ϵ^5 でまとめる.

$$
\begin{aligned}
O(\varepsilon^4)\quad : \quad &\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_3}+\frac{\partial\Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_2}\\
&-\frac{1}{2}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_1}h^2+\frac{1}{2}(\frac{\partial\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})^2\\
&+g\zeta_4(x_0,\cdots,t_0,\cdots)-\frac{T}{\rho}(-\frac{1}{g}\frac{\partial^3\Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2\partial t_1})=0, \qquad (185)
\end{aligned}
$$

$$
\begin{aligned}
O(\varepsilon^5) \quad : \quad & \frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_4} + \frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial t_3} \\
& -\frac{1}{2}\frac{\partial^3 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2 \partial t_2} h^2 + \frac{\partial \Phi_4(x_0,\cdots,z,t_0,\cdots)}{\partial t_1} \\
& \frac{1}{2}(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_2} \\
& +\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1}\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial x_1}) \\
& +g\zeta_5(x_0,\cdots,t_0,\cdots) \\
& -\frac{T}{\rho}(-2\frac{1}{g}\frac{\partial^3 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1 \partial x_1 \partial x_2} + \frac{\partial^2 \Psi_3}{\partial x_1^2}) = 0 \qquad (186)
\end{aligned}
$$

となる. ここで, (170) の両辺を h 倍する. そして, (177) と比べ,

$$
h\frac{\partial \Phi_4(x_0,\cdots,0,t_0,\cdots)^2}{\partial z^2} = \frac{\partial \Phi_4(x_0,\cdots,0,t_0,\cdots)}{\partial z}
$$

が同じとして考えて計算すると,

$$
(\frac{\partial^2}{\partial t_1^2} - gh\frac{\partial^2}{\partial x_1^2})\Psi_2(x_1,\cdots,t_1,\cdots) + 2(\frac{\partial^2}{\partial t_1 \partial t_2} - gh\frac{\partial^2}{\partial x_1 \partial x_2})\Psi_1(x_1,\cdots,t_1,\cdots) = 0 \qquad (187)
$$

を得る. 又, (171) の両辺を h 倍する. そして,(178) と比べ,

$$
h\frac{\partial \Phi_5(x_0,\cdots,0,t_0,\cdots)^2}{\partial z^2} = \frac{\partial \Phi_5(x_0,\cdots,0,t_0,\cdots)}{\partial z}
$$

が同じと考え計算する.(178) に代入する. この時,(185) も (178) 代入する.

$$
\begin{aligned}
& (\frac{\partial^2}{\partial t_1^2} - gh\frac{\partial^2}{\partial x_1^2})\Psi_3(x_1,\cdots,t_1,\cdots) + 2(\frac{\partial^2}{\partial t_1 \partial t_2} - gh\frac{\partial^2}{\partial x_1 \partial x_2})\Psi_1(x_1,\cdots,t_1,\cdots) \\
& +(\frac{\partial^2}{\partial t_2^2} - gh\frac{\partial^2}{\partial x_2^2})\Psi_1(x_1,\cdots,t_1,\cdots) + 2(\frac{\partial^2}{\partial t_1 \partial t_3} - gh\frac{\partial^2}{\partial x_1 \partial x_3})\Psi_1(x_1,\cdots,t_1,\cdots) \\
& +(\frac{gh^3}{3} - \frac{Th}{\rho})\frac{\partial^4 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^4} + 2(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1})\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1 \partial t_1} \\
& +(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial t_1})\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial x_1^2} = 0 \qquad (188)
\end{aligned}
$$

