Asymptotic Analysis of Stress Fields around Elastic/Elastic-Plastic Interface Edge of Dissimilar Materials Joints

（弾性/弾塑性異材接合端部近傍の応力場の漸近解析）

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A Dissertation submitted to the Saitama University in fulfillment of the requirement for the degree of Doctor of Philosophy

in

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by

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Dedicated to

KANIZ FATEMA BANI

My beloved wife

ABSTRACT

This dissertation presents the research work done for the degree of Doctor of philosophy. Considering the stress and displacement fields around an interface edge of an elastic and a power-law hardening materials joint, one single separable form solution of stress on the interface, $\sigma_{ij} \propto r^{\lambda-1} f_{ij}(\theta)_{\theta=0}$, gives stress continuity where σ_{ij} denotes stress components, r and θ are polar coordinates, λ is the eigenvalue and $f_{ij}(\theta)$ is the angular function. However, the displacement on the interface in the power-law material is $u_i \propto r^{1+n(\lambda-1)} g_i(\theta)_{\theta=0}$, and in the elastic material is $u_i \propto r^{\lambda} h_i(\theta)_{\theta=0}$, where u_i is the displacement component, n is power-law hardening exponent, $g_i(\theta)$ and $h_i(\theta)$ are angular functions. Due to the dissimilarity of power of r the displacement does not become continuous. The theoretical study on singularity around an interface free edge of elastic/power-law hardening materials joint has not solved yet. This thesis solved the singular stress fields around an interface edge of elastic/elastic-plastic materials joint. The objective of this thesis is to present an iteration method to determine the stress and displacement fields around an interface edge of a joint in which materials behaves as an elastic and a power-law hardening material. J_2-deformation plasticity theory under plane strain condition is assumed for the power-law hardening material. Both the balance of force and the continuity of displacements are satisfied on the interface iteratively. The stress fields are found to be singular with the type of r^{λ_i-1} singularity from the i-th order approximation, where r is the radial distance from the interface. Due to the increase of iteration, the discrepancy of the r dependence of the fields along the interface is decreased. The power of r in the stress equation depends on the hardening exponent n. $(i+1)$ or more singular terms exist in the i-th order approximation for $n<(i+1)/i$. As n is increased the absolute value of the i-th order of singularity, $|\lambda_i - 1|$, tends to be decreased to zero when $\lambda_i - 1 < 0$.

An asymptotic analysis for singular stress fields around an interface-edge of dissimilar power-law hardening materials joint has also been presented. Both the balance of force and the continuity of displacement are satisfied on the interface for two dissimilar power-law hardening materials joint having different power-law hardening exponent. In the higher order approximation, the nonlinear effective stress term was expanded by Taylor series. Our analyses show the order of stress singularity has a dependency with the combination of hardening exponents. Multiple stress singular terms exist for $\left(n_1 - n_2\right) < 1$ in the higher order approximation. The order of stress singularity has a dependency with the combination of hardening exponents, n_1 and n_2.

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LIST OF SYMBOLS

Symbol	Definition
-(bar)	With bar(-) all are in dimensionalized form
′(prime)	Derivative with respect to θ of the quantity it is used with A′-derivative of A with respect to θ
~(squiggle)	Angular part of the (r, θ)separable quantity it is used over $\tilde{\sigma}$ -angular part of σ [where σ=σ(r, θ)]
ϕ	Generalized airy stress function
ϕ_0	Airy stress function for zero-th order approximation
ϕ_1	Airy stress function for first order approximation
ϕ_2	Airy stress function for second order approximation
ϕ_i	Airy stress function for i-th order approximation
$\tilde{\phi}$	Angular variation of airy stress function
$\tilde{\phi}_0$	Angular variation of airy stress function for zero-th order approximation
$\tilde{\phi}_1$	Angular variation of airy stress function for first order approximation
$\tilde{\phi}_2$	Angular variation of airy stress function for second order approximation
$\tilde{\phi}_i$	Angular variation of airy stress function for i-th order approximation

α	Power-law hardening constant
δ_{ij}	Two dimensional Kronecker delta (i, j=r, θ)
n	Stress hardening exponent
ν	Generalized Poisson's ratio
ν^{I}	Poisson's ratio for elastic-plastic material
ν^{II}	Poisson's ratio for elastic material
E	Generalized Young's modulus
E^{I}	Young's modulus for elastic-plastic material
E^{II}	Young's modulus for elastic material
A	Generalized stress intensity factor
A_0	Stress intensity factor at zero-th order approximation
A_1	Stress intensity factor at first order approximation
A_2	Stress intensity factor at second order approximation
A_i	Stress intensity factor at i-th order approximation
λ	Generalized exponent of the stress singularity
λ_0	Exponent of the stress singularity for zero-th order approximation
λ_1	Exponent of the stress singularity for first order approximation
λ_2	Exponent of the stress singularity for second order approximation
λ_i	Exponent of the stress singularity for i-th order approximation
u_i	Displacement in i direction (i=r, θ)

$\tilde{u}_i$	Angular variation of displacements
u_i^I	Displacement in i direction (i=r, θ) of elastic-plastic material
u_i^{II}	Displacement in i direction (i=r, θ) of elastic material
u_{i0}	Incremental displacement in i direction (i=r, θ) of zero-th order approximation
u_{i1}	Incremental displacement in i direction (i=r, θ) of first order approximation
u_{i2}	Incremental displacement in i direction (i=r, θ) of second order approximation
u_{li}	Incremental displacement in l direction (l=r, θ) of i-th order approximation
$u_{i(1)}$	Total displacement in i direction (i=r, θ) of first order approximation
$u_{i(2)}$	Total displacement in i direction (i=r, θ) of second order approximation
$u_{l(i)}$	Total displacement in l direction (l=r, θ) of i-th order approximation
ε_{ij}^I	Strain component(i, j=r, θ) of elastic-plastic material
ε_{ij}^{II}	Strain component(i, j=r, θ) of elastic material
ε_{ij0}	Strain component(i, j=r, θ) of zero-th order approximation
ε_{ij1}	Incremental strain component(i, j=r, θ) of first order approximation

ε_{ij2}	Incremental strain component(i, j=r, θ) of second order approximation
ε_{lji}	Incremental strain component(l, j=r, θ) of i-th order approximation
$\varepsilon_{ij(1)}$	Total strain (i, j=r, θ) of first order approximation
$\varepsilon_{ij(2)}$	Total strain (i, j=r, θ) of second order approximation
$\varepsilon_{lj(i)}$	Total strain (i, j=r, θ) of i-th order approximation
σ_{ij}	Stress component(i, j=r, θ)
$\tilde{\sigma}_{ij}$	Angular portion of stress component
σ_{ij}^{I}	Stress component(i, j=r, θ) of elastic-plastic material
σ_{ij}^{II}	Stress component(i, j=r, θ) of elastic material
σ_{ij0}	Incremental stress component(i, j=r, θ) of zero-th order approximation
σ_{ij1}	Incremental stress component(i, j=r, θ) of first order approximation
σ_{ij2}	Incremental stress component(i, j=r, θ) of second order approximation
σ_{lji}	Incremental stress component(l, j=r, θ) of i-th order approximation
$\sigma_{ij(1)}$	Total stress (i, j=r, θ) of first order approximation

$\sigma_{ij(2)}$	Total stress (i, j=r, θ) of second order approximation
$\sigma_{lj(i)}$	Total stress (i, j=r, θ) of i-th order approximation

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

The material researched in our project is the characteristics of singular fields around an interface edge of elastic/elastic-plastic materials joint. Several industrial applications require advanced materials that will fulfill demanding thermal and mechanical conditions. In most cases, no unique class of materials can sustain these challenging conditions. Due to the brittle nature of some materials, machining is not advisable. Production requires materials which are able to survive for a long time at high temperatures. Due to the resistant of high temperatures and suitable for machining, this elastic/elastic-plastic materials joint have extensive promising applications in combustion engine, gas turbine and heat exchanger. Elastic and elastic-plastic joints also found as a bimetals, composite materials with hybrid matrices, bonded circuits in micro-electronics, bonded solid rocket propellant grains with binder, coating on ceramic substrates, adhesively bonded parts and welded parts. In these components, damage is often observed along the bimaterial interface at locations where there is either material discontinuity or geometry discontinuity or both. Fig. 1.1 shows interfacial geometries of interest in engineering. The presence of localized damage suggests that there may be intensive stress accentuation at this interfacial discontinuity. Knowledge of these locally accentuated stress distributions is essential for understanding the initiation and growth of damage. Such understanding is important to improve design, both in the material tailoring level and in the component level.

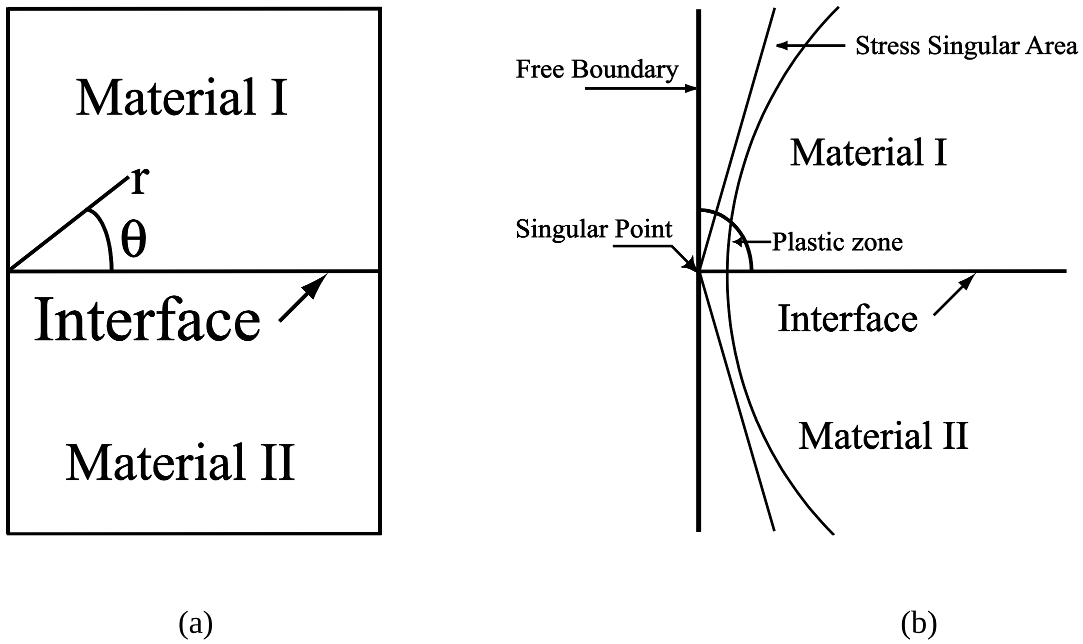


Fig. 1.1 Example of elastic-plastic /elastic material interface and illustration of stress singularity of elastic-plastic/elastic material joint.

1.2 BACKGROUND

With the increasing technological importance of joints with dissimilar materials, interface-edge stress fields and strength of jointed materials have also become a topic of major practical interest [1]. Elastic and power-law hardening materials joint is being increasingly used in engineering applications in order to profit from the advantages of each material [2]. The elasto-plastic interfacial problem has received considerable attention in the last decade enabling a thorough understanding to be developed. It is very important to clarify the fields for the engineering applications such as the strength evaluation of bonded elastic/power-law hardening materials joint and power-law hardening materials having different hardening exponent [3].

Due to the different mechanical properties of the jointed materials, very high stresses develop near the interface edge [4-6]. Stress singularities might exist in most cases around the interface edge of elastic/power-law hardening materials joint [4-6].

Many studies have been directed toward computing the order of the stress singularity for various single and multiple-phase notch/wedge/crack geometries in both isotropic and anisotropic media. The stress singularities at the vertex of an elastic plate under extension were investigated in detail by Williams [4,5]. The published work on bimaterial interfaces is primarily for plastic deformation and for crack geometries [7,8] and interface notch geometries [9-12].Some studies have considered elastic or elastic-plastic bimaterials, [13-22]. Bogy [1,6,23], Chen and Nisitani [24] Hein and Erdagon [25] and Dempsy and Sinclair [26] presented the order of stress singularity using Mellin transform method at the isotropic elastic bimaterial wedge apex. Considering the influence of the regular terms, the stress singularities at the interface edge in elastic bi-materials with edge tractions was analyzed by Yang and Munz [27, 28]. Dissimilar materials joint with arbitrary bonding angle was analyzed by Xu et al. [29, 30]. The aforementioned works focus on the analysis of singularities in the homogeneous elastic material and elastic/elastic bimaterial joint.

In order to account in greater detail for the development of plastic deformation in the vicinity of the crack tip, Hutchinson [31], and Rice and Rosengren [32] performed an asymptotic analysis of the crack tip fields in a homogenous power-law hardening material.

Recently, many researchers have investigated the elastic-plastic stress singularity of an interface crack between two bonded power-law hardening materials. Xia and Wang

[33,34] presented asymptotic analysis for interface crack in elastic-plastic material. Numerical solutions involving elasto-plastic behavior at an interface crack tip for a power-law hardening materials joint have been developed by Shih and Asaro [35-37]. In their work nearly separable singular fields have been characterized as small strain HRR type fields proposed by Hutchinson [31], and Rice and Rosengren [32]. They found that in bimaterial interface problems the stress and strain fields in the more compliant materials behave like those of a material with identical plastic properties bonded to a rigid substrate, and the near-tip stress fields in the higher hardening material are limited to those levels that can be attained in the lower hardening material. Xia and Wang [38] have made a higher-order asymptotic analysis on the plane strain interfacial crack problem in power-law hardening bimaterials which have different power, n, each other and obtained the asymptotic fields. They found that along the interface ahead of crack tip the stress fields are co-order continuous while the displacement fields are cross-order continuous. Lau and Delale [39], Sckuhr et al.[40] as well as Rudge [41] presented a separable asymptotic solution on a edge bonded wedges of power-law material having the same hardening exponent, n.

Considering the stress and displacement fields around an interface edge of an elastic and a power-law hardening materials joint, one single separable form solution of stress on the interface, $\sigma_{ij} \propto r^{\lambda-1} f_{ij}\left(\theta\right)_{\theta=0}$, gives stress continuity where σ_{ij} denotes stress components, r and θ are polar coordinates, λ is the eigenvalue and $f_{ij}\left(\theta\right)$ is the angular function. However, the displacement on the interface in the power-law material is $u_i \propto r^{1+n(\lambda-1)} g_i\left(\theta\right)_{\theta=0}$, and in the elastic material is $u_i \propto r^{\lambda} h_i\left(\theta\right)_{\theta=0}$, where u_i is the displacement component, n is power-law hardening exponent, $g_i\left(\theta\right)$ and $h_i\left(\theta\right)$ are angular functions. Due to the dissimilarity of power of r the displacement does not become continuous.

In the dissimilar power-law hardening materials joint, the dissimilarity of the power of r also exist in the displacement field on the interface due to the different hardening exponent. Due to the dissimilarity of power of r the displacement fields are not continuous at the interface. The question then arises how to satisfy the displacement continuity condition on the interface, which is an unsolved problem for the power-law hardening materials joint having different hardening exponent.

In order to satisfy the continuous conditions of displacement and the equilibrium of force at the interface edge, Duva[42], Rahman[43], Reedy[44] ,Wang[45] and Xu et al. [46] modelized the elastic/power-law hardening materials joint as a power law hardening material on a rigid substrate. They conducted the asymptotic analysis similar to the nonlinear crack problem developed by Hutchinson [31].

Following the previous studies, relatively little work is found in the literature concerning the determination of stress fields around interface-edge of dissimilar power-law materials joint.

However, as far as we know, the theoretical study on singularity around an interface free edge of elastic/power-law hardening materials joint and power-law hardening materials joint having different hardening exponent has not yet been reported.

1.3 SCOPE OF THE RESEARCH

The scopes of this research include the following aspects:

At the interface of two bonded materials the stresses and displacements should be continuous due to the following:

Two materials are jointed with each other. The traction which acts along the interface calculated from the power-law hardening material will be equal and opposite traction for the elastic material due to the mechanical equilibrium of forces (localization of balance of linear momentum on the interface).

After joining of two materials, all grain boundary of elastic and elastic-plastic material are rigidly bonded with each other. The movement of one grain will be the same as another so the displacement along the interface must be continuous for the deformation at the interface. Due to the deformation of the structure could be compatible, the non-traction strains are continuous. So, the displacement at the interface should be continuous.

In the previous study, the displacement field of elastic material was assumed as 0, i. e., the elastic material was assumed as rigid. Obviously, in reality, the assumption has some problem because of the displacement field of the elastic material should have engineering interest. In elastic-plastic/elastic materials joint, the elastic material is brittle in many cases and hence the singular stress field and the displacement field of elastic material are important to evaluate the characteristics of strength of elastic/ elastic-plastic materials joint. Due to the existing singular stress and displacement fields in the elastic material, rigid/power-law hardening material model could not be applicable for the strength evaluation of the joint. So it is important to clarify the fields which satisfy the continuity condition of displacement and the equilibrium of force along the interface of elastic and power-law hardening materials joint.

In this thesis, an iteration method is proposed for the determination of singular fields around an interface edge of an elastic and a power-law hardening materials joint. In the proposed iteration method, to overcome the problem we have considered at the interface boundary the additional stress fields are set in the elastic side to satisfy stress continuity, the additional displacement fields in the elastic-plastic side to satisfy displacement continuity, successively. A governing differential equation in the iterative form obtained from the compatibility condition is solved theoretically to satisfy the continuity of displacement and the balance of force on the interface between an elastic material and a power-law hardening material joint. In order to satisfy the condition of stresses and displacements on the interface, an asymptotic expansion of the solution in the summation form is used. Due to the increase of iteration, the discrepancy of the r dependence of the fields is decreased.

Using the same iterative method we satisfy the boundary conditions to determine the singular fields around an interface edge of two dissimilar power-law hardening materials joint having different hardening exponent.

Singular exponents can be determined from the theoretical iterative solution. To have the stress intensity factor singular fields are compared with stress fields by FEM. Using the numerical analysis by FEM, stress fields of jointed materials can be determined where the determination of the singularity and stress intensity factor are impossible. So, theoretical iteration method is also important to determine the singularity.

The aim of the present research is to contribute to a better understanding of determining the stress singularity of elastic-plastic and elastic materials joint interface by using asymptotic analysis.

1.4 OBJECTIVES

The main objectives of this research are as follows:

1. To present an iteration method by higher order asymptotic analysis for the determination of singular stress and displacement fields around an interface edge of two dissimilar materials joint in which materials behaves as an elastic and a power-law hardening material.
2. To show the stress fields around the interface free-edge of elastic/elastic-plastic materials joint with the proposed iteration method by higher order asymptotic analysis under plane strain condition.

1.5 OUTLINE OF THE THESIS

The research work conducted for this project is completely presented in this dissertation, which is organized as follows. Chapter 1 is an introduction of the research, which describes the general introduction, background, the motivation, the scopes and the objectives of the project. In chapter 2, the current research in the area of nonlinear elastic-plastic material is jointed with linear elastic material are reviewed. The iteration method of theoretical analysis was discussed. The numerical results of Finite Element Method (FEM) were also discussed, which included comparison of theoretical analysis with FEM. The applicability of theoretical iteration method to the materials joint of two power-law hardening materials having different power-law hardening exponent are discussed in chapter 3. Chapter 4 describes the concluding remarks and directions for future investigations regarding this research work.

CHAPTER 2

SINGULAR STRESS FIELDS IN ELASTIC/ POWER-LAW HARDENING MATERIALS JOINT

2.1 Introduction

In this chapter, we solved for the singular stress and displacement fields around an interface edge of a joint formed by quarter planes in which materials behaves as an elastic and a power-law hardening material. We have formulated and solved under the plane strain condition. J_2-deformation plasticity theory is assumed for the power-law hardening material. By taking the same wedge angle of two materials, our generic interface-edge model is as butt joint model with the interface-edge of two dissimilar elastic and power-law hardening materials.

The stress-strain behavior of most engineering materials, in particular metals and alloys, can be described by the Ramberg-Osgood model, which in uniaxial tensile deformation is expressed as,

$$\varepsilon = \frac{\sigma}{E} + \alpha\sigma^n \tag{2.1}$$

Where, E is the Young's modulus, α is a material constant called power-law hardening constant and n is the stress hardening exponent. For most engineering materials n ranges from 1 to 20 [47]. The first term in equation (2.1), representing elastic strain, varies linearly with σ. The second term, representing plastic strain, varies as the n-th power of σ. When large loads are applied to the material producing full scale plastic deformation, the plastic strain dominates over the elastic strain. Even when small loads are applied, such that the overall stress is below yielding (and the overall plastic strain in the material is negligible), the local stress is highly accentuated at the immediate vicinity of material discontinuity and geometric discontinuity. In these regions, the local plastic strain (which scales as σ^n) still dominates over the local elastic strain (which scales as σ). Consequently, to analyze these local accentuated stresses in the vicinity of discontinuity in elastic-plastic materials, the deformation can be modeled as being purely plastic[31].

As interface edge is approached, the elastic strain presented as the first term is much smaller compared with the plastic strain presented as the second term of Eqn.(2.1), so that the first term can be neglected in asymptotic analysis. Under this condition, the material law, Eqn. (2.1) is simply replaced by a purely power-law hardening model, that is:

$$\varepsilon = \alpha\sigma^n \tag{2.2}$$

Eqn. (2.2) is the basic form of the power-law hardening material model used in this chapter.

The thesis reported in this chapter is an asymptotic analysis for singular stress fields around an interface-edge of dissimilar power-law hardening materials joint under plane-strain condition and J_2 deformation plasticity theory.

In Section 2.2, we formulate the governing equations for the singular stress field under the plane strain when an elastic material is jointed with a power-law hardening material. An effective solution method of numerical shooting method with a fourth-order Runge-Kutta method is also presented. Numerical Analysis using Finite Element Method (FEM) on singular field around interface edge is presented in Section 2.3. Formulation of 0^{th} Order Approximation, 1^{st} Order Approximation and i^{th} Order Approximation are presented in Section 2.4, Section 2.5 and Section 2.6, respectively**.** Section 2.7 shows results for the interface-edge problem of two dissimilar elastic/power-law hardening materials joint. This chapter is concluded with summary in Section 2.8.

2.2 Formulation: Elastic/ Power-Law Hardening Materials Joint Case

Consider a joint plate of dissimilar materials shown in Fig. 2.1. Material I is considered as a power-law hardening material and material II is considered as an elastic material. Plane strain condition is assumed.

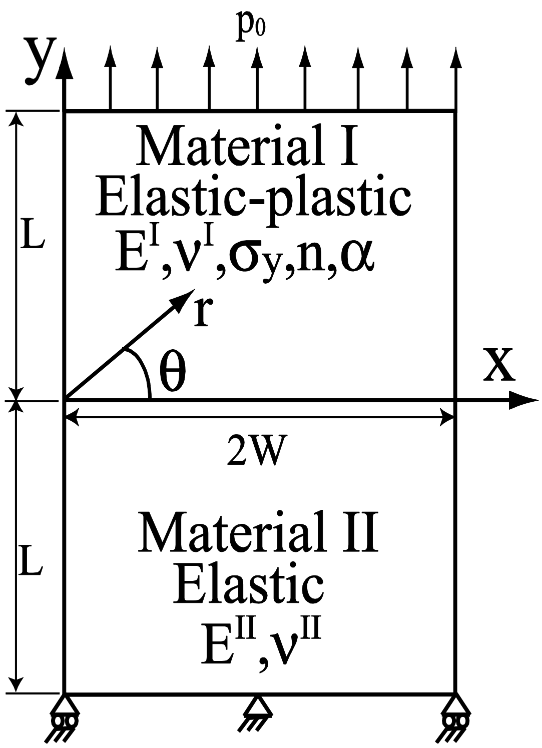


Fig. 2.1: Theoretical Model of Elastic-plastic/Elastic materials joint.

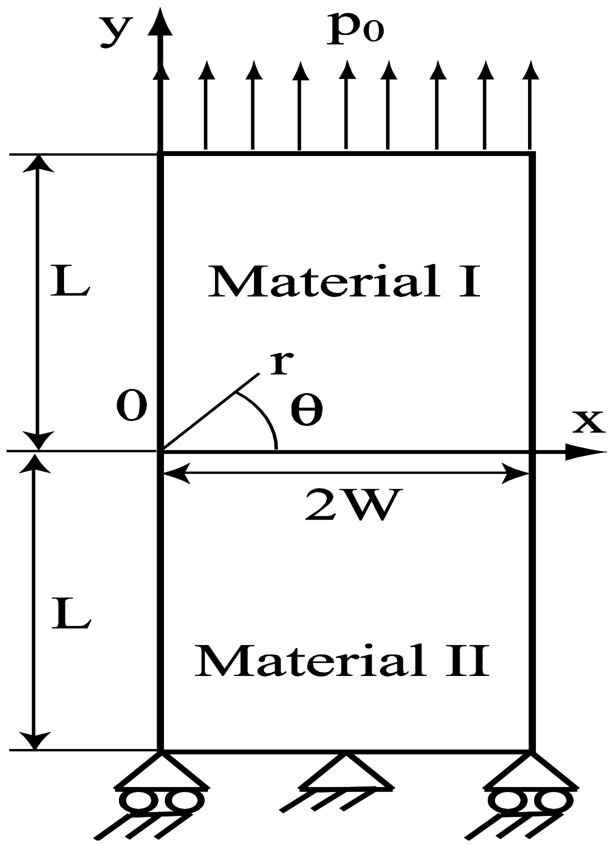


Fig. 2.2: Geometries considered and coordinates.

2.2.1 Stress and strain relationships

The generalized dimensionless relationship between strain and stress governed by a power law form and stress-strain relation in the elastic-plastic side is given by,

$$\varepsilon_{ij}^{I} = \frac{3}{2}\alpha\sigma_e^{n-1}s_{ij}^{I}. \tag{2.3}$$

In the elastic side the stress-strain relation is given by,

$$\varepsilon_{ij}^{II} = \frac{E^{I}}{E^{II}}\left\{\left(1+\nu^{II}\right)\sigma_{ij}^{II} - \nu^{II}\sigma_{kk}^{II}\delta_{ij}^{II}\right\}, \tag{2.4}$$

where, $\sigma_{kk} = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}$, $\varepsilon_{ij} = \dfrac{\bar{\varepsilon}_{ij}}{\varepsilon_y}$, $\varepsilon_y = \dfrac{\sigma_y}{E}$, and $\sigma_{ij} = \dfrac{\bar{\sigma}_{ij}}{\sigma_y}$, $\sigma_e = \sqrt{\dfrac{3}{2}s_{ij}s_{ij}}$, $s_{ij} = \sigma_{ij} - \dfrac{1}{3}\sigma_{kk}\delta_{ij}$.The stresses and displacements of power-law hardening material are referred to with a superscript “I” while those of the elastic material, with a superscript “II”. The dimensionless stress function ϕ and coordinate r are given in terms of dimensional quantities (barred) by $\phi = \bar{\phi}/\sigma_y w^2$ and $r = \bar{r}/w$, where w is the characteristic length which is taken as the half of the width of plate. It is noted that the barred quantities are the non-normalized field variables,(σ_y, ε_y) is a reference point for the uniaxial stress strain curve, $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$ are the stress and strain components, where i, j and k are used for subscript indicates r or θ. E is Young’s modulus, ν is the Poisson’s ratio, δ_{ij} is the two dimensional Kronecker delta symbol, α and n are hardening coefficient and hardening exponent, respectively. σ_e is the effective stress and s_{ij} is the stress deviator. Summation conversion is assumed.

2.2.2 Numerical Shooting Method

The shooting method is actually an adaptation of the initial value schemes to solve boundary value problems. It is an effective tool to solve two point boundary value problems [45] and has been used with success in solving the differential equations arising in fracture mechanics of interface edge. The first step in two-point shooting is to start an initial value scheme at one of the boundaries and march towards the second boundary. In an initial value problem the set of initial values of the dependent variable at the boundary is sufficient of find the solution at an increment of the dependent variable and hence the solution can march on. In the boundary value problem, however, not all the initial values are known at the start (since the problem is defined some boundary values too). The unknown initial values are guessed. After the second boundary is reached the mismatch of the given boundary conditions with the corresponding results from the initial value scheme with guessed initial values are computed. The mismatch is used to refine the guesses for the initial values and they are systematically changed until the boundary conditions are exactly satisfied. The success of shooting problem depends to a large extent on the guessed starting values [68, 69].

The joint problem of two materials joint is not truly a two point boundary value problem since boundary conditions are actually given in three points, namely the two free-surfaces and on the interface. The shooting technique though can be easily modified to accommodate this. To adapt the shooting technique to the joint problem of two materials joint, we simultaneously shoot from the two free surfaces (one on each component wedge) to the interface. At the interface we have to compute the mismatch in traction and displacement continuity conditions and adjust our guessed parameters accordingly. Here, in the first step we considered the traction only to adjust our guessed parameters.

