Chiral Symmetry Breaking and Phase Transitions in Holographic Gauge Theories

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Abstract

We discuss the spontaneous breaking of the chiral symmetry in QCD in the framework of the string/gauge duality. We use two different holographic models of QCD. One of them is the general intersecting Dq/Dp brane model consisting of N_c Dq-branes and a single probe Dp-brane. The other is the Dq/Dp-$\overline{\mathrm{D}p}$ brane model in which we use N_f Dp-$\overline{\mathrm{D}p}$-brane pairs as a probe. In particular we consider the D4/D8-$\overline{\mathrm{D8}}$ configuration. In both models there is an s-dimensional intersection between the Dq-branes and the probe branes. The theory localized at the intersection is an $(s+1)$-dimensional QCD-like theory (QCD$_{s+1}$) in certain cases. In terms of the string/gauge duality we study the dynamics of strongly coupled large N_c QCD$_{s+1}$ at zero and finite temperature (and also at finite chemical potential). The near horizon limit and the probe approximation allow us to treat the Dq-branes as a background geometry and the Dp-brane or Dp-$\overline{\mathrm{D}p}$-branes as a probe which does not affect this background. In both models the breaking of the chiral symmetry is closely related to configurations of the probe branes in the Dq background. The quark mass and the quark condensate can be read from the asymptotic behavior of the Dp(-$\overline{\mathrm{D}p}$)-brane embedding. In the Dq/Dp model we find that the chiral symmetry is spontaneously broken at zero temperature for certain (q, p, s). We also find that there appear massless Nambu-Goldstone (NG) bosons associated with this spontaneous symmetry breaking. In the D4/D8-$\overline{\mathrm{D8}}$ model we introduce a chemical potential for the baryon number symmetry as well as temperature. The chemical potential for the baryon number symmetry is introduced as a non-vanishing asymptotic value of the time component of U(1) gauge field on the probe brane. We analyze the phase structure of the model and find a chiral phase transition of the first order.

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Contents

Chapter 1

Introduction

The duality between string theory and gauge theory has been discussed with great interest. Recently this duality is often called the string/gauge duality. The basic idea of this relation was firstly pointed out by 't Hooft in 1974 [1]. His statement is as follows; the perturbative expansion of SU(N_c) gauge theory can be considered as string loop expansion under large N_c limit with fixed 't Hooft coupling $g_{YM}^2 N_c$, where g_{YM} is the gauge coupling of the gauge theory. Since the effects of non-planar diagrams can be ignored in the large N_c limit, the analysis of the large N_c gauge theory becomes much simpler. The large N_c gauge theory can be a good tool for analysis of quantum chromodynamics (QCD), which is the quantum field theory of the strong interaction. One of the most interesting properties of QCD is the asymptotic freedom. Because of this QCD becomes strongly coupled at low energy. So some non-perturbative methods have been needed to analyze the low energy aspects of QCD such as the confinement and the spontaneous chiral symmetry breaking and so on.

The AdS/CFT correspondence [2, 3, 4] (see [5] for a review) provides a new non-perturbative approach to strongly coupled gauge theories. This duality relates a weakly coupled string theory (or supergravity as the low energy effective theory of it) in $(d+1)$-dimensional anti de Sitter (AdS) spacetime (times a compact space) to a strongly coupled d-dimensional conformal field theory (CFT). In terms of the AdS/CFT correspondence one can obtain correlation functions of operators in CFT_d from the supergravity action of corresponding fields evaluated at the boundary of AdS_{d+1} [3, 4]. One can also obtain potential between static quark-antiquark pair by studying the configuration of fundamental strings in the AdS spacetime [6, 7]. The results of refs. [6, 7] agree with the one expected from conformal symmetry. It may be said that the AdS/CFT correspondence can be considered as a realization of the proposal of ref. [1].

It seems that the AdS/CFT correspondence can be used to analyze the non-perturbative aspects of QCD since this correspondence relates a strongly coupled gauge theory to a weakly coupled string or supergravity theory. There, however, are some difficulties in an application of the AdS/CFT correspondence to QCD. QCD is a non-supersymmetric and non-conformal field theory, and there are quarks belonging to the fundamental representation of the gauge group SU(3). On the other hand, in the framework of the standard AdS/CFT correspondence we can only construct QCD-like models, which are superconformal theories and have fermions belonging to the adjoint representation of $SU(N_c)$.

The AdS/CFT correspondence can be extended to the string/gauge duality, which is a generalization to non-conformal and non-supersymmetric theories. Since deformations of the AdS spacetime lead to the breaking of the supersymmetry and conformal symmetry of dual gauge theory, we can obtain more realistic models of QCD as in ref. [8]. In ref. [8] Witten considered N_c coincident D4-branes wrapped on $\mathbb{R}^4 \times S^1$. Imposing a supersymmetry breaking boundary condition for fields on the D4-branes along S^1 direction, fermion and scalar fields decouple from the system at low energy. Therefore low energy effective theory becomes a pure Yang-Mills (YM) theory (only contains gluon fields belonging to the adjoint representation of $SU(N_c)$). Using this method one can discuss the confinement/deconfinement phase transition [8], the static quark-antiquark potential at finite temperature [9, 10, 11] and the mass spectrum of glueballs [12, 13, 14, 15]. Although these approaches are interesting, there are still no dynamical quarks in the fundamental representation of the gauge group.

To introduce quarks in the fundamental representation of $SU(N_c)$ Karch and Katz considered N_c D3-branes and N_f Dp-branes [16]. Open strings connecting D3-branes and Dp-branes generate N_f flavored fermions in the fundamental representation of $SU(N_c)$. Such fermions can be regarded as quarks. In the probe approximation $N_f \ll N_c$ the dynamics of the dual gauge theory can be described by the dynamics of the probe Dp-branes in the D3-brane background geometry. Therefore the string/gauge duality can provide realistic models of QCD and methods for analysis of low energy behaviors of QCD. This approach is often called the holographic QCD [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] (and references therein).

One of the most interesting phenomena of the low energy QCD is the spontaneous breaking of the chiral symmetry. In the holographic approach the chiral symmetry can be realized in two different ways. One of them is based on an intersecting Dq/Dp brane system [18, 19, 20, 21, 22, 23, 32]. In this approach one introduces N_c color Dq-branes and N_f flavor Dp-branes. The $U(N_c)$ gauge field on the Dq-branes represents a gluon field of a QCD-like theory. Open strings connecting the Dq-branes and the Dp-branes represent quarks in the fundamental representation of $U(N_c)$. When these brane configurations have directions transverse to both of the Dq and Dp-branes, a rotational symmetry in these directions can be understood as a chiral symmetry of the dual gauge theories in certain cases. One can separate color branes and flavor branes in such directions. This is a holographic description of the explicit chiral symmetry breaking. The asymptotic distance between these branes is identified with a quark mass. So one can study the chiral symmetry breaking starting from a theory with a non-vanishing quark mass and taking the massless limit. So far only the Abelian chiral symmetry $U(1)_V \times U(1)_A$ is considered in this approach. In ref. [32] we considered all other types of Dq/Dp models and gave general discussions.

Alternatively, the chiral symmetry can be realized as a gauge symmetry on the flavor branes. Such a realization of the chiral symmetry is based on an intersecting Dq/Dp-$\overline{\mathrm{D}p}$ brane system [24, 25, 26, 27, 28, 29, 30, 31]. When Dp-$\overline{\mathrm{D}p}$-brane pairs are used as flavor branes, one can obtain a non-Abelian chiral symmetry $U(N_f)_L \times U(N_f)_R$.

These configurations of physical interest often do not have directions transverse to both of the color and flavor branes. Therefore, it is not obvious how to introduce a quark mass in these models. In both of these two approaches, the spontaneous breaking of the chiral symmetry is closely related to the configurations of the probe branes in the background geometry.

The chiral symmetry breaking was also discussed at finite temperature [18, 19, 33, 34, 35, 36, 37, 38] and at finite chemical potential [39, 40, 41, 42, 43]. The temperature T is related to a period δt_E of the S^1 compactified Euclidean time coordinate as $T = 1/\delta t_E$. The chemical potential μ is introduced as a non-vanishing asymptotic value of the time component of the gauge field on the probe brane $A_0 \sim \mu$. One can study a chiral phase transition and obtain a phase diagram of the QCD-like theories. Particularly in ref. [40] we obtained the phase diagram of the Sakai-Sugimoto model, which is based on the D4/D8-$\overline{\mathrm{D8}}$ brane model and gives a holographic dual of QCD. The phase diagram which we obtained looks like that of QCD at high temperature phase but different at low temperature phase. So far there are many related works for the holographic QCD or holographic gauge theories at finite chemical potential for the baryon number symmetry [44, 45, 46, 47, 48, 49, 50, 51, 52] and for the isospin symmetry [53, 54, 55, 56]. Historically, the chemical potential in the framework of the AdS/CFT correspondence was firstly introduced for R symmetry [57, 58, 59]. There are also many related works concerning the chemical potential for R symmetry [60, 61, 62, 63, 64, 65].

The purpose of the present paper is to study the chiral symmetry breaking in general intersecting D-brane model in terms of the string/gauge duality. We use two different models. One of them is the Dq/Dp brane model consisting of N_c Dq-branes and a single probe Dp-brane with an s-dimensional intersection. The other is the Dq/Dp-$\overline{\mathrm{D}p}$ brane model in which we use N_f Dp-$\overline{\mathrm{D}p}$ brane pairs as probe brane instead of a single Dp-brane in the Dq/Dp model. In particular we concentrate on a special configuration, D4/D8-$\overline{\mathrm{D8}}$ model with a three-dimensional intersection. These two brane models give holographic duals of QCD-like theories in $(s+1)$-dimensional spacetime QCD$_{s+1}$ in certain cases. In the Dq/Dp model approach there can be configurations having directions transverse to both of the Dq and Dp-branes. A rotational symmetry of such directions can be regarded as a chiral symmetry of dual gauge theory at the intersection in certain cases. We can obtain a non-Abelian chiral symmetry for certain values of (q, p, s) in contrast to the models in refs. [18, 19]. In the D4/D8-$\overline{\mathrm{D8}}$ model, on the other hand, there are no such directions. However, a gauge symmetry on the D8-$\overline{\mathrm{D8}}$-branes can be regarded as a chiral symmetry. In the near horizon limit and the probe approximation we can treat the N_c Dq-branes as a gravitational background geometry and the Dp-brane or N_f Dp-$\overline{\mathrm{D}p}$-branes as a probe which do not affect this background. We discuss the chiral symmetry breaking by analyzing the Dp or Dp-$\overline{\mathrm{D}p}$-brane dynamics in the Dq-brane background geometry.

The organization of the paper is as follows. Chapter 2 and 3 are devoted to brief reviews of the basic aspects of the chiral symmetry in QCD and the basic idea of the string/gauge duality, respectively.

In Chapter 4 we study the chiral symmetry breaking in the general intersecting Dq/Dp brane model. First we study the low energy spectrum at an s-dimensional intersection. In general dual theories are defect field theories [66, 67]. We are interested in field theories without defects. The Dq/D$(q+2)$ model with $s = q-1$ is dual to QCD$_{s+1}$ since one of the spatial directions of the Dq-branes is compactified on S^1 and then there is no defect at low energy limit. In particular, the D2/D4 model with $s=1$, the D3/D5 model with $s=2$ and the D4/D6 model with $s=3$ correspond to QCD$_2$, QCD$_3$ and QCD$_4$, respectively. For certain (q,p,s) the rotational symmetry of the transverse directions can be understood as a chiral symmetry in the QCD-like theories. These chiral symmetries are non-Abelian SU$(2)_L \times$ SU$(2)_R$ for QCD$_2$ and Abelian U$(1)_A$ for QCD$_4$. Next, we study the dynamics of the Dp-brane in the Dq-brane background geometry for the Dq/D$(q+2)$ model by using the string/gauge duality at zero and finite temperature. The quark mass m_q and the quark condensate $\langle\bar{\psi}\psi\rangle$ can be read from the asymptotic behavior of the Dp-brane embedding. As a result, in the zero temperature analysis, we find a non-zero quark condensate even for massless quark limit. This corresponds to a spontaneous chiral symmetry breaking in QCD$_2$ and QCD$_4$. Then we find massless Nambu-Goldstone (NG) bosons as the fluctuations around the vacuum Dp-brane embedding. If quarks are massive $m_q \neq 0$ there appear pseudo-NG bosons with a non-vanishing mass. In the finite temperature analysis the vacuum embeddings also breaks rotational symmetry. Then the chiral symmetry is also broken as in the zero temperature case. We find that quark condensate vanishes and the chiral symmetry restores at high temperature limit. Note that we also study the Dq/D$(q+4)$ model with $s=q$. This chapter is based on the work [32].

In Chapter 5 we study the chiral symmetry breaking in the general intersecting Dq/Dp-$\overline{\text{D}p}$ brane model. In particular we analyze the phase structure of the D4/D8-$\overline{\text{D8}}$ model at finite temperature and finite chemical potential. This model is called the Sakai-Sugimoto model. It has a manifest U$(N_f)_L \times$ U$(N_f)_R$ chiral symmetry as a gauge symmetry on the probe D8-$\overline{\text{D8}}$-brane pairs and gives a holographic dual of QCD$_4$. The chemical potential can be introduced as a non-vanishing Euclidean time component of U(1) gauge field on the probe D8-$\overline{\text{D8}}$-branes. As in Chapter 4 we study it in terms of the string/gauge duality. In the low temperature phase only U-shaped connected D8-$\overline{\text{D8}}$ embeddings are possible. In the high temperature phase, on the other hand, another type of embeddings is possible. They are straight disconnected D8 and $\overline{\text{D8}}$ embeddings which fall into the horizon. By comparing the Euclidean actions for each embeddings we can analyze the phase structure of the model at finite temperature and finite chemical potential. This chapter is based on the work [40].

Finally we summarize our results in Chapter 6. In Appendix we give a short discussion on a fluctuation of the radial part considered in Chapter 4.

Chapter 2

Spontaneous chiral symmetry breaking in QCD

A quantum field theory of the strong interaction can be described by quantum chromodynamics (QCD). QCD is based on an SU(3) gauge theory. The action of QCD with N_f flavored quarks is

$$S = \int d^4x \left[-\frac{1}{2g_{YM}^2} \mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) + \sum_{i,j=1}^{N_f} \bar{\psi}_i \left\{ i\delta_{ij}\gamma^\mu(\partial_\mu - iA_\mu) - M_{ij} \right\} \psi_j \right], \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the field strength of gluon fields $A_\mu = A_\mu^a T^a$ (T^a $(a = 1, 2. \cdots, 8)$ are generators of SU(3)) which belong to the adjoint representation of SU(3). ψ_i $(i = 1, 2, \cdots, N_f)$ are quark fields with N_f flavors and belong to the fundamental representation of SU(3). The trace "tr" is taken over color indices and the normalization is defined as $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. g_{YM} and M_{ij} are the gauge coupling constant and quark mass matrix respectively. We have ignored the ghost term and the gauge fixing term for simplicity.

If quarks are all massless the action (2.1) has the following global symmetry

$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A, \tag{2.2}$$

where $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ are a chiral symmetry, a baryon number symmetry and an axial rotational symmetry of quarks, respectively. Note that $\mathrm{U}(1)_A$ is a symmetry of the classical theory but it is broken by quantum anomaly. Indeed, at the classical level, the action (2.1) with $M_{ij} = 0$ is invariant under the following transformations

$$\psi_L \to e^{i\alpha_L^a t_L^a}\psi_L \quad (e^{i\alpha_L^a t_L^a} \in \mathrm{SU}(N_f)_L), \quad \psi_R \to e^{i\alpha_R^a t_R^a}\psi_R \quad (e^{i\alpha_R^a t_R^a} \in \mathrm{SU}(N_f)_R), \tag{2.3}$$

$$\psi \to e^{i\alpha}\psi \quad (e^{i\alpha}\psi \in \mathrm{U}(1)_V), \quad \psi \to e^{i\alpha_5\gamma^5}\psi \quad (e^{i\alpha_5\gamma^5} \in \mathrm{U}(1)_A), \tag{2.4}$$

where t_L^a and t_R^a $(a = 1, 2, \cdots, N_f^2 - 1)$ are generators of $\mathrm{SU}(N_f)_L$ and $\mathrm{SU}(N_f)_R$, respectively. α_L^a, α_R^a, α and α_5 are the real parameters. Then the corresponding conserved currents are

$$j_L^{a\mu} = \bar{\psi}_L \gamma^\mu t_L^a \psi_L, \qquad j_R^{a\mu} = \bar{\psi}_R \gamma^\mu t_R^a \psi_R, \tag{2.5}$$

$$j^\mu = \bar{\psi}\gamma^\mu \psi, \qquad j^{5\mu} = \bar{\psi}\gamma^\mu \gamma^5 \psi. \tag{2.6}$$

Defining the vector and axial-vector currents in terms of eqs. (2.5)

$$V^{a\mu} \equiv j_L^{a\mu} + j_R^{a\mu}, \qquad A^{a\mu} \equiv j_L^{a\mu} - j_R^{a\mu}, \tag{2.7}$$

conserved charges corresponding to the transformations (2.3) can be written as

$$Q_V^a = \int d^3x \; V^{a0}, \qquad Q_A^a = \int d^3x \; A^{a0}. \tag{2.8}$$

The vacuum of QCD is only symmetric with respect to the diagonal part of the chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$Q_V^a \left|0\right\rangle = 0, \qquad Q_A^a \left|0\right\rangle \neq 0. \tag{2.9}$$

Therefore the chiral symmetry is spontaneously broken to its diagonal subgroup

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V. \tag{2.10}$$

Then there appear $(N_f^2 - 1)$ massless Nambu-Goldstone (NG) bosons associated with this spontaneous symmetry breaking. The spontaneous chiral symmetry breaking requires a non-zero quark condensate

$$\left\langle 0|\bar{\psi}_i \psi_i|0\right\rangle \neq 0. \tag{2.11}$$

Because of the asymptotic freedom QCD is strongly coupled at low energy. The origin of the spontaneous chiral symmetry breaking seems to concern with the low energy dynamics of QCD, but so far it is not fully understood.