を得る. 今, (159) において, $\Psi_1(x_1,\cdots,t_1,\cdots)$ が

$$
\xi_1 \equiv x_1 - \sqrt{gh}t_1
$$

のみに依存し, $\Psi_2(x_1,\cdots,t_1,\cdots)$,$\Psi_3(x_1,\cdots,t_1,\cdots)$ も x_1, t_1 は, ξ_1 のみに依存する.
又,$\Psi_1(x_1,\cdots,t_1,\cdots)$,$\Psi_2(x_1,\cdots,t_1,\cdots)$ の x_2, t_2 は,

$$
\xi_2 \equiv x_2 - \sqrt{gh}t_2
$$

のみに依存する.
この時, (187) は,

$$
\begin{aligned}
& \frac{\partial}{\partial t_1}(\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\frac{\partial \xi_1}{\partial t_1}) - gh\frac{\partial}{\partial x_1}(\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1}) \\
& +2(\frac{\partial}{\partial t_1}(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_2}\frac{\partial \xi_2}{\partial t_2}) - gh\frac{\partial}{\partial x_1}(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_2})) = 0
\end{aligned}
$$

$$-\sqrt{gh}\frac{\partial}{\partial \xi_1}(\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\frac{\partial \xi_1}{\partial t_1}) - gh\frac{\partial}{\partial \xi_1}(\frac{\partial \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1})$$
$$+2(-\sqrt{gh}\frac{\partial}{\partial \xi_2}(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\frac{\partial \xi_1}{\partial t_1}) - gh\frac{\partial}{\partial \xi_2}(\frac{\partial \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1})) = 0$$

$$(gh)\frac{\partial^2 \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^2} - (gh)\frac{\partial^2 \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^2} + 2((gh)\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_2 \partial \xi_1}$$
$$-(gh)\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_2 \partial \xi_1} = 0$$

となり恒等的に満たされることがわかる. 又, (188) は (155) と

$$\frac{\partial \Psi_n(x_1,\cdots,t_1,\cdots)}{\partial x_1} = \frac{\partial \Psi_n(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}\ (n=1,2,3)$$

$$\frac{\partial \Psi_n(x_1,\cdots,t_1,\cdots)}{\partial t_1} = -\frac{\partial \Psi_n(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}(\sqrt{gh})$$
$$= -\frac{\partial \Psi_n(x_1,\cdots,t_1,\cdots)}{\partial x_1}(\sqrt{gh})\ (n=1,2,3) \tag{189}$$

を使って表せば,

$$(gh)\frac{\partial^2 \Psi_3(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^2} - (gh)\frac{\partial^2 \Psi_3(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^2} + 2((gh)\frac{\partial^2 \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1 \partial \xi_2}$$
$$-(gh)\frac{\partial^2 \Psi_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1 \partial \xi_2})$$
$$+(gh)\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_2^2} - (gh)\frac{\partial^2 \Psi_1(x_1,\cdots,t_1,\cdots)}{\partial \xi_2^2}$$
$$+2(\frac{\partial}{\partial t_3}(-g\eta_2(x_1,\cdots,t_1,\cdots)) - gh\frac{\partial}{\partial x_3}(\sqrt{\frac{g}{h}}\eta_2))$$
$$+(\frac{gh^3}{3} - \frac{Th}{\rho})\sqrt{\frac{g}{h}}\frac{\partial^3 \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^3} - 2g\sqrt{\frac{g}{h}}\eta_2(x_1,\cdots,t_1,\cdots)\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1}$$
$$-g\sqrt{\frac{g}{h}}\eta_2(x_1,\cdots,t_1,\cdots)\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1} = 0$$

$$-2g(\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_3} + \sqrt{gh}\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial x_3})$$
$$+(\frac{gh^3}{3} - \frac{Th}{\rho})\sqrt{\frac{g}{h}}\frac{\partial^3 \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^3}$$
$$-3g\sqrt{\frac{g}{h}}\eta_2(x_1,\cdots,t_1,\cdots)\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1} = 0$$

$$\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial t_3} + \sqrt{gh}\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial x_3}$$
$$-\frac{1}{2}\sqrt{gh}(\frac{1}{3} - \frac{T}{\rho g h^2})h^2\frac{\partial^3 \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1^3}$$
$$+\frac{3}{2}\sqrt{\frac{g}{h}}\eta_2(x_1,\cdots,t_1,\cdots)\frac{\partial \eta_2(x_1,\cdots,t_1,\cdots)}{\partial \xi_1} = 0$$

となり KdV 方程式として表示されることがわかる.