2.2.3 Equilibrium Equations

The equilibrium equations are automatically satisfied for all stresses derived from the Airy stress function ϕ when the stresses are defined in the following manner [48]:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \tag{2.5}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \tag{2.6}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}. \tag{2.7}$$

2.2.4 Compatibility and Strain-Displacement Equations

Using small deformation theory, the strain compatibility equation is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\varepsilon_{\theta\theta}\right)+\frac{1}{r^2}\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2}-\frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r}-\frac{2}{r^2}\frac{\partial^2}{\partial r\partial\theta}\left(r\varepsilon_{r\theta}\right)=0. \tag{2.8}$$

The small deformation strain-displacement relationships are written [48],

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}, \tag{2.9}$$

$$\varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}, \tag{2.10}$$

$$\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right). \tag{2.11}$$

2.2.5 Boundary Conditions

Stress free boundaries, balance of force and continuity of displacements on the interface are the boundary conditions of this problem. The boundary conditions can be expressed as follows in the polar coordinate system located on the interface edge for elastic/power-law hardening materials joint,

$$\begin{aligned}
&\left(\sigma^{I}_{\theta\theta}\right)_{\theta=\frac{\pi}{2}}=\left(\sigma^{II}_{\theta\theta}\right)_{\theta=-\frac{\pi}{2}}=0, &&\left(u^{I}_{r}\right)_{\theta=0}=\left(u^{II}_{r}\right)_{\theta=0}, &&\left(\sigma^{I}_{\theta\theta}\right)_{\theta=0}=\left(\sigma^{II}_{\theta\theta}\right)_{\theta=0},\\
&\left(\sigma^{I}_{r\theta}\right)_{\theta=\frac{\pi}{2}}=\left(\sigma^{II}_{r\theta}\right)_{\theta=-\frac{\pi}{2}}=0, &&\left(u^{I}_{\theta}\right)_{\theta=0}=\left(u^{II}_{\theta}\right)_{\theta=0}, &&\left(\sigma^{I}_{r\theta}\right)_{\theta=0}=\left(\sigma^{II}_{r\theta}\right)_{\theta=0}.
\end{aligned} \tag{2.12}$$

2.2.6 Solution Method

In the zero-th order approximation, the stress and displacement fields in the power-law hardening material are assumed to be the same as the ones in the plate jointed to rigid substrate instead of elastic material and subjected to the same tensile load.

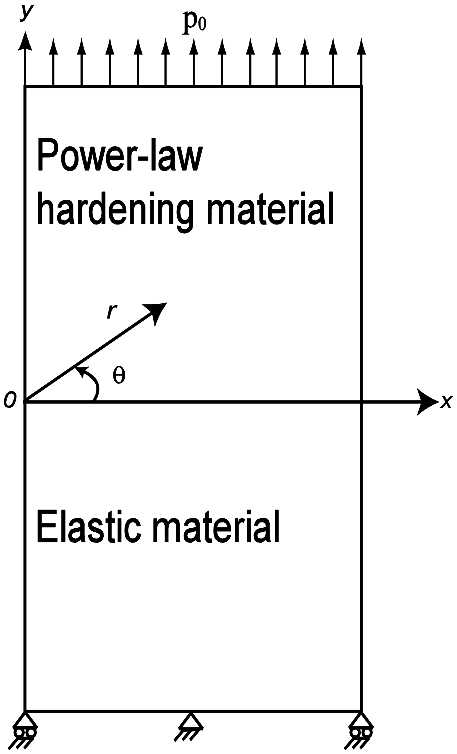


Fig. 2.3: Elastic-plastic/Elastic materials joint.

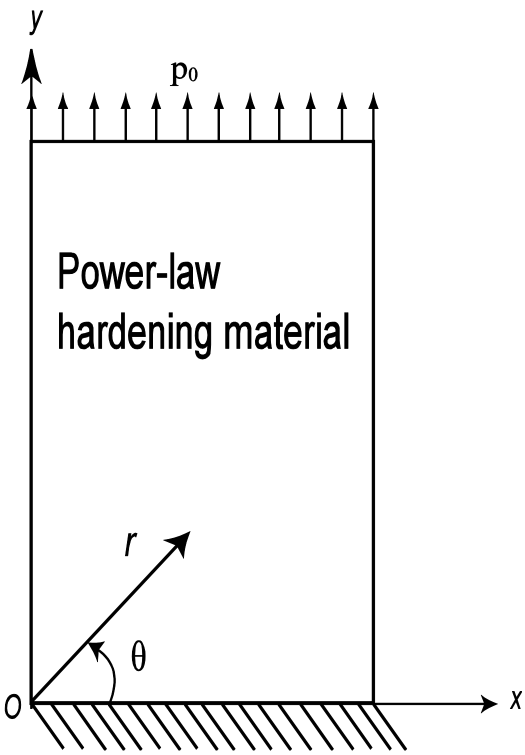
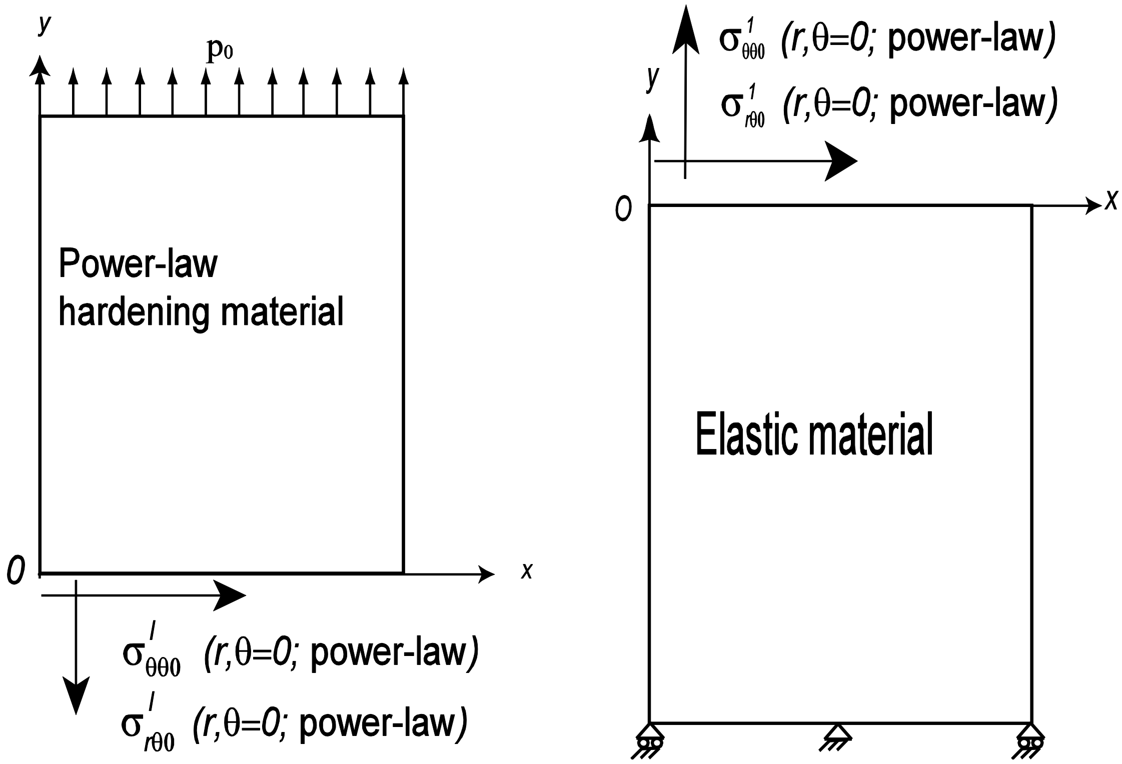


Fig. 2.4: Power-law hardening material bonded to rigid substrate.

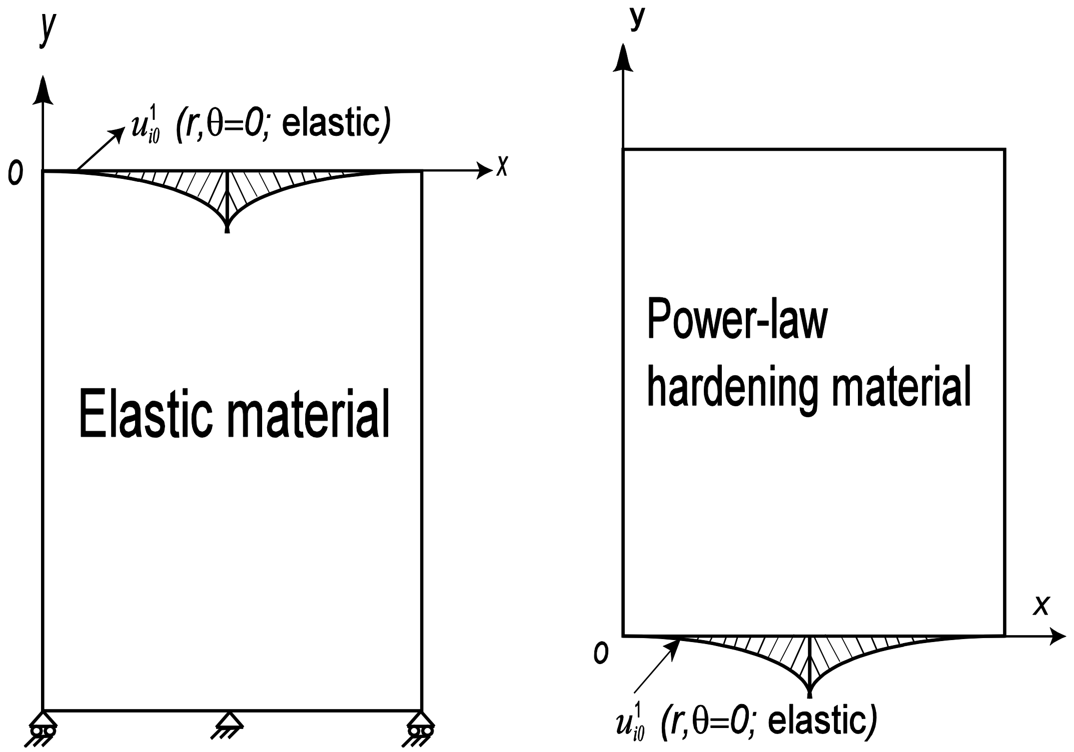
The stress fields in the elastic material can be described by the fields of an elastic wedge which is subjected to distributed tractions along the one edge. The tractions are the same as the stress distributions on the rigid/power-law interface.



(a) (b)

Fig. 2.5: Schematic diagram of applying Traction from Power-law hardening material to the elastic material.(a) Traction from power-law hardening material (b) Traction to elastic material.

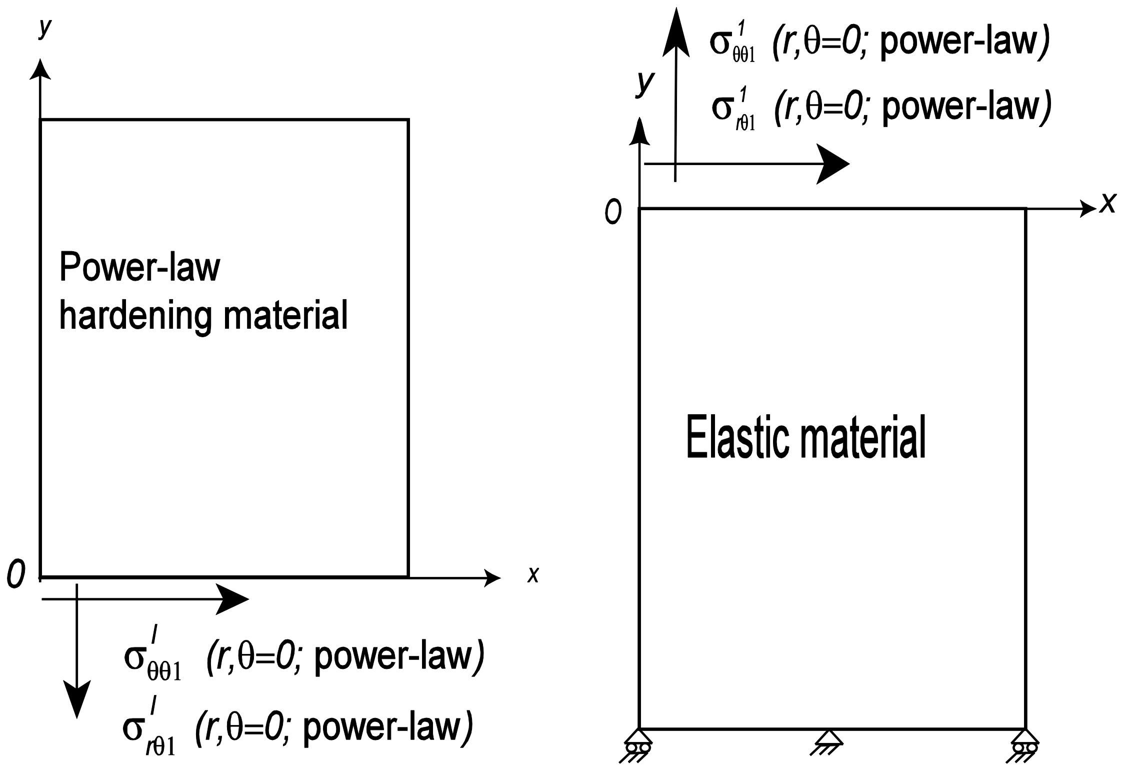
In the first order approximation, the power-law hardening material having the initial fields of the zero-th order approximation is subjected to a forced displacement which is the field on the edge of elastic material of the zero-th order approximation.



(a) (b)

Fig. 2.6: Schematic diagram of applying Forced displacement from elastic material to the power-law hardening material. (a) Forced displacement from elastic material (a) Forced displacement to power-law hardening material.

The increase of stress fields in the elastic material can be described by the fields of an elastic wedge which is subjected to distributed tractions along the one edge. The magnitudes of the traction are the same as the stress distributions on the power-law material wedge.



(a) (b)

Fig. 2.7: Schematic diagram of applying Traction from power-law hardening material to the elastic material.(a) Traction from power-law hardening material (b) Traction to elastic material.

2.2.7 Asymptotic Analysis

An asymptotic expansion of the Airy stress function in a separable form is assumed as,

$$\phi^k = \sum_i A_i r^{\lambda_i+1} \tilde{\phi}_i^k,\ i = 0,1,2,\ldots, \tag{2.13}$$

as $r \to 0$, where $k = I$ for power-law hardening material and $k = II$ for elastic material. λ_i is defined as the singular exponent in the i-th order of approximation. $\tilde{\phi}_i^k$ is the angular function of airy stress function in the i-th order of approximation. A_i is a constant which is proportional to the stress intensity factor of i-th order incremental fields. A_i is defined as,

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} = \sum_{i} A_i\, r^{\lambda_i - 1},\ i = 0,1,2,.... \tag{2.14}$$

A_0 is controlled by external loading. In the higher order approximation, to satisfy the displacement continuity condition on the interface A_1 depends on A_0, A_2 also depends on A_0 which means A_i has a dependency on A_0 or external loading.

2.2.8 Error Calculation for the Solution

The error value is calculated by using the equation,

$$\text{error} = \sqrt{\left(\text{error1}\right)^2 + \left(\text{error2}\right)^2}. \tag{2.15}$$

After integration, solution region is obtained with different error ranges which systematically cover the range of $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 for different error. From the minimum error region, the range of $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 is selected as the initial value for the integration of next step. The final region of the minimum error is taken with the region of $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 contains two equal digits after the decimal point. The final solution is obtained from this minimum error region and calculated numerically the solution point where the minimum error occurs. The minimum error region is integrated with 200 smaller divisions and minimum error point is selected as the solution point and the corresponding $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 is calculated as the solution. The iterative procedure stops when the error is less than 10^{-6}. After tentative solution, to get the exact solution it is necessary to correct the initial value of $\left(\tilde{\phi}_0^I\right)''$ which was assumed as 1. To get the exact value, it is necessary to satisfy $\left(\tilde{\sigma}_{\theta\theta}^{I}\right)_{\theta=0} = 1$. So initial value of $\left(\tilde{\phi}_0^I\right)''$ becomes, $\left(\tilde{\phi}_0^I\right)'' = 1.0 \big/ \left(\tilde{\sigma}_{\theta\theta}^{I}\right)_{\theta=0}^{T}$, where $\left(\tilde{\sigma}_{\theta\theta}^{I}\right)_{\theta=0}^{T}$ is the tentative solution obtained assuming $\left(\tilde{\phi}_0^I\right)'' = 1.0$.

Assume at $\theta = \dfrac{\pi}{2}$, $\tilde{\phi}_0^I = 0,$ $\left(\tilde{\phi}_0^I\right)' = 0,$ $\left(\tilde{\phi}_0^I\right)'' = \dfrac{1.0}{\left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0}^T},$ $\left(\tilde{\phi}_0^I\right)'''$, λ_0 and after i-th integration at $\theta = 0,$ $\tilde{\phi}_0^I = \tilde{\phi}_0^{I(i)},$ $\left(\tilde{\phi}_0^I\right)' = \left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}, \left(\tilde{\phi}_0^I\right)'' = \left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}, \left(\tilde{\phi}_0^I\right)''' = \left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$.

The solution procedure is same as described above for seeking the minimum error. Due to the change of $\left(\tilde{\phi}_0^I\right)'' = 1.0 \big/ \left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0}^T$, all angular function terms changes with the same ratio for the exact solution.

2.3 Numerical Analysis on Singular Field Around Interface Edge

In this section, we examine the numerical results of elastic and elastic-plastic materials joint interface. For this reason, a numerical model is considered to examine the singularity index of elastic and elastic-plastic materials joints interface. The numerical calculation is carried out on the basis of singularity theory. The power-law hardening prediction is considered on the numerical calculation for elastic-plastic material. The stress singularity fields of power-law hardening material are compared to examine the results. The continuity condition of displacement and stress components at the interface conflicts the existence of a separable form singular solution like $u_i \propto r^{\lambda} f_i\left(\theta\right)$ for the material pair. Applicability of the power-law/rigid materials joint model to ceramic/metal joint in regard to the interface-edge problem should be examined by numerical facts. In this section, we consider numerical model for elastic/power-law hardening butt joint to determine the stress and displacement distribution. One of the powerful numerical tools extensively used in fracture mechanics research in the Finite Element Method (FEM). In this chapter, we constructed an elastic and elastic-plastic butt joint model to analysis elastic and plastic deform material. For plastic deformation, we consider the power-law hardening materials prediction by small deformation and finite deformation theory. It was also analyzed that the stress and displacement distribution of elastic/power-law hardening butt joint interface is linear or nonlinear function for small deformation and finite deformation theory.

2.3.1 Finite Element Method (FEM)

2.3.1.1 Small Deformation Theory

The classical theory of isotropic plastic solids undergoing small deformations is based on the decomposition,

$$\Delta u = H^e + H^p, \qquad trH^p = 0 \tag{2.16}$$

of the displacement gradient into elastic and plastic parts, where H^e represents rotation and stretching of the material structure, while H^p,the plastic distorsion, characterizes the evolutions of dislocations and other defects through this structure. In this classical theory the plastic rotation W^p the skew tensor in the decomposition $H^p = E^p + W^p$ where, H^p into symmetric and skew parts-is essentially irrelevant, and it may be absorbed by its elastic counterpart without affecting the resulting field equations. Recent interest in the behavior of material at micron length scales has led to a growing literature concerned with strain gradient plasticity (Fleck and Hutchinson [49]; Gurtin [50]; Gudmundson [51]). A tracit central assumption of these gradient theories motivated, by experience with classical plasticity is that the constitutive theory not involve the plastic rotation-field W^p; consequently.

To understand the issue of whether or not an isotropic theory of plasticity should involve the rotation field, it is useful to bear in mind that, the Cauchy stress T expends power during plastic flow in consort with plastic-strain rate $\bar{E}^p$; the spin $\bar{W}^p$, whether it may be, involves no expenditure of power and, consequently, generates no dissipation. But the development of higher order (strain gradient) theory necessarily involves higher order stresses and this renders uncertain what form the underlying power expenditures should take.

The principal of virtual power then led us to account for power expanded by the field H^p, which we accomplish with the aid of second microscopic stress T^p. We complete this accounting with the assumption that power expended in stretching and rotating the material structure has the form $T : H^e$. So the power expended within any part P (sub region of the body), has the form,

$$\int_V \left(T : E^p + T^p : H^p + S : curlH^p \right) dV \,. \tag{2.17}$$

Consequences of the virtual power principal are that the classical macroscopic balances need be supplemented by micro force balance.

2.3.1.2 Finite Deformation Theory

A converted co-ordinate Lagrangian formulation of the field equations is employed with the initial unstressed state taken as reference. All fields quantities are considered to be functions of convected co-ordinates y^i. Based on the finite element analysis, body forces are neglected, the requirement of equilibrium is specified in terms of the principal of virtual work is written as;

$$\int_V \tau_{ij}\delta\varepsilon_{ij}dV - \int_S \tau_{ij}\delta u_{ij}dS = 0 \tag{2.18}$$

Here, τ_{ij} are the contravariant components of the Kirchhoff stress ($\tau = J\sigma$,where σ is the Cauchy stress and J is the ratio of current to reference volume of a material element) on the deformed converted coordinate net. The quantities V and S are the volume and surface, respectively.

The nominal traction components, T_i , and the Lagrangian strain components, ε_{ij} are given by

$$\begin{aligned} T_i &= \left(\tau_{ij} + \tau_{kj}, u_{i,k}\right)\eta_j \\ \varepsilon_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{ki}, u_{k,j}\right) \end{aligned} \tag{2.19}$$

where, η is the surface normal in the reference configuration, u_{ij} are the components of the displacement vector on the base vectors in the reference configuration and(), j denotes convariant differentiation in the reference frame.

2.3.2 Finite Element Model and Mesh

Theoretical elastic and elastic-plastic finite element model, rigid and power-law hardening model joint is shown in Fig.2.8 and Fig.2.9 was carried out using the finite element code ABAQUS V6.7 [52]. For elastic and elastic-plastic model, fixed boundary condition were applied to the bottom layer of the model and uniform tension was applied to the upper boundary of the model. For rigid and power-law hardening model bottom layer is fixed and uniform tension was applied to the upper boundary of the model. The length of elastic and elastic-plastic plate along y direction is 4.0 and 4.0 mm, respectively. Overall mesh division of the model is shown in Fig.2.10. Fine mesh is considered near to the interface. Near to the interface edge of model the mesh division is shown in Fig.2.11.

To calculate and determine the value of stress singularity and stress intensity factor, external load of 130 MPa is applied. To calculate the maximum amount of external distributed load, stepping method is used and load in increased by step by step by using the restarting method. During the numerical calculation we consider the following FEM method.

Consider a plane strain elastic-plastic plate under uniform tensile load is applied in the upper edge and fixed to the lower edge as shown in Fig.2.8. Elastic/power-law hardening materials joint is modelized as an elastic-plastic plate bonded with an elastic plate where lower edge of elastic plate is sliding and the uniform tension load is applied to the upper edge of elastic-plastic plate as shown in the same Fig 2.9.

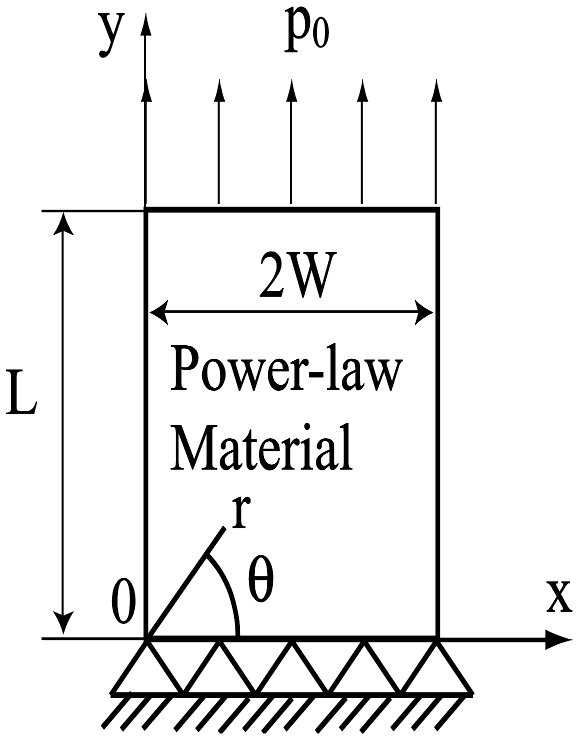


Fig.2.8: FEM model of Power-law material bonded to rigid substrate.

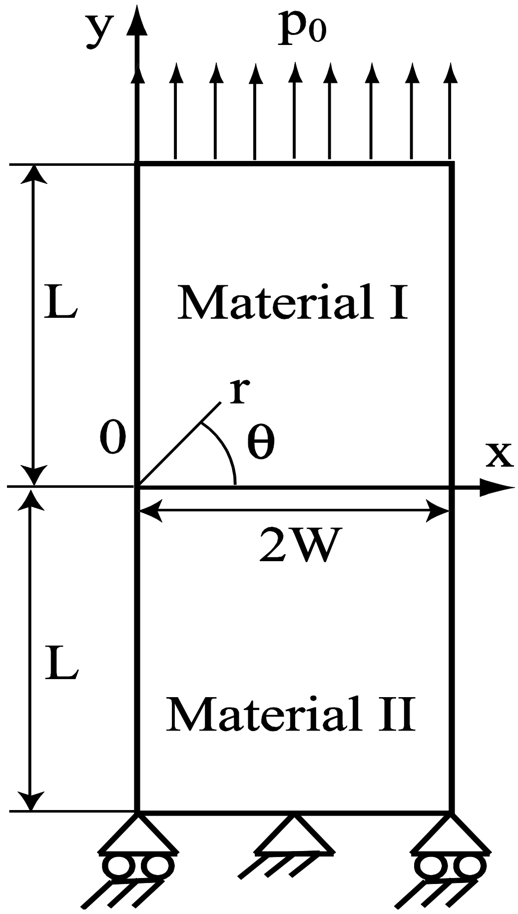


Fig.2.9: FEM model of Elastic/power-law hardening materials

In Fig.2.8, L is the characteristic length dimension, W is half of the width of plate, p_0 is the far field uniform tension and r, θ are the coordinate system. The finite element mesh we have used in our joint material model is given in Fig.2.9. The magnified mesh division near the interface corner is given in Fig.2.24.

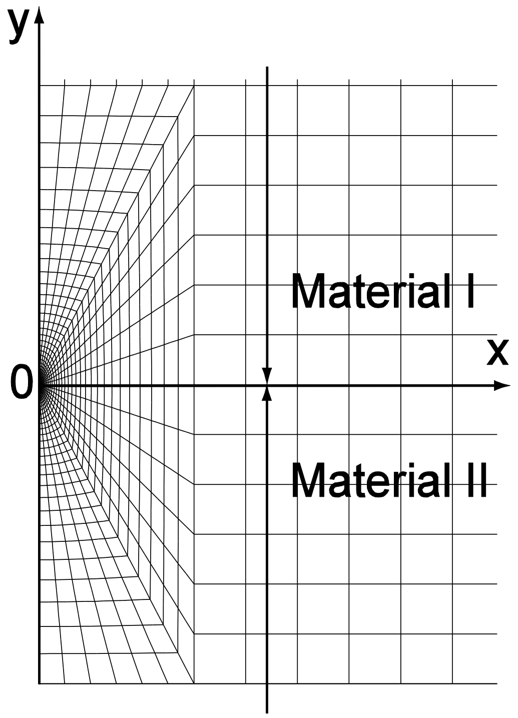


Fig.2.10: Mesh division near interface edge

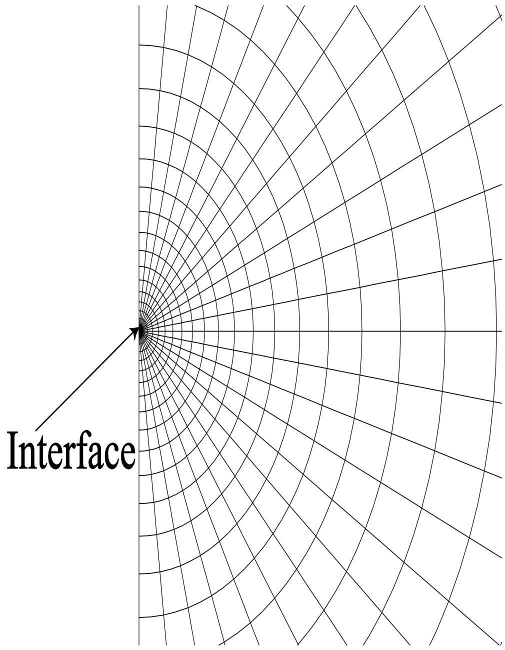


Fig.2.11: Magnified mesh division near interface corner

The stress fields were also calculated numerically by using elastoplastic finite element method [54]. In FEM calculation, the Cu, was assumed to be an elastic power-law hardening plastic material with power-law hardening constant, α and power-law hardening exponent *n*. Si_3N_4 was assumed to be elastic. Plane strain 8 nodes isoparametric elements were used. Finite element meshes were divided into 40 in the near edge area, $0 < r/t \le 0.1$. The length of elements along radial direction varies as following equations [53],

$$l_i = \frac{l_{i-1}}{0.9}, \qquad i = 1, 2, 3, \cdots, 30. \tag{2.20}$$

where l_i is a radial length of i-th element. The minimum length of elements, l_0/t, is 10^{-5}. For circumferential direction meshes are divided into 24 by equal angle. The total number of elements is 31347 and the total number of nodes is 62995. Aspect ratio [54] is kept constant, $L/W = 2.67$ where, $L = 4$ mm and $W = 1.5$ mm.

2.3.3 Stress-Strain Relationship

The elastic properties of material I (Si_3N_4)and material II (Cu)are shown on the Table 1. The plastic property of the Cu was measured experimentally in the previous study by Liton S.K et. al.[55-57] where the initial yielding strength was 30 MPa. It is noted that, our analyzing method is suitable for any value of n. Two or more singular terms exist only for those cases of $n < 2.0$. In what follows an interface edge problem with two different n, $n < 2.0$ and $n > 2.0$, will be investigated in detail. The Mechanical properties of power-law hardening material and elastic material assumed to illustrate the stress and displacement fields are listed in Table 2.1.

Table 2.1: Mechanical Properties of jointed materials for elastic/power-law hardening materials joint

Properties	E[GPa]	ν	σ_y [MPa]	n	α	P_0 [MPa]
Material I	108	0.33	30	1-20	10.1	130
Material II	304	0.27	-	-	-	-

2.3.4 Determination of the Stress Intensity Factor A_0

Stress intensity factor is an important mechanical parameter and fracture toughness is another important material parameter used in fracture mechanics

A parameter called the stress-intensity factor (A_i) is used to determine the fracture toughness of most materials. Fracture toughness is a property which describes the ability of a material containing an interface to resist fracture, and is one of the most important properties of any material for virtually all design applications

To prevent the material from failure/fracture, it is important to know the material parameter for the fracture at the interface before loading.