Let us consider two-point function of the axial-vector currents and that of the divergence of the axial-vector currents

$$i\Pi^{ab\mu\nu}(q) = \int d^4x \; e^{iq\cdot x} \left\langle 0 \left| T \left(A^{a\mu}(x) A^{b\nu}(0) \right) \right| 0 \right\rangle, \tag{2.12}$$

$$i\Psi^{ab}(q) = \int d^4x \; e^{iq\cdot x} \left\langle 0 \left| T \left(\partial_\mu A^{a\mu}(x) \partial_\nu A^{b\nu}(0) \right) \right| 0 \right\rangle. \tag{2.13}$$

Then one can find the following relation

$$\begin{aligned} iq_\mu q_\nu \Pi^{ab\mu\nu}(q) = i\Psi^{ab}(q) - iq_\mu \int d^4x \; e^{iq\cdot x} \left\langle 0 \left| \delta(x^0) \left[A^{b0}(0), A^{a\mu}(x) \right] \right| 0 \right\rangle \\ + \int d^4x \; e^{iq\cdot x} \left\langle 0 \left| \delta(x^0) \left[A^{a0}(x), \partial_\mu A^{b\mu}(0) \right] \right| 0 \right\rangle. \end{aligned} \tag{2.14}$$

Using the partially conserved axial current (PCAC) hypothesis

$$\partial_\mu A^{a\mu} = f_a M_a^2 \phi^a \qquad (a = 1, 2, \cdots, N_f^2 - 1), \tag{2.15}$$

where f_a, M_a and ϕ^a are the decay constants, masses and field operators of pseudo-scalar mesons, respectively, then the first term in the right hand side of eq. (2.14) represents the two-point function of pseudo-scalar mesons. The third term is the so-called σ-term. Taking a limit $q \to 0$ eq. (2.14) becomes

$$0 = f_a^2 M_a^4 \left(\left. \frac{i\delta^{ab}}{q^2 - M_a^2} \right|_{q\to 0} + \cdots \right) - i \left\langle 0 \left| \bar{\psi}_i \left\{ T^a, \left\{ T^b, M \right\} \right\}_{ij} \psi_j \right| 0 \right\rangle . \tag{2.16}$$

For simplicity we assume that $N_f = 2$ and the each quark mass are equivalent $M_{ij} = m_q \mathbf{1}_{ij}$. Therefore one can obtain the Gell-Mann-Oakes-Renner (GMOR) relation [68]

$$M_\pi^2 = -\frac{m_q \left\langle 0 \left| \left(\bar{u}u + \bar{d}d \right) \right| 0 \right\rangle}{f_\pi^2}. \tag{2.17}$$

Note that the case for $N_f = 3$ was studied in ref. [68]. If $m_q = 0$ a non-zero quark condensate $\left\langle \bar{\psi}\psi \right\rangle \neq 0$ leads the spontaneous breaking of the chiral symmetry. Then the NG-bosons associated with the symmetry breaking are massless. On the other hand, if $m_q \neq 0$ the quark mass term in the action (2.1) explicitly breaks the chiral symmetry. Then the NG-bosons become pseudo-NG bosons with mass M_a.

In the following discussions we consider more general QCD with N_c colors and N_f flavors rather than ordinary QCD with three-colors and six-flavors. We often take a limit $N_c \to \infty$ to simplify the analyses.

Chapter 3

A brief review of the string/gauge duality

We give a brief review of the string/gauge duality in this chapter. First we explain basic aspects of D-branes in string theory. The D-branes play an important role when we discuss the duality between string theory and gauge theory. The AdS/CFT correspondence is proposed by Maldacena [2] and formulated by Gubser et al. [3] and Witten [4]. It relates a weakly coupled string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime to a strongly coupled four-dimensional $\mathcal{N} = 4$ super Yang Mills theory. In terms of the AdS/CFT correspondence we can study non-perturbative aspects of strongly coupled gauge theories. The AdS/CFT correspondence can be extended to non-supersymmetric and non-conformal theories. It is called the string/gauge duality. The string/gauge duality also provides us a non-perturbative method for the analysis of strongly coupled gauge theories such as low energy QCD.

3.1 D-branes in string theory

String theory is based on one-dimensional extended compact objects, called strings. Since string sweeps a two-dimensional surface (world-sheet) in the D-dimensional spacetime ($D = 26$ for bosonic string theory and $D = 10$ for superstring theory), string world-sheet is parametrized by a set of two parameters (τ, σ). The position of the string in the spacetime is given by string coordinate $X^\mu(\tau, \sigma)$ $(\mu = 0, 1, \cdots, D-1)$. The string world-sheet action is

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta}(\sigma) \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X), \tag{3.1}$$

where $g_{\mu\nu}(X)$ and $h_{\alpha\beta}(\sigma)$ are the metrics of the the D-dimensional spacetime and two-dimensional string world-sheet, respectively. $h \equiv |\det h_{\alpha\beta}|$ is the absolute value of the determinant of $h_{\alpha\beta}$. α' is the Regge slope parameter and is related to the string length ℓ_s as $\alpha' = \ell_s^2$.

For a one-dimensional compact object two types of topology are possible. One is closed string without boundary, which is topologically equivalent to a circle. The closed

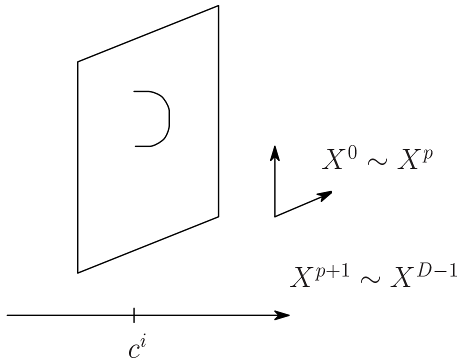


Figure 3.1: Dp-brane can be understood as a $(p+1)$-dimensional hypersurface on which open strings can end.

string boundary condition is

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi). \tag{3.2}$$

The other is open string with boundaries at the endpoints of the string, $\sigma = 0,\ \pi$, which is topologically equivalent to a line segment. We need to impose a boundary condition for both ends of open string coordinate $X^\mu(\tau,\sigma)$. Requiring the vanishing of the surface term in the variation of the string world-sheet action (3.1) [1]

$$\delta S = -\frac{1}{2\pi\alpha'}\int d\tau \left\{ \left(\partial_\sigma X_\mu\right)\delta X^\mu|_{\sigma=\pi} - \left(\partial_\sigma X_\mu\right)\delta X^\mu|_{\sigma=0} \right\}, \tag{3.3}$$

we have to adopt the Neumann boundary condition $\partial_\sigma X^\mu|_{\sigma=0,\pi} \equiv \frac{\partial}{\partial\sigma}X^\mu|_{\sigma=0,\pi} = 0$ or the Dirichlet boundary condition $\delta X^\mu|_{\sigma=0,\pi} = 0$.

For example we impose the Neumann boundary condition for the $01\cdots p$-directions of the open string coordinate

$$\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0 \quad (\mu = 0,\cdots,p) \tag{3.4}$$

and the Dirichlet one for the other directions

$$X^i(\tau,\sigma)|_{\sigma=0,\pi} = c^i \quad (i = p+1,\cdots,D-1), \tag{3.5}$$

where c^i is a constant. Then we have a $(p+1)$-dimensional hypersurface on which open strings can end (Fig. 3.1). It is called *Dirichlet p-brane* or *Dp-brane* in short [69]. If we do not need to specify the dimension of it, we simply call it *D-brane*. Dp-brane breaks the Poincaré invariance ISO$(1,D-1)$ of the D-dimensional spacetime in the directions transverse to the Dp-brane

$$\mathrm{ISO}(1,D-1) \to \mathrm{ISO}(1,p)\times\mathrm{SO}(D-p-1). \tag{3.6}$$

[1]Here, for simplicity, we use the flat Minkowski metric $g_{\mu\nu} = \mathrm{diag}(-1,1,\cdots,1)$ for the background spacetime and the conformal gauge $h_{\alpha\beta} = e^\phi\eta_{\alpha\beta}$ for the string world-sheet, respectively. e^ϕ is a conformal factor and $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$.

In the following we note some important properties of the D-branes in superstring theory without detailed discussions. Let us add open strings with the boundary conditions (3.4) and (3.5) to type II closed superstring theory. This is a consistent string theory only if p is even for type IIA theory and only if p is odd for type IIB theory. The physics on the Dp-brane can be described by open strings ending on the Dp-brane. Its low energy effective theory is a $(p+1)$-dimensional super Yang-Mills (SYM) theory. It can be obtained by a dimensional reduction from ten-dimensional $\mathcal{N}=1$ SYM theory. The physics far from the Dp-brane can be described by closed strings and it is locally equivalent to type II superstring theory. These sectors interact with ordinary open-closed string interactions.

D-brane breaks a half of the spacetime supersymmetry. In type II superstring theory there is $\mathcal{N}=2$ supersymmetry in ten-dimensional language. Corresponding to it there are two Majorana-Weyl supercharges with 16 real components. However the D-brane preserves only half of it since the open string boundary condition relates left-moving sectors to right-moving sectors. Therefore type II superstring theory coupled to the D-brane preserves only 16 real supercharges (D-brane is a so called BPS state). It is $\mathcal{N}=1$ supersymmetry in ten-dimensions.

The Dp-branes carry the Ramond-Ramond (RR) charges. There are $(p+1)$-form gauge fields C_{p+1} from the RR sector in type II superstring theories. The field strength of the RR gauge field is given by $F_{p+2}=dC_{p+1}$. The coupling of the RR gauge field to the Dp-brane world-volume is given by

$$\mu_p \int_{\mathrm{D}p} C_{p+1}, \tag{3.7}$$

where μ_p is the RR charge of the Dp-brane. The RR gauge fields of type IIA theory are C_1, C_3, C_5, C_7 and they couple to the Dp-brane for $p=0,\ 2,\ 4,\ 6$, respectively. The RR gauge fields of type IIB theory are C_0, C_2, C_4, C_6, C_8 and they couple to the Dp-brane for $p=-1\ ,\ 1,\ 3,\ 5,\ 7$, respectively. The field strength of the RR gauge field C_{p+1} is given by $F_{p+2}=dC_{p+1}$. Note that not all of the above RR gauge fields are independent since the Hodge duality $F_{8-p}=*F_{p+2}$ relates C_{7-p} to C_{p+1} in analogy with the electric-magnetic duality in electromagnetism. The D8-brane for type IIA theory and the D9-brane for type IIB theory also require the RR gauge fields C_9 and C_{10}, respectively.

The D-brane plays an important role to study the non-perturbative aspects of string theory since the D-brane itself is a non-perturbative dynamical object of string theory. The D-brane is also important for the analysis of gauge theories since the D-brane leads to a duality between string theory and gauge theory as discussed bellow.

3.2 The AdS/CFT correspondence

The AdS/CFT correspondence [2] is a powerful duality between string theory and gauge theory. Original statement of the AdS/CFT correspondence is concerned with type IIB string theory (or supergravity as the low energy effective theory of it) on $\mathrm{AdS}_5\times\mathrm{S}^5$ and four-dimensional $\mathcal{N}=4$ $\mathrm{U}(N_c)$ SYM theory.

Let us consider N_c coincident D3-branes in ten-dimensional bulk. The complete action of this system is not known well, however the effective action of the system in the low energy limit can be written as

$$S_{\text{eff}} = S_{\text{D3}} + S_{\text{bulk}} + S_{\text{int}}, \tag{3.8}$$

where S_{D3}, S_{bulk} and S_{int} are the actions for fields on the N_c D3-branes, fields in the ten-dimensional bulk and interactions between these fields, respectively. Taking the limit for the Regge slope parameter

$$\alpha' \to 0 \tag{3.9}$$

keeping the energy scale, the string coupling g_s and the number of D3-branes N_c fixed, then the interactions between the fields on the branes and the fields in the bulk S_{int} vanish. The field theory on the branes decouples from the bulk. So there are two decoupled descriptions of the system. In the above decoupling limit and the low energy limit, S_{D3} can be written by the lowest modes of open strings having both ends on D3-branes, which give four-dimensional $\mathcal{N} = 4$ U(N_c) SYM theory. S_{bulk} can be written by free type IIB supergravity.

An alternative picture of the system at low energy is possible. There is a three-brane solution in type IIB supergravity, which is the correspondent of D3-brane in string theory. The massless fields in the bulk decouple from the near horizon region due to the decoupling limit (3.9). So there are two decoupled descriptions of the system, i.e., free type IIB supergravity and the near horizon region of the geometry. The near horizon geometry of the three-brane solution can be written as $\text{AdS}_5 \times \text{S}^5$.

Note that we can trust classical supergravity approximation if string loop corrections can be ignored and typical length scale of the theory is much larger than the string length ℓ_s

$$g_s N_c = \text{fixed} \gg 1, \qquad N_c \to \infty. \tag{3.10}$$

Since the four-dimensional Yang-Mills coupling g_{YM} is related to the string coupling g_s as

$$g_s N_c = \frac{g_{YM}^2 N_c}{2\pi} \equiv \lambda, \tag{3.11}$$

the limit (3.10) requires large 't Hooft coupling λ and small string coupling g_s. In the above two pictures there are two decoupled descriptions and the one of them is free type IIB supergravity in both pictures. Therefore it seems that a weakly coupled type IIB supergravity on $\text{AdS}_5 \times \text{S}^5$ is related to a strongly coupled four-dimensional $\mathcal{N} = 4$ large N_c U(N_c) SYM theory.

There is no general proof of the AdS/CFT correspondence. However, a large number of examples which support this correspondence were found. In the remainder of this section we review the ansatz proposed by Gubser, Klebanov and Polyakov [3], and Witten [4] (GKP-W ansatz). The GKP-W ansatz relates the generating functional for the conformal field theory (especially $\mathcal{N} = 4$ SYM theory) to the partition function for the string theory on $\text{AdS}_5 \times \text{S}^5$ spacetime as

$$\left\langle \exp\left(\int_{\partial(\text{AdS})} \phi_0 \mathcal{O} \right) \right\rangle_{\text{CFT}} = Z_{\text{string}}\left[\phi_0\right]. \tag{3.12}$$

The left and right hand side of eq. (3.12) are the generating functional for $\mathcal{N}=4$ SYM theory and the partition function for the string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$, respectively. ϕ_0 is the value of a bulk field ϕ on the boundary of AdS spacetime $\partial(\mathrm{AdS})$ and $\mathcal{O}$ is an operator of $\mathcal{N}=4$ SYM theory.

According to the GKP-W ansatz (3.12) we can obtain the correlation function of the operator $\mathcal{O}$s in $\mathcal{N}=4$ SYM theory by evaluating the partition function of the string theory (or supergravity) as a function of boundary values of the fields. For example two point function of $\mathcal{O}$s is calculated as follows. The two point function of $\mathcal{O}$s is obtained by differentiating (3.12) with respect to ϕ_0

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle=\frac{\delta}{\delta\phi_0(x_1)}\frac{\delta}{\delta\phi_0(x_2)}Z_{\text{string}}\left[\phi_0\right]. \tag{3.13}$$

In the limit (3.10) supergravity approximation is valid then we can write

$$\begin{aligned}Z_{\text{string}}\left[\phi_0\right] &\sim \exp(-S_{\text{SUGRA}}[\phi])\\ &= \exp\left(-\int d^4x_1d^4x_2\;\phi_0(x_1)G(x_1-x_2)\phi_0(x_2)\right),\end{aligned} \tag{3.14}$$

where $G(x_1-x_2)$ is the Green function. Therefore the two point function of $\mathcal{O}$ is given by

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle=G(x_1-x_2). \tag{3.15}$$

Note that there is a one to one correspondence between the operators $\mathcal{O}$ in SYM theory and the fields ϕ in supergravity [4]. Although the above $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence are based on N_c coincident D3-branes, we can start with N_c coincident Dp-branes. In such case the correspondence can be extended to the $\mathrm{AdS}_{p+2}/\mathrm{CFT}_{p+1}$ correspondence.

3.3 The string/gauge duality

The AdS/CFT correspondence can be extended to the string/gauge duality. This is a generalization to non-conformal and non-supersymmetric theories. The deformations of the AdS spacetime leads to the breaking of the supersymmetry and conformal symmetry. Therefore there will be a relation between a string theory in the asymptotically $\mathrm{AdS}_{d+1}\times\mathrm{S}^{9-d}$ spacetime and a d-dimensional non-supersymmetric and non-conformal gauge theory.

Let us consider N_c coincident non-extremal Dp-branes, which correspond to the theory at finite temperature. Taking the decoupling limit (3.9) we can replace the N_c Dp-branes with the black p-brane solution. The near horizon geometry of the black p-brane solution is [70]

$$\begin{aligned}ds^2&=\left(\frac{U}{R}\right)^{\frac{7-p}{2}}\left(-f(U)dt^2+\sum_{i=1}^{p}\left(dx^i\right)^2\right)+\left(\frac{R}{U}\right)^{\frac{7-p}{2}}\left(\frac{dU^2}{f(U)}+U^2d\Omega_{8-p}^2\right),\\ f(U)&=1-\left(\frac{U_T}{U}\right)^{7-p},\qquad R^{7-p}=(4\pi)^{\frac{5-p}{2}}\,\Gamma\left(\tfrac{7-p}{2}\right)g_sN_c\,\ell_s^{\,7-p},\end{aligned} \tag{3.16}$$

where $d\Omega_{8-p}^2$ is the metric of a unit S^{8-p}. The dilaton field and the RR flux are given by

$$e^{\phi} = g_s \left(\frac{R}{U}\right)^{\frac{(7-p)(3-p)}{4}}, \qquad F_{8-p} = dC_{7-p} = \frac{2\pi N_c}{V_{8-p}}\epsilon_{8-p}, \tag{3.17}$$

where ϵ_{8-p} and V_{8-p} are the volume form and the volume of a unit S^{8-p}. The RR charge is measured in units of 2π as $\int_{\mathrm{S}^{8-p}} F_{8-p} = 2\pi N_c$. The relations between the coupling constant of the gauge theory and that of the string theory is

$$g_{p+1}^2 = (2\pi)^{p-2} g_s \ell_s^{p-3}, \tag{3.18}$$

where g_{p+1} is the $(p+1)$-dimensional gauge coupling. The $(p+1)$-dimensional 't Hooft coupling is defined as

$$\lambda_{p+1} = \frac{g_{p+1}^2 N_c}{(2\pi)^{p-2}}. \tag{3.19}$$

Note that the supergravity description is valid for [70]

$$1 \ll \lambda_{p+1} \left(\frac{U_T}{\ell_s^2}\right)^{p-3} \ll N_c^{\frac{4}{7-p}}. \tag{3.20}$$

For the extremal case $U_T = 0$ the low energy effective theory on the N_c Dp-branes is $(p+1)$-dimensional $\mathrm{U}(N_c)$ SYM theory with sixteen supercharges. The isometry of the metric (3.16) is

$$\mathrm{ISO}(1,p) \times \mathrm{SO}(9-p). \tag{3.21}$$

In the field theoretical view point $\mathrm{ISO}(1,p)$ and $\mathrm{SO}(9-p)$ symmetries correspond to the Poincaré symmetry and the R symmetry of the SYM theory on the Dp-branes, respectively.