5　振幅方程式

本章では和達 [2, p.26-28] に沿って振幅方程式について述べる．これまで, ばねの伸縮や水面の変位が局在する波動現象を考察してきた．しかし, 非線形波動において, もうひとつ重要な事がある．それは $e^{i(kx-\omega t)}$ が, 非線形性と分散性によってどのように変わっていくかを調べる事にある．

$$c_0 = \sqrt{\frac{\kappa}{m}}a$$

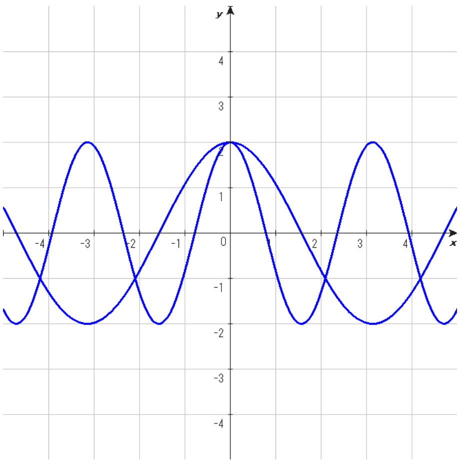
とする．今, $A(x,t)$ を振幅, $e^{i(kx-\omega t)}$ を搬送波とする. このとき, 波動場を

$$u(x,t) = A(x,t)e^{i(kx-\omega t)} + \bar{A}(x,t)e^{-i(kx-\omega t)} \tag{190}$$

とする. 線形波の場合, 搬送波が伝わる速度が位相速度であり, 振幅が伝わる速度が群速度である．
$k_1 = k + \delta k$,$k_2 = k - \delta k$,$\omega_1 = \omega + \delta\omega$,$\omega_2 = \omega - \delta\omega$ とおく．

$$\begin{aligned} u(x,t) &= a\cos(k_1x - \omega_1 t) + a\cos(k_2x - \omega_2 t) \\ &= 2a\cos\{\frac{1}{2}(k_1+k_2)x - \frac{1}{2}(\omega_1+\omega_2)t\}\cos\{\frac{1}{2}(k_1-k_2)x - \frac{1}{2}(\omega_1-\omega_2)t\} \\ &= 2a\cos(kx-\omega t)\cos(\Delta kx - \Delta\omega t) \end{aligned} \tag{191}$$

となり,$e^{i(kx-\omega t)} + e^{-i(kx-\omega t)} = 2\cos(kx-\omega t)$,$A(x,t) + \bar{A}(x,t) = a\cos(\Delta kx - \Delta\omega t)$ となる．



$2\cos(kx-\omega t)$ が激しいグラフ, $a\cos(\Delta kx - \Delta\omega t)(a, \Delta = 2)$ がなめらかなグラフである．
$\cos(kx-\omega t)$ は 2 つの線形波の合成の速さ, つまり位相速度として表され, $2a\cos(\Delta kx - \Delta\omega t)$ はゆっくりとした振幅の速さ, 郡速度として表されることがわかる．

又, 非線形効果による波の突っ立ちは分散効果で抑えられる. これを波数成分で考えると, 非線形効果による波数 $2k, 3k, \cdots$ の波の励起は無制限に続かず, 分散効果により, 抑える事ができる．v_g で動く座標系を導入すると, 振幅の変化はゆっくりしたものになる. この変化と非線形項が同じ大きさならば, そのオーダーで振幅に対する閉じた方程式が得られる．今までの考察から方程式を導いてみる．振幅は小さいが有限であるとする．今,

$$F_n(t) = \kappa(\Delta + \alpha^2\Delta)$$

で与えられる. 非線形のばねで結ばれた 1 次元格子で考える. KdV 方程式や変形 KdV 方程式を求めた時と同じ方法を用いる. 運動方程式は,

$$
\begin{aligned}
m\frac{d^2y_n(t)}{dt^2} &= F_n(t) \\
& \kappa(y_{n+1}(t)-y_n(t)+\alpha^2(y_{n+1}(t)-y_n(t))^3) \\
-& \kappa(y_n(t)-y_{n-1}(t)+\alpha^2(y_n(t)-y_{n-1}(t))^3)\ (n=0,\pm1,\pm2,\cdots) \qquad (192)
\end{aligned}
$$