The theoretical results will be applicable in materials engineering to design elastic/elastic-plastic material joint (for example ceramic/copper joint) to meet specific performance requirements.

The dominant stresses near the interface is of the r-θ form of : $\sigma_{ij0} = A_0 r^{\lambda_0 - 1}\tilde{\sigma}_{ij0}$, where $0 < \lambda_0 < 1$, so the stress becomes singular (unbounded) as r→0. The local asymptotic analyses determines λ_0 and $\tilde{\sigma}_{ij0}$, thus giving the spatial structure of the local stress distribution. The generalized stress intensity factor, A_i, scales the magnitude of the singular stress field. A_i links the remote applied loading to the local stress field, and can only be determined if the global full-field solution is solved.

The stress intensity factor, A_i is the only quantity not determined by the asymptotic analysis. A_i is determined by the full solution, and it depends on loading, geometry and mechanical properties. The value of A_i characterizes the magnitude of the stress state in the region of the interface corner. No matching of FEM with theory is needed in the higher order approximation to determine $A_i\,(i = 1, 2, 3\cdots)$ since the magnitude of higher order term $A_i\,(i = 1, 2, 3\cdots)$ is determined by A_0. For the determination of stress intensity factor, A_0 from the stresses obtained with FEM the least squares method has been used. The stress field can be written as[58,59]:

$$\sigma_{ij} = A_0\, r^{\lambda_0 - 1}\tilde{\sigma}_{ij} \tag{2.21}$$

Where, $\tilde{\sigma}_{ij}$ is the angular function. $ij = rr$, $\theta\theta$ or $r\theta$.

From the definition of stress intensity factor, $\tilde{\sigma}_{\theta\theta}(\theta = 0) = 1$[60,61]

$$(\sigma_{\theta\theta})_{\theta=0} = A_0\, r^{\lambda_0 - 1} \tag{2.22}$$

$$\Rightarrow \log(\sigma_{\theta\theta})_{\theta=0} = \log(A_0) + (\lambda_0 - 1)\log r \tag{2.23}$$

equation is same the equation of straight line like, $y = ax + b$,where, $a = (\lambda_0 - 1)$,called slope and $b = \log(A_0)$,called intercept of the line.

$$\log A_0 = \frac{Intercept}{\tilde{\sigma}_{theory}} \tag{2.24}$$

where, $\tilde{\sigma}_{\theta\theta}\left(\theta=0\right)=1 \quad \log A_0 = \dfrac{Intercept}{1} = Intercept = b \qquad A_0 = 10^b \qquad (2.25)$

To determine the stress intensity factor, logarithmic distribution of $\left(\sigma_{\theta\theta}\right)_{\theta=0}$ along r has been used from FEM of rigid/power law hardening material.

The order of the stress singularity, $\lambda_0 - 1$, is fully determined by the asymptotic singularity analysis and depends only on power-law hardening exponent, n for bonded power-law hardening material and rigid quarter planes.

The logarithm of radial distribution of stress component near the interface corner along direction at $\theta = 1.875^\circ$, is considered to calculate the stress intensity factor.

Least squares method was used to get slope and intercept for different ranges of r.

For different range of radial distance r, slope and intercept was evaluated numerically and slopes are compared with the order of stress singularity by theoretical analysis (theoretical slope) and most close value is selected. The slopes obtained by least squares method is compared with theoretical slope where, slopes are converged and also diverged for different range of *r*. Numerically calculated slopes which are converged to theoretical value after three decimal point is shown by the following figure and corresponding intercept also shown in this figure. Stress intensity factor is calculated from the intercept using, $A_0 = 10^b$.

Following criteria is considered to determine the stress intensity factor. r_{min} should be enough small as much as possible due to the singularity dominance near the interface edge.

There are numerical error found in the minimum r region, so here, r_{min} is selected far from the numerical error region and r_{max} is changed, stress singularity and stress intensity factor is determined for different ranges of r. Again, selected a new value of r_{min} and r_{max} is changed, stress singularity and stress intensity factor is determined for different ranges of r and the procedure is followed repeatedly.

Slope and stress intensity factor is determined for different ranges of r and plotted in a graph where allowable range from the theoretical slope is considered. For minimum values of r_{max} there are slope and stress intensity factor changes rapidly that means there are numerical error for smaller r_{max} region. For larger r_{max} region is far from the interface so it will give far field with the singular field. Suitable data has been selected after the error range within the allowable range of theoretical slope.

And hence, stress intensity factor is selected from the allowable range taken after three decimal points considering above criteria.

For n=2.4(rigid/power law hardening material), The theoretical calculated eigenvalue, $\lambda_0 = 0.753675$.The theoretical order of stress singularity is $\lambda_0 - 1 = -0.246325$.

The stress intensity factor is determined by numerical analysis using Finite Element Method. The determination procedure of stress intensity factor is shown in Figures (2.12-2.17).

Stress field can be explained on the interface as Eqn.(2.22)

which show the dependence of singular stress field on these two parameters, the stress intensity factor of singular stress field, A_0, and its order, $|\lambda_0 - 1|$, is different. A_0 affects proportionally and $|\lambda_0 - 1|$ affects exponentially.

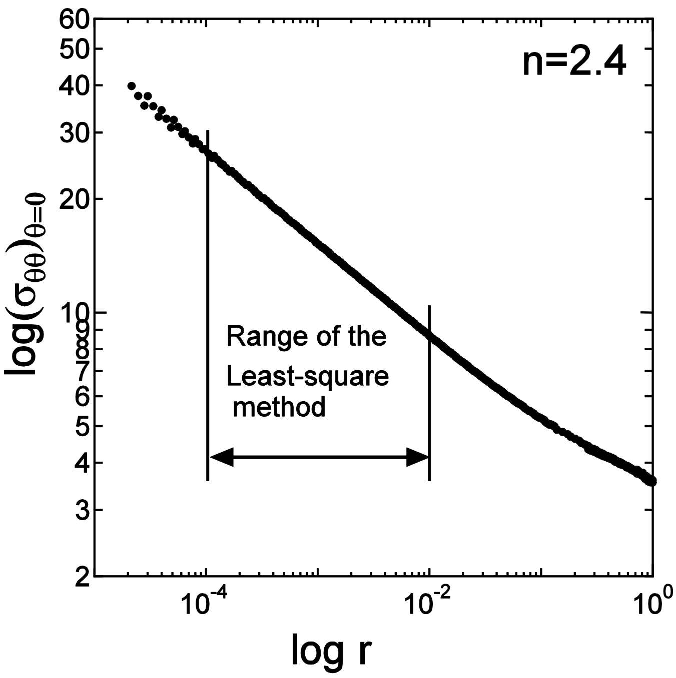


Fig.2.12: log-log plot of $\sigma_{\theta\theta}$ along r in rigid/power-law hardening material for n=2.4

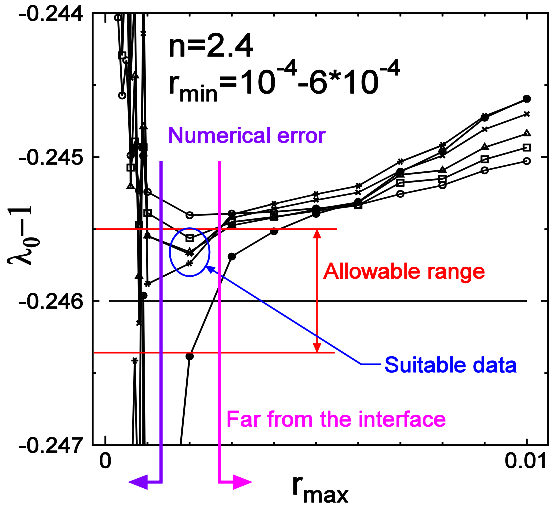


Fig.2.13: Convergence of $(\lambda_0 - 1)$ along r for n=2.4

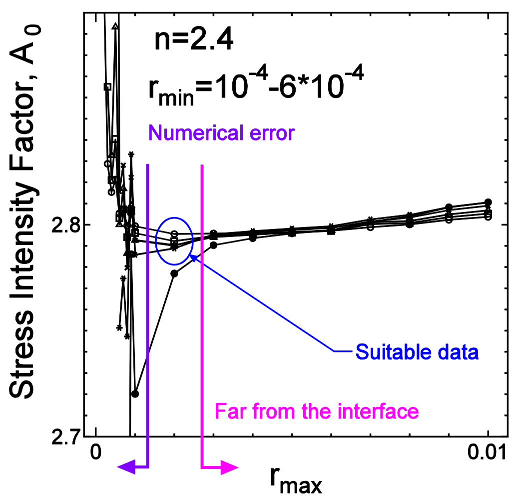


Fig.2.14: Convergence of A_0 along r for n=2.4

From the selected suitable data range, $A_0 = 2.792, 2.790, 2.790$ and 2.789

Finally, $A_0 = 2.790$

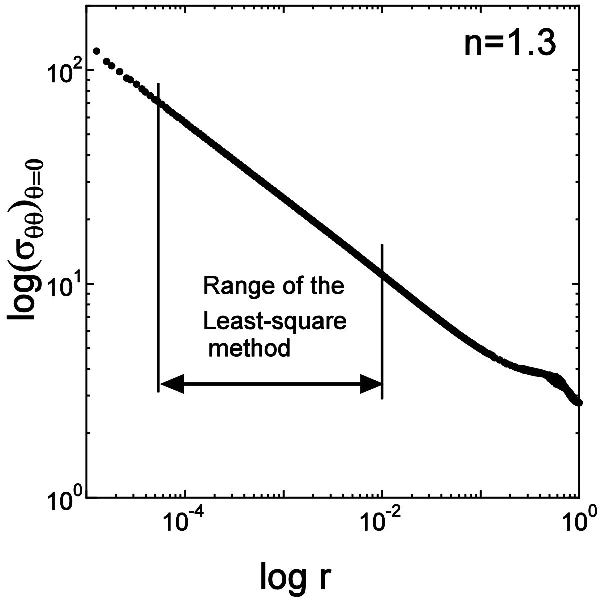


Fig.2.15: log-log plot of $\sigma_{\theta\theta}$ along r in rigid/power-law hardening material for n=1.3

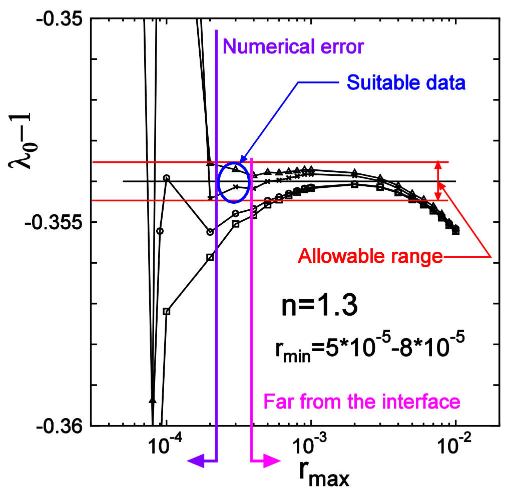


Fig.2.16: Convergence of (λ_0-1) along r for n=1.3

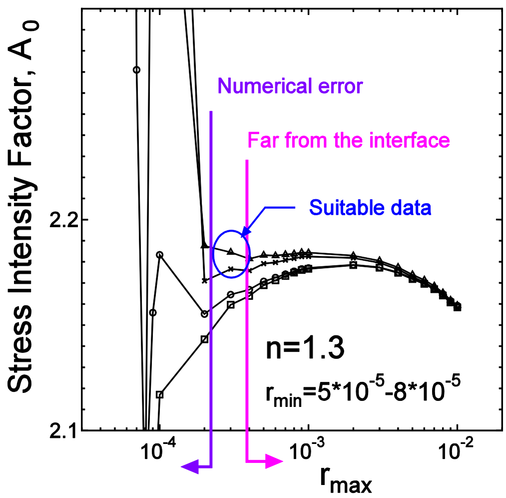


Fig.2.17: Convergence of A_0 along r for n=1.3

From the selected suitable data range, $A_0 = 2.177$ and 2.185. Finally, $A_0 = 2.185$.

2.4 Formulation of 0th Order Approximation

2.4.1 Formulation of 0th Order Approximation: Constitutive Equations in the Power-Law Hardening Material Bonded With Rigid Substrate

Invoking the plane strain condition stress-strain relation can be expressed as:

$$\varepsilon_{rr} = (1+\nu)\{(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{rr} \tag{2.26}$$

$$\varepsilon_{\theta\theta} = (1+\nu)\{(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{\theta\theta} \tag{2.27}$$

$$\varepsilon_{r\theta} = (1+\nu)\sigma_{r\theta} + \frac{3}{2}\alpha\sigma_e^{n-1}s_{r\theta} \tag{2.28}$$

In this approximation, the elastic strains in compared with the plastic strains are small and can be neglected in the asymptotic analysis. Hence according to the plastic deformation theory the three dimensional stress-strain relations take the form given in Eqn. (2.3). The nonlinear term of the strain components for plane strain is written as follows:

$$\varepsilon_{rr} = \alpha\sigma_e^{n-1}\left\{\frac{(2-\nu)}{2}\sigma_{rr} - \frac{(1+\nu)}{2}\sigma_{\theta\theta}\right\} \tag{2.29}$$

$$\varepsilon_{\theta\theta} = \alpha\sigma_e^{n-1}\left\{\frac{(2-\nu)}{2}\sigma_{\theta\theta} - \frac{(1+\nu)}{2}\sigma_{rr}\right\} \tag{2.30}$$

$$\varepsilon_{r\theta} = \frac{3}{2}\alpha\sigma_e^{n-1}\sigma_{r\theta} \tag{2.31}$$

The stress boundary conditions are that the free-edges are traction free and the displacement boundary conditions are given by Eqn. (2.32)

$$\begin{matrix} \left(\sigma^I_{\theta\theta(0)}\right)_{\theta=\frac{\pi}{2}} = 0 & & \left(u^I_{r(0)}\right)_{\theta=0} = 0 \\ & \text{and} & \\ \left(\sigma^I_{r\theta(0)}\right)_{\theta=\frac{\pi}{2}} = 0 & & \left(u^I_{\theta(0)}\right)_{\theta=0} = 0 \end{matrix} \tag{2.32}$$

We can express the stress distribution in terms of stress function following Rice [32]

Airy stress function in a separable form is assumed as,

$$\phi = \phi_0^I = A_0 r^{\lambda_0+1}\tilde{\phi}_0^I, \quad 0 \le \theta \le \frac{\pi}{2} \tag{2.33}$$

Assume, λ_0 depends on the deformation and substituting Eqn.(2.33) into Eqns.(2.5-2.7), we can express the stress distribution in terms of the stress function as [31,32,62]

$$\sigma_{rr0} = A_0 r^{\lambda_0-1}\tilde{\sigma}_{rr0} \tag{2.34}$$

$$\sigma_{\theta\theta0} = A_0 r^{\lambda_0-1}\tilde{\sigma}_{\theta\theta0} \tag{2.35}$$

$$\sigma_{r\theta0} = A_0 r^{\lambda_0-1}\tilde{\sigma}_{r\theta0} \tag{2.36}$$

where

$$\tilde{\sigma}_{rr0} = \tilde{\phi}_0^I(\lambda_0+1) + \left(\tilde{\phi}_0^I\right)'' \tag{2.37}$$

$$\tilde{\sigma}_{\theta\theta0} = \tilde{\phi}_0^I(\lambda_0+1)\lambda_0 \tag{2.38}$$

$$\tilde{\sigma}_{r\theta0} = -\left(\tilde{\phi}_0^I\right)'\lambda_0 \tag{2.39}$$

In this thesis, $(\ )'$ denotes differentiation with respect to θ, and (~) denotes the angular variation of ( ), respectively.

The effective stress term can be expressed in terms of deviatoric stress as,

$$\sigma_e^2 = \frac{3}{2}\left\{s_{ij}s_{ij}\right\} \tag{2.40}$$

$$\sigma_e^2 = \frac{3}{2}\left[s_{rr}^2 + s_{\theta\theta}^2 + s_{zz}^2 + s_{r\theta}^2 + s_{rz}^2 + s_{\theta r}^2 + s_{\theta z}^2 + s_{zr}^2 + s_{z\theta}^2\right] \tag{2.41}$$

Plane strain condition, $s_{zz} = s_{\theta z} = s_{zr} = s_{rz} = s_{z\theta} = 0$ (2.42)

Eqn. (2.41) yields,

$$\sigma_e^2 = \frac{3}{2}\left[s_{rr}^2 + s_{\theta\theta}^2 + 2s_{r\theta}^2 \right] \tag{2.43}$$

Deviatoric components can be written as,

$$s_{rr0} = \frac{1}{2}\left\{ r^{\lambda_0 - 1} f_{0rr} \right\}, \quad s_{\theta\theta 0} = \frac{1}{2}\left\{ r^{\lambda_0 - 1} f_{0\theta\theta} \right\}, \quad s_{r\theta 0} = r^{\lambda_0 - 1} f_{0r\theta} \tag{2.44}$$

where,

$$f_{0rr} = A_0 \left\{ \tilde{\phi}_0^I \left(\lambda_0 + 1\right)\left(1 - \lambda_0\right) + \left(\tilde{\phi}_0^I\right)'' \right\}, \; f_{0\theta\theta} = -A_0 \left\{ \tilde{\phi}_0^I \left(\lambda_0 + 1\right)\left(1 - \lambda_0\right) + \left(\tilde{\phi}_0^I\right)'' \right\}, \; f_{0r\theta} = -A_0 \left(\tilde{\phi}_0^I\right)' \lambda_0 \tag{2.45}$$

Substituting Eqn.(2.44) into Eqn.(2.43) yields an expression for the effective stress,

$$\sigma_e^{n-1} = \left[\frac{3}{8}\left(r^{2(\lambda_0 - 1)} \left(f_{0rr}^2 + f_{0\theta\theta}^2 + 8 f_{0r\theta}^2 \right)\right) \right]^{\frac{n-1}{2}} \tag{2.46}$$

Replacing Eqn.(2.45),Eqn.(2.46) is written as,

$$\sigma_e^{n-1} = A_0^{(n-1)} r^{(n-1)(\lambda_0 - 1)} \left(\frac{3}{8}\right)^{\frac{n-1}{2}} \left(\left\{ \tilde{\phi}_0^I \left(\lambda_0 + 1\right)\left(1 - \lambda_0\right) + \left(\tilde{\phi}_0^I\right)'' \right\}^2 + \left\{ \tilde{\phi}_0^I \left(\lambda_0 + 1\right)\left(1 - \lambda_0\right) + \left(\tilde{\phi}_0^I\right)'' \right\}^2 + 8\left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \tag{2.47}$$

Eqn.(2.47) can be rewritten as,

$$\sigma_e^{n-1} = A_0^{(n-1)} r^{(n-1)(\lambda_0 - 1)} \tilde{\sigma}_e^{n-1} \tag{2.48}$$

Where,

$$\tilde{\sigma}_e^{n-1} = \left(\frac{3}{8}\right)^{\frac{n-1}{2}} \left(\left\{ \tilde{\phi}_0^I (\lambda_0+1)(1-\lambda_0) + \left(\tilde{\phi}_0^I\right)'' \right\}^2 + \left\{ \tilde{\phi}_0^I (\lambda_0+1)(1-\lambda_0) + \left(\tilde{\phi}_0^I\right)'' \right\}^2 + 8\left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \tag{2.49}$$

Substituting all the strain components of nonlinear part of Equations (2.29-2.31) in the strain compatibility equation ,Eqn.(2.8) and replacing stresses as Eqns.(2.5-2.7) in terms of airy stress function the final form of the Eqn.(2.8) yields,

$$\begin{aligned}
&\frac{1}{r}\frac{\partial^2}{\partial r^2}\left\{\sigma_e^{n-1}\left((2-\nu)r\frac{\partial^2\phi}{\partial r^2}-(1+\nu)\frac{\partial\phi}{\partial r}-\frac{(1+\nu)}{r}\frac{\partial^2\phi}{\partial\theta^2}\right)\right\}+\frac{6}{r^2}\frac{\partial^2}{\partial r\partial\theta}\left\{\sigma_e^{n-1}\left(\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right\}\\
&+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left\{\sigma_e^{n-1}\left(\frac{(2-\nu)}{r}\frac{\partial\phi}{\partial r}+\frac{(2-\nu)}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-(1+\nu)\frac{\partial^2\phi}{\partial r^2}\right)\right\}\\
&-\frac{1}{r}\frac{\partial}{\partial r}\left\{\sigma_e^{n-1}\left(\frac{(2-\nu)}{r}\frac{\partial\phi}{\partial r}+\frac{(2-\nu)}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-(1+\nu)\frac{\partial^2\phi}{\partial r^2}\right)\right\}=0
\end{aligned} \tag{2.50}$$

Eqn.(2.50) includes four terms. Internal part of each terms can be calculated separately as,

$$\begin{aligned}
&(2-\nu)r\frac{\partial^2\phi}{\partial r^2}-(1+\nu)\frac{\partial\phi}{\partial r}-\frac{(1+\nu)}{r}\frac{\partial^2\phi}{\partial\theta^2}\\
&=A_0 r^{\lambda_0}\left\{\left\{(2-\nu)(\lambda_0+1)\lambda_0-(1+\nu)(\lambda_0+1)\right\}\tilde{\phi}_0^I-(1+\nu)\left(\tilde{\phi}_0^I\right)''\right\}
\end{aligned} \tag{2.51}$$

$$\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial\phi}{\partial\theta}=A_0 r^{\lambda_0}\lambda_0\left(\tilde{\phi}_0^I\right)' \tag{2.52}$$

$$\begin{aligned}
&\pm\left(\frac{(2-\nu)}{r}\frac{\partial\phi}{\partial r}+\frac{(2-\nu)}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-(1+\nu)\frac{\partial^2\phi}{\partial r^2}\right)\\
&=\pm\left\{A_0 r^{\lambda_0-1}\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right\}
\end{aligned} \tag{2.53}$$

Substituting Eqns.(2.51-2.53) into Eqn.(2.50), compatibility equation becomes,

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial^2}{\partial r^2}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0}\left\{\left\{(2-\nu)(\lambda_0+1)\lambda_0-(1+\nu)(\lambda_0+1)\right\}\tilde{\phi}_0^I-(1+\nu)\left(\tilde{\phi}_0^I\right)''\right\}\right\} \\
&+\frac{6}{r^2}\frac{\partial^2}{\partial r\partial\theta}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0}\lambda_0\left(\tilde{\phi}_0^I\right)'\right\} \\
&+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0-1}\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right\} \\
&-\frac{1}{r}\frac{\partial}{\partial r}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0-1}\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right\}=0
\end{aligned}
\tag{2.54}
$$

Derivatives of Eqn.(2.48) are as,

$$
\frac{\partial\sigma_e^{n-1}}{\partial r}=A_0^{n-1}(n-1)(\lambda_0-1)r^{(n-1)(\lambda_0-1)-1}\tilde{\sigma}_e^{n-1} \tag{2.55}
$$

$$
\frac{\partial^2\sigma_e^{n-1}}{\partial r^2}=A_0^{n-1}(n-1)(\lambda_0-1)\left\{(n-1)(\lambda_0-1)-1\right\}r^{(n-1)(\lambda_0-1)-2}\tilde{\sigma}_e^{n-1} \tag{2.56}
$$

$$
\frac{\partial^2\sigma_e^{n-1}}{\partial r\partial\theta}=A_0^{n-1}(n-1)(\lambda_0-1)r^{(n-1)(\lambda_0-1)-2}\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta} \tag{2.57}
$$

1st term of Eqn.(2.54) can be represented as:

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial^2}{\partial r^2}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0}\left\{\left\{(2-\nu)(\lambda_0+1)\lambda_0-(1+\nu)(\lambda_0+1)\right\}\tilde{\phi}_0^I-(1+\nu)\left(\tilde{\phi}_0^I\right)''\right\}\right\} \\
&=A_0^n r^{(n\lambda_0-n-2)}\Big[(\lambda_0-1)\Big((n-1)\left\{(n-1)(\lambda_0-1)-1\right\}+2\lambda_0(n-1)+\lambda_0\Big)\tilde{\sigma}_e^{n-1} \\
&\times\left\{\left\{(2-\nu)(\lambda_0+1)\lambda_0-(1+\nu)(\lambda_0+1)\right\}\tilde{\phi}_0^I-(1+\nu)\left(\tilde{\phi}_0^I\right)''\right\}\Big]
\end{aligned}
\tag{2.58}
$$

2nd term of Eqn.(2.54) can be represented as:

$$
\begin{aligned}
&\frac{6}{r^2}\frac{\partial^2}{\partial r\partial\theta}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0}\lambda_0\left(\tilde{\phi}_0^I\right)'\right\} \\
&=6A_0^n r^{(n\lambda_0-n-2)}\left\{\lambda_0(n\lambda_0-n+1)\left(\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\left(\tilde{\phi}_0^I\right)'+\tilde{\sigma}_e^{n-1}\left(\tilde{\phi}_0^I\right)''\right)\right\}
\end{aligned}
\tag{2.59}
$$

3rd term of Eqn.(2.54) can be represented as:

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0-1}\left\{\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right\}\right\}$$

$$\begin{aligned}
&= A_0^n r^{(n\lambda_0-n-2)}\Bigg[\Bigg\{\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right)\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\Bigg\}\\
&\quad+2\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\left(\tilde{\phi}_0^I\right)'+(2-\nu)\left(\tilde{\phi}_0^I\right)'''\right)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\\
&\quad+\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\left(\tilde{\phi}_0^I\right)''+(2-\nu)\left(\tilde{\phi}_0^I\right)^{(4)}\right)\tilde{\sigma}_e^{n-1}\Bigg]
\end{aligned} \tag{2.60}$$

4th term of Eqn.(2.54) can be represented as:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left\{\sigma_e^{n-1}A_0 r^{\lambda_0-1}\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right)\right\}$$

$$=-A_0^n r^{(n\lambda_0-n-2)}\left\{\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right)n(\lambda_0-1)\tilde{\sigma}_e^{n-1}\right\} \tag{2.61}$$

Replacing Eqns.(2.58-2.61) into Eqn. (2.54) compatibility equation is expressed as,

$$\begin{aligned}
&A_0^n r^{(n\lambda_0-n-2)}\left\{(\lambda-1)\left((n-1)\left\{(n-1)(\lambda_0-1)-1\right\}+2\lambda_0(n-1)+\lambda_0\right)\tilde{\sigma}_e^{n-1}\left\{\begin{Bmatrix}(2-\nu)(\lambda_0+1)\lambda_0\\-(1+\nu)(\lambda_0+1)\end{Bmatrix}\tilde{\phi}_0^I-(1+\nu)\left(\tilde{\phi}_0^I\right)''\right\}\right\}\\
&+6A_0^n r^{(n\lambda_0-n-2)}\left\{\lambda_0(n\lambda_0-n+1)\left(\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\left(\tilde{\phi}_0^I\right)'+\tilde{\sigma}_e^{n-1}\left(\tilde{\phi}_0^I\right)''\right)\right\}\\
&+A_0^n r^{(n\lambda_0-n-2)}\Bigg[\Bigg\{\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right)\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\Bigg\}\\
&\quad+2\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\left(\tilde{\phi}_0^I\right)'+(2-\nu)\left(\tilde{\phi}_0^I\right)'''\right)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\\
&\quad+\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\left(\tilde{\phi}_0^I\right)''+(2-\nu)\left(\tilde{\phi}_0^I\right)^{(4)}\right)\tilde{\sigma}_e^{n-1}\Bigg]\\
&-A_0^n r^{(n\lambda_0-n-2)}\left\{\left(\left\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\right\}\tilde{\phi}_0^I+(2-\nu)\left(\tilde{\phi}_0^I\right)''\right)n(\lambda_0-1)\tilde{\sigma}_e^{n-1}\right\}=0
\end{aligned} \tag{2.62}$$

$$
\begin{aligned}
&A_0^n r^{(n\lambda_0-n-2)} \\
&\times\Bigg[\Bigg\{(\lambda_0-1)\Big((n-1)\big\{(n-1)(\lambda_0-1)-1\big\}+2\lambda_0(n-1)+\lambda_0\Big)\tilde{\sigma}_e^{n-1}\Bigg\{\big\{(2-\nu)(\lambda_0+1)\lambda_0-(1+\nu)(\lambda_0+1)\big\}\tilde{\phi}_0^I-(1+\nu)\big(\tilde{\phi}_0^I\big)''\Bigg\}\Bigg\} \\
&\qquad+6\left\{\lambda_0(n\lambda_0-n+1)\left(\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\big(\tilde{\phi}_0^I\big)'+\tilde{\sigma}_e^{n-1}\big(\tilde{\phi}_0^I\big)''\right)\right\} \\
&\qquad+\left\{\left(\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\tilde{\phi}_0^I+(2-\nu)\big(\tilde{\phi}_0^I\big)''\right)\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\right\} \\
&\qquad+2\left(\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\big(\tilde{\phi}_0^I\big)'+(2-\nu)\big(\tilde{\phi}_0^I\big)'''\right)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta} \\
&\qquad+\left(\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\big(\tilde{\phi}_0^I\big)''+(2-\nu)\big(\tilde{\phi}_0^I\big)^{(4)}\right)\tilde{\sigma}_e^{n-1} \\
&\qquad-\left\{\left(\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\tilde{\phi}_0^I+(2-\nu)\big(\tilde{\phi}_0^I\big)''\tilde{\phi}_0^I\right)n(\lambda_0-1)\tilde{\sigma}_e^{n-1}\right\}\Bigg]=0
\end{aligned}
\tag{2.63}
$$

$$
\begin{aligned}
&(\lambda_0-1)\Big((n-1)\big\{(n-1)(\lambda_0-1)-1\big\}+2\lambda_0(n-1)+\lambda_0\Big)\big\{(2-\nu)(\lambda_0+1)\lambda_0-(1+\nu)(\lambda_0+1)\big\}\tilde{\sigma}_e^{n-1}\tilde{\phi}_0^I \\
&-\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}n(\lambda_0-1)\tilde{\sigma}_e^{n-1}\tilde{\phi}_0^I \\
&+\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\tilde{\phi}_0^I \\
&+6\lambda_0(n\lambda_0-n+1)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\big(\tilde{\phi}_0^I\big)'+2\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\big(\tilde{\phi}_0^I\big)' \\
&-(\lambda_0-1)(1+\nu)\Big((n-1)\big\{(n-1)(\lambda_0-1)-1\big\}+2\lambda_0(n-1)+\lambda_0\Big)\tilde{\sigma}_e^{n-1}\big(\tilde{\phi}_0^I\big)'' \\
&+6\lambda_0(n\lambda_0-n+1)\tilde{\sigma}_e^{n-1}\big(\tilde{\phi}_0^I\big)''+(2-\nu)\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\big(\tilde{\phi}_0^I\big)''-(2-\nu)n(\lambda_0-1)\tilde{\sigma}_e^{n-1}\big(\tilde{\phi}_0^I\big)'' \\
&+\big\{(2-\nu)(\lambda_0+1)-(1+\nu)(\lambda_0+1)\lambda_0\big\}\tilde{\sigma}_e^{n-1}\big(\tilde{\phi}_0^I\big)'' \\
&+2(2-\nu)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\big(\tilde{\phi}_0^I\big)'''+(2-\nu)\tilde{\sigma}_e^{n-1}\big(\tilde{\phi}_0^I\big)^{(4)}=0
\end{aligned}
\tag{2.64}
$$

$$
\begin{aligned}
&\left(2n^2\lambda_0^4 - n^2\nu\lambda_0^4 - 3n^2\lambda_0^3 + 3n\lambda_0^3 - n^2\lambda_0^2 - 3n\lambda_0^2 + 2n^2\nu\lambda_0^2 + 3n^2\lambda_0 - 3n\lambda_0 - n^2 + 3n + n^2\nu\right)\tilde{\sigma}_e^{n-1}\tilde{\phi}_0^I \\
&+\left\{-\nu\lambda_0^2 - 2\nu\lambda_0 - \lambda_0^2 + \lambda_0 - \nu + 2\right\}\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\tilde{\phi}_0^I \\
&+\left\{-2\nu\lambda_0^2 - 2\lambda_0^2 - 6n\lambda_0 + 6n\lambda_0^2 - 4\nu\lambda_0 + 8\lambda_0 - 2\nu + 4\right\}\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\left(\tilde{\phi}_0^I\right)' \\
&\left\{\begin{array}{l} -n^2\lambda_0^2 + 2\lambda_0 n^2 - \nu n^2\lambda_0^2 + 2\lambda_0\nu n^2 - \nu n^2 - n^2 - 9n\lambda_0 \\ +3n - \lambda_0^2 + 6\lambda_0^2 n + 7\lambda_0 - \nu\lambda_0^2 - 2\nu\lambda_0 - \nu + 2 \end{array}\right\}\tilde{\sigma}_e^{n-1}\left(\tilde{\phi}_0^I\right)'' \\
&+\left(2-\nu\right)\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}\left(\tilde{\phi}_0^I\right)'' \\
&+2\left(2-\nu\right)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\left(\tilde{\phi}_0^I\right)''' \\
&+\left(2-\nu\right)\tilde{\sigma}_e^{n-1}\left(\tilde{\phi}_0^I\right)^{(4)} = 0
\end{aligned}
$$