The string/gauge duality is a powerful tool for the analysis of non-perturbative aspects of strongly coupled gauge theories. It is also useful for the analysis of the low energy behaviors of QCD such as the confinement and the spontaneous chiral symmetry breaking.

Chapter 4

Dq/Dp brane model

In this chapter we study the chiral symmetry breaking in the general intersecting Dq/Dp brane model. This brane model gives a holographic description of QCD-like theory. The rotational symmetry in the directions transverse to both Dq-branes and Dp-brane can be regarded as chiral symmetry for certain cases. We study the breaking of these symmetries and related phenomena by using the string/gauge duality.

4.1 General setup

We consider an intersecting brane system consisting of N_c color Dq-branes and a single probe Dp-brane

$$\begin{array}{c|cccccccccccc} & x^0 & \cdots & x^s & x^{s+1} & \cdots & x^q & x^{q+1} & \cdots & x^{q+p-s} & x^{q+p-s+1} & \cdots & x^9 \\ \hline N_c\ \mathrm{D}q & \circ & \cdots & \circ & \circ & \cdots & \circ & - & \cdots & - & - & \cdots & - \\ \mathrm{D}p & \circ & \cdots & \circ & - & \cdots & - & \circ & \cdots & \circ & - & \cdots & - \end{array} \tag{4.1}$$

with x^q being a coordinate of S^1. It has an s-dimensional intersection in the directions $x^1, \cdots, x^s$. The configuration (4.1) is a T-dual of Ds'/D9 system with $s' = 9-(q+p-2s) \geq s$. Following ref. [30] we call it a *transverse intersection* if $s' = s$ $(q+p-s=9)$ and a *non-transverse intersection* if $s' > s$ $(q+p-s<9)$. Non-transverse intersections have directions transverse to both of the Dq-branes and the Dp-brane, while transverse intersections do not.

The configuration (4.1) has the following symmetries. The gauge symmetry of this system is $\mathrm{U}(N_c) \times \mathrm{U}(1)$. The $\mathrm{U}(1)$ gauge symmetry on the Dp-brane is regarded as a global symmetry (baryon number symmetry) of an $(s+1)$-dimensional field theory at the intersection. The ten-dimensional Lorentz symmetry $\mathrm{SO}(1,9)$ is broken to its subgroup by the configuration (4.1). Therefore the global symmetry preserved at the intersection contains

$$\mathrm{SO}(1,s) \times \mathrm{SO}(9-q-p+s) \times \mathrm{U}(1), \tag{4.2}$$

where $\mathrm{SO}(1,s)$ is the Lorentz symmetry at the intersection and $\mathrm{SO}(9-q-p+s)$ is the rotational symmetry in the directions $x^{q+p-s+1}, \cdots, x^9$.

The spectrum of the theory localized at the intersection is as follows. Massless fields generated by q-q strings (open strings having both ends on the Dq-branes) are a gauge field A_μ ($\mu = 0, 1, \cdots, s$), scalar fields Φ^i ($i = s+1, \cdots, 9$) and fermionic fields S. Imposing the periodic boundary condition for the bosonic fields and the anti-periodic one for the fermionic fields along the compact x^q direction, the fermions become massive at zero mode and supersymmetry is explicitly broken at low energy. Then the scalars acquire mass at one-loop level. Thus only the gauge field A_μ is massless at low energy. This gives a pure U(N_c) gauge theory.

To study the lowest modes generated by q-p strings (open strings connecting the Dq-branes and the Dp-brane), we note the zero-point energy in the R sector and the NS sector [71]

$$a^{\mathrm{R}} = 0, \qquad a^{\mathrm{NS}} = \frac{\#\mathrm{ND} - 4}{8}, \tag{4.3}$$

where $\#\mathrm{ND} = q + p - 2s = 9 - s'$ is the number of spatial coordinates of open strings which have the Neumann boundary condition for one end and the Dirichlet one for the other end. The lowest modes generated by q-p strings in the NS sector are massive for $\#\mathrm{ND} > 4$ ($\#\mathrm{ND} = 6,\ 8$), massless for $\#\mathrm{ND} = 4$ and tachyonic for $\#\mathrm{ND} < 4$ ($\#\mathrm{ND} = 0,\ 2$). We do not consider the tachyonic case $\#\mathrm{ND} < 4$. When $\#\mathrm{ND} \geq 4$, the lowest modes from the NS sector are massive (by loop effects for $\#\mathrm{ND} = 4$) and are decoupled at low energy. There are only massless fermions from the R sector. They belong to representations of the Clifford algebra for the NN and DD directions. These fermions belong to the fundamental representation of U(N_c) and are called “quarks”.

In general the Dq/Dp configuration (4.1) is dual to a defect field theory [66, 67]. We only consider the case $s+1 = q$, which corresponds to a theory without defects. We are interested in non-transverse intersections satisfying $s' > s$, which implies $s < 9 - \#\mathrm{ND}$. Possible cases are $s = 1,\ 2,\ 3,\ 4$ for $\#\mathrm{ND} = 4$ ($p = q+2$) and $s = 1,\ 2$ for $\#\mathrm{ND} = 6$ ($p = q+4$). The configurations with $\#\mathrm{ND} = 6$ do not preserve supersymmetry and are most likely unstable. We further restrict ourselves to the cases $s = 1,\ 2,\ 3$ since we are especially interested in theories in four and lower dimensions. To summarize, we consider the Dq/D($q+2$) configurations for $q = 2,\ 3,\ 4$ compactified on x^q shown in Table 4.1. The effective theory on the intersection at low energy is an $(s+1)$-dimensional non-supersymmetric U(N_c) gauge theory with quarks in the fundamental representation. We call this theory “QCD$_{s+1}$” for the sake of convenience.

Since these configurations are non-transverse intersections, there are directions transverse to both of the Dq-branes and the Dp-brane. In refs. [18, 19] the rotational symmetry SO($9-q-p+s$) of such directions is interpreted as a chiral symmetry in the dual gauge theory for certain sets of (q, p, s). When the Dq-branes and the Dp-brane are separated along these directions, quarks on the intersection become massive and the chiral symmetry is explicitly broken. As we will see in subsect. 4.2.2, only when $a^{\mathrm{NS}} = 0$ ($\#\mathrm{ND} = 4$) and $p - s - 2 > 0$, which are satisfied for the configurations in Table 4.1, an equation for probe brane embeddings derived from the Dirac-Born-Infeld (DBI) action has a solution for which the distance between the color branes and the probe brane asymptotically approaches a constant value. This distance is interpreted as a quark mass.

		0	1	2	3	4	5	6	7	8	9
color	D2	∘	∘	∘	–	–	–	–	–	–	–
probe	D4	∘	∘	–	∘	∘	∘	–	–	–	–
color	D3	∘	∘	∘	∘	–	–	–	–	–	–
probe	D5	∘	∘	∘	–	∘	∘	∘	–	–	–
color	D4	∘	∘	∘	∘	∘	–	–	–	–	–
probe	D6	∘	∘	∘	∘	–	∘	∘	∘	–	–

Table 4.1: The Dq/D$(q+2)$ brane configurations with #ND = 4. From top to bottom these are dual to QCD$_2$, QCD$_3$ and QCD$_4$, respectively.

We can explicitly write down the symmetry (4.2) for the configurations in Table 4.1. In the D2/D4 model, which has a one-dimensional intersection and is dual to QCD$_2$, we can identify the SO$(4)_{6789}$ rotational symmetry in the x^6, x^7, x^8, x^9 directions with an SU$(2)_L \times$SU$(2)_R$ chiral symmetry of quarks. Indeed, the GSO projection in the R sector of open strings requires that the chiralities of SO$(1,1)_{01}$ and SO$(4)_{6789}$ are correlated. Left-handed (right-handed) quarks of SO$(1,1)_{01}$ have the positive (negative) chirality of SO$(4)_{6789}$ and transform as $(\mathbf{2},\mathbf{1})$ $((\mathbf{1},\mathbf{2}))$ under SU$(2)_L \times$SU$(2)_R$. The gauge symmetry U(1) on the probe brane acts on quarks as a baryon number symmetry U$(1)_V$. Therefore the global symmetry (apart from the Lorentz symmetry) of QCD$_2$ at the intersection is

$$\mathrm{SO}(4)_{6789} \times \mathrm{U}(1) \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V. \tag{4.4}$$

Thus we can realize a non-Abelian chiral symmetry in a holographic model of this type, although spacetime is two-dimensional.

In the D3/D5 model, which has a two-dimensional intersection and is dual to QCD$_3$, we can identify the SO$(3)_{789}$ rotational symmetry in the x^7, x^8, x^9 directions with an SU(2) symmetry of QCD$_3$. Then the global symmetry of QCD$_3$ at the intersection is

$$\mathrm{SO}(3)_{789} \times \mathrm{U}(1) \sim \mathrm{SU}(2) \times \mathrm{U}(1). \tag{4.5}$$

Quarks transform as $\mathbf{2}$ under SU(2). Note that there is no chirality in QCD$_3$ and therefore the symmetry (4.5) is not a chiral symmetry.

Finally, in the D4/D6 model, which has a three-dimensional intersection and is dual to QCD$_4$, we can identify the SO$(2)_{89}$ rotational symmetry in the x^8, x^9 directions with an axial U$(1)_A$ symmetry of QCD$_4$ as discussed in refs. [18, 19]. The global symmetry of QCD$_4$ at the intersection is

$$\mathrm{SO}(2)_{89} \times \mathrm{U}(1) \sim \mathrm{U}(1)_A \times \mathrm{U}(1)_V. \tag{4.6}$$

4.2 Chiral symmetry breaking from supergravity analysis

The dynamics of a strongly coupled large N_c gauge theory can be analyzed by supergravity. We study the chiral symmetry breaking in this section. The near horizon limit and the large N_c limit $N_c \gg 1$ allow us to treat the Dq-branes as a background geometry and the Dp-brane as a probe which does not affect this background. We will find that the Dp-brane embedding breaks the SO$(9-q-p+s)$ rotational symmetry in the directions transverse to both of the branes. This can be interpreted as the chiral symmetry breaking in QCD$_2$ and QCD$_4$. We will calculate the quark condensate and find a non-zero value even in the massless quark limit. Although we are most interested in the configurations in Table 4.1, we will give formulae for the configuration (4.1) with general q, p, s.

4.2.1 Dq-brane background

The near horizon geometry of S^1 compactified N_c Dq-branes can be obtained by the double Wick rotation $t \to it$, $x^q \to ix^q$ and interchanging of $t \leftrightarrow x^q$ in the metric (3.16)

$$ds^2 = \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(-dt^2 + \sum_{i=1}^{q-1} \left(dx^i\right)^2 + f(U)(dx^q)^2\right) + \left(\frac{R}{U}\right)^{\frac{7-q}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_{8-q}^2\right),$$
$$f(U) = 1 - \left(\frac{U_{KK}}{U}\right)^{7-q}, \qquad R^{7-q} = (4\pi)^{\frac{5-q}{2}} \Gamma\left(\tfrac{7-q}{2}\right) g_s N_c \, \ell_s^{7-q}. \tag{4.7}$$

x^q is a coordinate of S^1 and its period is denoted as $\delta x^q = 2\pi/M_{KK}$. M_{KK} is a compactification scale. To avoid a conical singularity at $U = U_{KK}$ in the U-x^q plane the period must be related to a constant U_{KK} as

$$\delta x^q = \frac{4\pi \, R^{\frac{7-q}{2}}}{(7-q)\, U_{KK}^{\frac{5-q}{2}}}. \tag{4.8}$$

The dilaton field and the RR flux are given by eqs. (3.17)

The relations between the parameters in the gauge theory and those in the string theory are

$$g_{q+1}^2 = (2\pi)^{q-2} g_s \, \ell_s^{q-3}, \qquad M_{KK} = \frac{7-q}{2\,(4\pi)^{\frac{5-q}{4}}\, \Gamma(\frac{7-q}{2})^{\frac{1}{2}}} \frac{U_{KK}^{\frac{5-q}{2}}}{(g_s N_c)^{\frac{1}{2}} \, \ell_s^{\frac{7-q}{2}}}, \tag{4.9}$$

where g_{q+1} is the $(q+1)$-dimensional gauge coupling.

We introduce isotropic coordinates in the directions (U, Ω_{8-q}) to simplify the following analysis. Introducing a new radial coordinate ρ defined by

$$U = \left(\rho^{\frac{7-q}{2}} + \frac{U_{KK}^{7-q}}{4\rho^{\frac{7-q}{2}}}\right)^{\frac{2}{7-q}}, \qquad \rho^2 = \sum_{\alpha=q+1}^{9} (x^\alpha)^2 \tag{4.10}$$

the metric for the transverse space (U, Ω_{8-q}) in eq. (4.7) can be written as

$$\begin{aligned} \left(\frac{R}{U}\right)^{\frac{7-q}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_{8-q}^2\right) &= K(\rho)\left(d\rho^2 + \rho^2 d\Omega_{8-q}^2\right) \\ &= K(\rho) \sum_{\alpha=q+1}^{9} (dx^\alpha)^2, \end{aligned} \tag{4.11}$$

where

$$K(\rho) = \frac{R^{\frac{7-q}{2}} U^{\frac{q-3}{2}}}{\rho^2}. \tag{4.12}$$

We divide the coordinates $x^{q+1}, \cdots, x^9$ into two parts and introduce spherical coordinates $(\lambda, \Omega_{p-s-1})$ for the x^{q+1}, $\cdots$, x^{q+p-s} directions and $(r, \Omega_{8-q-p+s})$ for the $x^{q+p-s+1}$, $\cdots$, x^9 directions. Then the Dq background becomes

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(-dt^2 + \sum_{i=1}^{q-1}(dx^i)^2 + f(U)(dx^q)^2\right) \\ &\quad + K(\rho)\left(d\lambda^2 + \lambda^2 d\Omega_{p-s-1}^2 + dr^2 + r^2 d\Omega_{8-q-p+s}^2\right), \end{aligned} \tag{4.13}$$

where $\rho^2 = \lambda^2 + r^2$. We will wrap the probe Dp-brane around S^{p-s-1} in the next subsection.

4.2.2 Dp-brane embeddings

We study the dynamics of a Dp-brane in the Dq background. Taking the large N_c limit $N_c \to \infty$ with keeping $\lambda_{q+1} = g_s N_c \ell_s^{q-3} = \text{fixed} \gg 1$, the probe approximation $N_c \gg N_f$ (N_f is the number of probe Dp-branes and we set $N_f = 1$ in this chapter) requires

$$\lambda_{p+1} = g_s N_f \ell_s^{p-3} \to 0, \tag{4.14}$$

where λ_{p+1} is the 't Hooft coupling of the $(p+1)$-dimensional theory on the Dp-brane world-volume. Then the gauge theory on the Dp-branes decouples from the system and the backreaction of the N_f Dp-branes to the ten-dimensional bulk is too small comparing with that of the N_c Dq-branes. So the N_f Dp-branes can be introduced into the Dq background as a probe, which do not affect the background.

The dynamics of the probe Dp-brane in the background (4.13) is described by the DBI action

$$S_{\mathrm{D}p} = -T_p \int d^{p+1}x\, e^{-\phi} \sqrt{-\det g_{MN}}, \tag{4.15}$$

where g_{MN} $(M, N = 0, 1, \cdots, p)$ is the induced metric on the world-volume and T_p is the tension of the Dp-brane. For simplicity we have ignored the gauge field on the probe Dp-brane. The gauge field on the probe brane plays an important role when we introduce a chemical potential for the baryon number symmetry in eq. (4.2).

We use a physical gauge for Dp-brane world-volume reparametrizations and use the spacetime coordinates x^μ $(\mu = 0, 1, \cdots, s)$, λ, Ω_{p-s-1} as the world-volume coordinates. Then the configurations of the Dp-brane are determined by x^i $(i = s+1, \cdots, q)$, r and $\Omega_{8-q-p+s}$ as functions of those world-volume coordinates. We make an ansatz

$$x^{s+1}, \cdots, x^q = \text{constant}, \quad r = r(\lambda), \quad \theta^a = \text{constant}, \tag{4.16}$$

where θ^a $(a = 1, 2, \cdots, 8-q-p+s)$ are coordinates of $\mathrm{S}^{8-q-p+s}$.