である.$x=na$ を連続変数として連続体近似をする.

$$
y_{n\pm1}(t)=y(x\pm a,t)=y(x,t)\pm a\frac{\partial y(x,t)}{\partial x}+\frac{a^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2}\pm\frac{a^3}{6}\frac{\partial^3 y(x,t)}{\partial x^3}+\frac{a^4}{24}\frac{\partial^4 y(x,t)}{\partial x^4}\pm\cdots \qquad (193)
$$

を (192) に代入すると,

$$
\frac{d^2y(x,t)}{dt^2}={c_0}^2(\frac{\partial^2 y(x,t)}{\partial x^2}+\frac{a^2}{12}\frac{\partial^4 y(x,t)}{\partial x^4}+3\alpha^2a^2(\frac{\partial y(x,t)}{\partial x})^2\frac{\partial^2 y(x,t)}{\partial x^2}+\cdots) \qquad (194)
$$

となる.ϵ を小さい無次元パラメータとして, 独立変数 x,t を新しい独立変数 ξ,τ に変える. (ξ,τ を無次元になるように規格化する.)

$$
\xi=\epsilon^{\frac{1}{2}}(x-Vt),\tau=\epsilon t \quad (V \text{ は後で決まる事なので今は定数としておく}), \qquad (195)
$$

$$
\begin{aligned}
y(x,t) &= \epsilon^{\frac{1}{2}}(\Phi(\xi,\tau)e^{i(kx-\omega t)}+\bar{\Phi}(\xi,\tau)e^{-i(kx-\omega t)}) \\
&= \epsilon^{\frac{1}{2}}(\Phi(\xi,\tau)E+c.c.) \quad (E=e^{i(kx-\omega t)},\ c.c.=\text{複素共役}) \qquad (196)
\end{aligned}
$$

である.(192) に (194), (196) を代入する. 左辺は,

$$
\begin{aligned}
\frac{d^2y_n(t)}{dt^2} &= E\epsilon^{\frac{1}{2}}(-i\omega-\epsilon^{\frac{1}{2}}V\frac{\partial}{\partial\xi}+\epsilon\frac{\partial}{\partial\tau})^2\Phi(\xi,\tau)+c.c \\
&= E\{\epsilon^{\frac{1}{2}}(-\omega^2\Phi(\xi,\tau))+\epsilon(2i\omega V\frac{\partial\Phi(\xi,\tau)}{\partial\xi})+\epsilon^{\frac{3}{2}}(-2i\omega\frac{\partial\Phi(\xi,\tau)}{\partial\tau}+V^2\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2}) \\
&+ \cdots+c.c\}, \qquad (197)
\end{aligned}
$$

右辺は,

$$
\begin{aligned}
& Ec_0^2\{\epsilon^{\frac{1}{2}}(-k^2\Phi(\xi,\tau)+\frac{a^2}{12}k^4\Phi(\xi,\tau)+\cdots+c.c.)+\epsilon(2ik\frac{\partial\Phi(\xi,\tau)}{\partial\xi} \\
& -\frac{a^2}{3}ik^3\frac{\partial\Phi(\xi,\tau)}{\partial\xi}+\cdots+c.c) \\
& +\epsilon^{\frac{3}{2}}(\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2}-\frac{a^2}{2}k^2\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2}+3\alpha^2a^2k^4(\Phi^3(\xi,\tau)E^2-\Phi^2(\xi,\tau)\bar{\Phi}(\xi,\tau)+c.c) \\
& +\cdots\} \qquad (198)
\end{aligned}
$$