(2.65)

Co efficient are：

$$
\begin{aligned}
&\left(\tilde{\phi}_0^I\right)^{(4)} : \quad \left(2-\nu\right)\tilde{\sigma}_e^{n-1} \\
&\left(\tilde{\phi}_0^I\right)''' : \quad \left(4-2\nu\right)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta} \\
&\left(\tilde{\phi}_0^I\right)'' : \quad \left\{\begin{array}{l} -n^2\lambda_0^2 + 2\lambda_0 n^2 - \nu n^2\lambda_0^2 + 2\lambda_0\nu n^2 - \nu n^2 - n^2 - 9n\lambda_0 \\ +3n - \lambda_0^2 + 6\lambda_0^2 n + 7\lambda_0 - \nu\lambda_0^2 - 2\nu\lambda_0 - \nu + 2 \end{array}\right\}\tilde{\sigma}_e^{n-1} + \left(2-\nu\right)\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2} \\
&\left(\tilde{\phi}_0^I\right)' \quad : \quad \left\{-2\nu\lambda_0^2 - 2\lambda^2 - 6n\lambda_0 + 6n\lambda^2 - 4\nu\lambda_0 + 8\lambda_0 - 2\nu + 4\right\}\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta} \\
&\quad\tilde{\phi}_0^I \quad : \quad \left\{2n^2\lambda_0^4 - n^2\nu\lambda_0^4 - 3n^2\lambda_0^3 + 3n\lambda_0^3 - n^2\lambda_0^2 - 3n\lambda_0^2 + 2n^2\nu\lambda_0^2 + 3n^2\lambda_0 - 3n\lambda_0 - n^2 + 3n + n^2\nu\right\}\tilde{\sigma}_e^{n-1} \\
&\qquad\qquad + \left\{-\nu\lambda_0^2 - 2\nu\lambda_0 - \lambda_0^2 + \lambda_0 - \nu + 2\right\}\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}
\end{aligned}
$$

(2.66)

Effective stress term can be calculated applying plane strain condition as:

$$
\sigma_e^2 = \left(\nu^2 - \nu + 1\right)\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2 \qquad (2.67)
$$

$$\tilde{\sigma}_e^{n-1} = \Big[\left(\lambda_0 + 1\right)^2 \left\{ \left(1 + \lambda_0^2\right)\left(\nu^2 - \nu + 1\right) + \lambda_0 \left(2\nu^2 - 2\nu - 1\right) \right\} \left(\tilde{\phi}_0^I\right)^2$$

$$+ \left(\lambda_0 + 1\right) \left\{ 2\left(\nu^2 - \nu + 1\right) + \left(2\nu^2 - 2\nu - 1\right)\lambda_0 \right\} \tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)'' + \left(\nu^2 - \nu + 1\right) \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 3\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)' \right)^2 \Big]^{\frac{n-1}{2}}$$

(2.68)

$$\frac{\partial \tilde{\sigma}_e^{n-1}}{\partial \theta} = \frac{n-1}{2} \Big[2\left(\lambda_0 + 1\right)^2 \left\{ \left(1 + \lambda_0^2\right)\left(\nu^2 - \nu + 1\right) + \lambda_0 \left(2\nu^2 - 2\nu - 1\right) \right\} \left(\tilde{\phi}_0^I\right)' \tilde{\phi}_0^I$$

$$+ \left(\lambda_0 + 1\right) \left\{ 2\left(\nu^2 - \nu + 1\right) + \left(2\nu^2 - 2\nu - 1\right)\lambda_0 \right\} \left\{ \left(\tilde{\phi}_0^I\right)' \left(\tilde{\phi}_0^I\right)'' + \tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)''' \right\}$$

$$+ 2\left(\nu^2 - \nu + 1\right) \left(\tilde{\phi}_0^I\right)''' \left(\tilde{\phi}_0^I\right)'' + 6\lambda_0^2 \left(\tilde{\phi}_0^I\right)'' \left(\tilde{\phi}_0^I\right)' \Big]$$

$$\times \Big\{ \left(\lambda_0 + 1\right)^2 \left\{ \left(1 + \lambda_0^2\right)\left(\nu^2 - \nu + 1\right) + \lambda_0 \left(2\nu^2 - 2\nu - 1\right) \right\} \left(\tilde{\phi}_0^I\right)^2$$

$$+ \left(\lambda_0 + 1\right) \left\{ 2\left(\nu^2 - \nu + 1\right) + \left(2\nu^2 - 2\nu - 1\right)\lambda_0 \right\} \tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)'' + \left(\nu^2 - \nu + 1\right) \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 3\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)' \right)^2 \Big\}^{\frac{n-3}{2}}$$

(2.69)

$$
\begin{aligned}
\frac{\partial^2 \tilde{\sigma}_e^{n-1}}{\partial\theta^2} = \frac{n-1}{2}\Bigg[& 2\left(\lambda_0+1\right)^2\left\{\left(1+\lambda_0^2\right)\left(\nu^2-\nu+1\right)+\lambda_0\left(2\nu^2-2\nu-1\right)\right\}\left\{\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)''+\left(\left(\tilde{\phi}_0^I\right)'\right)^2\right\} \\
&+\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\left\{\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)'''+\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)^{(4)}\right\} \\
&+2\left(\nu^2-\nu+1\right)\left\{\left(\tilde{\phi}_0^I\right)^{(4)}\left(\tilde{\phi}_0^I\right)''+\left(\left(\tilde{\phi}_0^I\right)'''\right)^2\right\}+6\lambda_0^2\left\{\left(\tilde{\phi}_0^I\right)'''\left(\tilde{\phi}_0^I\right)'+\left(\left(\tilde{\phi}_0^I\right)''\right)^2\right\}\Bigg] \\
\times\Bigg[&\left(\lambda_0+1\right)^2\left\{\left(1+\lambda_0^2\right)\left(\nu^2-\nu+1\right)+\lambda\left(2\nu^2-2\nu-1\right)\right\}\left(\tilde{\phi}_0^I\right)^2 \\
&+\left(\lambda+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda\right\}\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)'' \\
&+\left(\nu^2-\nu+1\right)\left(\left(\tilde{\phi}_0^I\right)''\right)^2+3\lambda_0^2\left(\left(\tilde{\phi}_0^I\right)'\right)^2\Bigg]^{\frac{n-3}{2}} \\
+\frac{n-1}{2}\Bigg[& 2\left(\lambda_0+1\right)^2\left\{\left(1+\lambda_0^2\right)\left(\nu^2-\nu+1\right)+\lambda_0\left(2\nu^2-2\nu-1\right)\right\}\left(\tilde{\phi}_0^I\right)'\tilde{\phi}_0^I \\
&+\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\left\{\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)''+\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)'''\right\} \\
&+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)'''\left(\tilde{\phi}_0^I\right)''+6\lambda_0^2\left(\tilde{\phi}_0^I\right)''\left(\tilde{\phi}_0^I\right)'\Bigg] \\
\times\frac{n-3}{2}\Bigg[& 2\left(\lambda_0+1\right)^2\left\{\left(1+\lambda_0^2\right)\left(\nu^2-\nu+1\right)+\lambda_0\left(2\nu^2-2\nu-1\right)\right\}\left(\tilde{\phi}_0^I\right)'\tilde{\phi}_0^I \\
&+\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\left\{\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)''+\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)'''\right\} \\
&+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)'''\left(\tilde{\phi}_0^I\right)''+6\lambda_0^2\left(\tilde{\phi}_0^I\right)''\left(\tilde{\phi}_0^I\right)'\Bigg] \\
\times\Bigg[&\left(\lambda_0+1\right)^2\left\{\left(1+\lambda_0^2\right)\left(\nu^2-\nu+1\right)+\lambda_0\left(2\nu^2-2\nu-1\right)\right\}\left(\tilde{\phi}_0^I\right)^2 \\
&+\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)''+\left(\nu^2-\nu+1\right)\left(\left(\tilde{\phi}_0^I\right)''\right)^2+3\lambda_0^2\left(\left(\tilde{\phi}_0^I\right)'\right)^2\Bigg]^{\frac{n-5}{2}}
\end{aligned}
\tag{2.70}
$$

From $\frac{\partial^2 \tilde{\sigma}_e^{n-1}}{\partial\theta^2}$, the terms included $\left(\tilde{\phi}_0^I\right)^{(4)}$

$$\left[\frac{n-1}{2}\left\{\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)''\right\}f_1\left(\theta\right)^{\frac{n-3}{2}}\right]\times\left(\tilde{\phi}_0^I\right)^{(4)}$$

(2.71)

Where,

$$f_1\left(\theta\right)=\Big[\left(\lambda_0+1\right)^2\left\{\left(1+\lambda_0^2\right)\left(\nu^2-\nu+1\right)+\lambda_0\left(2\nu^2-2\nu-1\right)\right\}\left(\tilde{\phi}_0^I\right)^2$$

$$+\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)''+\left(\nu^2-\nu+1\right)\left(\left(\left(\tilde{\phi}_0^I\right)''\right)^2+3\lambda_0^2\left(\left(\tilde{\phi}_0^I\right)'\right)^2\right)\Big]$$

(2.72)

The coefficient of $\left(\tilde{\phi}_0^I\right)^{(4)}$ becomes,

$$\left[\frac{n-1}{2}\left\{\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)''\right\}f_1\left(\theta\right)^{\frac{n-3}{2}}\right]$$

(2.73)

From the compatibility equation, the coefficient of $\left(\tilde{\phi}_0^I\right)^{(4)}$ becomes from the $\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2}$ term,

$$\left[\left(2-\nu\right)\left(\tilde{\phi}_0^I\right)''+\left\{-\nu\lambda_0^2-2\nu\lambda_0-\lambda_0^2+\lambda_0-\nu+2\right\}\tilde{\phi}_0^I\right]\times\frac{\partial^2\tilde{\sigma}_e^{n-1}}{\partial\theta^2} \quad (2.74)$$

So, the term included $\left(\tilde{\phi}_0^I\right)^{(4)}$ in the compatibility equation,

$$\left[\left(2-\nu\right)\left(\tilde{\phi}_0^I\right)''+\left\{-\nu\lambda_0^2-2\nu\lambda_0-\lambda_0^2+\lambda_0-\nu+2\right\}\tilde{\phi}_0^I\right]$$

$$\times\left[\frac{n-1}{2}\left\{\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)''\right\}f_1\left(\theta\right)^{\frac{n-3}{2}}\right]\times\left(\tilde{\phi}_0^I\right)^{(4)}$$

(2.75)

From the compatibility equation, the term included $\left(\tilde{\phi}_0^I\right)^{(4)}$ becomes from the $\tilde{\sigma}_e^{n-1}$ term is,

$$\left(2-\nu\right)\tilde{\sigma}_e^{n-1}\times\left(\tilde{\phi}_0^I\right)^{(4)}=\left(2-\nu\right)f_1\left(\theta\right)^{\frac{n-3}{2}}\times\left(\tilde{\phi}_0^I\right)^{(4)} \tag{2.76}$$

where the coefficient of $\left(\tilde{\phi}_0^I\right)^{(4)}$ is, $\left(2-\nu\right)f_1\left(\theta\right)^{\frac{n-3}{2}}$

So, the total terms included $\left(\tilde{\phi}_0^I\right)^{(4)}$ becomes,

$$\begin{aligned}&\left\{\left(2-\nu\right)f_1\left(\theta\right)^{\frac{n-3}{2}}+\left[\left(2-\nu\right)\left(\tilde{\phi}_0^I\right)''+\left\{-\nu\lambda_0^2-2\nu\lambda_0-\lambda_0^2+\lambda_0-\nu+2\right\}\tilde{\phi}_0^I\right]\right.\\&\left.\times\left[\frac{n-1}{2}\left\{\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)''\right\}f_1\left(\theta\right)^{\frac{n-3}{2}}\right]\right\}\end{aligned} \tag{2.77}$$

So, Compatibility equation becomes in the form of

$$\left(\tilde{\phi}_0^I\right)^{(4)}=-\frac{C}{B}\left(\left(\tilde{\phi}_0^I\right)''',\left(\tilde{\phi}_0^I\right)'',\left(\tilde{\phi}_0^I\right)',\tilde{\phi}_0^I,n,\lambda_0\right) \tag{2.78}$$

Where,

$$\begin{aligned}B=&\left(2-\nu\right)f_1\left(\theta\right)^{\frac{n-3}{2}}+\left[\left(2-\nu\right)\left(\tilde{\phi}_0^I\right)''+\left\{-\nu\lambda_0^2-2\nu\lambda_0-\lambda_0^2+\lambda_0-\nu+2\right\}\tilde{\phi}_0^I\right]\\&\times\left[\frac{n-1}{2}\left\{\left(\lambda_0+1\right)\left\{2\left(\nu^2-\nu+1\right)+\left(2\nu^2-2\nu-1\right)\lambda_0\right\}\tilde{\phi}_0^I+2\left(\nu^2-\nu+1\right)\left(\tilde{\phi}_0^I\right)''\right\}f_1\left(\theta\right)^{\frac{n-3}{2}}\right]\end{aligned} \tag{2.79}$$

And,

$$\begin{aligned}C=&\left(4-2\nu\right)\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\times\left(\tilde{\phi}_0^I\right)'''+\left\{\left(-n^2\lambda_0^2+2\lambda_0n^2-\nu n^2\lambda_0^2+2\lambda_0\nu n^2-\nu n^2-n^2-9n\lambda_0\right.\right.\\&\left.\left.+3n-\lambda_0^2+6\lambda_0^2n+7\lambda_0-\nu\lambda_0^2-2\nu\lambda_0-\nu+2\right)\tilde{\sigma}_e^{n-1}+\left(2-\nu\right)k\right\}\left(\tilde{\phi}_0^I\right)''\\&+\left\{-2\nu\lambda_0^2-2\lambda_0^2-6n\lambda_0+6n\lambda_0^2-4\nu\lambda_0+8\lambda_0-2\nu+4\right\}\frac{\partial\tilde{\sigma}_e^{n-1}}{\partial\theta}\times\left(\tilde{\phi}_0^I\right)'\\&+\left\{\left(2n^2\lambda_0{}^4-n^2\nu\lambda_0^4-3n^2\lambda_0^3+3n\lambda_0^3-n^2\lambda_0^2-3n\lambda_0^2+2n^2\nu\lambda_0^2+3n^2\lambda_0-3n\lambda_0-n^2+3n+n^2\nu\right)\tilde{\sigma}_e^{n-1}\right.\\&\left.+\left\{-\nu\lambda_0^2-2\nu\lambda_0-\lambda_0^2+\lambda_0-\nu+2\right\}k\right\}\times\tilde{\phi}_0^I\end{aligned} \tag{2.80}$$

Where, $f_1(\theta)$ is expressed in Eqn.(2.72)

and

$$
\begin{aligned}
k = \frac{n-1}{2}\Bigg[& 2(\lambda_0+1)^2\left\{(1+\lambda_0^2)(\nu^2-\nu+1)+\lambda_0(2\nu^2-2\nu-1)\right\}\left\{\tilde{\phi}_0^I\times(\tilde{\phi}_0^I)''+\left((\tilde{\phi}_0^I)'\right)^2\right\}\\
&+(\lambda_0+1)\left\{2(\nu^2-\nu+1)+(2\nu^2-2\nu-1)\lambda_0\right\}\left\{\left((\tilde{\phi}_0^I)''\right)^2+2(\tilde{\phi}_0^I)'\times(\tilde{\phi}_0^I)'''\right\}\\
&+2(\nu^2-\nu+1)\left((\tilde{\phi}_0^I)'''\right)^2+6\lambda_0^2\left\{(\tilde{\phi}_0^I)'''\times(\tilde{\phi}_0^I)'+\left((\tilde{\phi}_0^I)''\right)^2\right\}\Bigg]\\
\times\Bigg[& (\lambda_0+1)^2\left\{(1+\lambda_0^2)(\nu^2-\nu+1)+\lambda_0(2\nu^2-2\nu-1)\right\}(\tilde{\phi}_0^I)^2\\
&+(\lambda_0+1)\left\{2(\nu^2-\nu+1)+(2\nu^2-2\nu-1)\lambda_0\right\}\tilde{\phi}_0^I\times(\tilde{\phi}_0^I)''\\
&+(\nu^2-\nu+1)\left((\tilde{\phi}_0^I)''\right)^2+3\lambda_0^2\left((\tilde{\phi}_0^I)'\right)^2\Bigg]^{\frac{n-3}{2}}\\
+\frac{n-1}{2}\Bigg[& 2(\lambda_0+1)^2\left\{(1+\lambda_0^2)(\nu^2-\nu+1)+\lambda_0(2\nu^2-2\nu-1)\right\}(\tilde{\phi}_0^I)'\times\tilde{\phi}_0^I\\
&+(\lambda_0+1)\left\{2(\nu^2-\nu+1)+(2\nu^2-2\nu-1)\lambda_0\right\}\left\{(\tilde{\phi}_0^I)'\times(\tilde{\phi}_0^I)''+\tilde{\phi}_0^I\times(\tilde{\phi}_0^I)'''\right\}\\
&+2(\nu^2-\nu+1)(\tilde{\phi}_0^I)'''\times(\tilde{\phi}_0^I)''+6\lambda_0^2(\tilde{\phi}_0^I)''\times(\tilde{\phi}_0^I)'\Bigg]\\
\times\frac{n-3}{2}\Bigg[& 2(\lambda_0+1)^2\left\{(1+\lambda_0^2)(\nu^2-\nu+1)+\lambda_0(2\nu^2-2\nu-1)\right\}(\tilde{\phi}_0^I)'\times\tilde{\phi}_0^I\\
&+(\lambda_0+1)\left\{2(\nu^2-\nu+1)+(2\nu^2-2\nu-1)\lambda_0\right\}\left\{(\tilde{\phi}_0^I)'\times(\tilde{\phi}_0^I)''+\tilde{\phi}_0^I\times(\tilde{\phi}_0^I)'''\right\}\\
&+2(\nu^2-\nu+1)(\tilde{\phi}_0^I)'''\times(\tilde{\phi}_0^I)''+6\lambda_0^2(\tilde{\phi}_0^I)''\times(\tilde{\phi}_0^I)'\Bigg]\\
\times\Bigg[& (\lambda_0+1)^2\left\{(1+\lambda_0^2)(\nu^2-\nu+1)+\lambda_0(2\nu^2-2\nu-1)\right\}(\tilde{\phi}_0^I)^2\\
&+(\lambda_0+1)\left\{2(\nu^2-\nu+1)+(2\nu^2-2\nu-1)\lambda_0\right\}\tilde{\phi}_0^I\times(\tilde{\phi}_0^I)''+(\nu^2-\nu+1)\left((\tilde{\phi}_0^I)''\right)^2+3\lambda_0^2\left((\tilde{\phi}_0^I)'\right)^2\Bigg]^{\frac{n-5}{2}}
\end{aligned}
\tag{2.81}
$$

Equation (2.78) is fourth-order ordinary differential equation. For the solution of fourth-order equation, the equation is reduced into a system of first-order equations. And, therefore, equation is solved using the Runge-Kutta method.

Stress fields are expressed in Eqns. (2.34-2.36) where the effective stress as,

$$\sigma_{e(0)}^{I} = A_0\, r^{\lambda_0 - 1} \times 2^{1-n} \times 3^{\frac{n-1}{2}} \left\{ \left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\left(\tilde{\phi}_0^I\right)''\right) + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \right\}^{\frac{1}{2}} \tag{2.82}$$

Applying plane strain condition strain components are expressed as:

$$\varepsilon_{rr0}^{I} = A_0^n\, r^{n(\lambda_0 - 1)} \tilde{\varepsilon}_{rr0}^{I} \tag{2.83}$$

$$\varepsilon_{\theta\theta 0}^{I} = A_0^n\, r^{n(\lambda_0 - 1)} \tilde{\varepsilon}_{\theta\theta 0}^{I} \tag{2.84}$$

$$\varepsilon_{r\theta 0}^{I} = A_0^n\, r^{n(\lambda_0 - 1)} \tilde{\varepsilon}_{r\theta 0}^{I} \tag{2.85}$$

where,

$$\tilde{\varepsilon}_{rr0}^{I} = 2^{-1-n} 3^{\frac{n+1}{2}} \alpha \left\{ \left(1 - \lambda_0^2\right)\tilde{\phi}_0^I + \left(\tilde{\phi}_0^I\right)'' \right\} \left(\left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)'' + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \right)^{\frac{n-1}{2}} \tag{2.86}$$

$$\tilde{\varepsilon}_{\theta\theta 0}^{I} = 2^{-1-n} 3^{\frac{n+1}{2}} \alpha \left\{ \left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I - \left(\tilde{\phi}_0^I\right)'' \right\} \left(\left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)'' + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \right)^{\frac{n-1}{2}} \tag{2.87}$$

$$\tilde{\varepsilon}_{r\theta 0}^{I} = -2^{-n} 3^{\frac{n+1}{2}} \alpha \lambda_0 \left(\tilde{\phi}_0^I\right)' \left(\left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)'' + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \right)^{\frac{n-1}{2}} \tag{2.88}$$

From Equation (2.9) displacement can be written as,

$$u_{r0}^{I} = A_0^n\, r^{n\lambda_0 - n + 1} \tilde{u}_{r0}^{I} \tag{2.89}$$

where,

$$\tilde{u}_{r0}^{I} = \frac{2^{-1-n} 3^{\frac{n+1}{2}} \alpha}{\left(n\lambda_0 - n + 1\right)} \times \left\{ \left(1-\lambda_0^2\right)\tilde{\phi}_0^I + \left(\tilde{\phi}_0^I\right)'' \right\} \left(\left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)'' + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \right)^{\frac{n-1}{2}} \tag{2.90}$$

Eqn.(2.11) can be rewritten as, $r\dfrac{\partial u_\theta}{\partial r} - u_\theta = 2r\varepsilon_{r\theta} - \dfrac{\partial u_r}{\partial \theta}$ (2.91)

We assume,

$u_\theta = kr^{n\lambda_0 - n+1} f\left(\theta\right) \therefore \dfrac{\partial u_\theta}{\partial r} = \left(n\lambda_0 - n + 1\right) kr^{n\lambda_0 - n+1-1} f\left(\theta\right);\ or,\ \ r\dfrac{\partial u_\theta}{\partial r} = \left(n\lambda_0 - n + 1\right) kr^{n\lambda_0 - n+1} f\left(\theta\right)$

So equation becomes, When $\lambda_0 \neq 1$,

$$kr^{n\lambda_0 - n+1} f\left(\theta\right)\left\{n\left(\lambda_0 - 1\right)\right\} = 2r\varepsilon_{r\theta} - \frac{\partial u_r}{\partial \theta} \tag{2.92}$$

$$u_{\theta 0}^{I} = A_0^n r^{n\lambda_0 - n+1} \tilde{u}_{\theta 0}^{I} \tag{2.93}$$

where,

$$\begin{aligned}
\tilde{u}_{\theta 0}^{I} = & \frac{-2^{-1-n} 3^{\frac{n+1}{2}} \alpha}{\left\{n\left(\lambda_0 - 1\right)\right\}\left(n\lambda_0 - n + 1\right)} \left\{ \left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\left(\tilde{\phi}_0^I\right)''\right) + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \right\}^{\frac{n-3}{2}} \\
& \times \Bigg[4\lambda_0^2 \left\{ \left(4n\left(\lambda_0 - 1\right) - \lambda_0 + 4\right)\lambda_0 + 1 \right\} \left(\tilde{\phi}_0^I\right)''' + \left\{ \left(7n - 4\right)\lambda_0^2 - 4\left(n-1\right)\lambda_0 + n \right\} \left(\left(\tilde{\phi}_0^I\right)''\right)^2 \left(\tilde{\phi}_0^I\right)' \\
& + \left\{ 4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 + n\left(\left(\tilde{\phi}_0^I\right)''\right)^2 \right\} \left(\tilde{\phi}_0^I\right)''' + 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\left(\tilde{\phi}_0^I\right)''\right) \left\{ -\left\{ \left(5n - 2\right)\lambda_0^2 - 4\left(n-1\right)\lambda_0 + n \right\} \left(\tilde{\phi}_0^I\right)' - n\left(\tilde{\phi}_0^I\right)''' \right\} \\
& + \left(\lambda_0^2 - 1\right)^2 \left(\tilde{\phi}_0^I\right)^2 \left\{ \left(3n\lambda_0^2 - 4n\lambda_0 + 4\lambda_0 + n\right)\left(\tilde{\phi}_0^I\right)' + n\left(\tilde{\phi}_0^I\right)''' \right\} \Bigg]
\end{aligned} \tag{2.94}$$

When $\lambda_0 = 1$, the displacement is infinite. Obviously, infinite displacement does not occur in reality. To determine the displacement field we have to use another expression.