With this ansatz, the induced metric on the Dp-brane is

$$ds^2 = \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(-dt^2 + \sum_{i=1}^{s}(dx^i)^2\right) + K(\rho)\left[\left(1+(r')^2\right)d\lambda^2 + \lambda^2 d\Omega_{p-s-1}^2\right], \quad (4.17)$$

where $r' = \frac{dr}{d\lambda}$. Then the DBI action of the Dp-brane becomes

$$S_{\mathrm{D}p} = -\tilde{T}_p V_{p-s-1} \int d^{s+1}x \int d\lambda\, \rho^\alpha \left(1 + \frac{U_{KK}^{7-q}}{4\rho^{7-q}}\right)^\beta \lambda^{p-s-1}\sqrt{1+(r')^2}, \quad (4.18)$$

where $\tilde{T}_p \equiv g_s^{-1} T_p R^{-\alpha}$ and V_{p-s-1} is the volume of S^{p-s-1}. The parameters α and β are defined as

$$\alpha = \frac{1}{4}(7-q)(4+2s-q-p)\,, \quad \beta = \frac{1}{2}(4+2s-q-p) + \frac{2(p-s)}{7-q}. \quad (4.19)$$

The action (4.18) leads to the equation of motion for $r(\lambda)$

$$\frac{d}{d\lambda}\left[\rho^\alpha \left(1 + \frac{U_{KK}^{7-q}}{4\rho^{7-q}}\right)^\beta \frac{\lambda^{p-s-1}\, r'}{\sqrt{1+(r')^2}}\right] = \frac{\partial}{\partial r}\left[\rho^\alpha \left(1 + \frac{U_{KK}^{7-q}}{4\rho^{7-q}}\right)^\beta\right] \lambda^{p-s-1}\sqrt{1+(r')^2}. \quad (4.20)$$

As in refs. [18, 19] we are interested in the situation in which the asymptotic distance between the Dq-branes and the Dp-brane is a finite constant r_∞. This constant is proportional to the quark mass. Therefore we impose the boundary conditions for $\lambda \to \infty$

$$r(\lambda)|_{\lambda\to\infty} = r_\infty, \quad r'(\lambda)|_{\lambda\to\infty} = 0. \quad (4.21)$$

Then, eq.(4.20) can be linearized at large λ as

$$\frac{d}{d\lambda}\left(\lambda^{\alpha+p-s-1} r'\right) = \alpha\, \lambda^{\alpha+p-s-3}\, r, \quad (4.22)$$

and the asymptotic behavior of the solution is

$$r(\lambda) \sim a\lambda^{k_+} + b\lambda^{k_-}, \quad (4.23)$$

where a, b are constants and

$$k_\pm = \frac{-(\alpha+p-s-2) \pm \sqrt{(\alpha+p-s-2)^2 + 4\alpha}}{2}. \quad (4.24)$$

For the boundary condition (4.21) to be satisfied, we must require $\alpha = 0$ and $p-s-2 > 0$. The first condition implies that the ground states of the NS sector of q-p strings are massless since $\alpha = -2(7-q)a^{\mathrm{NS}}$ as seen from eq. (4.3). Then, the asymptotic behavior of $r(\lambda)$ is

$$r(\lambda) \sim r_\infty + c\,\lambda^{-(p-s-2)}, \quad (4.25)$$

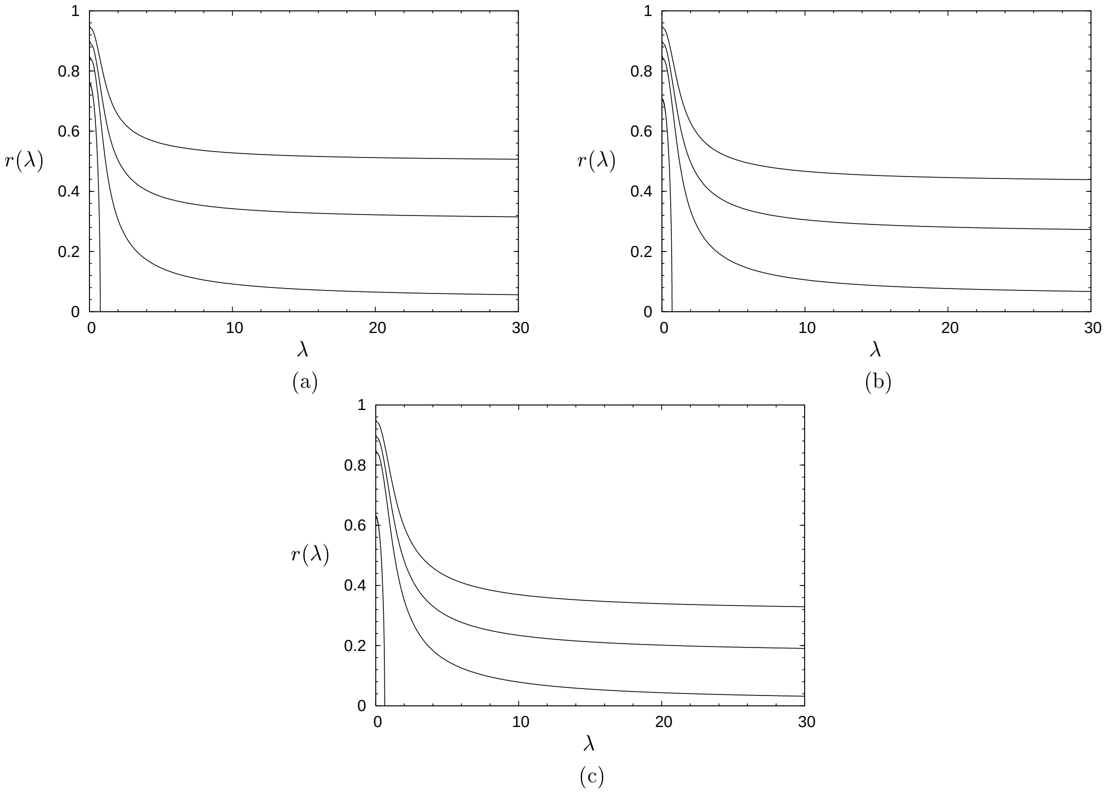


Figure 4.1: Solutions of eq. (4.20) for various values of r_∞ in (a) the D2/D4 model with $s = 1$, (b) the D3/D5 model with $s = 2$ and (c) the D4/D6 model with $s = 3$.

where c is a constant. As in ref. [19] the quark condensate $\langle\bar{\psi}\psi\rangle$ can be calculated by differentiating the vacuum energy density derived from the DBI action (4.18) with respect to the quark mass m_q. Thus we obtain the quark mass and the quark condensate in terms of the constants r_∞ and c as

$$m_q = \frac{r_\infty}{2\pi\ell_s^2}, \qquad \langle\bar{\psi}\psi\rangle = -2\pi(p-s-2)\,\ell_s^2\,\tilde{T}_p V_{p-s-1}\,c. \tag{4.26}$$

We have numerically solved eq. (4.20) for all possible values of q, p, s satisfying $\alpha = 0$, $p - s - 2 > 0$, $s \leq 3$. The solutions of the D2/D4 model with $s = 1$, the D3/D5 model with $s = 2$ and the D4/D6 model with $s = 3$ are plotted in Fig. 4.1 for various values of r_∞. The variables λ and r in these figures denote dimensionless ones rescaled by appropriate powers of U_{KK}. The leftmost curve in these figures represents $U = U_{KK}$. Its interior $U < U_{KK}$ is not a part of the space that we are considering. All the solutions have similar behaviors to those of the D4/D6 model with $s = 3$ (Fig. 4.1 (c)), which was studied in ref. [19]. The solutions approach a constant value r_∞ for $\lambda = \infty$, while they reach a point outside of the curve $U = U_{KK}$ at $\lambda = 0$. The solutions break the rotational symmetry $\mathrm{SO}(9-q-p+s)$ in the $(r,\,\Omega_{8-q-p+s})$ space to $\mathrm{SO}(8-q-p+s)$.

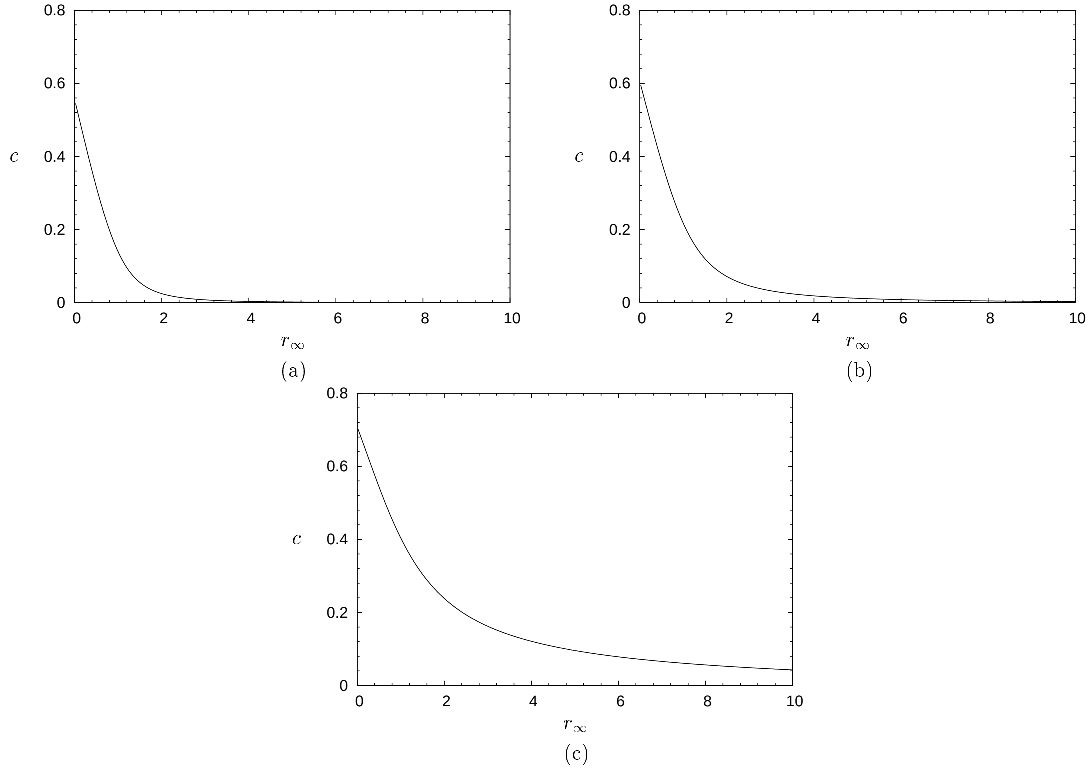


Figure 4.2: The quark condensate as a function of the quark mass for (a) the D2/D4 model with $s = 1$, (b) the D3/D5 model with $s = 2$ and (b) the D4/D6 model with $s = 3$.

We have also numerically calculated the quark condensate as a function of the quark mass $c = c(r_\infty)$ for all possible values of q, p, s satisfying $\alpha = 0$, $p - s - 2 > 0$, $s \leq 3$. It is plotted in Fig. 4.2 for the D2/D4 model with $s = 1$, the D3/D5 model with $s = 2$ and the D4/D6 model with $s = 3$. The variables r_∞ and c in these figures denote dimensionless ones rescaled by appropriate powers of U_{KK}. For all cases we find a non-zero quark condensate for $r_\infty = 0$. This agrees with a field theoretical view point. In QCD we expect that the chiral symmetry is spontaneously broken by the non-zero quark condensate even for $m_q = 0$.

Finally, we write down a pattern of the symmetry breaking explicitly. For the D2/D4 model with $s = 1$ it is

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V \to \mathrm{SU}(2)_V \times \mathrm{U}(1)_V, \tag{4.27}$$

for the D3/D5 model with $s = 2$

$$\mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{U}(1) \times \mathrm{U}(1), \tag{4.28}$$

and for the D4/D6 model with $s = 3$ [19]

$$\mathrm{U}(1)_A \times \mathrm{U}(1)_V \to \mathrm{U}(1)_V. \tag{4.29}$$

4.3 NG bosons as fluctuations of the probe brane

In this section we study fluctuations of the probe brane around the vacuum embedding. In the previous section we have seen that the vacuum embedding breaks SO$(9-q-p+s)$ rotational symmetry in the $(r, \Omega_{8-q-p+s})$ space to SO$(8-q-p+s)$. This symmetry breaking corresponds to the chiral symmetry breaking for certain sets of (q, p, s). Therefore, there should be $(8-q-p+s)$ NG bosons associated with the symmetry breaking. If quarks are massless these bosons are massless NG bosons. On the other hand, if quarks are massive these are pseudo-NG bosons with a non-vanishing mass. We will show that these pseudo-NG bosons satisfy the GMOR relation (2.17) for a small quark mass m_q. We will also give the effective action of the fluctuations at quartic order. These results are a generalization of those of the D4/D6 system studied in ref. [19] to the Dq/Dp systems.

4.3.1 Fluctuations around the vacuum embeddings

We study fluctuation modes around the vacuum Dp-brane embedding

$$x^{s+1}, \cdots, x^q = \text{constant}, \quad r = r_{\text{vac}}(\lambda) + \delta r(x^M), \quad \theta^a = 0 + \delta\theta^a(x^M), \tag{4.30}$$

where r_{vac} is the vacuum embedding determined numerically in the previous section. For simplicity we concentrate on fluctuations of r and θ^a. In general, these fluctuations depend on all of the world-volume coordinates x^M of the Dp-brane. We will see that the fluctuations $\delta\theta^a$ are identified with the (pseudo-)NG bosons for the breaking of the rotational symmetry of S$^{8-q-p+s}$ (a subspace of the $(r, \Omega_{8-q-p+s})$ space).

The induced metric on the Dp-brane world-volume is

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(-dt^2 + \sum_{i=1}^{s}(dx^i)^2\right) + K(\rho)\left[\left(1+(r'_{\text{vac}})^2\right)d\lambda^2 + \lambda^2 d\Omega^2_{p-s-1}\right] \\ &\quad + K(\rho)\left[2r'_{\text{vac}}\partial_M\delta r d\lambda dx^M + \left(\partial_M\delta r\partial_N\delta r + r^2_{\text{vac}}\gamma_{ab}\partial_M\delta\theta^a\partial_N\delta\theta^b\right)dx^M dx^N\right], \end{aligned} \tag{4.31}$$

where γ_{ab} is the metric of a unit S$^{8-q-p+s}$. Then the DBI action of the Dp-brane (4.15) becomes

$$\begin{aligned} S_{\text{D}p} &= -\tilde{T}_p \int d^{p+1}x \sqrt{\det\gamma_{\alpha\beta}} \left(1 + \frac{U_{KK}^{7-q}}{4\rho^{7-q}}\right)^{\beta} \lambda^{p-s-1}\sqrt{1+(r'_{\text{vac}})^2} \\ &\quad \times \left[1 + \frac{r'_{\text{vac}}\partial_\lambda\delta r}{1+(r'_{\text{vac}})^2} + \frac{K(\rho)}{2}g^{MN}\left(\frac{\partial_M\delta r\partial_N\delta r}{1+(r'_{\text{vac}})^2} + r^2_{\text{vac}}\gamma_{ab}\partial_M\delta\theta^a\partial_N\delta\theta^b\right) + \cdots\right], \end{aligned} \tag{4.32}$$

where g_{MN} is the induced metric without the fluctuations (4.17) and the dots represent terms higher than quadratic order in the fluctuations. Note that $\rho^2 = \lambda^2 + (r_{\text{vac}} + \delta r)^2$

still contains the fluctuation δr. Expanding eq. (4.32) in the fluctuations we obtain the action to quadratic order as

$$S_{\mathrm{D}p} = S_{\mathrm{vac}} + S_{\delta r} + S_{\delta\theta}, \tag{4.33}$$

where S_{vac} is the action for the vacuum embedding, i.e., eq. (4.18) for $r = r_{\mathrm{vac}}$. $S_{\delta r}$ and $S_{\delta\theta}$ are the actions for the fluctuations δr and $\delta\theta^a$, respectively. We concentrate on the fluctuations $\delta\theta^a$. The fluctuation δr can be studied in a similar way (see Appendix A). After some simple calculations we obtain the action for $\delta\theta^a$

$$\begin{aligned} S_{\delta\theta} &= -\tilde{T}_p \int d^{p+1}x \sqrt{\det \gamma_{\alpha\beta}}\; \lambda^{p-s-1} \sqrt{1+(r'_{\mathrm{vac}})^2} \\ &\qquad \times \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\mathrm{vac}}^{7-q}}\right)^{\beta} \frac{K}{2} g^{MN} r_{\mathrm{vac}}^2 \gamma_{ab} \partial_M \delta\theta^a \partial_N \delta\theta^b \end{aligned} \tag{4.34}$$

and the equation of motion

$$\begin{aligned} &\left(\frac{7-q}{2}\right)^2 \frac{U_{KK}^{5-q}}{M_{KK}^2} \rho_{\mathrm{vac}}^{-(7-q)} \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\mathrm{vac}}^{7-q}}\right)^{\beta-\frac{2(5-q)}{7-q}} r_{\mathrm{vac}}^2 \partial_\mu \partial^\mu \delta\theta^a \\ &\quad + \frac{1}{\lambda^{p-s-1}\sqrt{1+(r'_{\mathrm{vac}})^2}} \frac{\partial}{\partial\lambda}\left[\left(1+\frac{U_{KK}^{7-q}}{4\rho_{\mathrm{vac}}^{7-q}}\right)^{\beta} \frac{\lambda^{p-s-1} r_{\mathrm{vac}}^2}{\sqrt{1+(r'_{\mathrm{vac}})^2}} \frac{\partial}{\partial\lambda}\delta\theta^a\right] \\ &\quad + \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\mathrm{vac}}^{7-q}}\right)^{\beta} \frac{r_{\mathrm{vac}}^2}{\lambda^2} \nabla^2 \delta\theta^a = 0, \end{aligned} \tag{4.35}$$

where $\gamma_{\alpha\beta}$ and ∇^2 are the metric and the Laplacian on a unit S^{p-s-1}.

We can write a solution of the equation of motion (4.35) in a form

$$\delta\theta^a = F^a(\lambda)\, Y(\Omega_{p-s-1})\, e^{ik\cdot x}, \tag{4.36}$$

where $Y(\Omega_{p-s-1})$ is the spherical harmonics on S^{p-s-1}. We consider the zero (constant) mode of Y and study only lowest-mass modes for simplicity. Substituting eq. (4.36) into eq. (4.35) we obtain an eigenvalue equation for the $(s+1)$-dimensional mass $M_\theta^2 = -k^\mu k_\mu$. Although we can solve eq. (4.35) by numerical calculations as in ref. [19], here we are content with asymptotic solutions of a linearized equation of motion for $\lambda \to \infty$. Taking account of the asymptotic behavior of r_{vac} in eq. (4.25) the first term of eq. (4.35) is sub-leading and the linearized equation for $\lambda \to \infty$ becomes

$$\frac{\partial}{\partial\lambda}\left(\lambda^{p-s-1} r_{\mathrm{vac}}^2 \frac{\partial}{\partial\lambda}\delta\theta^a\right) = 0. \tag{4.37}$$

Depending on the value r_∞ in eq. (4.25) the general solution is

$$\delta\theta^a \sim \begin{cases} a\lambda^{p-s-2} + b & (r_\infty = 0) \\ a + b\lambda^{-(p-s-2)} & (r_\infty \neq 0), \end{cases} \tag{4.38}$$

where a, b are independent of λ. Since $p-s-2>0$, these solutions are normalizable when $a=0$.