である. 以上から, ϵ のオーダーでまとめる. $c.c$ は省略する.

$$
\begin{aligned}
\epsilon^{\frac{1}{2}}:-\omega^2\Phi(\xi,\tau) &= c_0^2(-k^2\Phi(\xi,\tau)+\frac{a^2}{12}k^4\Phi(\xi,\tau)+\cdots), \\
\omega^2 &= c_0^2(-k^2+\frac{a^2}{12}k^4+\cdots) \qquad (199)
\end{aligned}
$$

となり, 分散関係式を得る.

$$\begin{aligned}\epsilon : 2i\omega V\frac{\partial\Phi(\xi,\tau)}{\partial\xi} &= c_0^2(2ik\frac{\partial\Phi(\xi,\tau)}{\partial\xi} - \frac{a^2}{3}ik^3\frac{\partial\Phi(\xi,\tau)}{\partial\xi} + \cdots) \\ V &= \frac{c_0^2k}{\omega}(1-\frac{a^2k^2}{6}+\cdots) \qquad (200)\end{aligned}$$

ここで, (199) から,

$$\begin{aligned}\frac{1}{\omega} &= \frac{1}{c_0k}\sqrt(1-\frac{a^2k^2}{12}) \\ &= \frac{1}{c_0k}(1+\frac{a^2k^2}{24}+\cdots)\end{aligned}$$

より,

$$\begin{aligned}V &= c_0^2k(1-\frac{a^2k^2}{6}+\cdots)\frac{1}{c_0k}(1+\frac{a^2k^2}{24}+\cdots) \\ &= c_0(1-\frac{a^2k^2}{8}+\cdots) \qquad (201)\end{aligned}$$

となる. 又, (199) の両辺を k で微分すると,

$$\begin{aligned}2\omega\frac{\partial\omega(k)}{\partial k} &= c_0^2(2k-\frac{a^2k^3}{3}+\cdots) \\ \frac{\partial\omega(k)}{\partial k} &= \frac{c_0^2}{2\omega}(2k-\frac{a^2k^3}{3}+\cdots) \\ &= \frac{c_0^2k}{\omega}(1-\frac{a^2k^2}{6}+\cdots) \\ &= c_0(1-\frac{a^2k^2}{8}+\cdots)\end{aligned}$$

となる. よって

$$V = v_g$$

となる. 又,

$$\begin{aligned}\epsilon^{\frac{3}{2}} : -2i\omega\frac{\partial\Phi(\xi,\tau)}{\partial\tau} + V^2\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2} &= c_0^2(\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2} - \frac{a^2k^2}{2}\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2} \\ &+ 3\alpha^2a^2k^4(\Phi^3(\xi,\tau)E^2 - \Phi^2(\xi,\tau)\bar{\Phi}(\xi,\tau)) \qquad (202)\end{aligned}$$

となる. ここで,

$$E(x,t) = e^{i(kx-\omega t)}, E^2(x,t) = e^{2i(kx-\omega t)}$$

とする.

二つを比べると, 大きさが異なるので,$\Phi^3(\xi,\tau)E^2$ は無視して考えても変わらないことが分かる. 以上からまとめると,

$$2i\omega\frac{\partial\Phi(\xi,\tau)}{\partial\tau} + \{c_0^2(1-\frac{a^2k^2}{2}) - V^2\}\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2}) - 3\alpha^2a^2k^4\Phi^2(\xi,\tau)\bar{\Phi}(\xi,\tau) = 0$$

$$i\frac{\partial\Phi(\xi,\tau)}{\partial\tau} + \{\frac{c_0^2(1-\frac{a^2k^2}{2}) - V^2}{2\omega}\}\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2} - \frac{3\alpha^2a^2c_0^2k^4}{2\omega}|\Phi(\xi,\tau)|^2\Phi(\xi,\tau) = 0 \qquad (203)$$

となり, 非線形シュレディンガー方程式が導く事ができた.

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