We know from the strain displacement relation from Eqn.(2.10):

$$u_{\theta(0)}^{I} = A_0^n r^{n\lambda_0 - n+1} \tilde{u}_{\theta(0)}^{I} \tag{2.95}$$

where,

$$\tilde{u}^I_{\theta(0)} = -2^{-n}3^{\frac{n+1}{2}}\alpha\left(\left(\tilde{\phi}^I_0\right)''\right)\int_{\frac{\pi}{2}}^{0}\left\{\left(\left(\tilde{\phi}^I_0\right)''\right)^2+4\left(\left(\tilde{\phi}^I_0\right)'\right)^2\right\}^{\frac{n-1}{2}}d\theta-\int_{\frac{\pi}{2}}^{0}\left(\left(\tilde{\phi}^I_0\right)'''\right)\left[\int_{\frac{\pi}{2}}^{0}\left\{\left(\left(\tilde{\phi}^I_0\right)''\right)^2+4\left(\left(\tilde{\phi}^I_0\right)'\right)^2\right\}^{\frac{n-1}{2}}d\theta\right]d\theta \tag{2.96}$$

The boundary condition for traction free and clamped condition is given in Eqn. (2.32) and applying boundary condition, Initial conditions at $\theta=\dfrac{\pi}{2}$: From equation (2.38) & (2.39)

$$\left(\lambda_0+1\right)\lambda_0\tilde{\phi}^I_0=0,\quad \tilde{\phi}^I_0=0,\ \text{ if }\quad \lambda_0\neq -1,\ \lambda_0\neq 0 \tag{2.97}$$

$$-\lambda_0\left(\tilde{\phi}^I_0\right)'=0,\quad \left(\tilde{\phi}^I_0\right)'=0,\ \text{ if }\ \lambda_0\neq 0 \tag{2.98}$$

Unknown are: $\left(\tilde{\phi}^I_0\right)''$, $\left(\tilde{\phi}^I_0\right)'''$, λ_0 After integration final conditions, at $\theta=0\,,\tilde{u}^I_{r(0)}=0$ and $\tilde{u}^I_{\theta(0)}=0$

$$\begin{aligned}&\frac{2^{-1-n}3^{\frac{n+1}{2}}\alpha}{\left(n\lambda_0-n+1\right)}\left(\left(1-\lambda_0^2\right)\tilde{\phi}^I_0+\left(\tilde{\phi}^I_0\right)''\right)\\&\times\left\{\left(\lambda_0^2-1\right)^2\left(\tilde{\phi}^I_0\right)^2-2\left(\lambda_0^2-1\right)\tilde{\phi}^I_0\left(\left(\tilde{\phi}^I_0\right)''\right)+\left(\left(\tilde{\phi}^I_0\right)''\right)^2+4\lambda_0^2\left(\left(\tilde{\phi}^I_0\right)'\right)^2\right\}^{\frac{n-1}{2}}=0\end{aligned} \tag{2.99}$$

$$\begin{aligned}&-\frac{2^{-1-n}3^{\frac{n+1}{2}}\alpha}{\left\{n\left(\lambda_0-1\right)\right\}\left(n\lambda_0-n+1\right)}\left\{\left(\lambda_0^2-1\right)^2\left(\tilde{\phi}^I_0\right)^2-2\left(\lambda_0^2-1\right)\tilde{\phi}^I_0\left(\left(\tilde{\phi}^I_0\right)''\right)+\left(\left(\tilde{\phi}^I_0\right)''\right)^2+4\lambda_0^2\left(\left(\tilde{\phi}^I_0\right)'\right)^2\right\}^{\frac{n-3}{2}}\\&\times\Bigg[4\lambda_0^2\left\{\left(4n\left(\lambda_0-1\right)-\lambda_0+4\right)\lambda_0+1\right\}\left(\tilde{\phi}^I_0\right)'''+\left\{\left(7n-4\right)\lambda_0^2-4\left(n-1\right)\lambda_0+n\right\}\left(\left(\tilde{\phi}^I_0\right)''\right)^2\left(\tilde{\phi}^I_0\right)'\\&+\left\{4\lambda_0^2\left(\left(\tilde{\phi}^I_0\right)'\right)^2+n\left(\left(\tilde{\phi}^I_0\right)''\right)^2\right\}\left(\tilde{\phi}^I_0\right)'''+2\left(\lambda_0^2-1\right)\tilde{\phi}^I_0\left(\left(\tilde{\phi}^I_0\right)''\right)\left\{-\left\{\left(5n-2\right)\lambda_0^2-4\left(n-1\right)\lambda_0+n\right\}\left(\tilde{\phi}^I_0\right)'-n\left(\tilde{\phi}^I_0\right)'''\right\}\\&+\left(\lambda_0^2-1\right)^2\left(\tilde{\phi}^I_0\right)^2\left\{\left(3n\lambda_0^2-4n\lambda_0+4\lambda_0+n\right)\left(\tilde{\phi}^I_0\right)'+n\left(\tilde{\phi}^I_0\right)'''\right\}\Bigg]=0\end{aligned} \tag{2.100}$$

Assume at $\theta = \dfrac{\pi}{2}$, $\tilde{\phi}_0^I = 0, \left(\tilde{\phi}_0^I\right)' = 0, \left(\tilde{\phi}_0^I\right)'' = 1.0,$ $\left(\tilde{\phi}_0^I\right)'''$, λ_0 and after i-th integration

at $\theta = 0,$ $\tilde{\phi}_0^I = \tilde{\phi}_0^{I(i)}$, $\left(\tilde{\phi}_0^I\right)' = \left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}, \left(\tilde{\phi}_0^I\right)'' = \left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}$ and $\left(\tilde{\phi}_0^I\right)''' = \left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$

$$\text{error1} = \frac{2^{-1-n}3^{\frac{n+1}{2}}}{\left(n\lambda_0 - n + 1\right)}\left(\left(1-\lambda_0^2\right)\tilde{\phi}_0^{I(i)} + \left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}\right)$$
$$\times\left\{\left(\lambda_0^2-1\right)^2\left(\tilde{\phi}_0^{I(i)}\right)^2 - 2\left(\lambda_0^2-1\right)\tilde{\phi}_0^{I(i)}\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)} + \left(\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}\right)^2 + 4\lambda_0^2\left(\left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}\right)^2\right\}^{\frac{n-1}{2}}$$
(2.101)

$$\text{error2} = -\frac{2^{-1-n}3^{\frac{n+1}{2}}}{\left\{n\left(\lambda_0-1\right)\right\}\left(n\lambda_0 - n + 1\right)}$$
$$\times\left\{\left(\lambda_0^2-1\right)^2\left(\tilde{\phi}_0^{I(i)}\right)^2 - 2\left(\lambda_0^2-1\right)\tilde{\phi}_0^{I(i)}\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)} + \left(\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}\right)^2 + 4\lambda_0^2\left(\left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}\right)^2\right\}^{\frac{n-3}{2}}$$
$$\times\left[4\lambda_0^2\left\{\left(4n\left(\lambda_0-1\right)-\lambda_0+4\right)\lambda_0+1\right\}\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)} + \left\{\left(7n-4\right)\lambda_0^2 - 4\left(n-1\right)\lambda_0 + n\right\}\left(\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}\right)^2\left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}\right.$$
$$+\left\{4\lambda_0^2\left(\left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}\right)^2 + n\left(\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}\right)^2\right\}\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)} + 2\left(\lambda_0^2-1\right)\tilde{\phi}_0^{I(i)}\left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}$$
$$\times\left\{-\left\{\left(5n-2\right)\lambda_0^2 - 4\left(n-1\right)\lambda_0 + n\right\}\left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)} - n\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}\right\}$$
$$\left.+\left(\lambda_0^2-1\right)^2\left(\tilde{\phi}_0^{I(i)}\right)^2\left\{\left(3n\lambda_0^2 - 4n\lambda_0 + 4\lambda_0 + n\right)\left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)} + n\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}\right\}\right]$$
(2.102)

The error value is then calculated by using the Equation (2.15) applying following procedure: After integration, solution region is obtained with different error ranges which systematically cover the range of $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 for different error. From the minimum error region, the range of $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 is selected as the initial value for the integration of next step. The final region of the minimum error is taken with the region of $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 contains two equal digits after the decimal point. The final solution is obtained from this minimum error region and calculated numerically the solution point where the minimum error occurs. The minimum error region is integrated with 200 smaller divisions and minimum error point is selected as the solution point and the corresponding $\left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$ and λ_0 is calculated as the solution.

After tentative solution, to get the exact solution it is necessary to correct the initial value of $\left(\tilde{\phi}_0^I\right)''$ which was assumed as 1. To get the exact value, it is necessary to satisfy $\left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0} = 1$.So initial value of $\left(\tilde{\phi}_0^I\right)''$ becomes, $\left(\tilde{\phi}_0^I\right)'' = 1.0 \big/ \left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0}^T$, where $\left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0}^T$ is the tentative solution obtained assuming $\left(\tilde{\phi}_0^I\right)'' = 1.0$.

Assume at $\theta = \dfrac{\pi}{2}$, $\tilde{\phi}_0^I = 0$, $\left(\tilde{\phi}_0^I\right)' = 0$, $\left(\tilde{\phi}_0^I\right)'' = \dfrac{1.0}{\left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0}^T}$, $\left(\tilde{\phi}_0^I\right)'''$, λ_0 and after i-th integration at $\theta = 0$, $\tilde{\phi}_0^I = \tilde{\phi}_0^{I(i)}$, $\left(\tilde{\phi}_0^I\right)' = \left(\left(\tilde{\phi}_0^I\right)'\right)^{(i)}$, $\left(\tilde{\phi}_0^I\right)'' = \left(\left(\tilde{\phi}_0^I\right)''\right)^{(i)}$ and $\left(\tilde{\phi}_0^I\right)''' = \left(\left(\tilde{\phi}_0^I\right)'''\right)^{(i)}$. The solution procedure is same as described above for seeking the minimum error. Due to the change of $\left(\tilde{\phi}_0^I\right)'' = \dfrac{1.0}{\left(\tilde{\sigma}_{\theta\theta}^I\right)_{\theta=0}^T}$, all angular function terms changes with the same ratio for the exact solution.

Once singular exponent, λ_0 is known the angular variation of stresses can be computed. To compute the stresses we need to calculate unknown angular functions to satisfy traction on the interface.

2.4.2 Formulation of 0th Order Approximation: Constitutive Equations in the Elastic Material Subjected to Traction

The traction boundary conditions are that the free-edges are traction free and that the $\sigma_{\theta\theta,}\sigma_{r\theta}$ are continuous at the interface. Mathematically this can be expressed by:

$$\left(\sigma^{II}_{\theta\theta(0)}\right)_{\theta=-\frac{\pi}{2}}=0 \qquad \text{and} \qquad \left(\sigma^{II}_{\theta\theta(0)}\right)_{\theta=0}=\left(\sigma^{I}_{\theta\theta(0)}\right)_{\theta=0}$$
$$\left(\sigma^{II}_{r\theta(0)}\right)_{\theta=-\frac{\pi}{2}}=0 \qquad\qquad \left(\sigma^{II}_{r\theta(0)}\right)_{\theta=0}=\left(\sigma^{I}_{r\theta(0)}\right)_{\theta=0} \tag{2.103}$$

Assumed, Airy stress function $\phi=\phi_0^{II}=A_0\,r^{\lambda_0+1}\tilde{\phi}_0^{II}$ (2.104)

Compatibility equation becomes in the form of,

$$\frac{d^4\tilde{\phi}_0^{II}}{d\theta^4}=-\left(1-\lambda_0^2\right)^2\tilde{\phi}_0^{II}-2\left(\lambda_0^2+1\right)\frac{d^2\tilde{\phi}_0^{II}}{d\theta^2}\,. \tag{2.105}$$

Equation (2.105) is the fourth-order ordinary differential equation. For the solution of fourth-order equation, the equation is reduced into a system of first-order equations. And, therefore, equation is solved using the Runge-Kutta method.

Stress fields can be expressed as:

$$\sigma^{II}_{rr(0)}=A_0\,r^{\lambda_0-1}\tilde{\sigma}^{II}_{rr(0)} \tag{2.106}$$

$$\sigma^{II}_{\theta\theta(0)}=A_0\,r^{\lambda_0-1}\tilde{\sigma}^{II}_{\theta\theta(0)} \tag{2.107}$$

$$\sigma^{II}_{r\theta(0)}=A_0\,r^{\lambda_0-1}\tilde{\sigma}^{II}_{r\theta(0)} \tag{2.108}$$

where,

$$\tilde{\sigma}^{II}_{rr(0)}=\left(\lambda_0+1\right)\tilde{\phi}_0^{II}+\left(\tilde{\phi}_0^{II}\right)'' \tag{2.109}$$

$$\tilde{\sigma}^{II}_{\theta\theta(0)}=\left(\lambda_0+1\right)\lambda_0\tilde{\phi}_0^{II} \tag{2.110}$$

$$\tilde{\sigma}^{II}_{r\theta(0)}=-\lambda_0\left(\tilde{\phi}_0^{II}\right)' \tag{2.111}$$

Applying plane strain condition strain fields are expressed as,

$$\varepsilon_{rr}^{II} = \frac{E^{I}}{E^{II}}\left\{\left(1+\nu^{II}\right)\left\{\left(1-\nu^{II}\right)\sigma_{rr}^{II} - \nu^{II}\sigma_{\theta\theta}^{II}\right\}\right\} \tag{2.112}$$

$$\varepsilon_{\theta\theta}^{II} = \frac{E^{I}}{E^{II}}\left\{\left(1+\nu^{II}\right)\left\{\left(1-\nu^{II}\right)\sigma_{\theta\theta}^{II} - \nu^{II}\sigma_{rr}^{II}\right\}\right\} \tag{2.113}$$

$$\varepsilon_{r\theta}^{II} = \frac{E^{I}}{E^{II}}\left\{\left(1+\nu^{II}\right)\sigma_{r\theta}^{II}\right\} \tag{2.114}$$

Displacement fields are also expressed by the following equations,

$$u_{r(0)}^{II} = \int_{0}^{r}\varepsilon_{rr(0)}^{II}dr = \frac{E^{I}}{E^{II}}\left\{\left(1+\nu^{II}\right)\left[\left(1-\nu^{II}\right)\int\sigma_{rr(0)}^{II}dr - \nu^{II}\int\sigma_{\theta\theta(0)}^{II}dr\right]\right\} + \left(u_{r(0)}^{II}\right)_{r=0} \tag{2.115}$$

We assume $\left(u_{r(0)}^{II}\right)_{r=0} = 0.$ Displacement fields are,

$$u_{r(0)}^{II} = A_{0}\, r^{\lambda_0}\tilde{u}_{r(0)}^{II} \tag{2.116}$$

where,

$$\tilde{u}_{r(0)}^{II} = \frac{E^{I}}{E^{II}}\left\{\frac{\left(1+\nu^{II}\right)}{\lambda_0}\left[\left(\lambda_0+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_0\right)\tilde{\phi}_0^{II} + \left(1-\nu^{II}\right)\left(\tilde{\phi}_0^{II}\right)''\right]\right\} \tag{2.117}$$

From equation (2.10) we can deduce the displacement equation as,

$$u_{\theta}^{II} = r\int_{-\frac{\pi}{2}}^{\theta}\varepsilon_{\theta\theta}^{II}d\theta - \int_{-\frac{\pi}{2}}^{\theta}u_{r}^{II}d\theta + \left(u_{\theta(0)}^{II}\right)_{\theta=-\frac{\pi}{2}} \tag{2.118}$$

$\left(u_{\theta(0)}^{II}\right)_{\theta=-\frac{\pi}{2}}$ can be a function of r or a constant. We assume $\left(u_{\theta(0)}^{II}\right)_{r=0} = 0$,

$\left(u^{II}_{\theta(0)}\right)_{\theta=-\frac{\pi}{2}} = C_0 r^d \quad , d \neq 0$ and $\left(\varepsilon^{II}_{\theta\theta}\right)_{\theta=-\frac{\pi}{2}} = \left(\frac{1}{r}u^{II}_{r(0)} + \frac{1}{r}\frac{\partial u^{II}_{\theta(0)}}{\partial\theta}\right)_{\theta=-\frac{\pi}{2}}$ If

$d=1,\ \left(u^{II}_{\theta(0)}\right)_{\theta=-\frac{\pi}{2}} = C_0 r$.

This is rigid body rotation. Neither strain nor stress occurs due to the rotation. So we can neglect it for the discussion of stress or strain. But, for the displacement we have to consider rigid body rotation.

$$u^{II}_{\theta(0)} = A_0\, r^{\lambda_0} \tilde{u}^{II}_{\theta(0)} \tag{2.119}$$

where,

$$\tilde{u}^{II}_{\theta(0)} = \frac{E^I}{E^{II}} \frac{\left(1+\nu^{II}\right)}{\lambda_0} \left[\left(\lambda_0+1\right)\left(\lambda_0^2-1\right)\left(1-\nu^{II}\right) \int_{-\frac{\pi}{2}}^{\theta} \tilde{\phi}^{II}_0 d\theta - \left(\nu^{II}\lambda_0 + 1 - \nu^{II}\right)\left(\tilde{\phi}^{II}_0\right)' \right] \tag{2.120}$$

Using Equation (2.11) another expression of strain, displacement can be written as:

$$\left(\varepsilon^{II}_{r\theta}\right)_{\theta=-\frac{\pi}{2}} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u^{II}_{r(0)}}{\partial\theta} + C_0 d r^{d-1} - C_0 r^{d-1}\right)_{\theta=-\frac{\pi}{2}} \tag{2.121}$$

Or, $$r\frac{\partial u^{II}_{\theta}}{\partial r} - u^{II}_{\theta} = 2r\varepsilon^{II}_{r\theta} - \frac{\partial u^{II}_{r}}{\partial\theta} \tag{2.122}$$

Rigid body rotation is depends on r and angular function. To overcome the rigid body rotation we have to assume the displacement as a function of r and the angular function term.

Assume, $u^{II}_{\theta} = kr^{\lambda_0} f(\theta) \therefore \frac{\partial u^{II}_{\theta}}{\partial r} = \lambda_0 k r^{\lambda_0 - 1} f(\theta);\ or,\ \ r\frac{\partial u^{II}_{\theta}}{\partial r} = \lambda_0 k r^{\lambda_0} f(\theta)$

When $\lambda_0 \neq 1$, equation becomes, $$kr^{\lambda_0} f(\theta)\left(\lambda_0 - 1\right) = 2r\varepsilon^{II}_{r\theta} - \frac{\partial u^{II}_{r}}{\partial\theta} \tag{2.123}$$

Displacement u_θ can be expressed as,

$$u^{II}_{\theta(0)} = A_0\, r^{\lambda_0} \tilde{u}^{II}_{\theta(0)} \tag{2.124}$$

where,

$$\tilde{u}^{II}_{\theta(0)} = -\frac{E^{I}}{E^{II}}\frac{\left(1+\nu^{II}\right)}{\lambda_0\left(\lambda_0-1\right)}\left[\left\{2\lambda_0^2+\left(\lambda_0+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_0\right)\right\}\left(\tilde{\phi}_0^{II}\right)'+\left(1-\nu^{II}\right)\left(\tilde{\phi}_0^{II}\right)'''\right] \quad (2.125)$$

Boundary equation on the interface at $\theta = 0$ can be expressed as,

$$\sigma^{II}_{\theta\theta(0)} = A_0\, r^{\lambda_0-1}\left(\lambda_0+1\right)\lambda_0\tilde{\phi}_0^{II} = A_0\, r^{\lambda_0-1}\left(\lambda_0+1\right)\lambda_0\tilde{\phi}_0^{I}, \quad (2.126)$$

$$\sigma^{II}_{r\theta(0)} = -A_0\, r^{\lambda_0-1}\lambda_0\left(\tilde{\phi}_0^{II}\right)' = -A_0\, r^{\lambda_0-1}\lambda_0\left(\tilde{\phi}_0^{I}\right)'. \quad (2.127)$$

Finally, governing differential equation (Eqn. (2.105)) is solved to satisfy the traction boundary condition on the interface.

Assume at $\theta = -\frac{\pi}{2}$, $\tilde{\phi}_0^{II} = 0, \left(\tilde{\phi}_0^{II}\right)' = 0, \left(\tilde{\phi}_0^{II}\right)''$ and $\left(\tilde{\phi}_0^{II}\right)'''$ and after i-th integration

at $\theta = 0,\ \tilde{\phi}_0^{II} = \tilde{\phi}_0^{II(i)},\ \left(\tilde{\phi}_0^{II}\right)' = \left(\left(\tilde{\phi}_0^{II}\right)'\right)^{(i)}, \left(\tilde{\phi}_0^{II}\right)'' = \left(\left(\tilde{\phi}_0^{II}\right)''\right)^{(i)}$ and $\left(\tilde{\phi}_0^{II}\right)''' = \left(\left(\tilde{\phi}_0^{II}\right)'''\right)^{(i)}$

where,

$$\mathrm{error1} = \tilde{\phi}_0^{I} - \left(\tilde{\phi}_0^{II}\right)^{(i)} \quad (2.128)$$

and

$$\mathrm{error2} = \left(\left(\tilde{\phi}_0^{I}\right)'\right) - \left(\left(\tilde{\phi}_0^{II}\right)'\right)^{(i)} \quad (2.129)$$

The error value is then calculated by using the Equation (2.15) and solution is obtained for the minimum error.

2.5 Formulation of 1st Order Approximation

2.5.1 Formulation of 1st Order Approximation: Constitutive Equations in the Power-Law Hardening Material Subjected To Forced Displacement

In the following the Airy stress function and other equations assuming only to the first order approximation can be described for the illustration of solution method. Assumed, λ_0 and λ_1 depend on the deformation and

$$\phi = \phi_0^I + \phi_1^I = A_0\, r^{\lambda_0+1}\tilde{\phi}_0^I + A_1\, r^{\lambda_1+1}\tilde{\phi}_1^I \,. \tag{2.130}$$

Substituting Eqn. (2.130) into Eqns.(2.5-2.8), stress fields are expressed as,

$$\sigma_{rr} = A_0 r^{\lambda_0-1}\left\{ \tilde{\phi}_0^I\left(\lambda_0+1\right)+\left(\tilde{\phi}_0^I\right)''\right\} + A_1\, r^{\lambda_1-1}\left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)+\left(\tilde{\phi}_1^I\right)''\right\} \tag{2.131}$$

$$\sigma_{\theta\theta} = A_0\, r^{\lambda_0-1}\tilde{\phi}_0^I\left(\lambda_0+1\right)\lambda_0 + A_1\, r^{\lambda_1-1}\tilde{\phi}_1^I\left(\lambda_1+1\right)\lambda_1 \tag{2.132}$$

$$\sigma_{r\theta} = -\left(A_0\, r^{\lambda_0-1}\left(\tilde{\phi}_0^I\right)'\lambda_0 + A_1 r^{\lambda_1-1}\left(\tilde{\phi}_1^I\right)'\lambda_1 \right) \tag{2.133}$$

$$s_{rr} = \frac{1}{2}\left\{ r^{\lambda_0-1} f_{0rr} + A_1\, r^{\lambda_1-1} f_{1rr} \right\} \tag{2.134}$$

$$s_{\theta\theta} = \frac{1}{2}\left\{ r^{\lambda_0-1} f_{0\theta\theta} + A_1\, r^{\lambda_1-1} f_{1\theta\theta} \right\} \tag{2.135}$$

$$s_{r\theta} = r^{\lambda_0-1} f_{0r\theta} + A_1\, r^{\lambda_1-1} f_{1r\theta} \tag{2.136}$$

where,

$$f_{0rr} = A_0\left\{\tilde{\phi}_0^I\left(\lambda_0+1\right)\left(1-\lambda_0\right)+\left(\tilde{\phi}_0^I\right)''\right\}, \quad f_{1rr} = \left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)\left(1-\lambda_1\right)+\left(\tilde{\phi}_1^I\right)''\right\}$$

$$f_{0\theta\theta} = -A_0\left\{\tilde{\phi}_0^I\left(\lambda_0+1\right)\left(1-\lambda_0\right)+\left(\tilde{\phi}_0^I\right)''\right\}, \quad f_{1\theta\theta} = -\left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)\left(1-\lambda_1\right)+\left(\tilde{\phi}_1^I\right)''\right\}$$

$$f_{0r\theta} = \left\{-A_0\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}, \quad f_{1r\theta} = \left\{-\left(\tilde{\phi}_1^I\right)'\lambda_1\right\} \tag{2.137}$$

Substituting Eqns.(2.134 -2.136) Eqn.(2.43) yields,

$$\sigma_e^{n-1}=\left[\frac{3}{8}\left(r^{2(\lambda_0-1)}\left(f_{0rr}^2+f_{0\theta\theta}^2+8f_{0r\theta}^2\right)+2A_1\,r^{(\lambda_0+\lambda_1-2)}\left(f_{0rr}\,f_{1rr}+f_{0\theta\theta}\,f_{1\theta\theta}+8f_{0r\theta}\,f_{1r\theta}\right)+A_1^2\,r^{2(\lambda_1-1)}\left(f_{1rr}^2+f_{1\theta\theta}^2+8f_{1r\theta}^2\right)\right)\right]^{\frac{n-1}{2}}$$

(2.138)

From the Taylor series expansion,

$$y+dy=\left(x+dx\right)^{\frac{n-1}{2}}=x^{\frac{n-1}{2}}+\frac{n-1}{2}x^{\frac{n-3}{2}}dx+\frac{\left(n-1\right)\left(n-3\right)}{8}x^{\frac{n-5}{2}}\left(dx\right)^2+\cdots$$

(2.139)

In the 1st order approximation, nonlinear effective stress term σ_e^{n-1} is expanded by Taylor series and the first two terms are considered for further calculations. Before expansion this term is written as,

$$\sigma_e^{n-1}=\left[\frac{3}{8}\left(r^{2(\lambda_0-1)}f_0+2A_1\,r^{(\lambda_0+\lambda_1-2)}f_1+A_1^2\,r^{2(\lambda_1-1)}f_2\right)\right]^{\frac{n-1}{2}}, \tag{2.140}$$

where,

$$f_0=\left(f_{0rr}^2+f_{0\theta\theta}^2+8f_{0r\theta}^2\right),\; f_1=\left(f_{0rr}\,f_{1rr}+f_{0\theta\theta}\,f_{1\theta\theta}+8f_{0r\theta}\,f_{1r\theta}\right),\; f_2=\left(f_{1rr}^2+f_{1\theta\theta}^2+8f_{1r\theta}^2\right),$$

$$f_{0rr}=A_0\left\{\tilde{\phi}_0^I\left(\lambda_0+1\right)\left(1-\lambda_0\right)+\left(\tilde{\phi}_0^I\right)''\right\},\; f_{0\theta\theta}=-A_0\left\{\tilde{\phi}_0^I\left(\lambda_0+1\right)\left(1-\lambda_0\right)+\left(\tilde{\phi}_0^I\right)''\right\},\; f_{0r\theta}=\left\{-A_0\left(\tilde{\phi}_0^I\right)'\lambda_0\right\},$$

$$f_{1rr}=\left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)\left(1-\lambda_1\right)+\left(\tilde{\phi}_1^I\right)''\right\},\; f_{1\theta\theta}=-\left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)\left(1-\lambda_1\right)+\left(\tilde{\phi}_1^I\right)''\right\},\; f_{1r\theta}=\left\{-\left(\tilde{\phi}_1^I\right)'\lambda_1\right\}.$$

(2.141)

Before expansion Eqn. (2.140) can be written as,

$$\sigma_e^{n-1}=r^{(\lambda_0-1)(n-1)}\left[\frac{3}{8}\left(f_0+2A_1\,r^{(\lambda_1-\lambda_0)}f_1+A_1^2\,r^{2(\lambda_1-\lambda_0)}f_2\right)\right]^{\frac{n-1}{2}}. \tag{2.142}$$

Assuming smaller range of r $\left(r<1\right)$ near the interface edge it is reasonable to have the singular exponent of incremental stress λ_1 which is larger than the zero-th order singular exponent λ_0, i.e., $\lambda_0<\lambda_1$. The order of r of the terms in the part powered by

$(n-1)/2$ in Eqn. (2.140) is $0, (\lambda_1-\lambda_0)$ and $2(\lambda_1-\lambda_0)$ respectively which means the order of r in the second and third terms are positive in magnitude. Positive power of small r gives the value smaller than 1. Also $A_1<1$, $f_1<1$ and $f_2<1$. The summation of the second and third terms has the smaller magnitude than 1. This satisfies the convergence condition for the Taylor expansion. Assuming the first term as the leading term and remaining terms as the incremental term Taylor expansion is applied to Eqn. (2.140). After expansion and neglecting the higher order term of A_1 the equation becomes,

$$\sigma_e^{n-1} \approx \left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-1}{2}} + \frac{n-1}{2}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2} f_1\right). \tag{2.143}$$

Substituting effective stress term from Eqn.(2.143) into Eqn. (2.3), strain components are expressed as:

$$\begin{aligned}
\varepsilon_{rr} \approx \frac{3}{4}\alpha\Bigg[\Bigg\{\Bigg(&\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-1}{2}} r^{\lambda_0-1} f_{0rr} + A_1 \times \frac{n-1}{2}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} r^{(\lambda_0+\lambda_1-2)} f_1 r^{\lambda_0-1} f_{0rr} \\
&+A_1^2 \times \Bigg(\frac{n-1}{2}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{8} r^{2(\lambda_1-1)} f_2 \\
&+\frac{(n-1)(n-3)}{8}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-5}{2}} \times \left\{\frac{9}{16} r^{2(\lambda_0+\lambda_1-2)} \left(f_1\right)^2\right\}\Bigg) r^{\lambda_0-1} f_{0rr}\Bigg\} \\
&+\Bigg\{A_1\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-1}{2}} r^{\lambda_1-1} f_{1rr} + A_1^2 \times \frac{n-1}{2}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} r^{(\lambda_0+\lambda_1-2)} f_1 r^{\lambda_1-1} f_{1rr} \\
&+A_1^3 \times \Bigg(\frac{n-1}{2}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{8} r^{2(\lambda_1-1)} f_2 \\
&+\frac{(n-1)(n-3)}{8}\left[\frac{3}{8} r^{2(\lambda_0-1)} f_0\right]^{\frac{n-5}{2}} \times \left\{\frac{9}{16} r^{2(\lambda_0+\lambda_1-2)} \left(f_1\right)^2\right\}\Bigg) r^{\lambda_1-1} f_{1rr}\Bigg\}\Bigg]
\end{aligned} \tag{2.144}$$

A_1^3 includes cube of small magnitudes. Which means the terms are very small in compared with the initial term. So, A_1^3 term will be neglected.

$$\varepsilon_{rr} \approx \frac{3}{4}\alpha \left[\left[\frac{3}{8} f_0\right]^{\frac{n-1}{2}} r^{n(\lambda_0-1)} f_{0rr} \right.$$
$$+A_1 \times r^{n\lambda_0-n-\lambda_0+\lambda_1} \left\{ \frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} f_1 f_{0rr} + \left[\frac{3}{8} f_0\right]^{\frac{n-1}{2}} f_{1rr} \right\}$$
$$+A_1^2 \times \left\{ r^{n\lambda_0-3\lambda_0-n+1+2\lambda_1} \left(\frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{8} f_2 + \frac{(n-1)(n-3)}{8}\left[\frac{3}{8} f_0\right]^{\frac{n-5}{2}} \times \frac{9}{16}(f_1)^2 f_{0rr} \right) \right.$$
$$\left.\left. + \frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} r^{n\lambda_0-2\lambda_0-n+2\lambda_1} f_1 f_{1rr} \right\} \right] \tag{2.145}$$