The non-linear equation of motion (4.35) has exact solutions $\delta\theta^a = F^a e^{ik\cdot x}$ ($F^a =$ constant), which have an eigenvalue $M_\theta = 0$. From the above results on the asymptotic behaviors these solutions are normalizable only when $r_\infty = 0$. Since $r_\infty = 0$ means vanishing quark mass $m_q = 0$, the normalizable solutions with $M_\theta = 0$ can be regarded as the NG bosons associated with the spontaneous breaking of the rotational symmetry $\mathrm{SO}(9-q-p+s)$. When $r_\infty \neq 0$ ($m_q \neq 0$), the quark mass term explicitly breaks the chiral symmetry and we do not expect massless bosons. However, for small quark mass m_q there should exist pseudo-NG bosons with a small mass M_θ, which we consider in the next subsection.

In two dimensions there exists no massless NG boson associated with a spontaneous symmetry breaking [72]. We have seen that there appear massless bosons even in the D2/D4 model with a one-dimensional intersection when quarks are massless. These massless bosons should be an artifact of the large N_c limit and should become massive if we take into account contributions from higher orders in the $1/N_c$ expansion. The situation is similar to the case of the Gross-Neveu model in two dimensions [73], in which massless bosons appear in the large N limit.

4.3.2 Light pseudo-NG bosons and the GMOR relation

As we have seen above, the embeddings with $r_\infty = 0$ and those with $r_\infty \neq 0$ have different properties. For the $r_\infty = 0$ embeddings the asymptotic distance between Dq and Dp-branes is zero and the quarks at the intersection are massless. There are $(8-q-p+s)$ massless scalars $\delta\theta^a$ in the spectrum, which can be identified with the NG bosons associated with the spontaneous symmetry breaking $\mathrm{SO}(9-q-p+s) \to \mathrm{SO}(8-q-p+s)$. We call these NG bosons *pions*. On the other hand, for the $r_\infty \neq 0$ embeddings quarks are massive and the vacuum embedding explicitly breaks the rotational symmetry $\mathrm{SO}(9-q-p+s)$ even for the asymptotic region $\lambda \to \infty$. In this case the fluctuations $\delta\theta^a$ are pseudo-NG bosons with a non-vanishing mass M_θ.

We can show the GMOR relation [68]

$$M_\theta^2 = -\frac{m_q \left\langle \bar{\psi}\psi \right\rangle}{f_\pi^2}. \tag{4.39}$$

for a small quark mass m_q by using the holographic method [19]. Here, f_π is the pion decay constant. We begin with the $r_\infty = 0$ embedding and make a small change $r_\infty = \delta r_\infty$. This gives a small mass to quarks. As shown in ref. [19] the mass of the pseudo-NG bosons M_θ can be obtained by using a standard perturbation theory in quantum mechanics and is written as

$$M_\theta^2 = (p-s-2)\,\frac{\bar{c}\,\delta r_\infty}{\int d\lambda\, \bar{\mu}}, \tag{4.40}$$

where $\bar{c}$ is the coefficient in eq. (4.25) and $\bar{\mu}$ is given by

$$\bar{\mu} = \left(\frac{7-q}{2}\right)^2 \frac{U_{KK}^{5-q}}{M_{KK}^2} \bar{\rho}_{\text{vac}}^{-(7-q)} \left(1 + \frac{U_{KK}^{7-q}}{4\bar{\rho}_{\text{vac}}^{7-q}}\right)^{\beta - \frac{2(5-q)}{7-q}} \bar{r}_{\text{vac}}^2 \, \lambda^{p-s-1} \sqrt{1 + (\bar{r}'_{\text{vac}})^2}. \tag{4.41}$$

Here and in the following, quantities with a bar denote those for the $r_\infty = 0$ embedding. The quantities δr_∞ and $\bar{c}$ are related to the quark mass and the quark condensate as in eq. (4.26). The pion decay constant f_π can be read from the effective action of $\delta\theta^a$. Assuming that $\delta\theta^a$ depend only on the coordinates of the intersection x^μ ($\mu = 0, 1, \cdots, s$) and integrating over λ and Ω_{p-s-1} in eq. (4.34) we obtain

$$S_{\delta\theta} = -f_\pi^2 \int d^{s+1}x \, \frac{1}{2} \gamma_{ab} \, \partial_\mu \delta\theta^a \partial^\mu \delta\theta^b, \tag{4.42}$$

where f_π is given by

$$f_\pi^2 = \tilde{T}_p V_{p-s-1} \int_0^\infty d\lambda \, \bar{\mu}. \tag{4.43}$$

Using eqs. (4.26), (4.43) in eq. (4.40), we obtain the GMOR relation (4.39).

4.3.3 The pion effective action

We can write down the effective action of the pion fields $\delta\theta^a$ at the intersection beyond the quadratic order. We assume that $\delta\theta^a$ depend only on the coordinates of the intersection x^μ ($\mu = 0, 1, \cdots, s$). By expanding the DBI action (4.15) for the induced metric (4.31) to quartic order in $\delta\theta^a$ we obtain

$$\begin{aligned} S_{\delta\theta} \;=\; & -\int d^{s+1}x \, \Bigg(\frac{f_\pi^2}{2} \gamma_{ab} \partial_\mu(\delta\theta^a) \partial^\mu(\delta\theta^b) + \frac{h_1}{4} \left[\gamma_{ab} \partial_\mu(\delta\theta^a) \partial^\mu(\delta\theta^b)\right]^2 \\ & -\frac{h_2}{4} \left[\gamma_{ab} \partial_\mu(\delta\theta^a) \partial_\nu(\delta\theta^b)\right] \left[\gamma_{cd} \partial^\mu(\delta\theta^c) \partial^\nu(\delta\theta^d)\right] \Bigg), \end{aligned} \tag{4.44}$$

where f_π is the pion decay constant (4.43) and the constants h_1, h_2 are given by

$$\begin{aligned} 2h_1 = h_2 \;=\; & \tilde{T}_p V_{p-s-1} \left(\frac{7-q}{2}\right)^4 \frac{U_{KK}^{2(5-q)}}{M_{KK}^4} \int d\lambda \, \rho_{\text{vac}}^{-2(7-q)} \\ & \times \left(1 + \frac{U_{KK}^{7-q}}{4\rho_{\text{vac}}^{7-q}}\right)^{\beta - \frac{4(5-q)}{7-q}} r_{\text{vac}}^4 \, \lambda^{p-s-1} \sqrt{1 + (r'_{\text{vac}})^2}. \end{aligned} \tag{4.45}$$

The relative coefficients of the quartic terms are different from those assumed in the Skyrme model [74, 75] $h_1 = h_2$. This is in contrast with another approach [24] to the holographic QCD, in which the relation $h_1 = h_2$ of the Skyrme model was obtained.

4.4 Finite temperature analysis

To study the theory at finite temperature we introduce a periodic Euclidean time coordinate $t_E \equiv it \sim t_E + \delta t_E$. The period of t_E is the inverse temperature $\delta t_E = 1/T$. Then there are two periodic coordinates t_E and x^q. There exist two possible Euclidean geometries which have an appropriate asymptotic behavior. One of them is the Euclidean version of the metric (4.7)

$$ds^2 = \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(dt_E^2 + \sum_{i=1}^{q-1} \left(dx^i\right)^2 + f(U)(dx^q)^2\right) + \left(\frac{R}{U}\right)^{\frac{7-q}{2}} \left(\frac{dU^2}{\tilde{f}(U)} + U^2 d\Omega_{8-q}^2\right),$$
$$f(U) = 1 - \left(\frac{U_{KK}}{U}\right)^{7-q}, \qquad U_{KK} = \left(\frac{2}{7-q}\right)^{\frac{2}{5-q}} R^{\frac{7-q}{5-q}} M_{KK}^{\frac{2}{5-q}}. \tag{4.46}$$

The other is the Euclidean version of the non-extremal black Dq-brane geometry (3.16)

$$ds^2 = \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(\tilde{f}(U) dt_E^2 + \sum_{i=1}^{q-1} \left(dx^i\right)^2 + (dx^q)^2\right) + \left(\frac{R}{U}\right)^{\frac{7-q}{2}} \left(\frac{dU^2}{\tilde{f}(U)} + U^2 d\Omega_{8-q}^2\right),$$
$$\tilde{f}(U) = 1 - \left(\frac{U_T}{U}\right)^{7-q}, \qquad U_T = \left(\frac{4\pi}{7-q}\right)^{\frac{2}{5-q}} R^{\frac{7-q}{5-q}} T^{\frac{2}{5-q}} \tag{4.47}$$

with the dilaton and the RR flux given in eq. (3.17). The parameter U_T must be related to T as above to avoid a conical singularity at $U = U_T$ in the U-t_E plane. The period δt_E is fixed as

$$\delta t_E = \frac{4\pi R^{\frac{7-q}{2}}}{(7-q)\, U_T^{\frac{5-q}{2}}}, \tag{4.48}$$

and it leads the relation in eq. (4.47). It is obvious that these two backgrounds are related by interchanging x^q, U_{KK} and t_E, U_T.

It was shown [8, 19, 34] that the background (4.46) is dominant at low temperature, while the background (4.47) is dominant at high temperature by comparing values of the Euclidean supergravity action for these backgrounds. A phase transition between these backgrounds occurs at the temperature for $U_T = U_{KK}$, i.e. $T_c = M_{KK}/(2\pi)$. This phase transition corresponds to a confinement/deconfinement transition in the dual gauge theory [8].

We consider the probe brane dynamics in the high temperature phase. The probe brane dynamics in the low temperature phase is essentially the same as at zero temperature. We only consider the models with $\alpha = 0$, $p - s - 2 > 0$, $s \leq 3$ as in the zero temperature phase. With the ansatz (4.16) the induced metric on the Dp-brane in the background (4.47) can be written as

$$ds^2 = \left(\frac{U}{R}\right)^{\frac{7-q}{2}} \left(\tilde{f}(U) dt_E^2 + \sum_{i=1}^{s} (dx^i)^2\right) + K(\rho) \left[\left(1 + (r')^2\right) d\lambda^2 + \lambda^2 d\Omega_{p-s-1}^2\right]. \tag{4.49}$$

Then the DBI action of the probe Dp-brane becomes

$$S_{\mathrm{D}p} = \tilde{T}_p V_{p-s-1} \int d^{s+1}x \int d\lambda \left(1 + \frac{U_T^{7-q}}{4\rho^{7-q}}\right)^{\beta-1} \left(1 - \frac{U_T^{7-q}}{4\rho^{7-q}}\right) \lambda^{p-s-1} \sqrt{1+(r')^2}, \tag{4.50}$$

which leads to the equation of motion for $r(\lambda)$

$$\begin{aligned} &\frac{d}{d\lambda}\left[\left(1 + \frac{U_T^{7-q}}{4\rho^{7-q}}\right)^{\beta-1} \left(1 - \frac{U_T^{7-q}}{4\rho^{7-q}}\right) \frac{\lambda^{p-s-1}\, r'}{\sqrt{1+(r')^2}}\right] \\ &\qquad = \frac{\partial}{\partial r}\left[\left(1 + \frac{U_T^{7-q}}{4\rho^{7-q}}\right)^{\beta-1} \left(1 - \frac{U_T^{7-q}}{4\rho^{7-q}}\right)\right] \lambda^{p-s-1}\sqrt{1+(r')^2}. \end{aligned} \tag{4.51}$$

The asymptotic behavior of the solution $r(\lambda)$ of eq. (4.51) for large λ is the same as in the zero temperature case (4.25). The parameters r_∞ and c are related to the quark mass m_q and the quark condensate $\langle\bar{\psi}\psi\rangle$ as in eq. (4.26). We have numerically solved eq. (4.51) for all possible values of q, p, s. All the solutions have similar behaviors to those for the D4/D6 model with $s=3$ discussed in refs. [19, 36, 38]. The solutions for the D2/D4 model with $s=1$, the D3/D5 model with $s=2$ and the D4/D6 model with $s=3$ are plotted in Fig. 4.3 for various values of r_∞. The variables λ and r in these figures (and Fig. 4.4 below) denote dimensionless ones rescaled by appropriate powers of U_T. The leftmost curve in these figures represents $U = U_T$.

We have also numerically calculated the quark condensate $c = c(r_\infty)$. Here we are interested in the phase structure of the system when the temperature T is varied for fixed quark mass m_q. The relation between T and r_∞ can be obtained from eqs. (4.8), (4.9), (4.48) as

$$T = \frac{\bar{M}}{\sqrt{r_\infty^{5-q}}}, \qquad \bar{M}^2 = \frac{(7-q)^2 m_q^{5-q} M_{KK}}{4(4\pi)^{\frac{5-q}{2}}\Gamma(\frac{7-q}{2}) g_q^2 N_c}, \tag{4.52}$$

where $g_q = g_{q+1}/\delta x^q$ is the q-dimensional Yang-Mills coupling and r_∞ is the dimensionless variable rescaled by U_T. Using this relation the quark condensate as a function of the temperature is plotted in Fig. 4.4. Note that the region near $T=0$ in these figures is not valid since the background (4.46) is dominant at low temperature $T < T_c$. All the condensates have similar behaviors to those of the D4/D6 model with $s=3$ discussed in refs. [19, 36, 38].

As was discussed in refs. [19, 36, 38] there are two types of embeddings. For sufficiently large r_∞ the probe brane does not reach the horizon $U = U_T$. On the other hand, for sufficiently small r_∞ it reaches the horizon $U = U_T$. For an intermediate region of r_∞ more than one embeddings, which can be either type of embeddings, are possible. The physically realized embedding is the one with a minimal energy. Varying the value of r_∞ a phase transition between these two types of embeddings occurs at a certain temperature $T = T_{\mathrm{fund}}$. This phase transition, however, does not affect the chiral symmetry of the quarks in QCD$_{s+1}$ because of the non-zero value of c for all temperature region except for $T \to \infty$.

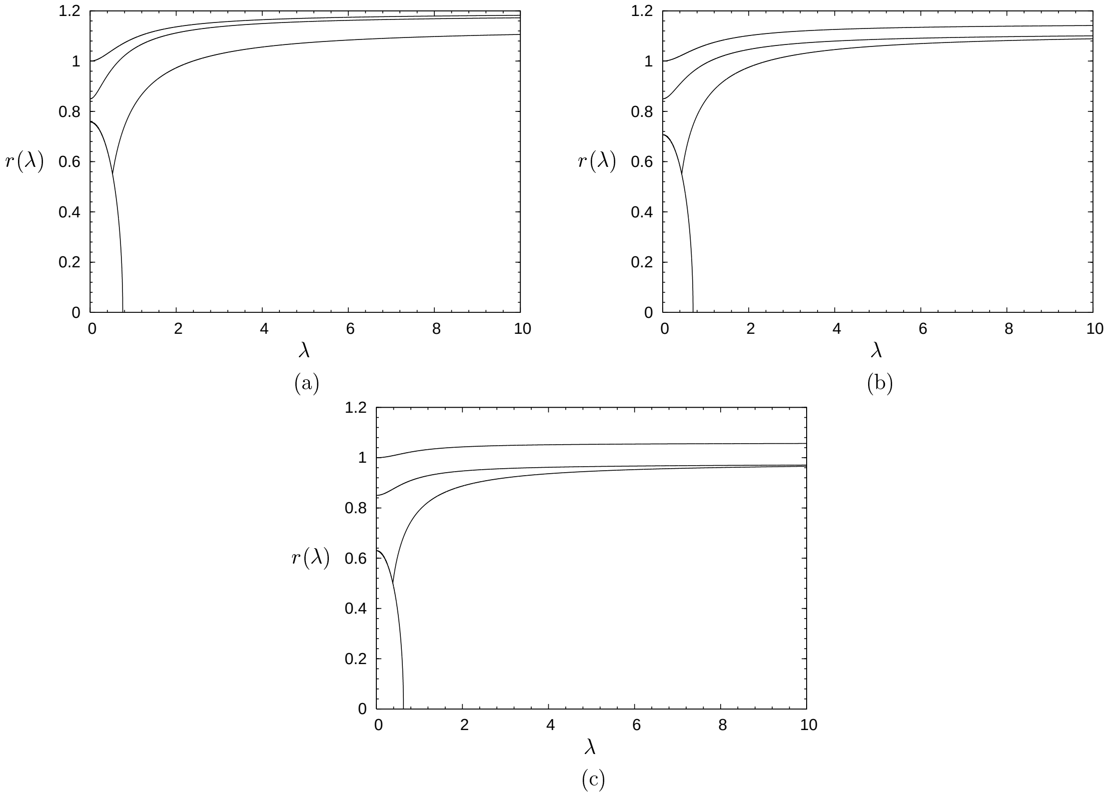
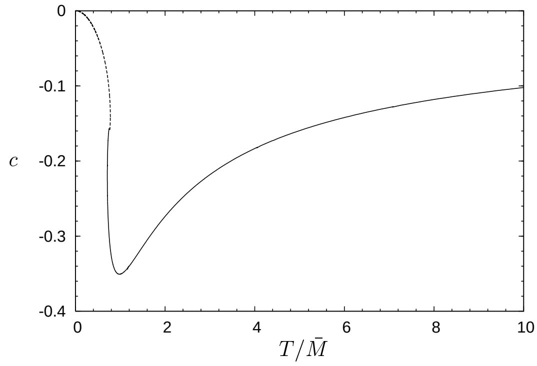


Figure 4.3: Solutions of eq. (4.51) for various values of r_∞ in (a) the D2/D4 model with $s = 1$, (b) the D3/D5 model with $s = 2$ and (c) the D4/D6 model with $s = 3$.