Similarly,

$$\varepsilon_{\theta\theta} \approx \frac{3}{4}\alpha \left[\left[\frac{3}{8} f_0\right]^{\frac{n-1}{2}} r^{n(\lambda_0-1)} f_{0\theta\theta} \right.$$
$$+A_1 \times r^{n\lambda_0-n-\lambda_0+\lambda_1} \left\{ \frac{n-1}{2}\left[\frac{3}{8} f_1\right]^{\frac{n-3}{2}} \times \frac{3}{4} f_1 f_{0\theta\theta} + \left[\frac{3}{8} f_0\right]^{\frac{n-1}{2}} f_{1\theta\theta} \right\}$$
$$+A_1^2 \times \left\{ r^{n\lambda_0-3\lambda_0-n+1+2\lambda_1} \left(\frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{8} f_2 + \frac{(n-1)(n-3)}{8}\left[\frac{3}{8} f_0\right]^{\frac{n-5}{2}} \times \frac{9}{16}(f_1)^2 f_{0\theta\theta} \right) \right.$$
$$\left.\left. + \frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} r^{n\lambda_0-2\lambda_0-n+2\lambda_1} f_1 f_{1\theta\theta} \right\} \right] \tag{2.146}$$

$$\varepsilon_{r\theta} \approx \frac{3}{2}\alpha \Bigg[\left[\frac{3}{8} f_0\right]^{\frac{n-1}{2}} r^{n(\lambda_0-1)} f_{0r\theta}$$

$$+A_1 \times r^{n\lambda_0-n-\lambda_0+\lambda_1} \left\{ \frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} f_1 f_{0r\theta} + \left[\frac{3}{8} f_0\right]^{\frac{n-1}{2}} f_{1r\theta} \right\}$$

$$+A_1^2 \times \Bigg\{ r^{n\lambda_0-3\lambda_0-n+1+2\lambda_1} \left(\frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{8} f_2 + \frac{(n-1)(n-3)}{8}\left[\frac{3}{8} f_0\right]^{\frac{n-5}{2}} \times \frac{9}{16}\left(f_1\right)^2 f_{0r\theta} \right)$$

$$+\frac{n-1}{2}\left[\frac{3}{8} f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4} r^{n\lambda_0-2\lambda_0-n+2\lambda_1} f_1 f_{1r\theta} \Bigg\} \Bigg\} \Bigg] \tag{2.147}$$

Strain components are in the following form according to the order of A_1:

$$\varepsilon_{rr} = \varepsilon_{rr0} + \varepsilon_{rr1}\left\{O(A_1)\right\} + \varepsilon_{rr2}\{O(A_1^2)\} \tag{2.148}$$

$$\varepsilon_{\theta\theta} = \varepsilon_{\theta\theta0} + \varepsilon_{\theta\theta1}\left\{O(A_1)\right\} + \varepsilon_{\theta\theta2}\{O(A_1^2)\} \tag{2.149}$$

$$\varepsilon_{r\theta} = \varepsilon_{r\theta0} + \varepsilon_{r\theta1}\left\{O(A_1)\right\} + \varepsilon_{r\theta2}\{O(A_1^2)\} \tag{2.150}$$

In these expressions strain components will have three terms with respect to the power of r. $r^{n(\lambda_0-1)}$, $r^{n\lambda_0-n-\lambda_0+\lambda_1}$ and $r^{n\lambda_0-3\lambda_0-n+1+2\lambda_1}$. From zero-th order approximation solution it is clear that the terms of $r^{n(\lambda_0-1)}$ satisfy the compatibility condition. To solve the compatibility condition on the remaining terms, we assume the two terms satisfy the conditions independently. At first we will consider it neglecting the third term.

$$\varepsilon_{ij(1)} \approx \varepsilon_{ij0} + \varepsilon_{ij1}\left\{O(A_1)\right\} \tag{2.151}$$

Strain components are in the following summation form:

$$\varepsilon^I_{rr(1)} \approx \varepsilon^I_{rr0} + \varepsilon^I_{rr1} \tag{2.152}$$

$$\varepsilon^I_{\theta\theta(1)} \approx \varepsilon^I_{\theta\theta0} + \varepsilon^I_{\theta\theta1} \tag{2.153}$$

$$\varepsilon^I_{r\theta(1)} \approx \varepsilon^I_{r\theta 0} + \varepsilon^I_{r\theta 1} \tag{2.154}$$

Initial part of strain components are expressed in Eqns. (2.83-2.85). Strain components and derivatives of strain components contains first order term of A_1:

$$\varepsilon^I_{rr1} = A_0^{n-1}\ r^{n\lambda_0 - n - \lambda_0 + \lambda_1}\tilde{\varepsilon}^I_{rr1} \tag{2.155}$$

$$\varepsilon^I_{\theta\theta 1} = A_0^{n-1}\ r^{n\lambda_0 - n - \lambda_0 + \lambda_1}\tilde{\varepsilon}^I_{\theta\theta 1} \tag{2.156}$$

$$\varepsilon^I_{r\theta 1} = A_0^{n-1}\ r^{n\lambda_0 - n - \lambda_0 + \lambda_1}\tilde{\varepsilon}^I_{r\theta 1} \tag{2.157}$$

where,

$$\begin{aligned}\tilde{\varepsilon}^I_{rr1} = 2^{-1-n}3^{\frac{n+1}{2}}\alpha\Bigg[&(n-1)\left(\left\{\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}^I_0\right)''\right)^2 + 2\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}^I_0\right)''\right)\right\} + 4\left\{\left(\tilde{\phi}^I_0\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}}\\
&\times\Bigg(\left\{\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}^I_0\right)''\right)^2 + 2\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}^I_0\right)''\right)\right\}\times(-1)\left\{A_1\tilde{\phi}^I_1\left(\lambda_1+1\right)\left(\lambda_1-1\right) - A_1\left(\tilde{\phi}^I_1\right)''\right\}\\
&+4\left\{\tilde{\phi}^I_0\left(1-\lambda_0^2\right) + \left(\tilde{\phi}^I_0\right)''\right\}\left\{\left(\tilde{\phi}^I_0\right)'\lambda_0\right\}\times\left\{A_1\left(\tilde{\phi}^I_1\right)'\lambda_1\right\}\Bigg)\\
&+\left(\left\{\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}^I_0\right)''\right)^2 + 2\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}^I_0\right)''\right)\right\} + 4\left\{\left(\tilde{\phi}^I_0\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times(-1)\left\{A_1\tilde{\phi}^I_1\left(\lambda_1+1\right)\left(\lambda_1-1\right) - A_1\left(\tilde{\phi}^I_1\right)''\right\}\Bigg]\end{aligned} \tag{2.158}$$

$$\begin{aligned}\tilde{\varepsilon}^I_{\theta\theta 1} = 2^{-1-n}3^{\frac{n+1}{2}}\alpha\Bigg[&(n-1)\left(\left\{\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}^I_0\right)''\right)^2 + 2\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}^I_0\right)''\right)\right\} + 4\left\{\left(\tilde{\phi}^I_0\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}}\\
&\times\Bigg(\left\{\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}^I_0\right)''\right)^2 + 2\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}^I_0\right)''\right)\right\}\times\left\{A_1\tilde{\phi}^I_1\left(\lambda_1+1\right)\left(\lambda_1-1\right) - A_1\left(\tilde{\phi}^I_1\right)''\right\}\\
&+4\left\{\tilde{\phi}^I_0\left(1-\lambda_0^2\right) + \left(\tilde{\phi}^I_0\right)''\right\}\left\{\left(\tilde{\phi}^I_0\right)'\lambda_0\right\}\times(-1)\left\{A_1\left(\tilde{\phi}^I_1\right)'\lambda_1\right\}\Bigg)\\
&+\left(\left\{\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}^I_0\right)''\right)^2 + 2\left(\tilde{\phi}^I_0\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}^I_0\right)''\right)\right\} + 4\left\{\left(\tilde{\phi}^I_0\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times\left\{A_1\tilde{\phi}^I_1\left(\lambda_1+1\right)\left(\lambda_1-1\right) - A_1\left(\tilde{\phi}^I_1\right)''\right\}\Bigg]\end{aligned} \tag{2.159}$$

$$\tilde{\varepsilon}_{r\theta 1}^{I} = 2^{-n}3^{\frac{n+1}{2}}\alpha\left[(n-1)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}}\right.$$

$$\times\left(\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right)+\tilde{\phi}_0''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times\left\{A_1\tilde{\phi}_1^I(\lambda_1+1)(\lambda_1-1)-A_1\left(\tilde{\phi}_1^I\right)''\right\}+4\left\{-\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\times(-1)\left\{A_1\left(\tilde{\phi}_1^I\right)'\lambda_1\right\}\right)$$

$$\left.+\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times(-1)\left\{A_1\left(\tilde{\phi}_1^I\right)'\lambda_1\right\}\right]$$

(2.160)

Derivatives of strain components with respect to r can be written as,

$$\frac{\partial\varepsilon_{rr1}^I}{\partial r} = A_0^{n-1}\ \alpha(n\lambda_0-n-\lambda_0+\lambda_1)r^{(n\lambda_0-n-\lambda_0+\lambda_1-1)}\tilde{\varepsilon}_{rr1}^I \tag{2.161}$$

$$\frac{\partial\varepsilon_{\theta\theta 1}^I}{\partial r} = A_0^{n-1}\ \alpha(n\lambda_0-n-\lambda_0+\lambda_1)r^{(n\lambda_0-n-\lambda_0+\lambda_1-1)}\tilde{\varepsilon}_{\theta\theta 1}^I \tag{2.162}$$

$$\frac{\partial\varepsilon_{r\theta 1}^I}{\partial r} = A_0^{n-1}\ \alpha(n\lambda_0-n-\lambda_0+\lambda_1)r^{(n\lambda_0-n-\lambda_0+\lambda_1-1)}\tilde{\varepsilon}_{r\theta 1}^I \tag{2.163}$$

$$\frac{\partial^2\varepsilon_{rr1}^I}{\partial r^2} = A_0^{n-1}\ (n\lambda_0-n-\lambda_0+\lambda_1)\times\left(n\lambda_0-n-\lambda_0+\lambda_1-1\right)r^{\left(n\lambda_0-n-\lambda_0+\lambda_1-2\right)}\tilde{\varepsilon}_{rr1}^I \tag{2.164}$$

$$\frac{\partial^2\varepsilon_{\theta\theta 1}^I}{\partial r^2} = A_0^{n-1}\ (n\lambda_0-n-\lambda_0+\lambda_1)\times\left(n\lambda_0-n-\lambda_0+\lambda_1-1\right)r^{\left(n\lambda_0-n-\lambda_0+\lambda_1-2\right)}\tilde{\varepsilon}_{\theta\theta 1}^I \tag{2.165}$$

$$\frac{\partial^2\varepsilon_{r\theta 1}^I}{\partial r^2} = A_0^{n-1}\ (n\lambda_0-n-\lambda_0+\lambda_1)\times\left(n\lambda_0-n-\lambda_0+\lambda_1-1\right)r^{\left(n\lambda_0-n-\lambda_0+\lambda_1-2\right)}\tilde{\varepsilon}_{r\theta 1}^I \tag{2.166}$$

From zero-th order approximation solution it is clear that the terms of $r^{n(\lambda_0-1)}$ satisfy the compatibility condition. To solve the compatibility condition on the remaining terms, we assumed the two terms (zero-th order term and first order term of A_1) of strain components satisfy the conditions independently. Finally, initial part of compatibility equation is same as the equation which is satisfied in zero-th order approximation. So, remaining part of compatibility equation should be satisfied independently. Using Eqn. (2.143), the compatibility equation will have three terms with respect to the power of r ,

$r^{n(\lambda_0-1)-2}$, $r^{n\lambda_0-n-\lambda_0+\lambda_1-2}$ and $r^{n\lambda_0-3\lambda_0-n+2\lambda_1-1}$. To solve the compatibility condition, we assume that those three terms satisfy the conditions order by order. Here we will neglect the third term. Assuming second term as the incremental part the compatibility equation includes the exponent of r as $r^{n\lambda_0-n-\lambda_0+\lambda_1-2}$. Hence, in the first order approximation, compatibility equation becomes in the form of,

$$B\times A_1\frac{d^4\tilde{\phi}_1^I}{d\theta^4}=-C\left(A_1\frac{d^3\tilde{\phi}_1^I}{d\theta^3},A_1\frac{d^2\tilde{\phi}_1^I}{d\theta^2},A_1\frac{d\tilde{\phi}_1^I}{d\theta},\ A_1\tilde{\phi}_1^I,\frac{d^4\tilde{\phi}_0^I}{d\theta^4},\frac{d^3\tilde{\phi}_0^I}{d\theta^3},\frac{d^2\tilde{\phi}_0^I}{d\theta^2},\frac{d\tilde{\phi}_0^I}{d\theta},\ \tilde{\phi}_0^I,n,\lambda_0,\lambda_1\right). \tag{2.167}$$

Equation (2.167) is the fourth-order ordinary differential equation of $\tilde{\phi}_1^I$. Within the first order approximation unknowns are $A_1\tilde{\phi}_1^I, A_1\tilde{\phi}_1^{I\prime}, A_1\tilde{\phi}_1^{I\prime\prime}, A_1\tilde{\phi}_1^{I\prime\prime\prime}$ and λ_1. B and C are derived using Mathematica (symbolic mathematics software) [63] and all terms of compatibility equation are presented in appendix; the equation is solved using Runge-Kutta method. Incremental stress fields are expressed as,

$$\sigma_{rr1}^I=A_1 r^{\lambda_1-1}\tilde{\sigma}_{rr1}^I \tag{2.168}$$

$$\sigma_{\theta\theta1}^I=A_1 r^{\lambda_1-1}\tilde{\sigma}_{\theta\theta1}^I \tag{2.169}$$

$$\sigma_{r\theta1}^I=A_1 r^{\lambda_1-1}\tilde{\sigma}_{r\theta1}^I \tag{2.170}$$

where,

$$\tilde{\sigma}_{rr1}^I=\tilde{\phi}_1^I\left(\lambda_1+1\right)+\left(\tilde{\phi}_1^I\right)'' \tag{2.171}$$

$$\tilde{\sigma}_{\theta\theta1}^I=\tilde{\phi}_1^I\left(\lambda_1+1\right)\lambda_1 \tag{2.172}$$

$$\tilde{\sigma}_{r\theta1}^I=-\left(\tilde{\phi}_1^I\right)'\lambda_1 \tag{2.173}$$

Strain components are expressed in Eqn. (2.155-2.157). Displacement component can be written in terms of strain components as,

$$\begin{aligned} u_r &= \int \varepsilon_{rr}dr \\ \Rightarrow u_{r0}^I+u_{r1}^I &\approx \int \varepsilon_{rr0}^I dr+\int \varepsilon_{rr1}^I dr \end{aligned} \tag{2.174}$$

Initial part of displacement components and initial strain components has a relationship like: $u_{r0}^{I} = \int \varepsilon_{rr0}^{I} dr$ expressed in Eqn. (2.89).

So, $u_{r1}^{I} = \int \varepsilon_{rr1}^{I} dr = A_0^{n-1}\ r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} \tilde{u}_{r1}^{I}$ (2.175)

where,

$$\tilde{u}_{r1}^{I} = \frac{2^{-1-n} 3^{\frac{n+1}{2}} \alpha}{(n\lambda_0 - n - \lambda_0 + \lambda_1 + 1)} \Bigg[(n-1) \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right) \right\} + 4\left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-3}{2}}$$

$$\times \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right) \right\} \times (-1) \left\{ A_1 \tilde{\phi}_1^I (\lambda_1+1)(\lambda_1-1) - A_1 \left(\tilde{\phi}_1^I\right)'' \right\} \right.$$

$$\left. + 4\left\{ \tilde{\phi}_0^I \left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I\right)'' \right\} \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\} \times \left\{ A_1 \left(\tilde{\phi}_1^I\right)' \lambda_1 \right\} \right)$$

$$+ \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right) \right\} + 4\left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \times (-1) \left\{ A_1 \tilde{\phi}_1^I (\lambda_1+1)(\lambda_1-1) - A_1 \left(\tilde{\phi}_1^I\right)'' \right\} \Bigg]$$

(2.176)

Finally, $u_{r(1)}^{I} = u_{r0}^{I} + u_{r1}^{I}$ (2.177)

From zero-th order approximation: $\left(u_{r0}^{I}\right)_{\theta=0} = 0$ So, $\left(u_{r(1)}^{I}\right)_{\theta=0} = u_{r1}^{I}$ (2.178)

Displacement at any arbitrary value of θ, $u_{r(1)}^{I} = u_{r0}^{I} + u_{r1}^{I} = A_0^{n}\ r^{n\lambda_0 - n + 1} \tilde{u}_{r0}^{I} + A_0^{n-1}\ r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} \tilde{u}_{r1}^{I}$

(2.179)

Displacement on the interface,

$$\left(u_{r(1)}^{I}\right)_{\theta=0} = \left(u_{r1}^{I}\right)_{\theta=0} = A_0^{n-1}\ r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} \tilde{u}_{r1}^{I} \quad (2.180)$$

Strain displacement relation in Eqn.(2.11)substituting $\varepsilon_{r\theta} \approx \varepsilon_{r\theta 0} + \varepsilon_{r\theta 1}$ and $u_\theta = u_{\theta 0} + u_{\theta 1}$ Strain displacement relation becomes,

$$r\frac{\partial u_{\theta 0}}{\partial r} - u_{\theta 0} + r\frac{\partial u_{\theta 1}}{\partial r} - u_{\theta 1} \approx 2r\varepsilon_{r\theta 0} - \frac{\partial u_{r0}}{\partial \theta} + 2r\varepsilon_{r\theta 1} - \frac{\partial u_{r1}}{\partial \theta} \quad (2.181)$$

Initial part of displacement components and initial part of strain components has a relationship like Eqn.(2.91)

So incremental part,

$$r\frac{\partial u_{\theta 1}}{\partial r} - u_{\theta 1} \approx 2r\varepsilon_{r\theta 1} - \frac{\partial u_{r1}}{\partial \theta} \tag{2.182}$$

Assume, $u_{\theta 1} = Kr^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} f(\theta) \therefore \frac{\partial u_{\theta 1}}{\partial r} = (n\lambda_0 - n - \lambda_0 + \lambda_1 + 1)Kr^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1-1} f(\theta);$

Equation becomes,

$$Kr^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} f(\theta)\{n\lambda_0 - n - \lambda_0 + \lambda_1\} = 2r\varepsilon_{r\theta 1} - \frac{\partial u_{r1}}{\partial \theta} \tag{2.183}$$

When $n\lambda_0 - n - \lambda_0 + \lambda_1 \neq 0$,

$$u_{\theta 1} = \frac{1}{\{n\lambda_0 - n - \lambda_0 + \lambda_1\}}\left\{2r\varepsilon_{r\theta 1} - \frac{\partial u_{r1}}{\partial \theta}\right\} \tag{2.184}$$

When $n\lambda_0 - n - \lambda_0 + \lambda_1 = 0$, the displacement is infinite. Obviously, infinite displacement does not occur in reality. To determine the displacement field we have to use another expression.

where,

$$
\begin{aligned}
&\frac{\partial u_{r1}}{\partial \theta} = \frac{2^{1-n} 3^{\frac{n+1}{2}} A_0^{n-1} \ \alpha r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1}}{(n\lambda_0 - n - \lambda_0 + \lambda_1 + 1)} \\
&\times \Bigg[\frac{(n-3)}{2^3} \Bigg(\Big\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right) \Big\} + 4\left(\left(\tilde{\phi}_0^I\right)' \lambda_0\right)^2 \Bigg)^{\frac{n-3}{2}-1} \\
&\times \Bigg(-4(n-1)\lambda_0\lambda_1 \left(\tilde{\phi}_0^I\right)' \left(\tilde{\phi}_1^I\right)' \left(\tilde{\phi}_0^I\right)'' + n\left(1-\lambda_0^2\right)^2 \left(\tilde{\phi}_0^I\right)^2 \left(\left(\lambda_1^2 - 1\right) A_1 \tilde{\phi}_1^I - A_1 \left(\tilde{\phi}_1^I\right)''\right) \\
&+4\left(\left(\tilde{\phi}_0^I\right)' \lambda_0\right)^2 \left(\left(-1+\lambda_1^2\right) A_1 \tilde{\phi}_1^I - A_1 \left(\tilde{\phi}_1^I\right)''\right) \\
&+n\left(\left(\tilde{\phi}_0^I\right)''\right)^2 \left(\left(-1+\lambda_1^2\right) A_1 \tilde{\phi}_1^I - A_1 \left(\tilde{\phi}_1^I\right)''\right) + 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \Big(2(n-1)\lambda_0\lambda_1 \left(\tilde{\phi}_0^I\right)' A_1 \left(\tilde{\phi}_1^I\right)' + n\left(\tilde{\phi}_0^I\right)'' \left(\left(1-\lambda_1^2\right) A_1 \tilde{\phi}_1^I + A_1 \left(\tilde{\phi}_1^I\right)''\right)\Big)\Bigg) \\
&\times \Big(2\left(1-\lambda_0^2\right)^2 \tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)' + 8\lambda_0^2 \left(\tilde{\phi}_0^I\right)' \left(\tilde{\phi}_0^I\right)'' - 2\left(-1+\lambda_0^2\right)\left(\tilde{\phi}_0^I\right)' \left(\tilde{\phi}_0^I\right)'' - 2\left(\lambda_0^2 - 1\right)\tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)^{(3)} + 2\left(\tilde{\phi}_0^I\right)'' \left(\tilde{\phi}_0^I\right)^{(3)} \Big) \\
&-\Bigg(\frac{1}{2^2} \Bigg(\Big\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right) \Big\} + 4\left(\left(\tilde{\phi}_0^I\right)' \lambda_0\right)^2 \Bigg)^{\frac{n-3}{2}} \\
&\times \Bigg\{ -4(n-1)\lambda_0\lambda_1 \tilde{\phi}_1' \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2n\left(1-\lambda_0^2\right)^2 \tilde{\phi}_0^I \left(\tilde{\phi}_0^I\right)' \left(\left(\lambda_1^2 - 1\right)\tilde{\phi}_1^I - \left(\tilde{\phi}_1^I\right)'' + 8\lambda_0^2 \left(\tilde{\phi}_0^I\right)' \left(\tilde{\phi}_0^I\right)'' \left(\left(\lambda_1^2 - 1\right) A_1 \tilde{\phi}_1^I - A_1 \left(\tilde{\phi}_1^I\right)''\right) \\
&-4(n-1)\lambda_0\lambda_1 \left(\tilde{\phi}_0^I\right)' \left(\tilde{\phi}_0^I\right)'' \left(\tilde{\phi}_1^I\right)'' + 2\left(-1+\lambda_0^2\right)\left(\tilde{\phi}_0^I\right)' \times \Big\{ 2(n-1)\lambda_0\lambda_1 \left(\tilde{\phi}_0^I\right)' \tilde{\phi}_1' + n\left(\tilde{\phi}_0^I\right)'' \left(\left(1-\lambda_1^2\right) A_1 \tilde{\phi}_1^I + A_1 \left(\tilde{\phi}_1^I\right)''\right)\Big\} \\
&-4(n-1)\lambda_0\lambda_1 \left(\tilde{\phi}_0^I\right)' A_1 \left(\tilde{\phi}_1^I\right)' \left(\tilde{\phi}_0^I\right)^{(3)} + 2n\left(\tilde{\phi}_0^I\right)'' \left(\left(-1+\lambda_1^2\right) A_1 \tilde{\phi}_1^I - A_1 \tilde{\phi}_1''\right)\left(\tilde{\phi}_0^I\right)^{(3)} + n\left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right)\right)^2 \left(\left(\lambda_1^2 - 1\right) A_1 \tilde{\phi}_1' - A_1 \tilde{\phi}_1^{(3)}\right) \\
&+4\lambda_0^2 \left(\left(\tilde{\phi}_0^I\right)'\right)^2 \left(\left(-1+\lambda_1^2\right) A_1 \left(\tilde{\phi}_1^I\right)' - A_1 \tilde{\phi}_1^{(3)}\right) + n\left(\left(\tilde{\phi}_0^I\right)''\right)^2 \left(\left(\lambda_1^2 - 1\right) A_1 \left(\tilde{\phi}_1^I\right)' - A_1 \tilde{\phi}_1^{(3)}\right) \\
&+2\left(1-\lambda_0^2\right)^2 \tilde{\phi}_0^I \times \Big[2(n-1)\lambda_0\lambda_1 A_1 \left(\tilde{\phi}_1^I\right)' \left(\tilde{\phi}_0^I\right)'' \\
&+2(n-1)\lambda_0\lambda_1 \left(\tilde{\phi}_0^I\right)' A_1 \left(\tilde{\phi}_1^I\right)'' + n\left(-\left(-1+\lambda_1^2\right) A_1 \tilde{\phi}_1^I + A_1 \left(\tilde{\phi}_1^I\right)''\right)\left(\tilde{\phi}_0^I\right)^{(3)} + n\left(\tilde{\phi}_0^I\right)'' \left(\left(1-\lambda_1^2\right) A_1 \left(\tilde{\phi}_1^I\right)' + A_1 \left(\tilde{\phi}_1^I\right)^{(3)}\right)\Big]\Bigg\}\Bigg]
\end{aligned}
\tag{2.185}
$$

The displacement equation can be expressed as,

$$u_{\theta 1}^I = A_0^{n-1} \ r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} \tilde{u}_{\theta 1}^I \tag{2.186}$$

where,

$$
\begin{aligned}
&\tilde{u}^{I}_{\theta 1} = \frac{2^{1-n}3^{\frac{n+1}{2}}\alpha}{\left(n\lambda_0 - n - \lambda_0 + \lambda_1\right)(n\lambda_0 - n - \lambda_0 + \lambda_1 + 1)} \\
&\Bigg\{\Bigg[(n-1)(n\lambda_0 - n - \lambda_0 + \lambda_1 + 1)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\} + 4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}} \\
&\times\left(\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I\right)''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times\left\{A_1\tilde{\phi}_1^I(\lambda_1+1)(\lambda_1-1) - A_1\left(\tilde{\phi}_1^I\right)''\right\} + 4\left\{-\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\times(-1)\left\{A_1\left(\tilde{\phi}_1^I\right)'\lambda_1\right\}\right) \\
&+\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\} + 4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times(-1)\left\{\left(\tilde{\phi}_0^I\right)'\lambda_1\right\}\Bigg] \\
&-\Bigg[\frac{(n-3)}{2^3}\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\} + 4\left(\left(\tilde{\phi}_0^I\right)'\lambda_0\right)^2\right)^{\frac{n-3}{2}-1} \\
&\times\Bigg(-4(n-1)\lambda_0\lambda_1\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_1^I\right)'\left(\tilde{\phi}_0^I\right)'' + n\left(1-\lambda_0^2\right)^2\left(\tilde{\phi}_0^I\right)^2\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1^I - A_1\left(\tilde{\phi}_1^I\right)''\right) + 4\left(\left(\tilde{\phi}_0^I\right)'\lambda_0\right)^2\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1^I - A_1\left(\tilde{\phi}_1^I\right)''\right) \\
&+n\left(\left(\tilde{\phi}_0^I\right)''\right)^2\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1^I - A_1\left(\tilde{\phi}_1^I\right)''\right) + 2\left(\lambda_0^2-1\right)\tilde{\phi}_0^I\left(2(n-1)\lambda_0\lambda_1\left(\tilde{\phi}_0^I\right)'A_1\left(\tilde{\phi}_1^I\right)' + n\left(\tilde{\phi}_0^I\right)''\left(\left(1-\lambda_1^2\right)A_1\tilde{\phi}_1^I + A_1\left(\tilde{\phi}_1^I\right)''\right)\right)\Bigg) \\
&\times\left(2\left(1-\lambda_0^2\right)^2\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)' + 8\lambda_0^2\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)'' - 2\left(\lambda_0^2-1\right)\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)'' - 2\left(\lambda_0^2-1\right)\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)^{(3)} + 2\left(\tilde{\phi}_0^I\right)''\left(\tilde{\phi}_0^I\right)^{(3)}\right)\Bigg) \\
&-\left(\frac{1}{2^2}\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\} + 4\left(\left(\tilde{\phi}_0^I\right)'\lambda_0\right)^2\right)^{\frac{n-3}{2}}\right. \\
&\times\Bigg\{-4(n-1)\lambda_0\lambda_1A_1\tilde{\phi}_1'\left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2n\left(1-\lambda_0^2\right)^2\tilde{\phi}_0^I\left(\tilde{\phi}_0^I\right)'\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1^I - A_1\left(\tilde{\phi}_1^I\right)'' + 8\lambda_0^2\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)''\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1^I - A_1\left(\tilde{\phi}_1^I\right)''\right)\right. \\
&-4(n-1)\lambda_0\lambda_1\left(\tilde{\phi}_0^I\right)'\left(\tilde{\phi}_0^I\right)''A_1\left(\tilde{\phi}_1^I\right)'' + 2\left(\lambda_0^2-1\right)\left(\tilde{\phi}_0^I\right)'\times\left\{2(n-1)\lambda_0\lambda_1\left(\tilde{\phi}_0^I\right)'\tilde{\phi}_1' + n\left(\tilde{\phi}_0^I\right)''\left(\left(1-\lambda_1^2\right)A_1\tilde{\phi}_1^I + A_1\left(\tilde{\phi}_1^I\right)''\right)\right\} \\
&-4(n-1)\lambda_0\lambda_1\left(\tilde{\phi}_0^I\right)'A_1\left(\tilde{\phi}_1^I\right)'\left(\tilde{\phi}_0^I\right)^{(3)} + 2n\left(\tilde{\phi}_0^I\right)''\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1^I - A_1\tilde{\phi}_1''\right)\left(\tilde{\phi}_0^I\right)^{(3)} + n\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2\left(\left(\lambda_1^2-1\right)A_1\tilde{\phi}_1' - A_1\tilde{\phi}_1^{(3)}\right) \\
&+4\lambda_0^2\left(\left(\tilde{\phi}_0^I\right)'\right)^2\left(\left(\lambda_1^2-1\right)A_1\left(\tilde{\phi}_1^I\right)' - A_1\tilde{\phi}_1^{(3)}\right) + n\left(\left(\tilde{\phi}_0^I\right)''\right)^2\left(\left(\lambda_1^2-1\right)A_1\left(\tilde{\phi}_1^I\right)' - A_1\tilde{\phi}_1^{(3)}\right) + 2\left(1-\lambda_0^2\right)^2\tilde{\phi}_0^I\Big[2(n-1)\lambda_0\lambda_1A_1\left(\tilde{\phi}_1^I\right)'\left(\tilde{\phi}_0^I\right)'' \\
&+2(n-1)\lambda_0\lambda_1\left(\tilde{\phi}_0^I\right)'A_1\left(\tilde{\phi}_1^I\right)'' + n\left(\left(1-\lambda_1^2\right)A_1\tilde{\phi}_1^I + A_1\left(\tilde{\phi}_1^I\right)''\right)\left(\tilde{\phi}_0^I\right)^{(3)} + n\left(\tilde{\phi}_0^I\right)''\left(\left(1-\lambda_1^2\right)A_1\left(\tilde{\phi}_1^I\right)' + A_1\left(\tilde{\phi}_1^I\right)^{(3)}\right)\Big]\Bigg\}\Bigg]\Bigg\}
\end{aligned}
\tag{2.187}
$$