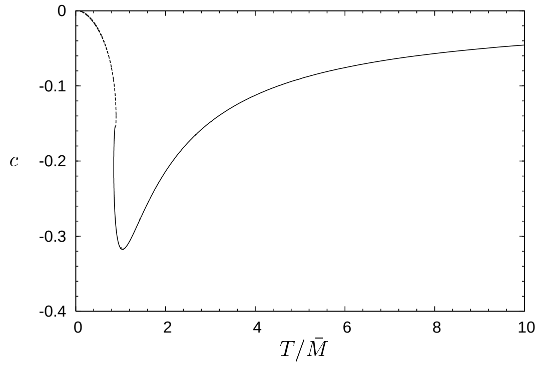
		0	1	2	3	4	5	6	7	8	9
color	D1	○	○	—	—	—	—	—	—	—	—
probe	D5	○	○	○	○	○	○	—	—	—	—
color	D2	○	○	○	—	—	—	—	—	—	—
probe	D6	○	○	○	○	○	○	○	—	—	—
color	D3	○	○	○	○	—	—	—	—	—	—
probe	D7	○	○	○	○	○	○	○	○	—	—

Table 4.2: The Dq/D$(q + 4)$ brane configurations with #ND = 4.

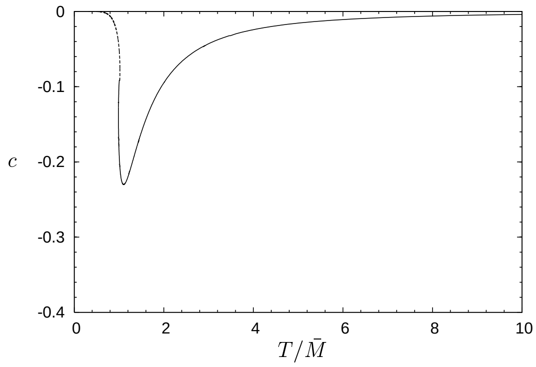
The above finite temperature analysis can be applied to another type of brane configurations. Here we consider a non-compact limit $M_{KK} \to 0$ of the Dq/D$(q+4)$ model with $s = q$ at finite temperature. At zero temperature it is dual to a supersymmetric gauge theory in $(q + 1)$-dimensions. The case $q = 3$ is the D3/D7 model at finite temperature discussed in refs. [18, 36, 38]. The Dq/D$(q + 4)$ configurations for $q = 1$, 2, 3 are shown in Table 4.2. The rotational symmetry SO$(9 - p)$ in the directions transverse to both branes is interpreted as a chiral symmetry for certain cases. In particular, the SO$(4)_{6789}$ rotational symmetry of the D1/D5 model with $s = 1$ is regarded



(a)



(b)



(c)

Figure 4.4: The quark condensate as a function of temperature for (a) the D2/D4 model with $s = 1$, (b) the D3/D5 model with $s = 2$ and (c) the D4/D6 model with $s = 3$. The solid (dashed) lines represent the contributions from the embeddings which do (not) reach the horizon $U = U_T$.

as an $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry and the $\mathrm{SO}(2)_{89}$ rotational symmetry of the D3/D7 model with $s = 3$ is regarded as an axial $\mathrm{U}(1)_A$ symmetry [18].

In the present case there are two possible background geometries. One of them is the Euclidean $\mathrm{AdS}_{q+2} \times \mathrm{S}^{8-q}$, which is obtained by setting $U_{KK} = 0$ in the Euclidean version of the metric (4.7). The other is the Euclidean version of the Schwarzschild $\mathrm{AdS}_{q+2} \times \mathrm{S}^{8-q}$, which is given by the geometry (4.47) with non-compact x^q. The phase transition occurs between these two backgrounds [8]. The Euclidean $\mathrm{AdS}_{q+2} \times \mathrm{S}^{8-q}$ is dominant at low temperature, while the Euclidean Schwarzschild $\mathrm{AdS}_{q+2} \times \mathrm{S}^{8-q}$ is dominant at high temperature.

We consider the probe D(q+4)-brane dynamics in the high temperature phase. The induced metric and the equation of motion have the same form as (4.49) and (4.51) with $p = q + 4$, $s = q$. The conditions $\alpha = 0$, $p - s - 2 > 0$, $s \leq 3$ require $q = 1, 2, 3$. We have numerically solved eq. (4.51) for these configurations. The solutions for the D1/D5 model with $s = 1$, the D2/D6 model with $s = 2$ and the D3/D7 model with $s = 3$ are plotted in Fig. 4.5 for various values of r_∞. All the solutions have similar behaviors to those for the D3/D7 model with $s = 3$ (Fig. 4.5 (c)) [18, 36, 38]. We have

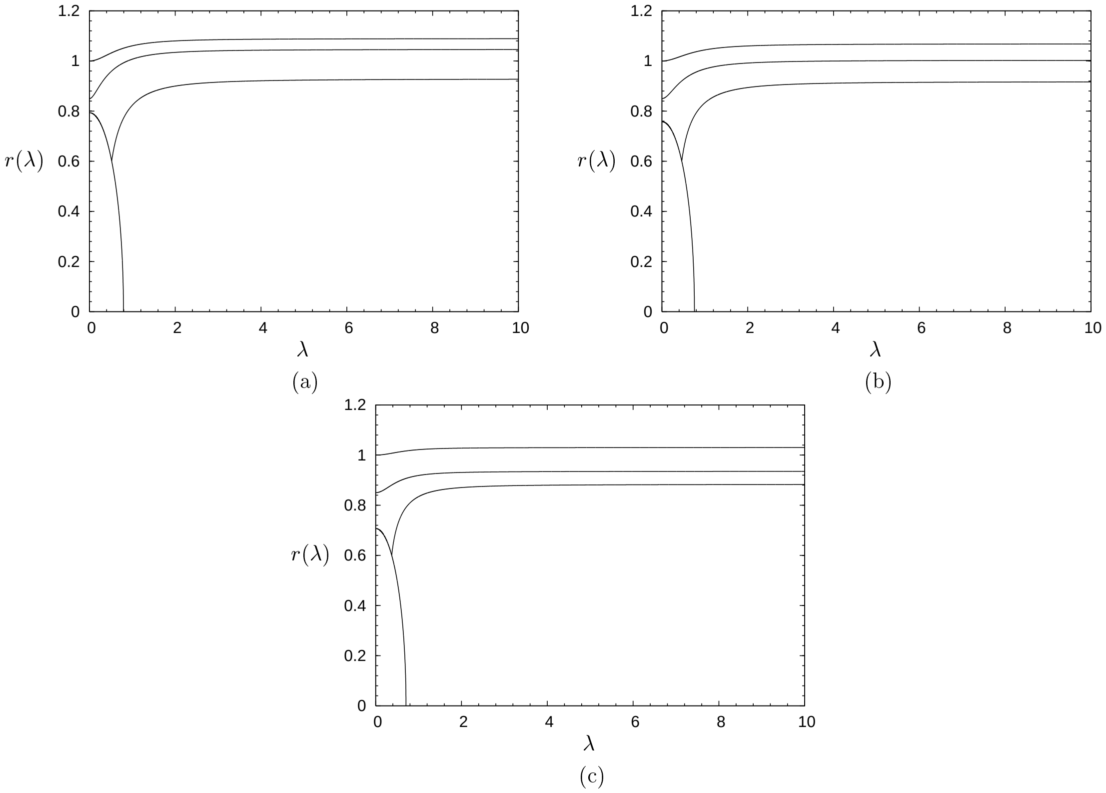


Figure 4.5: Solutions of eq. (4.51) for various values of r_∞ in (a) the D1/D5 model with $s = 1$, (b) the D2/D6 model with $s = 2$ and (c) the D3/D7 model with $s = 3$.

also numerically calculated the quark condensate as a function of the temperature. The results are plotted in Fig. 4.6. All the condensates have similar behaviors to those of the D3/D7 model with $s = 3$ (Fig. 4.6 (c)) discussed in refs. [18, 36, 38]

Finally we note that a chemical potential for the baryon number can be introduced by considering the U(1) gauge field A_M on the probe Dp-brane [39, 40]

$$S_{\mathrm{D}p} = T_p \int d^{p+1}x e^{-\phi} \sqrt{\det\left(g_{MN} + 2\pi\alpha' F_{MN}\right)} + S_{CS}, \tag{4.53}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength of the U(1) gauge field A_M on the Dp-brane world-volume. S_{CS} is the Chern-Simons (CS) term. In addition to the ansatz (4.16) we make an ansatz

$$A = A_0(\lambda) dt_E. \tag{4.54}$$

Then we can solve the equations of motion derived from the action (4.53) for $r(\lambda)$ and $A_0(\lambda)$. An asymptotically non-vanishing Euclidean time component of the U(1) gauge field can be understood as a chemical potential [39, 40]. The Dq/Dp brane models at finite chemical potential were discussed for certain sets of (q, p, s) in refs. [44, 45, 46, 47, 48, 49, 50, 51, 52]. It will be interesting to give a general discussion for the phase

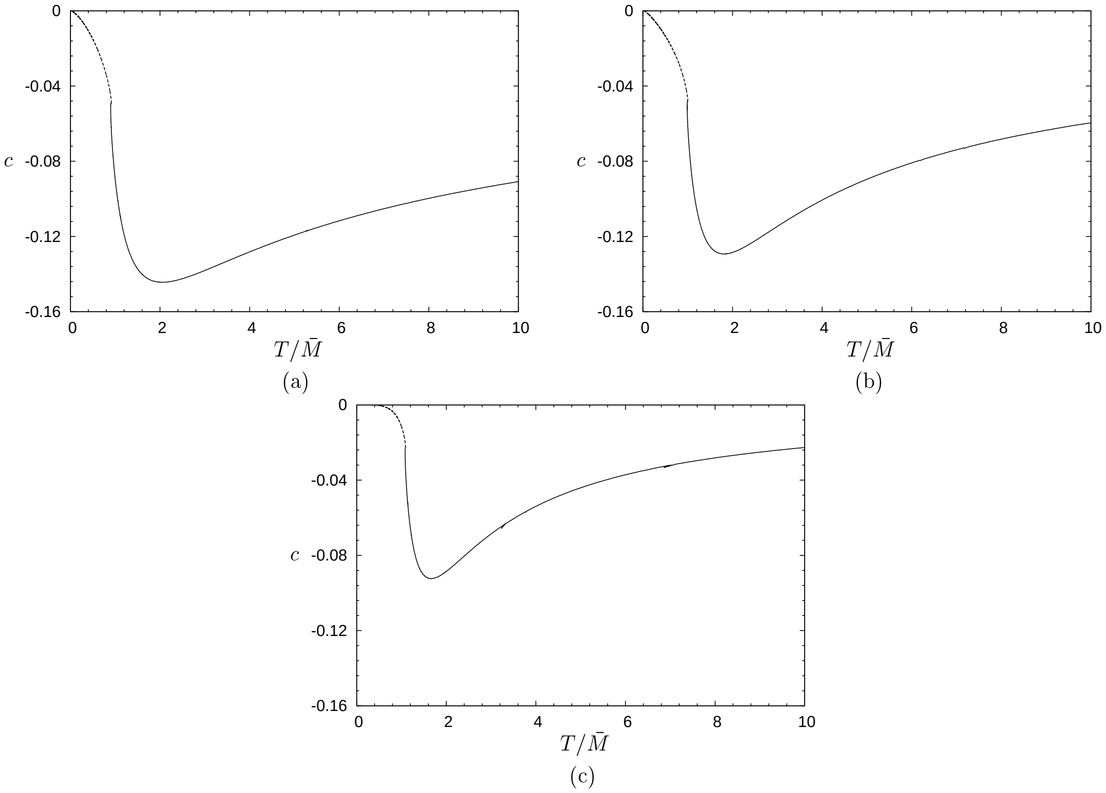


Figure 4.6: The quark condensate as a function of temperature for (a) the D1/D5 model with $s = 1$, (b) the D2/D6 model with $s = 2$ and (c) the D3/D7 model with $s = 3$. The solid (dashed) lines represent the contributions from the embeddings which do (not) reach the horizon $U = U_T$.

structure in μ-T space of the general intersecting Dq/Dp brane model by using this chemical potential.

Chapter 5

Dq/Dp-$\overline{\mathrm{D}p}$ brane model

As we saw in the previous chapter we could not construct QCD$_{s+1}$ having a manifest $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ chiral symmetry from the Dq/Dp brane model although we constructed QCD$_2$ having a non-Abelian chiral symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ from the D2/D4 model with $s = 1$. In order to realize a manifest $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ chiral symmetry we study a general intersecting Dq/Dp-$\overline{\mathrm{D}p}$ brane model [30].

5.1 General setup

We consider an intersecting brane system consisting of N_c color Dq-branes and N_f probe Dp-$\overline{\mathrm{D}p}$ brane pairs

	x^0	$\cdots$	x^s	x^{s+1}	$\cdots$	x^q	x^{q+1}	$\cdots$	x^{q+p-s}	$x^{q+p-s+1}$	$\cdots$	x^9
N_c Dq	$\circ$	$\cdots$	$\circ$	$\circ$	$\cdots$	$\circ$	$-$	$\cdots$	$-$	$-$	$\cdots$	$-$
N_f Dp-$\overline{\mathrm{D}p}$	$\circ$	$\cdots$	$\circ$	$-$	$\cdots$	$-$	$\circ$	$\cdots$	$\circ$	$-$	$\cdots$	$-$

(5.1)

with x^q being a coordinate of S^1. It has an s-dimensional intersection in the directions $x^1, \cdots, x^s$. We impose the supersymmetry breaking boundary condition along this S^1 as in the Dq/Dp model. The configuration (5.1) is a T-dual of Ds'/D9-$\overline{\mathrm{D}9}$ system with $s' = 9 - (q + p - 2s) \geq s$.

The configuration (5.1) has the following symmetries. The gauge symmetry of this system is $\mathrm{U}(N_c) \times \mathrm{U}(N_f)_{\mathrm{D}p} \times \mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$ in contrast with the Dq/Dp model. The $\mathrm{U}(N_f)_{\mathrm{D}p} \times \mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$ gauge symmetry on the Dp-$\overline{\mathrm{D}p}$-branes is regarded as a global symmetry of an $(s+1)$-dimensional field theory at the intersection. As in the Dq/Dp model the ten-dimensional Lorentz symmetry $\mathrm{SO}(1, 9)$ is broken to its subgroup by the configuration (5.1). Therefore the global symmetry preserved at the intersection contains

$$\mathrm{SO}(1, s) \times \mathrm{SO}(9 - q - p + s) \times \mathrm{U}(N_f)_{\mathrm{D}p} \times \mathrm{U}(N_f)_{\overline{\mathrm{D}p}}. \tag{5.2}$$

In contrast with the symmetry (4.2) in the Dq/Dp model there is a $\mathrm{U}(N_f)_{\mathrm{D}p} \times \mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$ symmetry.

The spectrum of the theory localized at the intersection is as follows. Fields generated by q-q and q-p strings are the same as those in the Dq/Dp model. As in the previous chapter we only consider configurations having #ND ≥ 4. Then there are only massless fermions, called quarks, from R sector of q-p and q-$\bar{p}$ strings. Note that quarks generated by q-$\bar{p}$ strings have opposite chirality of $\mathrm{SO}(1,s)$ to that of quarks generated by q-p strings. These quarks belong to the fundamental representation of $\mathrm{U}(N_c)$. Quarks generated by q-p strings (q-$\bar{p}$ strings) transform as $(\boldsymbol{N_f},\mathbf{1})$ $((\mathbf{1},\boldsymbol{N_f}))$ under $\mathrm{U}(N_f)_{\mathrm{D}p}\times\mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$. Therefore we can regard the symmetry $\mathrm{U}(N_f)_{\mathrm{D}p}\times\mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$ as a flavor chiral symmetry $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$ of quarks. Therefore we may construct QCD_{s+1} having a manifest chiral symmetry $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$.

5.2 An interesting case: D4/D8-$\overline{\text{D8}}$ brane model

There is an interesting configuration which is dual to QCD_4 having a manifest $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$ chiral symmetry. It is the D4/D8-$\overline{\text{D8}}$ brane system with $s=3$ called the Sakai-Sugimoto model [24, 25]. The brane configuration of the system is

	t	x^1	x^2	x^3	τ	U	θ^1	θ^2	θ^3	θ^4
N_c D4	○	○	○	○	○	–	–	–	–	–
N_f D8-$\overline{\text{D8}}$	○	○	○	○	–	○	○	○	○	○

(5.3)

with $\tau\equiv x^4$ and θ's being coordinates of S^1 and S^4 respectively. The period of τ is denoted as $\delta\tau=2\pi/M_{KK}$. We can explicitly write down the symmetry (5.2) for the configuration (5.3). The GSO projection for the R sector of 4-8 strings (4-$\bar{8}$ strings) requires that only left-handed quarks (right-handed quarks) of $\mathrm{SO}(1,3)_{0123}$ are physical states. Left-handed (right-handed) quarks transform as $(\boldsymbol{N_f},\mathbf{1})$ $((\mathbf{1},\boldsymbol{N_f}))$ under $\mathrm{U}(N_f)_{\mathrm{D}p}\times\mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$. Then the symmetry $\mathrm{U}(N_f)_{\mathrm{D}p}\times\mathrm{U}(N_f)_{\overline{\mathrm{D}p}}$ can be regarded as a flavor chiral symmetry $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$ of quarks. Therefore the global symmetry of the theory at the intersection contains

$$\mathrm{SO}(1,3)\times\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R. \tag{5.4}$$

5.2.1 D4/D8-$\overline{\text{D8}}$ brane model at finite temperature and finite chemical potential

The D4/D8-$\overline{\text{D8}}$ brane model with $s=3$ gives a holographic dual of QCD which contains N_f flavored left and right-handed quarks in the fundamental representation of $\mathrm{U}(N_c)$. The model at zero temperature was well studied by Sakai and Sugimoto [24, 25]. So we concentrate on the model at finite temperature and finite chemical potential. The finite temperature behavior of the Sakai-Sugimoto model was discussed in [34, 35, 37].