Finally, $u^{I}_{\theta(1)} = u^{I}_{\theta 0} + u^{I}_{\theta 1} \quad \Rightarrow \left(u^{I}_{\theta(1)}\right)_{\theta=0} = u^{I}_{\theta\;0} + u^{I}_{\theta}$ (2.188)

From zero-th order approximation: $\left(u^{I}_{\theta 0}\right)_{\theta=0} = 0$ So, $\left(u^{I}_{\theta(1)}\right)_{\theta=0} = u^{I}_{\theta 1}$ (2.189)

Displacement at any arbitrary value of θ,

Displacement at any arbitrary value of θ, $u^{I}_{\theta(1)} = u^{I}_{\theta 0} + u^{I}_{\theta 1} = A_0^n\, r^{n\lambda_0 - n + 1}\tilde{u}^{I}_{\theta 0} + A_0^{n-1}\, r^{(n\lambda_0 - n - \lambda_0 + \lambda_1) + 1}\tilde{u}^{I}_{\theta 1}$

(2.190)

Displacement on the interface,

$$\left(u^{I}_{\theta(1)}\right)_{\theta=0} = \left(u^{I}_{\theta 1}\right)_{\theta=0} = A_0^{n-1}\; r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1} \tilde{u}^{I}_{\theta 1} \tag{2.191}$$

Displacement of Elastic material side after zero-th order Approximation which will be used as the forced displacement on the interface:

$$u^{II}_{r0} = A_0\, r^{\lambda_0} \tilde{u}^{II}_{r0} \tag{2.192}$$

$$u^{II}_{\theta 0} = A_0\, r^{\lambda_0} \tilde{u}^{II}_{\theta 0} \tag{2.193}$$

where,

$$\tilde{u}^{II}_{r0} = \frac{E^{I}}{E^{II}} \left\{ \frac{\left(1+\nu^{II}\right)}{\lambda_0} \left[(\lambda_0 + 1)\left(1-\nu^{II} - \nu^{II}\lambda_0\right) \tilde{\phi}^{II}_0 + \left(1-\nu^{II}\right)\left(\tilde{\phi}^{II}_0\right)'' \right] \right\} \tag{2.194}$$

$$\tilde{u}^{II}_{\theta 0} = -\frac{E^{I}}{E^{II}} \frac{\left(1+\nu^{II}\right)}{\lambda_0\left(\lambda_0 - 1\right)} \left[\left\{ 2\lambda_0^2 + (\lambda_0 + 1)\left(1-\nu^{II} - \nu^{II}\lambda_0\right) \right\} \left(\tilde{\phi}^{II}_0\right)' + \left(1-\nu^{II}\right)\left(\tilde{\phi}^{II}_0\right)''' \right] \tag{2.195}$$

For the case of traction free edges and forced displacement from elastic material side Boundary conditions should be satisfied as:

$$\begin{aligned} &\left(\sigma^{I}_{\theta\theta(1)}\right)_{\theta=\frac{\pi}{2}} = 0 \\ &\left(\sigma^{I}_{r\theta(1)}\right)_{\theta=\frac{\pi}{2}} = 0 \end{aligned} \quad \text{and} \quad \begin{aligned} &\left(u^{I}_{r(1)}\right)_{\theta=0} = \left(u^{II}_{r(0)}\right)_{\theta=0} = A_0\, r^{\lambda_0} \tilde{u}^{II}_{r(0)} \left(\theta = 0; \text{elastic}\right) \\ &\left(u^{I}_{\theta(1)}\right)_{\theta=0} = \left(u^{II}_{\theta(0)}\right)_{\theta=0} = A_0\, r^{\lambda_0} \tilde{u}^{II}_{\theta(0)} \left(\theta = 0; \text{elastic}\right) \end{aligned} \tag{2.196}$$

Due to the forced displacement on the interface, displacement of first order approximation in the power law material side should be the same as the displacement of zero-th order approximation in the elastic material side.

The iterative boundary condition on the interface can be expressed as,

$$\left(u^{I}_{r(1)}\right)_{\theta=0} = \left(u^{II}_{r(0)}\right)_{\theta=0}, \quad \left(u^{I}_{\theta(1)}\right)_{\theta=0} = \left(u^{II}_{\theta(0)}\right)_{\theta=0}, \tag{2.197}$$

Within the first order approximation in the power-law material side we have,

Where $\left(u^{I}_{r(1)}\right)_{\theta=0} = \left(u^{I}_{r0} + u^{I}_{r1}\right)_{\theta=0}$ and $\left(u^{I}_{\theta(1)}\right)_{\theta=0} = \left(u^{I}_{\theta 0} + u^{I}_{\theta 1}\right)_{\theta=0}$ (2.198)

From zero-th order approximation we have,

$$\left(u_{r0}^{I}\right)_{\theta=0} = 0 \text{ and } \left(u_{\theta 0}^{I}\right)_{\theta=0} = 0 \tag{2.199}$$

So, remaining term $\left(u_{r1}^{I}\right)_{\theta=0} = \left(u_{r0}^{II}\right)_{\theta=0}$ and $\left(u_{\theta 1}^{I}\right)_{\theta=0} = \left(u_{\theta 0}^{II}\right)_{\theta=0}$ (2.200)

where u_{r0}^{II} and $u_{\theta 0}^{II}$ are the displacements of zero-th order approximation in the elastic material side, u_{r1}^{I} and $u_{\theta 1}^{I}$ are the incremental displacements of the first order approximation in the power-law hardening material side. To derive the expressions for displacement, small deformation strain-displacement relations, Eq.(2.9)-Eq.(2.11), have been used. Strain components are derived using Eq.(2.3) and Eq.(2.4) for power-law material and elastic material, respectively. From Eq.(2.9) the strain component, ε_{rr}, is integrated by r to derive the expression of displacement u_r and the expression of u_θ is also derived from $\varepsilon_{r\theta}$ using Eq.(2.11).

Displacement functions of elastic material side within zero-th order approximation which is used as the forced displacement on the power-law material side in the first order approximation are,

$$u_{r0}^{II} = A_0 r^{\lambda_0} \tilde{u}_{r0}^{II}, \quad u_{\theta 0}^{II} = A_0 r^{\lambda_0} \tilde{u}_{\theta 0}^{II} \tag{2.201}$$

where $\tilde{u}_{ri}^{k}$ and $\tilde{u}_{\theta i}^{k}$ are the angular function terms of displacement component, $i = 0$ and $1, k = I$ and II. $\tilde{u}_{r1}^{I}$ is a function of $A_1\tilde{\phi}_1^{I}, A_1\tilde{\phi}_1^{I\prime}, A_1\tilde{\phi}_1^{I\prime\prime}, A_1\tilde{\phi}_1^{I\prime\prime\prime}$ and λ_1.

To satisfy the boundary condition on the interface the power of r should be equal. Equating the power of r we have,

$$\lambda_1 = \lambda_0 + (1-n)(\lambda_0 - 1). \tag{2.202}$$

It seems the first order singularity is depends on hardening exponent n and zero-th order singularity λ_0 .

To satisfy the displacement continuity condition on the interface order by order the boundary equation can be expressed as,

$$A_0^{n-1}\ r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\left(\tilde{u}_{r1}^{I}\right)_{\theta=0}=A_0\,r^{\lambda_0}\left(\tilde{u}_{r0}^{II}\right)_{\theta=0} \tag{2.203}$$

and

$$A_0^{n-1}\ r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\left(\tilde{u}_{\theta1}^{I}\right)_{\theta=0}=A_0\,r^{\lambda_0}\left(\tilde{u}_{\theta0}^{II}\right)_{\theta=0} \tag{2.204}$$

Equating the power of r in the boundary equation, Radial part is equal in both sides so remaining part means the angular function part should be the same

Where, the magnitudes of L.H.S and R.H.S should be same due to forced displacement on the interface.

$$\Rightarrow A_0^{n-1}\left(\tilde{u}_{r1}^{I}\right)_{\theta=0}=A_0\left(\tilde{u}_{r0}^{II}\right)_{\theta=0}\quad\text{and}\quad A_0^{n-1}\left(\tilde{u}_{\theta1}^{I}\right)_{\theta=0}=A_0\left(\tilde{u}_{\theta0}^{II}\right)_{\theta=0} \tag{2.205}$$

$$\Rightarrow A_1=A_0^{2-n}\frac{\left(\tilde{u}_{r0}^{II}\right)_{\theta=0}}{\left(\tilde{u}_{r1}^{I}\right)_{\theta=0}\Big/A_1}\quad\text{and}\quad\Rightarrow A_1=A_0^{2-n}\frac{\left(\tilde{u}_{\theta0}^{II}\right)_{\theta=0}}{\left(\tilde{u}_{\theta1}^{I}\right)_{\theta=0}\Big/A_1} \tag{2.206}$$

$$\Rightarrow A_1=A_0^{2-n}\times C \tag{2.207}$$

where C is constant value numerically known from the angular function term.

$$C=\frac{\left(\tilde{u}_{r0}^{II}\right)_{\theta=0}}{\left(\tilde{u}_{r1}^{I}\right)_{\theta=0}\Big/A_1}\quad\text{and}\quad C=\frac{\left(\tilde{u}_{\theta0}^{II}\right)_{\theta=0}}{\left(\tilde{u}_{\theta1}^{I}\right)_{\theta=0}\Big/A_1} \tag{2.208}$$

Using Eqn. (2.208) the determination of constant term without the determination of stress intensity factor is impossible. Only one can have the relationship between the stress intensity factor of the first order term in terms of the zero-th order term by Eqn. (2.207).

To calculate Error following equation is followed from Eqn. (2.196):

$$
\begin{aligned}
&\text{error1} = A_0 \tilde{u}_{r0}^{II}\left(\theta\text{=0;elastic}\right) - A_0^{n-1}\,\tilde{u}_{r1}^{I}\left(\theta = 0;\text{power-law}\right) \\
&\text{error2} = A_0 \tilde{u}_{\theta 0}^{II}\left(\theta = 0;\text{elastic}\right) - A_0^{n-1}\,\tilde{u}_{\theta 1}^{I}\left(\theta = 0;\text{power-law}\right)
\end{aligned}
\tag{2.209}
$$

In the first order approximation, compatibility equation presented in Eqn. (2.167).

Equation (2.167) is the fourth-order ordinary differential equation of $\tilde{\phi}_1^I$. For the solution of fourth-order equation, the equation is reduced into a system of first-order equations:

Assume, $w1(1) = A_1\tilde{\phi}_1^I, \quad w1(2) = A_1\dfrac{d\tilde{\phi}_1^I}{d\theta}, \quad w1(3) = A_1\dfrac{d^2\tilde{\phi}_1^I}{d\theta^2}, \quad w1(4) = A_1\dfrac{d^3\tilde{\phi}_1^I}{d\theta^3}$

where, $\tilde{\phi}_0, \dfrac{d\tilde{\phi}_0}{d\theta}, \dfrac{d^2\tilde{\phi}_0}{d\theta^2}, \dfrac{d^3\tilde{\phi}_0}{d\theta^3}$ and $\dfrac{d^4\tilde{\phi}_0}{d\theta^4}$ are known from the solution of 0-th order approximation.

$$
\begin{cases}
w1(1)' = w1(2) \\
w1(2)' = w1(3), \\
w1(3)' = w1(4), \\
w1(4)' = A_1\dfrac{d^4\tilde{\phi}_1^I}{d\theta^4} \\
= f\left(A_1\dfrac{d^3\tilde{\phi}_1^I}{d\theta^3},\, A_1\dfrac{d^2\tilde{\phi}_1^I}{d\theta^2}, A_1\dfrac{d\tilde{\phi}_1^I}{d\theta}, A_1\tilde{\phi}_1^I, \dfrac{d^4\tilde{\phi}_0^I}{d\theta^4}, \dfrac{d^3\tilde{\phi}_0^I}{d\theta^3}, \dfrac{d^2\tilde{\phi}_0^I}{d\theta^2}, \dfrac{d\tilde{\phi}_0^I}{d\theta}, \tilde{\phi}_0^I, n, \lambda_0, \lambda_1\right) \\
= -\dfrac{C}{B}\left(A_1\dfrac{d^3\tilde{\phi}_1^I}{d\theta^3},\, A_1\dfrac{d^2\tilde{\phi}_1^I}{d\theta^2},\, A_1\dfrac{d\tilde{\phi}_1^I}{d\theta},\, A_1\tilde{\phi}_1^I, \dfrac{d^4\tilde{\phi}_0^I}{d\theta^4}, \dfrac{d^3\tilde{\phi}_0^I}{d\theta^3}, \dfrac{d^2\tilde{\phi}_0^I}{d\theta^2}, \dfrac{d\tilde{\phi}_0^I}{d\theta}, \tilde{\phi}_0^I, n, \lambda_0, \lambda_1\right)
\end{cases}
\tag{2.210}
$$

B and C are derived using Mathematica software and checked the initial part (zero-th order term) of compatibility equation which is same as the compatibility equation derived theoretically for zero-th order approximation. Within the first order approximation unknowns are $\tilde{\phi}_1^I, \tilde{\phi}_1^{I\prime}, \tilde{\phi}_1^{I\prime\prime}, \tilde{\phi}_1^{I\prime\prime\prime}, A_1$ and λ_1.

And, therefore, equation is solved using the Runge-Kutta method.

Applying boundary conditions on the stress free edge:

$$\left(\sigma^I_{\theta\theta(1)}\right)_{\theta=\frac{\pi}{2}} = \left(\sigma^I_{\theta\theta 0}\right)_{\theta=\frac{\pi}{2}} + \left(\sigma^I_{\theta\theta 1}\right)_{\theta=\frac{\pi}{2}} = 0 \tag{2.211}$$

From zero-th order approximation we know $\left(\sigma^I_{\theta\theta 0}\right)_{\theta=\frac{\pi}{2}} = 0$, so $\left(\sigma^I_{\theta\theta 1}\right)_{\theta=\frac{\pi}{2}} = 0$ (2.212)

similarly, $\left(\sigma^I_{r\theta(1)}\right)_{\theta=\frac{\pi}{2}} = 0 \quad \Rightarrow \sigma^I_{r\theta(1)} = \sigma^I_{r\theta 0} + \sigma^I_{r\theta 1}$ (2.213)

From zero-th order approximation we know $\left(\sigma^I_{r\theta 0}\right)_{\theta=\frac{\pi}{2}} = 0$, so $\left(\sigma^I_{r\theta 1}\right)_{\theta=\frac{\pi}{2}} = 0$ (2.214)

Incremental stress fields are expressed in Eqns. (2.168-2.170)

Initial conditions at $\theta = \frac{\pi}{2}$: From equation (2.169) & (2.170) dividing by $r^{\lambda_1 - 1}$

Or $\frac{\sigma^I_{\theta\theta 1}}{r^{\lambda_1 - 1}} = (\lambda_1 + 1)\lambda_1 \left(A_1 \tilde{\phi}^I_1\right) = 0, \qquad \left(A_1 \tilde{\phi}^I_1\right) = 0, \text{if } \lambda_1 \neq -1,\ \lambda_1 \neq 0$ (2.215)

And $\frac{\sigma^I_{r\theta 1}}{r^{\lambda_1 - 1}} = -\lambda_1 \left(A_1 \tilde{\phi}^{II}_1\right)' = 0, or \ \left(A_1 \tilde{\phi}^{II}_1\right)' = 0, \text{if } \quad \lambda_1 \neq 0$ (2.216)

We know λ_1 from Eqn.(2.202) and Unknown are: $\left(A_1 \tilde{\phi}^{II}_1\right)''$, $\left(A_1 \tilde{\phi}^{II}_1\right)'''$

After integration final conditions, at $\theta = 0$,

Assume at $\theta = \frac{\pi}{2}$, $A_1 \tilde{\phi}^I_1 = 0, \left(A_1 \tilde{\phi}^{II}_1\right)' = 0, \left(A_1 \tilde{\phi}^{II}_1\right)''$ and $\left(A_1 \tilde{\phi}^{II}_1\right)'''$ hence, after i-th

integration at $\theta = 0, \left(A_1 \tilde{\phi}^I_1\right) = \left(A_1 \tilde{\phi}^I_1\right)^{(i)}, \left(A_1 \tilde{\phi}^{II}_1\right)' = \left(\left(A_1 \tilde{\phi}^{II}_1\right)'\right)^{(i)}, \left(A_1 \tilde{\phi}^{II}_1\right)'' = \left(\left(A_1 \tilde{\phi}^{II}_1\right)''\right)^{(i)}$

and $\left(A_1 \tilde{\phi}^{II}_1\right)''' = \left(\left(A_1 \tilde{\phi}^{II}_1\right)'''\right)^{(i)}$ (2.217)

$$
\begin{aligned}
\text{error1} &= A_0 \tilde{u}_{r0}^{II}\left(\theta=0;\text{elastic}\right) - \frac{2^{-1-n}3^{\frac{n+1}{2}}A_0^{n-1}}{(n\lambda_0 - n - \lambda_0 + \lambda_1 + 1)} \\
&\times\alpha\left[(n-1)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\tilde{\phi}_0^I\right)''\right\}+4\left\{\left(A_1\tilde{\phi}_1^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}}\right. \\
&\times\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\tilde{\phi}_0^I\right)''\right\}\times(-1)\left\{\left(\left(A_1\tilde{\phi}_1^I\right)\right)^{(i)}(\lambda_1+1)(\lambda_1-1)-\left(\left(A_1\tilde{\phi}_1^I\right)''\right)^{(i)}\right\}\right. \\
&\left.+4\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right)+\left(\tilde{\phi}_0^I\right)''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times\left\{\left(\left(A_1\tilde{\phi}_1^I\right)'\right)^{(i)}\lambda_1\right\}\right) \\
&\left.+\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\tilde{\phi}_0^I\right)''\right\}+4\left\{\left(\left(A_1\tilde{\phi}_1^I\right)'\right)^{(i)}\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times(-1)\left\{\left(A_1\tilde{\phi}_1^I\right)^{(i)}(\lambda_1+1)(\lambda_1-1)-\left(\left(A_1\tilde{\phi}_1^I\right)''\right)^{(i)}\right\}\right]
\end{aligned}
\tag{2.218}
$$

$$
\begin{aligned}
&\text{error}2 = A_0\tilde{u}^{II}_{\theta 0}(\theta=0;\text{elastic}) - \frac{2^{1-n}3^{\frac{n+1}{2}}\alpha A_0^{n-1}}{(n\lambda_0-n-\lambda_0+\lambda_1)(n\lambda_0-n-\lambda_0+\lambda_1+1)} \\
&\Bigg\{\Bigg[(n-1)(n\lambda_0-n-\lambda_0+\lambda_1+1)\bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\big((\tilde{\phi}_0^I)''\big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\big((\tilde{\phi}_0^I)''\big)\Big\}+4\Big\{(\tilde{\phi}_0^I)'\lambda_0\Big\}^2\bigg)^{\frac{n-3}{2}} \\
&\times\bigg(\Big\{\tilde{\phi}_0^I(1-\lambda_0^2)+(\tilde{\phi}_0^I)''\Big\}\Big\{(\tilde{\phi}_0^I)'\lambda_0\Big\}\times\Big\{\big((A_1\tilde{\phi}_1^I)\big)^{(i)}(\lambda_1+1)(\lambda_1-1)-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big\}+4\Big\{-(\tilde{\phi}_0^I)'\lambda_0\Big\}^2\times(-1)\Big\{\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}\lambda_1\Big\}\bigg) \\
&+\bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\big((\tilde{\phi}_0^I)''\big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\big((\tilde{\phi}_0^I)''\big)\Big\}+4\Big\{(\tilde{\phi}_0^I)'\lambda_0\Big\}^2\bigg)^{\frac{n-1}{2}}\times(-1)\Big\{\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}\lambda_1\Big\}\Bigg] \\
&-\Bigg[\frac{(n-3)}{2^3}\bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\big((\tilde{\phi}_0^I)''\big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\big((\tilde{\phi}_0^I)''\big)\Big\}+4\Big((\tilde{\phi}_0^I)'\lambda_0\Big)^2\bigg)^{\frac{n-3}{2}-1} \\
&\times\bigg(-4(n-1)\lambda_0\lambda_1(\tilde{\phi}_0^I)'\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}(\tilde{\phi}_0^I)''+n(1-\lambda_0^2)^2(\tilde{\phi}_0^I)^2\Big((-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)+4\Big((\tilde{\phi}_0^I)'\lambda_0\Big)^2\Big((-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big) \\
&+n\Big((\tilde{\phi}_0^I)''\Big)^2\Big((-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)+2(-1+\lambda_0^2)\tilde{\phi}_0^I\Big(2(n-1)\lambda_0\lambda_1(\tilde{\phi}_0^I)'\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}+n(\tilde{\phi}_0^I)''\Big(-(-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}+\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)\Big)\bigg) \\
&\times\bigg(2(1-\lambda_0^2)^2\tilde{\phi}_0^I(\tilde{\phi}_0^I)'+8\lambda_0^2(\tilde{\phi}_0^I)'(\tilde{\phi}_0^I)''-2(-1+\lambda_0^2)(\tilde{\phi}_0^I)'(\tilde{\phi}_0^I)''-2(-1+\lambda_0^2)\tilde{\phi}_0^I(\tilde{\phi}_0^I)^{(3)}+2(\tilde{\phi}_0^I)''(\tilde{\phi}_0^I)^{(3)}\bigg)\bigg) \\
&-\bigg(\frac{1}{2^2}\bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\big((\tilde{\phi}_0^I)''\big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\big((\tilde{\phi}_0^I)''\big)\Big\}+4\Big((\tilde{\phi}_0^I)'\lambda_0\Big)^2\bigg)^{\frac{n-3}{2}} \\
&\times\bigg\{-4(n-1)\lambda_0\lambda_1\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}\big((\tilde{\phi}_0^I)''\big)^2+2n(1-\lambda_0^2)^2\tilde{\phi}_0^I(\tilde{\phi}_0^I)'\Big((-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)+8\lambda_0^2(\tilde{\phi}_0^I)'(\tilde{\phi}_0^I)''\Big((-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big) \\
&-4(n-1)\lambda_0\lambda_1(\tilde{\phi}_0^I)'(\tilde{\phi}_0^I)''\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}+2(-1+\lambda_0^2)(\tilde{\phi}_0^I)'\times\Big\{2(n-1)\lambda_0\lambda_1(\tilde{\phi}_0^I)'\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}+n(\tilde{\phi}_0^I)''\Big(-(-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}+\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)\Big\} \\
&-4(n-1)\lambda_0\lambda_1(\tilde{\phi}_0^I)'\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}(\tilde{\phi}_0^I)^{(3)}+2n(\tilde{\phi}_0^I)''\Big((-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}-\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)(\tilde{\phi}_0^I)^{(3)}+n\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2\Big((-1+\lambda_1^2)\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}-\big((A_1\tilde{\phi}_1^I)^{(3)}\big)^{(i)}\Big) \\
&+4\lambda_0^2\big((\tilde{\phi}_0^I)'\big)^2\Big((-1+\lambda_1^2)\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}-\big((A_1\tilde{\phi}_1^I)^{(3)}\big)^{(i)}\Big)+n\big((\tilde{\phi}_0^I)''\big)^2\Big((-1+\lambda_1^2)\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}-\big((A_1\tilde{\phi}_1^I)^{(3)}\big)^{(i)}\Big)+2(1-\lambda_0^2)^2\tilde{\phi}_0^I\times\Big[2(n-1)\lambda_0\lambda_1(A_1\tilde{\phi}_1^I)'(\tilde{\phi}_0^I)'' \\
&+2(n-1)\lambda_0\lambda_1(\tilde{\phi}_0^I)'\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}+n\Big(-(-1+\lambda_1^2)(A_1\tilde{\phi}_1^I)^{(i)}+\big((A_1\tilde{\phi}_1^I)''\big)^{(i)}\Big)(\tilde{\phi}_0^I)^{(3)}+n(\tilde{\phi}_0^I)''\Big(-(-1+\lambda_1^2)\big((A_1\tilde{\phi}_1^I)'\big)^{(i)}+\big((A_1\tilde{\phi}_1^I)^{(3)}\big)^{(i)}\Big)\Big]\bigg\}\bigg]\Bigg\}
\end{aligned}
\tag{2.219}
$$

The error value is calculated by using the equation (2.15)

2.5.1.1 Determination of the Stress Intensity Factor A_1

Stress can be expressed as,

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} = A_0\, r^{\lambda_0 - 1}\left(\tilde{\sigma}_{\theta\theta 0}^{I}\right)_{\theta=0} + A_1\, r^{\lambda_1 - 1}\left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} \tag{2.220}$$

Substituting, $\left(\tilde{\sigma}_{\theta\theta 0}^{I}\right)_{\theta=0} = 1$ in Eqn. (2.220)

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} = A_0\, r^{\lambda_0 - 1} + A_1\, r^{\lambda_1 - 1}\left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} \tag{2.221}$$

From FEM for joint material (power law hardening/elastic material joint), $\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}$ is known and from zero-th order approximation $A_0\, r^{\lambda_0 - 1}$ is also known

$$A_1\, r^{\lambda_1 - 1}\left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} = \left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} - A_0\, r^{\lambda_0 - 1} \tag{2.222}$$

From first order approximation after solution of differential equation the numerical value of $\left(A_1\tilde{\phi}_1^{I}\right)$ is known.

we have, $$A_1\, r^{\lambda_1 - 1}\left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} = \left(A_1\tilde{\phi}_1^{I}\right)_{\theta=0} r^{\lambda_1 - 1}\left(\lambda_1 + 1\right)\lambda_1 \tag{2.223}$$

As the definition of stress intensity factor A_1, $$\left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} = 1 \tag{2.224}$$

Substituting Eqn (2.224) into Eqn. (2.223) yields,

$$\Rightarrow A_1 = \left(A_1\tilde{\phi}_1^{I}\right)_{\theta=0}\left(\lambda_1 + 1\right)\lambda_1 \tag{2.225}$$

From first order approximation after solution of differential equation the numerical value of $\left(A_1\tilde{\phi}_1^{I}\right)$ and all of its derivatives are known.

From FEM, A_1 is known numerically fro Eqn. (2.207)

so, $\tilde{\phi}_1^{I} = \left(A_1\tilde{\phi}_1^{I}\right)/A_1$ and similarly all of its derivatives are known.

For $n > 2$, power of A_0 is smaller than 0 (means negative power shows opposite behavior) and for $n < 2$, power of A_0 is larger than 0 (means positive power shows same behavior). For $n < 2$, two singular term exist up to the first order approximation and for $n > 2$ one singular term exist.

When $n = 2,\ A_1 = A_0^{2-2} \times C$ or $A_1 = C$.

When $n = 2,\ \lambda_1 = 1$ stress fields are, $\sigma_{ij} = A_0 r^{\lambda_0 - 1} \tilde{\sigma}_{ij0} + \tilde{\sigma}_{ij1}$ (last part is independent of r)

Therefore, $n = 2$ is a trivial solution.

From FEM of joint material can be written as

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} = A_0\, r^{\lambda_0 - 1} \left(\tilde{\sigma}_{\theta\theta(0)}^{I}\right)_{\theta=0} + A_1\, r^{\lambda_1 - 1} \left(\Delta\tilde{\sigma}_{\theta\theta(1)}^{I}\right)_{\theta=0} \qquad (2.226)$$

Substituting, $\left(\tilde{\sigma}_{\theta\theta 0}^{I}\right)_{\theta=0} = 1$ in Eqn. (2.236)

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} = A_0\, r^{\lambda_0 - 1} + A_1\, r^{\lambda_1 - 1} \left(\Delta\tilde{\sigma}_{\theta\theta(1)}^{I}\right)_{\theta=0} \qquad (2.227)$$

From FEM for joint material (power law hardening/elastic material joint), $\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}$ is known and from zero-th order approximation $A_0\, r^{\lambda_0 - 1}$ is also known

$$A_1\, r^{\lambda_1 - 1} \left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} = \left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} - A_0\, r^{\lambda_0 - 1} \qquad (2.228)$$

As the definition of stress intensity factor A_1, $\left(\tilde{\sigma}_{\theta\theta 1}^{I}\right)_{\theta=0} = 1$

$$A_1\, r^{\lambda_1 - 1} = \left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} - A_0\, r^{\lambda_0 - 1} \qquad (2.229)$$

Taking logarithmic distribution,

$$\log A_1 + (\lambda_1 - 1)\log r = \log\left\{\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} - A_0\, r^{\lambda_0 - 1}\right\} \qquad (2.230)$$

Right hand side have the term $\left\{\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} - A_0\, r^{\lambda_0-1}\right\}$ is calculated at first and then logarithmic distribution is calculated. $\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}$ is numerically known along radial distance r by FEM for joint material and $A_0\, r^{\lambda_0-1}$ is calculated and subtracted for the same radial distance where A_0 and $\left(\lambda_0 - 1\right)$ is known from rigid/power law hardening material by FEM.