In the large N_c limit and the near horizon limit the D4-branes are described by a bulk background geometry, which is a classical solution of the type IIA supergravity in ten dimensions. Assuming $N_f\ll N_c$ the D8-$\overline{\text{D8}}$ pairs are treated as a probe which does not affect the bulk background. The bulk background geometry is represented by a metric with a periodic Euclidean time coordinate $t_E \equiv it \sim t_E + \delta t_E$ in addition to the periodic τ. The period of t_E is the inverse temperature $\delta t_E = 1/T$. There are two such Euclidean solutions which have an appropriate asymptotic boundary behavior. One of them is the Euclidean version of the S^1 compactified D4-brane geometry, which is given by the metric (4.46) with $q = 4$

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(dt_E^2 + \sum_{i=1}^{3}(dx^i)^2 + f(U)d\tau^2\right) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \\ & f(U) = 1 - \frac{U_{KK}^3}{U^3}, \qquad U_{KK} = \frac{4}{9}R^3 M_{KK}^2, \end{aligned} \tag{5.5}$$

where $d\Omega_4^2$ is the metric of S^4 and $R^3 = \pi g_s N_c l_s^3$. The parameter U_{KK} must be related to M_{KK} as above to avoid a singularity of the metric at $U = U_{KK}$. With this relation the τ-U submanifold has a cigar-like form with a tip at $U = U_{KK}$. The dilaton ϕ and the RR 3-form C_3 are given by eqs. (3.17) with $q = 4$

$$e^\phi = g_s \left(\frac{U}{R}\right)^{\frac{3}{4}}, \qquad F_4 = dC_3 = \frac{2\pi N_c}{V_4}\epsilon_4, \tag{5.6}$$

where ϵ_4 and V_4 are the volume form and the volume of S^4. The other solution is the Euclidean version of the non-extremal D4-brane geometry, which is given by the metric (4.47) with $q = 4$

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(\tilde{f}(U)dt_E^2 + \sum_{i=1}^{3}(dx^i)^2 + d\tau^2\right) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \left(\frac{dU^2}{\tilde{f}(U)} + U^2 d\Omega_4^2\right), \\ & \tilde{f}(U) = 1 - \frac{U_T^3}{U^3}, \qquad U_T = \frac{16\pi^2}{9}R^3 T^2 \end{aligned} \tag{5.7}$$

with the dilaton and the RR 3-form given in eq. (5.6). The parameter U_T must be related to T as above to avoid a singularity of the metric at $U = U_T$. The t_E-U submanifold has a cigar-like form with a tip at $U = U_T$. It is obvious that these two backgrounds are related by interchanging τ, U_{KK} and t_E, U_T. A confinement/deconfinement phase transition between these backgrounds occurs at the temperature $T_c = M_{KK}/(2\pi)$ as in the Dq/Dp brane model.

In refs. [34, 35] N_f D8-$\overline{\text{D8}}$ pairs were introduced as a probe in the backgrounds (5.5), (5.7). The effective action of the D8-branes consists of the DBI action and the CS term

$$S_{\text{D8}} = T_8 \int d^9x\, e^{-\phi}\, \mathrm{Tr}\sqrt{\det(g_{MN} + 2\pi\alpha' F_{MN})} - \frac{i}{48\pi^3}\int C_3\, \mathrm{Tr}F^3, \tag{5.8}$$

where g_{MN} and $F_{MN} = \partial_M A_N - \partial_N A_M - i\,[A_M, A_N]$ $(M, N = 0, 1, \cdots, 8)$ are the induced metric and the field strength of the $\mathrm{U}(N_f)$ gauge field A_M on the D8-branes. T_8 is the tension of the D8-brane. The effective action for the $\overline{\text{D8}}$-branes has a similar form. The total effective action has a gauge symmetry

$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A, \tag{5.9}$$

where $\mathrm{U}(N_f)_L$ and $\mathrm{U}(N_f)_R$ are symmetries of N_f D8 and $\overline{\text{D8}}$-branes respectively. It was argued in ref. [24] that this gauge symmetry corresponds to a flavor chiral symmetry of quarks. The total effective action can be written in the form (5.8) with the integrations being over the whole of the D8-$\overline{\text{D8}}$ world-volume. We use this form of the effective action in the following.

In refs. [24, 34] the gauge fields on the probe branes are treated as fluctuations representing the hadron spectrum. In this paper we consider a background gauge field. We assume that only the Euclidean time component of the U(1) gauge field has a non-vanishing background. We will see that it corresponds to an introduction of the baryon number chemical potential. We use a physical gauge for D8-brane world-volume reparametrizations and use the spacetime coordinates other than τ as the world-volume coordinates. Then, D8 and $\overline{\text{D8}}$-brane configurations are determined by τ as a function of those world-volume coordinates. We make an ansatz that A_0 and τ depend only on the coordinate U

$$\tau = \tau(U), \qquad A_0 = A_0(U). \tag{5.10}$$

By this ansatz the CS term in eq. (5.8) vanishes and does not concern us.

5.2.2 Low temperature phase

In the low temperature phase the geometry (5.5) is dominant. Using the ansatz (5.10) the induced metric g_{MN} on the probe D8-branes is given by

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(dt_E^2 + \sum_{i=1}^{3}(dx^i)^2\right) \\ &\quad + \left[\left(\frac{U}{R}\right)^{\frac{3}{2}} f(U)(\tau'(U))^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} \frac{1}{f(U)}\right] dU^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} U^2 d\Omega_4^2, \end{aligned} \tag{5.11}$$

where $\tau' = \frac{d\tau}{dU}$. Then, the effective action of the D8-branes (5.8) becomes

$$S_{\text{D8}} = \frac{N_f T_8 V_4}{g_s} \int d^4x\, dU U^4 \left[f\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(f^{-1} - (2\pi\alpha' A_0')^2\right)\right]^{\frac{1}{2}}, \tag{5.12}$$

where $A_0' = \frac{dA_0}{dU}$.

This action leads to equations of motion for $\tau(U)$ and $A_0(U)$

$$\begin{aligned} \frac{d}{dU}\left[\frac{U^4 f\,\tau'}{\sqrt{f\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(f^{-1} - (2\pi\alpha' A_0')^2\right)}}\right] &= 0, \\ \frac{d}{dU}\left[\frac{U^4 \left(\frac{R}{U}\right)^3 A_0'}{\sqrt{f\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(f^{-1} - (2\pi\alpha' A_0')^2\right)}}\right] &= 0, \end{aligned} \tag{5.13}$$

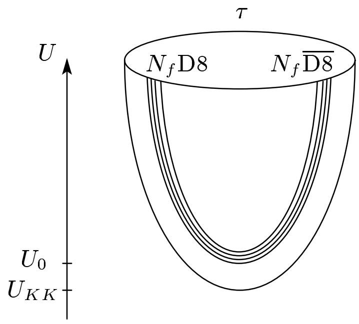


Figure 5.1: A D8-$\overline{\text{D8}}$-brane configuration in the low temperature phase.

which can be easily integrated once. We obtain

$$
\begin{aligned}
(\tau'(U))^2 &= \frac{\left(U_0^8 + C^2\left(\frac{U_0}{R}\right)^3\right) f(U_0)\left(\frac{R}{U}\right)^6}{f(U)^2\left[\left(\frac{R}{U}\right)^3 (U^8 f(U) - U_0^8 f(U_0)) + C^2\left(f(U) - f(U_0)\left(\frac{U_0}{U}\right)^3\right)\right]}, \\
(2\pi\alpha' A_0'(U))^2 &= \frac{C^2}{\left(\frac{R}{U}\right)^3 (U^8 f(U) - U_0^8 f(U_0)) + C^2\left(f(U) - f(U_0)\left(\frac{U_0}{U}\right)^3\right)},
\end{aligned}
\tag{5.14}
$$

where C and U_0 are integration constants. As in the zero temperature case [24] and the low temperature phase in ref. [34] we have imposed a condition $\tau'(U_0) = \infty$. A typical configuration of $\tau(U)$ is shown in Fig. 5.1. Since there is no place for the D8 and $\overline{\text{D8}}$-branes to end, they are connected at $U = U_0$. We also impose the boundary condition $A_0(\infty) = \mu$ at the both ends of the D8-$\overline{\text{D8}}$ world-volume, where μ is a constant. We will identify this constant with the chemical potential for the baryon number later. Solving eq. (5.14) with this boundary condition we find for $U \sim U_0$

$$A_0(U) \sim A_0(U_0) + \text{const.} \times C\,|U - U_0|^{\frac{1}{2}}. \tag{5.15}$$

This solution is singular at $U = U_0$ and does not actually satisfy the original equation (5.13) unless $C = 0$. Therefore, we must choose $C = 0$ and obtain $A_0(U) = \mu$. Because of the connected configuration of the D8 and $\overline{\text{D8}}$-branes the chiral symmetry $\text{U}(N_f)_L \times \text{U}(N_f)_R$ on the probe D8-$\overline{\text{D8}}$ pairs is always broken to a diagonal subgroup $\text{U}(N_f)_V$ in the low temperature phase. The situation is the same as in the cases without the gauge field background [24, 34].

Instead of using the constant U_0 to parametrize the solution we can also use the $U = \infty$ asymptotic separation L between the D8 and $\overline{\text{D8}}$-branes in the τ-direction. It is related to U_0 by

$$L = 2\int_{U_0}^{\infty} dU\,\tau'(U), \tag{5.16}$$

where $\tau'(U)$ is given in eq. (5.14) with $C = 0$.

Substituting eq. (5.14) with $C = 0$ into the action (5.12) and introducing new variables $u = U/U_0$, $u_{KK} = U_{KK}/U_0$ and $f(u) = 1 - u_{KK}^3/u^3$ the effective action becomes

$$S_{\rm D8} = \bar{T}_8 \int_1^\infty du\, u^5 \sqrt{\frac{u^3}{u^8 f(u) - f(1)}}, \tag{5.17}$$

where

$$\bar{T}_8 = \frac{N_f T_8 V_4}{g_s} \left(R^3 U_0^7\right)^{\frac{1}{2}} \int d^4x. \tag{5.18}$$

Note that this reproduces the result in ref. [34].

5.2.3 High temperature phase

In the high temperature phase the geometry (5.7) is dominant. Using the ansatz (5.10) the induced metric on the probe D8-branes is

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(\tilde{f}(U) dt_E^2 + \sum_{i=1}^{3} (dx^i)^2\right) \\ &\quad + \left[\left(\frac{U}{R}\right)^{\frac{3}{2}} (\tau'(U))^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} \frac{1}{\tilde{f}(U)}\right] dU^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} U^2 d\Omega_4^2 \end{aligned} \tag{5.19}$$

and the effective action of the D8-branes (5.8) becomes

$$S_{\rm D8} = \frac{N_f T_8 V_4}{g_s} \int d^4x\, dU U^4 \left[\tilde{f}\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(1 - (2\pi\alpha' A_0')^2\right)\right]^{\frac{1}{2}}. \tag{5.20}$$

This action leads to equations of motion for $\tau(U)$ and $A_0(U)$

$$\begin{aligned} \frac{d}{dU}\left[\frac{U^4 \tilde{f}\,\tau'}{\sqrt{\tilde{f}\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(1 - (2\pi\alpha' A_0')^2\right)}}\right] &= 0, \\ \frac{d}{dU}\left[\frac{U^4 \left(\frac{R}{U}\right)^3 A_0'}{\sqrt{\tilde{f}\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(1 - (2\pi\alpha' A_0')^2\right)}}\right] &= 0, \end{aligned} \tag{5.21}$$

which can be easily integrated once as before. As in the case without the gauge field [34, 35] there are two types of solutions in the high temperature phase.

One solution is similar to the one in the low temperature phase. The integration of eq. (5.21) gives

$$(\tau'(U))^2 = \frac{U_0^8 \tilde{f}(U_0)}{\left(\frac{U}{R}\right)^3 \tilde{f}(U) \left(U^8 \tilde{f}(U) - U_0^8 \tilde{f}(U_0)\right)}, \qquad A_0(U) = \mu. \tag{5.22}$$

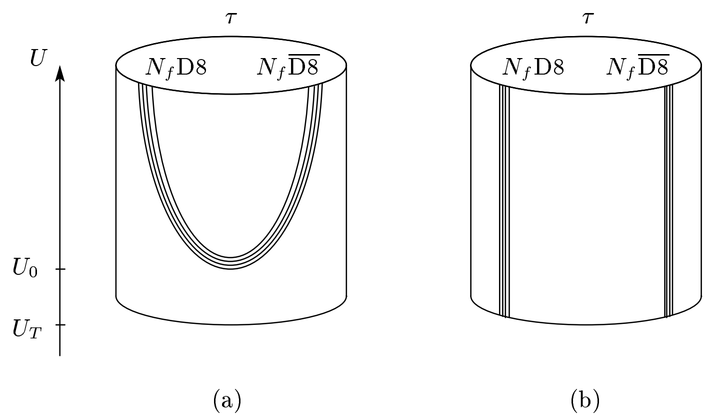


Figure 5.2: D8-$\overline{\text{D8}}$-brane configurations in the high temperature phase.

where U_0 is an integration constant. As before we have imposed the boundary conditions $\tau'(U_0) = \infty$ and $A_0(\infty) = \mu$. A typical configuration of $\tau(U)$ is shown in Fig. 5.2 (a). The chiral symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ is broken to a diagonal subgroup $\mathrm{U}(N_f)_V$. Substituting eq. (5.22) into eq. (5.20) the effective action becomes

$$S^{\mathrm{U}}_{\mathrm{D8}} = \bar{T}_8 \int_1^\infty du\ u^5 \sqrt{\frac{u^3 \tilde{f}(u)}{u^8 \tilde{f}(u) - \tilde{f}(1)}}, \tag{5.23}$$

where we have rescaled the variables as $u = U/U_0$, $u_T = U_T/U_0$, $\tilde{f}(u) = 1 - u_T^3/u^3$, and $\bar{T}_8$ is given in eq. (5.18).

Instead of using U_0 we can also use the asymptotic separation L in eq. (5.16) to parametrize the solution, which is more convenient when comparing this solution to the other one. The relation between L and U_0 is obtained from eqs. (5.16) and (5.22) as

$$L = \left(\frac{R^3}{U_0}\right)^{\frac{1}{2}} F(u_T), \tag{5.24}$$

where

$$F(u_T) = 2\int_1^\infty du \sqrt{\frac{\tilde{f}(1)}{u^3 \tilde{f}(u)\left(u^8 \tilde{f}(u) - \tilde{f}(1)\right)}}. \tag{5.25}$$

For the other solution the first integration of eq. (5.21) gives

$$\tau'(U) = 0, \qquad (2\pi\alpha' A_0'(U))^2 = \frac{C^2}{U^8 \left(\frac{R}{U}\right)^3 + C^2}, \tag{5.26}$$

where C is an integration constant. $\tau'(U) = 0$ is the trivial solution of (5.21). A typical configuration is shown in Fig. 5.2 (b). It describes a situation that the probe D8 and $\overline{\text{D8}}$-branes separately extend along the U-direction in straight lines. The separation between the D8 and $\overline{\text{D8}}$-branes is chosen to be the same as the asymptotic separation

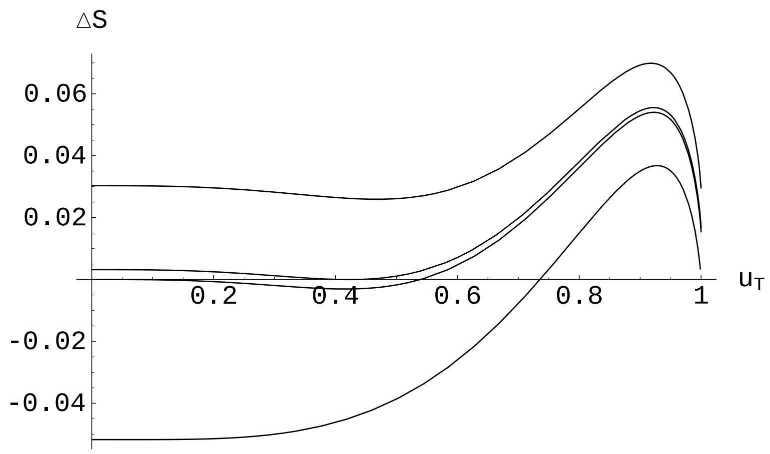


Figure 5.3: ΔS as a function of u_T for various values of c. From the bottom to the top each line represents the case for $c = 0$, 0.2158, 0.2252, 0.3 respectively.

L in the previous solution. The chiral symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ is unbroken in this case. Substituting eq. (5.26) into eq. (5.20) and using the rescaled variables as in eq. (5.23) the effective action becomes

$$S^{||}_{\mathrm{D8}} = \bar{T}_8 \int_{u_T}^{\infty} du \, \frac{u^5}{\sqrt{u^5 + c^2}}, \tag{5.27}$$

where

$$c^2 = \frac{C^2}{R^3 U_0^5}. \tag{5.28}$$

To determine which of the two solutions is dominant we compare the values of the effective action. From eqs. (5.23), (5.27) we obtain the difference as

$$\begin{aligned} \Delta S \;&\equiv\; \frac{S^{\mathrm{U}}_{\mathrm{D8}} - S^{||}_{\mathrm{D8}}}{\bar{T}_8} \\ &=\; \int_1^{\infty} du \; u^5 \left[\sqrt{\frac{u^3 \tilde{f}(u)}{u^8 \tilde{f}(u) - \tilde{f}(1)}} - \frac{1}{\sqrt{u^5 + c^2}} \right] - \int_{u_T}^{1} du \; \frac{u^5}{\sqrt{u^5 + c^2}}. \end{aligned} \tag{5.29}$$

For $\Delta S < 0$ the curved configuration (5.22) is dominant and the chiral symmetry is broken, while for $\Delta S > 0$ the straight configuration (5.26) is dominant and the chiral symmetry is unbroken. Although the integrals in eqs. (5.23), (5.27) are divergent at $U = \infty$, the difference is finite due to the same asymptotic behaviors of $\tau(U)$ and $A_0(U)$. We evaluate eq. (5.29) by numerical calculations. For that purpose it is more convenient to change an integration variable to $z = u^{-3}$, which has a finite interval $0 \le z \le 1$ for $1 \le u < \infty$. The result of the calculations is shown in Fig. 5.3. The behaviors of ΔS as a function of u_T for various values of c are given. The special

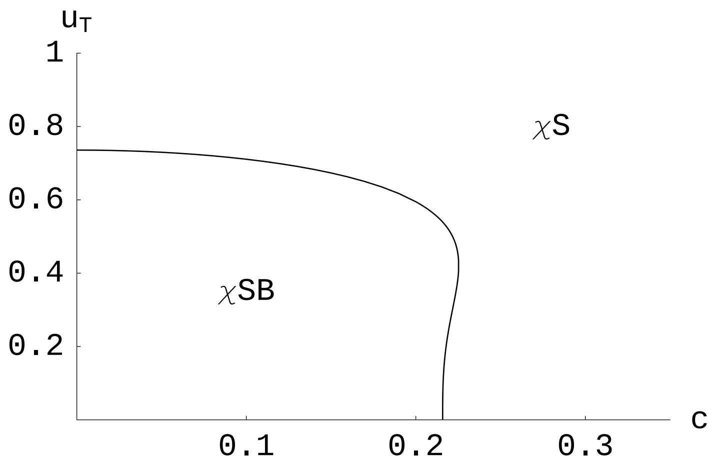


Figure 5.4: The phase diagram in the c-u_T space.

case $c = 0$ reduces to the result in ref. [34]. In this case ΔS is positive for u_T larger than a certain value u_{T0} and negative for $u_T < u_{T0}$. The chiral symmetry is broken for $u_T < u_{T0}$ and unbroken for $u_T > u_{T0}$. The point $u_T = u_{T0}$ is a phase transition point. This phase transition is of the first order since two different configurations in Fig. 5.2 are possible at the transition point. As c increases, the transition point u_{T0} decreases. When $c > 0.2158$, there appears a new region near $u_T = 0$ in which $\Delta S > 0$. When $c > 0.2252$, ΔS is positive for all values of u_T and the chiral symmetry is always unbroken.

From these results we can draw a phase diagram in the c-u_T space as shown in Fig. 5.4. The chiral symmetry is broken in the region of small c and small u_T and unbroken outside of it. Note that we are considering here the high temperature phase of the confinement/deconfinement transition and only the part $u_T > u_{KK}$ of this diagram is valid.

It is more appropriate, however, to draw it in the space of the temperature T and the baryon number chemical potential μ. From eq. (5.7) the temperature T is related to u_T as

$$T = \frac{3}{4\pi}\left(\frac{U_0}{R^3}\right)^{\frac{1}{2}}\sqrt{u_T} = \frac{3}{4\pi}\frac{\sqrt{u_T}}{L}F(u_T), \tag{5.30}$$

where we have used eq. (5.24).

The relation of the chemical potential μ to u_T and c can be obtained as follows. From eq. (5.26) the large U behavior of $A_0(U)$ has a form

$$A_0(U) \sim \mu + \frac{v}{U^{\frac{3}{2}}}, \tag{5.31}$$

where μ and v are constants. We have chosen the same value μ for the constant term as in the curved solution (5.22). According to the AdS/CFT dictionary [5] for a massless vector field in a six-dimensional bulk, μ is a source coupled to an operator of dimension four $\mathcal{O}_4$ on a five-dimensional boundary. The U(1) gauge field A_0 defined on the whole of the D8-$\overline{\text{D8}}$ world-volume contains the gauge fields for both of $\text{U}(1)_V$ and $\text{U}(1)_A$ in the flavor symmetry (5.9). The part of A_0 which is symmetric for an interchange of D8 and $\overline{\text{D8}}$ corresponds to $\text{U}(1)_V$, while the part which is antisymmetric corresponds to $\text{U}(1)_A$ [24, 39]. Since the constant term μ is symmetric, it is a background value of the $\text{U}(1)_V$ gauge field coupled to the baryon number density $\mathcal{O}_4$, and μ is the baryon number chemical potential.

Integration of eq. (5.26) determines $A_0(U)$ up to a constant term (μ in eq. (5.31)). We can fix this constant term as follows. We first require that $A_0(U)$ vanishes at $U = U_T$ because of the regularity. To see this we first change the coordinates from (U, t_E) to (r, θ) defined by

$$U^3 = U_T^3 + U_T r^2, \qquad \theta = \frac{3}{2}\left(\frac{U_T}{R^3}\right)^{\frac{1}{2}} t_E. \tag{5.32}$$

From the induced metric (5.19) with $\tau'(U) = 0$ we see that (r, θ) are the polar coordinates near the point $U = U_T$. The point $U = U_T$ corresponds to the origin $r = 0$ and should be treated with care since the polar coordinates are not good coordinates near the origin. It is better to use the Cartesian coordinates

$$y = r\cos\theta, \qquad z = r\sin\theta. \tag{5.33}$$

The relation between A_0 and the components A_y, A_z in the coordinates (y, z) is obtained from $A_0 dt_E = A_y dy + A_z dz$ as

$$A_0 = \frac{3}{2}\left(\frac{U_T}{R^3}\right)^{\frac{1}{2}} r\left(-A_y \sin\theta + A_z\cos\theta\right). \tag{5.34}$$

Since we require that A_y and A_z are regular at the origin $r = 0$, $A_0(U)$ must vanish at $U = U_T$. We also note that although $A_0(U)$ is a gauge dependent quantity, it must vanish at $U = U_T$ in any gauge. Only the gauge transformations which preserve the condition $A_0(U_T) = 0$ are allowed.

The vanishing of $A_0(U)$ at $U = U_T$ fixes the constant term in this case and we find

$$A_0(U) = \frac{U_0}{2\pi\alpha'}\int_{u_T}^{u} du' \sqrt{\frac{c^2}{u'^5 + c^2}}. \tag{5.35}$$

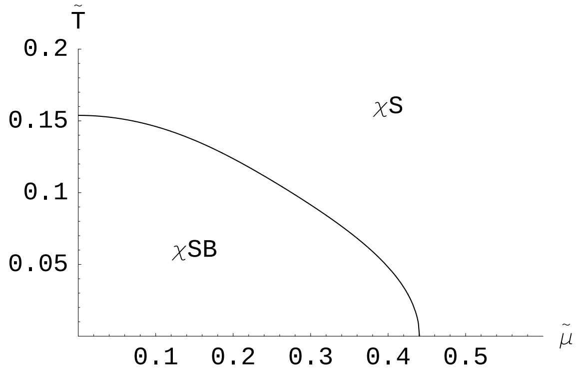
The chemical potential μ is obtained as the asymptotic value for $U = \infty$

$$\mu = A_0(\infty) = \frac{R^3}{2\pi\alpha' L^2}\left(F(u_T)\right)^2 \int_{u_T}^{\infty} du \sqrt{\frac{c^2}{u^5 + c^2}}, \tag{5.36}$$

where we have used eq. (5.24) to eliminate U_0. This gives an expression of the chemical potential in terms of u_T and c.

Using eqs. (5.30), (5.36) we can convert the phase diagram in Fig. 5.4 to that in the μ-T space by numerical calculations. Using dimensionless variables

$$\tilde{T} = LT, \qquad \tilde{\mu} = \frac{2\pi\alpha' L^2}{R^3}\mu \tag{5.37}$$



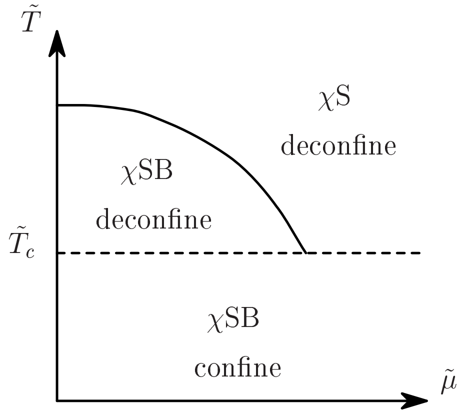


Figure 5.5: The phase diagram in the $\tilde{\mu}$-$\tilde{T}$ space.

Figure 5.6: The qualitative feature of the full phase diagram in the $\tilde{\mu}$-$\tilde{T}$ space.

the phase diagram in the $\tilde{\mu}$-$\tilde{T}$ space is shown in Fig. 5.5. Only the part of this diagram for the high temperature phase of the confinement/deconfinement transition, i.e. $\tilde{T} > \tilde{T}_c = LM_{KK}/(2\pi)$ is valid. Therefore, our result of the phase diagram looks like Fig. 5.6. The orders of T and μ at the transition points can be estimated from eq. (5.37). Using $R^3 = g_{YM}^2 N_c l_s^2/(2M_{KK})$ and $\tilde{\mu}, \tilde{T} = \mathcal{O}(1)$ we obtain

$$T = \mathcal{O}(L^{-1}), \qquad \mu = \mathcal{O}(g_{YM}^2 N_c L^{-2} M_{KK}^{-1}). \tag{5.38}$$

If we assume $L = \mathcal{O}(M_{KK}^{-1})$, the transition temperature is of the order of the compactification scale M_{KK}, and the chemical potential is of the order $g_{YM}^2 N_c M_{KK}$, which is much larger than M_{KK} since $g_{YM}^2 N_c \gg 1$.

Chapter 6

Summary

We discussed the chiral symmetry breaking in two different general intersecting brane models in the framework of the string/gauge duality. One of them is the Dq/Dp brane model consisting of N_c Dq-branes and a single probe Dp-brane with an s-dimensional intersection. The other is the Dq/Dp-$\overline{\mathrm{D}p}$ brane model in which we use N_f Dp-$\overline{\mathrm{D}p}$-brane pairs as probe brane instead of a single Dp-brane used in the Dq/Dp model. In both models we constructed $(s+1)$-dimensional QCD-like theories, QCD_{s+1}, at the intersection for certain (q, p, s) and obtained holographic descriptions of the chiral symmetry breaking. The chiral symmetry breaking can be understood as geometrical configurations of the probe branes.

In the Dq/Dp model approach we obtained QCD_2, QCD_3 and QCD_4 from the D2/D4 model with $s = 1$, the D3/D5 model with $s = 2$ and the D4/D6 model with $s = 3$, respectively. There is a rotational symmetry $\mathrm{SO}(9-q-p+s)$ of the directions transverse to Dq-branes and Dp-brane in the above models. We found that the rotational symmetry can be regarded as the chiral symmetry of QCD_{s+1} for certain cases. In particular this symmetry is non-Abelian $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ for QCD_2 and Abelian $\mathrm{U}(1)_A$ for QCD_4.

We studied the breaking of the chiral symmetry of QCD_{s+1} at zero and finite temperature in terms of the string/gauge duality. At zero temperature we found that the Dp-brane embeddings in the Dq-brane background geometry break the rotational symmetry. This corresponds to the chiral symmetry breaking in QCD_{s+1}. We calculated the quark mass m_q dependence of the quark condensate $\langle\bar{\psi}\psi\rangle$ numerically from the asymptotic behavior of the vacuum embedding and found a spontaneous breaking of the chiral symmetry.

We studied the fluctuations around the vacuum embeddings. We found that the fluctuations of the angular coordinates of the vacuum embedding are regarded as $(8-q-p+s)$ massless NG bosons associated with the spontaneous symmetry breaking $\mathrm{SO}(9-q-p+s) \to \mathrm{SO}(8-q-p+s)$. If quarks have a small mass they are pseudo-NG bosons with a non-vanishing mass. We showed that the mass of these pseudo-NG bosons satisfies the GMOR relation by the supergravity calculations. We also obtained the effective action of the NG bosons at quartic order.

At finite temperature we found that the rotational symmetry in the transverse space is also broken by the vacuum embedding as in the zero temperature case. This corresponds to the chiral symmetry breaking in QCD$_{s+1}$. We found a non-zero quark condensate except for the high temperature limit. At the high temperature limit the quark condensate vanishes and the chiral symmetry is restored. It is interesting to introduce chemical potential into the Dq/Dp model as well as temperature and discuss the phases of the dual gauge theories.

In the alternative approach, the Dq/Dp-$\overline{\mathrm{D}p}$ model, we especially studied the D4/D8-$\overline{\mathrm{D8}}$ model with $s = 3$. The great advantage of the D4/D8-$\overline{\mathrm{D8}}$ model is that we can construct QCD$_4$ with a manifest $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ chiral symmetry. The chiral symmetry is realized as the gauge symmetry $\mathrm{U}(N_f)_{\mathrm{D8}} \times \mathrm{U}(N_f)_{\overline{\mathrm{D8}}}$ on the N_f D8-$\overline{\mathrm{D8}}$-brane pairs. We studied this model at finite temperature and finite chemical potential. The chemical potential was introduced as a non-vanishing asymptotic value of the time component of the U(1) gauge field on the probe brane pairs. We studied the dynamics of the probe D8-$\overline{\mathrm{D8}}$-brane pairs in the D4-brane background geometry in terms of the string/gauge duality as in the Dq/Dp brane model. In the low temperature phase we found only U-shaped embeddings for which D8-branes and $\overline{\mathrm{D8}}$-branes are connected each other. This corresponds to the chiral symmetry breaking in QCD$_4$. Thus the chiral symmetry is always broken in the low temperature phase. In the high temperature phase another type of embeddings is possible. They are straight disconnected D8 and $\overline{\mathrm{D8}}$-brane embeddings. Such configurations respect the chiral symmetry. Comparing the values of the Euclidean action for each type of embeddings in the high temperature phase, we found a chiral phase transition of the first order. We obtained the phase diagram of QCD$_4$ in μ-T space.

It seems that we have good holographic descriptions of QCD. So far the D4/D8-$\overline{\mathrm{D8}}$ model is the best candidate for the holographic dual of QCD. There, however, are still many things to be resolved. For example, it is not clear how to introduce quark mass to the D4/D8-$\overline{\mathrm{D8}}$ model. In the D4/D8-$\overline{\mathrm{D8}}$ model quarks are always massless since there is no direction in which we can separate D4-branes and D8-$\overline{\mathrm{D8}}$-branes. For work toward an introduction of a quark mass to the model see refs. [76, 77, 78, 79, 80]. As for the finite temperature analysis, the phase diagram which we obtained (Fig. 5.6) seems different from the one expected from QCD (particularly in the low temperature phase) [81]. This is because of the probe approximation. We have ignored the contribution from the probe branes to the background geometry. The analysis including the backreaction of the probe branes is needed to clarify the phase structure bellow the temperature of the confinement/deconfinement phase transition. Therefore it can be said that the studies toward the real holographic dual of QCD have only just begun!

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Appendix A

Fluctuation of the radial part

In this Appendix we discuss the fluctuation δr in eq. (4.30). The action for δr can be obtained from eq. (4.32) as

$$
\begin{aligned}
S_{\delta r} \;=\; & -\tilde{T}_p \int d^{p+1}x \sqrt{\det \gamma_{\alpha\beta}}\; \lambda^{p-s-1}\sqrt{1+(r'_{\rm vac})^2} \\
& \times \left[\left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta} \frac{K}{2} g^{MN} \frac{\partial_M \delta r \partial_N \delta r}{1+(r'_{\rm vac})^2} + \frac{\partial}{\partial r_{\rm vac}}\left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta} \frac{r'_{\rm vac}\partial_\lambda (\delta r)^2}{2(1+(r'_{\rm vac})^2)} \right. \\
& \left. + \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta-2} \left(\frac{U_{KK}^{7-q}(c_1 r_{\rm vac}^2 + c_2\lambda^2)}{8\rho_{\rm vac}^{11-q}} + \frac{U_{KK}^{2(7-q)}(c_3 r_{\rm vac}^2 + c_2\lambda^2)}{32\rho_{\rm vac}^{18-2q}} \right) (\delta r)^2 \right],
\end{aligned}
\tag{A.1}
$$

where c's are constants

$$
c_1 = \beta(7-q)(8-q), \quad c_2 = -\beta(7-q), \quad c_3 = \beta(7-q)[\beta(7-q)+1]. \tag{A.2}
$$

From this action we obtain the equation of motion for δr

$$
\begin{aligned}
& \left(\frac{7-q}{2}\right)^2 \frac{U_{KK}^{5-q}}{M_{KK}^2} \rho_{\rm vac}^{-(7-q)} \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta-\frac{2(5-q)}{7-q}} \frac{\lambda^{p-s-1}}{\sqrt{1+(r'_{\rm vac})^2}}\,\partial_\mu\partial^\mu \delta r \\
& + \frac{\partial}{\partial\lambda}\left[\left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta} \frac{\lambda^{p-s-1}\partial_\lambda \delta r}{(1+(r'_{\rm vac})^2)^{\frac{3}{2}}} \right] + \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta} \frac{\lambda^{p-s-3}}{\sqrt{1+(r'_{\rm vac})^2}} \nabla^2 \delta r \\
& + \frac{\partial}{\partial\lambda}\left[\frac{\partial}{\partial r_{\rm vac}} \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta} \frac{\lambda^{p-s-1} r'_{\rm vac}}{\sqrt{1+(r'_{\rm vac})^2}} \right] \delta r \\
& - \lambda^{p-s-1}\sqrt{1+(r'_{\rm vac})^2} \left(1+\frac{U_{KK}^{7-q}}{4\rho_{\rm vac}^{7-q}}\right)^{\beta-2} \\
& \times \left\{ \frac{U_{KK}^{7-q}(c_1 r_{\rm vac}^2 + c_2\lambda^2)}{4\rho_{\rm vac}^{11-q}} + \frac{U_{KK}^{2(7-q)}(c_3 r_{\rm vac}^2 + c_2\lambda^2)}{16\rho_{\rm vac}^{18-2q}} \right\} \delta r = 0.
\end{aligned}
\tag{A.3}
$$

As in subsect. 4.3.1 we write a solution of the equation of motion in a form

$$\delta r = G(\lambda) Y(\Omega_{p-s-1})\, e^{ik\cdot x}, \tag{A.4}$$

and take the zero (constant) mode for $Y(\Omega_{p-s-1})$. Substituting eq. (A.4) into eq. (A.3) we obtain an eigenvalue equation for the $(s+1)$-dimensional mass $M_r^2 = -k^\mu k_\mu$. The linearized equation of motion for $\lambda \to \infty$ is

$$\frac{\partial}{\partial \lambda}\left(\lambda^{p-s-1}\frac{\partial}{\partial \lambda}\delta r\right) = 0, \tag{A.5}$$

whose general solution is

$$\delta r \sim a + b\lambda^{-(p-s-2)}. \tag{A.6}$$

In contrast to the case of $\delta\theta$ the asymptotic behavior of δr does not depend on the value r_∞. The normalizable solution corresponds to $a = 0$. Using this asymptotic behavior as a boundary condition one can obtain the mass spectrum by numerical calculations as in ref. [19]. We do not expect a massless particle in the spectrum.

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