For example, $n = 2.4$ is presented here: $A_0 = 2.790$ and $\left(\lambda_0 - 1\right) = -0.246$

$\left\{\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0} - A_0\, r^{\lambda_0-1}\right\}$ is calculated along radial distance *r*.

To determine the stress intensity factor, logarithmic distribution of stress along r has been used from FEM of joint material.

we have, from Eqn. (2.230)

Where slope is $\left(\lambda_1 - 1\right)$ and $\log A_1$ is intercept. Least squares method was used to calculate slope and intercept for an r range the r range is known and used to determine the stress intensity factor for rigid power law hardening material (zero-th order approximation). calculated $\left(\lambda_1 - 1\right) = -0.2893$ and $A_1 = 0.02738$. From first order approximation theoretical slope is $\left(\lambda_1 - 1\right) = 0.09853$ and theoretically calculated stress intensity factor is $A_1 = 0.059$.These values show some deviation of theoretical result from FEM result. This is due to the numerically calculated result by FEM includes higher order terms but, theoretically only zero-th order and first order terms are now considered.

2.5.1.2 Alternative Way To Express The Definition Of Stress Intensity Factor A_1

Assume $\phi = \phi_0^I + \phi_1^I = A_0 r^{\lambda_0+1}\tilde{\phi}_0^I + r^{\lambda_1+1}\tilde{\Phi}_1^I$ (2.231)

$$u_r^I = A_0^n r^{n(\lambda_0+1)+1}\tilde{u}_{r(0)}^I + A_0^{n-1} r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\tilde{U}_{r(0)}^I \quad (2.232)$$

$$u_\theta^I = A_0^n r^{n(\lambda_0+1)+1}\tilde{u}_{\theta(0)}^I + A_0^{n-1} r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\tilde{U}_{\theta(0)}^I \quad (2.233)$$

The magnitude of $\tilde{U}_{r(0)}^I$ and $\tilde{U}_{\theta(0)}^I$ can be determined based on the displacement continuity conditions,

$$A_0^{n-1} r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\left(\tilde{U}_{r(0)}^I\right)_{\theta=0} = A_0 r^{\lambda_0}\left(\tilde{u}_{r(0)}^{II}\right)_{\theta=0} \quad (2.234)$$

$$A_0^{n-1} r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\left(\tilde{U}_{\theta(0)}^I\right)_{\theta=0} = A_0 r^{\lambda_0}\left(\tilde{u}_{\theta(0)}^{II}\right)_{\theta=0} \quad (2.235)$$

Then we know from the solution of differential equation $\tilde{\Phi}_1^I$, $\tilde{\Phi}_1^{I\prime}$, $\tilde{\Phi}_1^{I\prime\prime}$ and $\tilde{\Phi}_1^{I\prime\prime\prime}$.

Stress can be expressed as

$$\sigma_{\theta\theta 1}^I = (\lambda_1+1)\lambda_1 r^{\lambda_1-1}\tilde{\Phi}_1^I \quad (2.236)$$

If we define A_1 as $\left(\sigma_{\theta\theta 1}^I\right)_{\theta=0} = A_1 r^{\lambda_1-1}$ (2.237)

Then, $A_1 = (\lambda_1+1)\lambda_1\left(\tilde{\Phi}_1^I\right)_{\theta=0}$ (2.238)

If we define $\tilde{\phi}_1^I$ as $r^{\lambda_1+1}\tilde{\Phi}_1^I = A_1 r^{\lambda_1+1}\tilde{\phi}_1^I$ (2.239)

Then $\tilde{\Phi}_1^I = A_1\tilde{\phi}_1^I$ (2.240)

$$\sigma_{\theta\theta 1}^I = (\lambda_1+1)\lambda_1 A_1 r^{\lambda_1-1}\tilde{\phi}_1^I \quad (2.241)$$

When $\theta = 0$ this should be, $\left(\sigma_{\theta\theta 1}^I\right)_{\theta=0} = A_1 r^{\lambda_1-1}$, as given in Eqn.(2.241)

so $(\lambda_1+1)\lambda_1\left(\tilde{\phi}_1^I\right)_{\theta=0} = 1$ (2.242)

If we divide $\sigma_{\theta\theta 1}^{I}$ by $(\lambda_1+1)\lambda_1\left(\tilde{\Phi}_1^I\right)_{\theta=0}$ and multiply A_1,

$$\sigma_{\theta\theta 1}^{I}\frac{A_1}{(\lambda_1+1)\lambda_1\left(\tilde{\Phi}_1^I\right)_{\theta=0}} \text{ where } \frac{A_1}{(\lambda_1+1)\lambda_1\left(\tilde{\Phi}_1^I\right)_{\theta=0}}=1 \text{ or } A_1=\left(\tilde{\Phi}_1^I\right)_{\theta=0}(\lambda_1+1)\lambda_1 \quad (2.243)$$

$$\sigma_{\theta\theta 1}^{I}\frac{A_1}{(\lambda_1+1)\lambda_1\left(\tilde{\Phi}_1^I\right)_{\theta=0}}=A_1\frac{(\lambda_1+1)\lambda_1 r^{\lambda_1-1}\tilde{\Phi}_1^I}{(\lambda_1+1)\lambda_1\left(\tilde{\Phi}_1^I\right)_{\theta=0}}=A_1 r^{\lambda_1-1}\frac{\tilde{\Phi}_1^I}{\left(\tilde{\Phi}_1^I\right)_{\theta=0}} \quad (2.244)$$

From Eq.(2.241), $\tilde{\phi}_1^I=\dfrac{\tilde{\Phi}_1^I}{\left(\tilde{\Phi}_1^I\right)_{\theta=0}(\lambda_1+1)\lambda_1}$ or $\dfrac{\tilde{\Phi}_1^I}{\left(\tilde{\Phi}_1^I\right)_{\theta=0}}=\tilde{\phi}_1^I(\lambda_1+1)\lambda_1$ (2.245)

From Eqn. (2.241) and Eqn. (245)

Then $r^{\lambda_1+1}\tilde{\Phi}_1^I=r^{\lambda_1+1}\left(\tilde{\Phi}_1^I\right)_{\theta=0}\tilde{\phi}_1^I(\lambda_1+1)\lambda_1=r^{\lambda_1+1}A_1\tilde{\phi}_1^I$ (2.246)

where $A_1=\left(\tilde{\Phi}_1^I\right)_{\theta=0}(\lambda_1+1)\lambda_1$ (2.247)

Once singular exponent, λ_1 is known the angular variation of stresses can be computed.

2.5.2 Formulation of 1st Order Approximation: Constitutive Equations in the Elastic Material Subjected to Traction

Invoking the plane strain condition strain can be expressed in terms of stresses as,

$$\varepsilon_{rr}^{II}=\frac{E^I}{E^{II}}\left\{\left(1+\nu^{II}\right)\left\{\left(1-\nu^{II}\right)\sigma_{rr}^{II}-\nu^{II}\sigma_{\theta\theta}^{II}\right\}\right\} \quad (2.248)$$

$$\varepsilon_{\theta\theta}^{II}=\frac{E^I}{E^{II}}\left\{\left(1+\nu^{II}\right)\left\{\left(1-\nu^{II}\right)\sigma_{\theta\theta}^{II}-\nu^{II}\sigma_{rr}^{II}\right\}\right\} \quad (2.249)$$

$$\varepsilon_{r\theta}^{II}=\frac{E^I}{E^{II}}\left\{\left(1+\nu^{II}\right)\sigma_{r\theta}^{II}\right\} \quad (2.250)$$

Compatibility equation in terms of stress component may be written as:

$$\frac{1}{r^3}\frac{\partial\phi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}+\frac{4}{r^4}\frac{\partial^2\phi}{\partial\theta^2}+\frac{2}{r}\frac{\partial^3\phi}{\partial r^3}-\frac{2}{r^3}\frac{\partial^3\phi}{\partial r\partial\theta^2}+\frac{2}{r^2}\frac{\partial^4\phi}{\partial r^2\partial\theta^2}+\frac{\partial^4\phi}{\partial r^4}+\frac{1}{r^4}\frac{\partial^4\phi}{\partial\theta^4}=0 \qquad (2.251)$$

For the case of stress free-edges and traction on the interface, boundary condition:

$$\begin{matrix}\left(\sigma^{II}_{\theta\theta(1)}\right)_{\theta=-\frac{\pi}{2}}=0 \\ \left(\sigma^{II}_{r\theta(1)}\right)_{\theta=-\frac{\pi}{2}}=0\end{matrix} \quad \text{and} \quad \begin{matrix}\left(\sigma^{I}_{\theta\theta1}\right)_{\theta=0}=\left(\sigma^{II}_{\theta\theta1}\right)_{\theta=0} \\ \left(\sigma^{I}_{r\theta1}\right)_{\theta=0}=\left(\sigma^{II}_{r\theta1}\right)_{\theta=0}\end{matrix} \qquad (2.252)$$

Assumed, Airy stress function, $\phi=\phi_0^{II}+\phi_1^{II}=A_0\,r^{\lambda_0+1}\tilde{\phi}_0^{II}+A_1\,r^{\lambda_1+1}\tilde{\phi}_1^{II}$ (2.253)

So, compatibility equation becomes,

$$\begin{aligned}&\frac{1}{r^3}\left(A_0\left(\lambda_0+1\right)r^{\lambda_0}\tilde{\phi}_0^{II}+A_1\left(\lambda_1+1\right)r^{\lambda_1}\tilde{\phi}_1^{II}\right)-\frac{1}{r^2}\left(A_0\left(\lambda_0+1\right)\lambda_0 r^{\lambda_0-1}\tilde{\phi}_0^{II}+A_1\left(\lambda_1+1\right)\lambda_1 r^{\lambda_1-1}\tilde{\phi}_1^{II}\right)\\&+\frac{4}{r^4}\left(r^{\lambda_0+1}A_0\left(\tilde{\phi}_0^{II}\right)''+r^{\lambda_1+1}A_1\left(\tilde{\phi}_1^{II}\right)''\right)+\frac{2}{r}\left(A_0\left(\lambda_0+1\right)\lambda_0\left(\lambda_0-1\right)r^{\lambda_0-2}\tilde{\phi}_0^{II}+A_1\left(\lambda_1+1\right)\lambda_1\left(\lambda_1-1\right)r^{\lambda_1-2}\tilde{\phi}_1^{II}\right)\\&-\frac{2}{r^3}\left(A_0\left(\lambda_0+1\right)r^{\lambda_0}\left(\tilde{\phi}_0^{II}\right)''+A_1\left(\lambda_1+1\right)r^{\lambda_1}\left(\tilde{\phi}_1^{II}\right)''\right)\\&+\frac{2}{r^2}\Bigg(A_0\left(\lambda_0+1\right)\lambda_0 r^{\lambda_0-1}\left(\tilde{\phi}_1^{II}\right)''+A_0\left(\lambda_0+1\right)\lambda_0\left(\lambda_0-1\right)\left(\lambda_0-2\right)r^{\lambda_0-3}\tilde{\phi}_0^{II}\\&\qquad+A_1\left(\lambda_1+1\right)\lambda_1 r^{\lambda_1-1}\left(\tilde{\phi}_1^{II}\right)''+A_1\left(\lambda_1+1\right)\lambda_1\left(\lambda_1-1\right)\left(\lambda_1-2\right)r^{\lambda_1-3}\tilde{\phi}_1^{II}\Bigg)\\&+\frac{1}{r^4}\left(A_0\,r^{\lambda_0+1}\left(\tilde{\phi}_0^{II}\right)^{(4)}+r^{\lambda_1+1}A_1\left(\tilde{\phi}_1^{II}\right)^{(4)}\right)=0\end{aligned}$$

(2.254)

From zero-th order approximation solution it is clear that the terms of r^{λ_0-3} satisfy the compatibility condition. To solve the compatibility condition, remaining terms satisfy the condition independently. So the compatibility equation for remaining terms,

$$\frac{d^4\tilde{\phi}_1}{d\theta^4}=-\left(1-\lambda_1^2\right)^2\tilde{\phi}_1^{II}-2\left(\lambda_1^2+1\right)\frac{\partial^2\tilde{\phi}_1^{II}}{\partial\theta^2} \qquad (2.255)$$

Equation (2.255) is the fourth-order ordinary differential equation. For the solution of fourth-order equation, the equation is reduced into a system of first-order equations:

Assume, $w1(1)=\tilde{\phi}_1^{II}$, $w1(2)=\dfrac{d\tilde{\phi}_1^{II}}{d\theta}$, $w1(3)=\dfrac{d^2\tilde{\phi}_1^{II}}{d\theta^2}$, $w1(4)=\dfrac{d^3\tilde{\phi}_1^{II}}{d\theta^3}$

Where, $w1(1)'=w1(2)$, $w1(2)'=w1(3)$, $w1(3)'=w1(4)$

$$w1(4)'=\frac{d^4\tilde{\phi}_1^{II}}{d\theta^4}=f\left(\frac{d^3\tilde{\phi}_1^{II}}{d\theta^3},\frac{d^2\tilde{\phi}_1^{II}}{d\theta^2},\frac{d\tilde{\phi}_1^{II}}{d\theta},\tilde{\phi}_1^{II},\lambda_1\right) \tag{2.256}$$

And, therefore, equation is solved using the Runge-Kutta method.

Where incremental part of stress field can be written as,

$$\sigma_{rr1}^{II}=A_1 r^{\lambda_1-1}\tilde{\sigma}_{rr1}^{II} \tag{2.257}$$

$$\sigma_{\theta\theta1}^{II}=A_1 r^{\lambda_1-1}\tilde{\sigma}_{\theta\theta1}^{II} \tag{2.258}$$

$$\sigma_{r\theta1}^{II}=A_1 r^{\lambda_1-1}\tilde{\sigma}_{r\theta1}^{II} \tag{2.259}$$

where,

$$\tilde{\sigma}_{rr1}^{II}=(\lambda_1+1)\tilde{\phi}_1^{II}+\left(\tilde{\phi}_1^{II}\right)'' \tag{2.260}$$

$$\tilde{\sigma}_{\theta\theta1}^{II}=(\lambda_1+1)\lambda_1\tilde{\phi}_1^{II} \tag{2.261}$$

$$\tilde{\sigma}_{r\theta1}^{II}=-\lambda_1\left(\tilde{\phi}_1^{II}\right)' \tag{2.262}$$

Displacement can be calculated as:

$$u_{r0}^{II}+u_{r1}^{II}=\int\varepsilon_{rr0}^{II}dr+\int\varepsilon_{rr1}^{II}dr \tag{2.263}$$

Initial part of displacement components and initial strain components has a relationship. Remaining term can be expressed as,

$$u_{r1}^{II}=A_1 r^{\lambda_1}\tilde{u}_{r1}^{II} \tag{2.264}$$

where,

$$\tilde{u}_{r1}^{II} = \frac{E^{I}}{E^{II}} \left\{ \frac{\left(1+\nu^{II}\right)}{\lambda_1} \left[\left(\lambda_1 + 1\right)\left(1-\nu^{II} - \nu^{II}\lambda_1\right)\tilde{\phi}_1^{II} + \left(1-\nu^{II}\right)\left(\tilde{\phi}_1^{II}\right)'' \right] \right\} \tag{2.265}$$

Displacement equation of first order approximation of elastic material side can be expressed as,

$$u_{r(1)}^{II} = u_{r0}^{II} + u_{r1}^{II} = A_0\, r^{\lambda_0} \tilde{u}_{r0}^{II} + A_1\, r^{\lambda_1} \tilde{u}_{r1}^{II} \tag{2.266}$$

And again we know from Eqn. (2.10), substituting , $\varepsilon_{\theta\theta 1}^{II} = \varepsilon_{\theta\theta 0}^{II} + \varepsilon_{\theta\theta 1}^{II}$, $u_{\theta(1)}^{II} = u_{\theta 0}^{II} + u_{\theta 1}^{II}$ and $u_{r(1)}^{II} = u_{r0}^{II} + u_{r1}^{II}$

Strain displacement relation becomes,

$$u_{\theta 0}^{II} + u_{\theta 1}^{II} = r\int_{-\frac{\pi}{2}}^{0} \varepsilon_{\theta\theta 0}^{II} d\theta - \int_{-\frac{\pi}{2}}^{0} u_{r0}^{II} d\theta + r\int_{-\frac{\pi}{2}}^{0} \varepsilon_{\theta\theta 1}^{II} d\theta - \int_{-\frac{\pi}{2}}^{0} u_{r1}^{II} d\theta \tag{2.267}$$

Initial part of displacement components and initial part of strain components has a relationship. Remaining term can be expressed as,

$$u_{\theta 1}^{II} = r\int_{-\frac{\pi}{2}}^{0} \varepsilon_{\theta\theta 1}^{II} d\theta - \int_{-\frac{\pi}{2}}^{0} u_{r1}^{II} d\theta \tag{2.268}$$

Finally,

$$u_{\theta(1)}^{II} = u_{\theta 0}^{II} + u_{\theta 1}^{II} \tag{2.269}$$

We know from the strain displacement relation given in Eqn.(2.10) from the incremental part strain displacement relation can be written as:

$$u_{\theta 1}^{II} = A_1\, r^{\lambda_1} \tilde{u}_{\theta 1}^{II} \tag{2.270}$$

where

$$\tilde{u}_{\theta 1}^{II} = \frac{E^I}{E^{II}} \frac{A_1 r^{\lambda_1}}{\lambda_1} \left(1+\nu^{II}\right) \left[\left(\lambda_1+1\right)\left(\lambda_1^2-1\right)\left(1-\nu^{II}\right) \int_{-\frac{\pi}{2}}^{0} \tilde{\phi}_1^{II} d\theta - \left(\nu^{II}\lambda_1 + 1 - \nu^{II}\right)\left(\tilde{\phi}_1^{II}\right)' \right] \quad (2.271)$$

Eqn. (2.269) can be rewritten as,

$$u_{\theta(1)}^{II} = u_{\theta 0}^{II} + u_{\theta 1}^{II} = A_0 r^{\lambda_0} \tilde{u}_{\theta 0}^{II} + A_1 r^{\lambda_1} \tilde{u}_{\theta 1}^{II} \quad (2.272)$$

Using expression of Eqn.(2.11)strain, displacement can be calculated.

Rigid body rotation is depends on r and angular function. To overcome the rigid body rotation we have to assume the displacement as a function of r and the angular function term.

Assume, $u_{\theta 1}^{II} = kr^{\lambda_1} f(\theta) \therefore \frac{\partial u_{\theta 1}^{II}}{\partial r} = \lambda_1 k r^{\lambda_1 - 1} f(\theta);\ or,\ \ r\frac{\partial u_{\theta 1}^{II}}{\partial r} = \lambda_1 k r^{\lambda_1} f(\theta)$

So equation becomes, When $\lambda_1 \neq 1, kr^{\lambda_1} f(\theta)(\lambda_1 - 1) = 2r\varepsilon_{r\theta 1}^{II} - \frac{\partial u_{r1}^{II}}{\partial \theta}$ (2.273)

$$u_{\theta 1}^{II} = A_1 r^{\lambda_1} \tilde{u}_{\theta 1}^{II} \quad (2.274)$$

where,

$$\tilde{u}_{\theta 1}^{II} = -\frac{E^I}{E^{II}} \frac{\left(1+\nu^{II}\right)}{\lambda_1\left(\lambda_1-1\right)} \left[\left\{2\lambda_1^2 + \left(\lambda_1+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_1\right)\right\}\left(\tilde{\phi}_1^{II}\right)' + \left(1-\nu^{II}\right)\left(\tilde{\phi}_1^{II}\right)''' \right] \quad (2.275)$$

Substituting into Eqn. (2.269) Total displacement fields can be expressed in Eqn. (2.272)

Applying boundary conditions traction free-edge can be expressed as,

$$\left(\sigma_{\theta\theta(1)}^{II}\right)_{\theta=-\frac{\pi}{2}} = \left(\sigma_{\theta\theta 0}^{II}\right)_{\theta=-\frac{\pi}{2}} + \left(\sigma_{\theta\theta 1}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0 \quad (2.276)$$

and

$$\left(\sigma_{r\theta(1)}^{II}\right)_{\theta=-\frac{\pi}{2}} = \left(\sigma_{r\theta 0}^{II}\right)_{\theta=-\frac{\pi}{2}} + \left(\sigma_{r\theta 1}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0 \quad (2.277)$$

From zero-th order approximation we know $\left(\sigma_{\theta\theta 0}^{II}\right)_{\theta=-\frac{\pi}{2}}=0\,,\left(\sigma_{r\theta 0}^{II}\right)_{\theta=-\frac{\pi}{2}}=0$

So $\left(\sigma_{\theta\theta 1}^{II}\right)_{\theta=-\frac{\pi}{2}}=0$ and $\left(\sigma_{r\theta 1}^{II}\right)_{\theta=-\frac{\pi}{2}}=0$ (2.278)

Initial conditions at $\theta=-\dfrac{\pi}{2}$:

$$\frac{\sigma_{\theta\theta 1}^{II}}{r^{\lambda_1-1}}=\left(\lambda_1+1\right)\lambda_1\left(A_1\tilde{\phi}_1^{II}\right)=0, \qquad \left(A_1\tilde{\phi}_1^{II}\right)=0,\text{if} \quad \lambda_1\neq -1,\ \lambda_1\neq 0 \tag{2.279}$$

And $$\frac{\sigma_{r\theta 1}^{II}}{r^{\lambda_1-1}}=-\lambda_1\frac{\partial\left(A_1\tilde{\phi}_1^{II}\right)}{\partial\theta}=0, or\ \frac{\partial\left(A_1\tilde{\phi}_1^{II}\right)}{\partial\theta}=0,\text{if} \quad \lambda_1\neq 0 \tag{2.280}$$

Assume at $\theta=-\dfrac{\pi}{2}$, $A_1\tilde{\phi}_1^{II}=0,\left(A_1\tilde{\phi}_1^{II}\right)'=0,\left(A_1\tilde{\phi}_1^{II}\right)''$ and $\left(A_1\tilde{\phi}_1^{II}\right)'''$ hence, after i-th integration at $\theta=0,\left(A_1\tilde{\phi}_1^{II}\right)=\left(A_1\tilde{\phi}_1^{II}\right)^{(i)}$, $\left(A_1\tilde{\phi}_1^{II}\right)'=\left(\left(A_1\tilde{\phi}_1^{II}\right)'\right)^{(i)},\left(A_1\tilde{\phi}_1^{II}\right)''=\left(\left(A_1\tilde{\phi}_1^{II}\right)''\right)^{(i)}$

and $\left(A_1\tilde{\phi}_1^{II}\right)'''=\left(\left(A_1\tilde{\phi}_1^{II}\right)'''\right)^{(i)}$

$$\text{error1}=\tilde{\phi}_1^{I}-\left(\tilde{\phi}_1^{II}\right)^{(i)} \tag{2.281}$$

$$\text{error2}=\left(\left(\tilde{\phi}_1^{I}\right)'\right)-\left(\left(\tilde{\phi}_1^{II}\right)'\right)^{(i)} \tag{2.282}$$

The error value is calculated by using the equation (2.15)

2.6 Formulation of ith Order Approximation

2.6.1 Formulation of ith Order Approximation: Constitutive Equations in the Power-Law Hardening Material Subjected To Forced Displacement

We can express the stress functions in the form of an infinite power series have been given by Hutchinson [31] in the solution of a nonlinear power-law hardening material crack tip field:

Let, $$\phi=\phi_0^I+\phi_1^I+\phi_2^I+\cdots+\phi_i^I=A_0\,r^{\lambda_0+1}\tilde{\phi}_0^I+A_1\,r^{\lambda_1+1}\tilde{\phi}_1^I+A_2\,\tilde{\phi}_2^I r^{\lambda_2+1}+\cdots+A_i\,\tilde{\phi}_i^I r^{\lambda_i+1} \tag{2.283}$$

where, $\phi_0^I + \phi_1^I + \phi_2^I = A_0\, r^{\lambda_0+1}\tilde{\phi}_0^I + A_1\, r^{\lambda_1+1}\tilde{\phi}_1^I + A_2\,\tilde{\phi}_2^I r^{\lambda_2+1}$ and $\phi_i^I = A_i\,\tilde{\phi}_i^I r^{\lambda_i+1}$

Assume, $\lambda_0, \lambda_1, \lambda_2 \cdots$ and λ_i depends on the deformation

Substituting Eqn. (2.283) into Eqns. (2.5-2.7), we obtain stress fields of i-th order approximation as,

$$\sigma_{rr} = r^{\lambda_0-1}\left\{A_0\,\tilde{\phi}_0^I\left(\lambda_0+1\right) + A_0\left(\tilde{\phi}_0^I\right)''\right\} + r^{\lambda_1-1}\left\{A_1\,\tilde{\phi}_1^I\left(\lambda_1+1\right) + A_1\left(\tilde{\phi}_1^I\right)''\right\} + r^{\lambda_2-1}\left\{A_2\,\tilde{\phi}_2^I\left(\lambda_2+1\right) + A_2\left(\tilde{\phi}_2^I\right)''\right\}$$
$$+\cdots + r^{\lambda_i-1}\left\{A_i\,\tilde{\phi}_i^I\left(\lambda_i+1\right) + A_i\left(\tilde{\phi}_i^I\right)''\right\} \tag{2.284}$$

$$\sigma_{\theta\theta} = A_0\,\tilde{\phi}_0^I\left(\lambda_0+1\right)\lambda_0 r^{\lambda_0-1} + A_1\,\tilde{\phi}_1^I\left(\lambda_1+1\right)\lambda_1 r^{\lambda_1-1} + A_2\,\tilde{\phi}_2^I\left(\lambda_2+1\right)\lambda_2 r^{\lambda_2-1} + \cdots + A_i\,\tilde{\phi}_i^I\left(\lambda_i+1\right)\lambda_i r^{\lambda_i-1} \tag{2.285}$$

$$\sigma_{r\theta} = -\left(r^{\lambda_0-1}A_0\left(\tilde{\phi}_0^I\right)'\lambda_0 + r^{\lambda_1-1}A_1\left(\tilde{\phi}_1^I\right)'\lambda_1 + r^{\lambda_2-1}A_2\left(\tilde{\phi}_2^I\right)'\lambda_2 + \cdots + r^{\lambda_i-1}A_i\left(\tilde{\phi}_i^I\right)'\lambda_i\right) \tag{2.286}$$

$$s_{rr} = \frac{1}{2}\left\{r^{\lambda_0-1}f_{0rr} + A_1\, r^{\lambda_1-1}f_{1rr} + A_2\, r^{\lambda_2-1}f_{2rr} + \cdots + A_i\, r^{\lambda_i-1}f_{irr}\right\} \tag{2.287}$$

$$s_{\theta\theta} = \frac{1}{2}\left\{r^{\lambda_0-1}f_{0\theta\theta} + A_1\, r^{\lambda_1-1}f_{1\theta\theta} + A_2\, r^{\lambda_2-1}f_{2\theta\theta} + \cdots + A_i\, r^{\lambda_i-1}f_{i\theta\theta}\right\} \tag{2.288}$$

$$s_{r\theta} = r^{\lambda_0-1}f_{0r\theta} + A_1\, r^{\lambda_1-1}f_{1r\theta} + A_2\, r^{\lambda_2-1}f_{2r\theta} + \cdots + A_i\, r^{\lambda_i-1}f_{ir\theta} \tag{2.289}$$

where,

$$f_{0rr} = A_0\left\{\tilde{\phi}_0^I\left(\lambda_0+1\right)\left(1-\lambda_0\right) + \left(\tilde{\phi}_0^I\right)''\right\},\quad f_{1rr} = \left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)\left(1-\lambda_1\right) + \left(\tilde{\phi}_1^I\right)''\right\}$$

$$f_{2rr} = \left\{\tilde{\phi}_2^I\left(\lambda_2+1\right)\left(1-\lambda_2\right) + \left(\tilde{\phi}_2^I\right)''\right\},\quad f_{irr} = \left\{\tilde{\phi}_i^I\left(\lambda_i+1\right)\left(1-\lambda_i\right) + \left(\tilde{\phi}_i^I\right)''\right\}$$

$$f_{0\theta\theta} = -\left\{\tilde{\phi}_0^I\left(\lambda_0+1\right)\left(1-\lambda_0\right) + \left(\tilde{\phi}_0^I\right)''\right\},\quad f_{1\theta\theta} = -\left\{\tilde{\phi}_1^I\left(\lambda_1+1\right)\left(1-\lambda_1\right) + \left(\tilde{\phi}_1^I\right)''\right\}$$

$$f_{2\theta\theta} = -\left\{ \tilde{\phi}_2^I \left(\lambda_2 + 1\right)\left(1 - \lambda_2\right) + \left(\tilde{\phi}_2^I\right)'' \right\}, \quad f_{i\theta\theta} = -\left\{ \tilde{\phi}_i^I \left(\lambda_i + 1\right)\left(1 - \lambda_i\right) + \left(\tilde{\phi}_i^I\right)'' \right\}$$

$$f_{0r\theta} = \left\{ -A_0 \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}, \quad f_{1r\theta} = \left\{ -\left(\tilde{\phi}_1^I\right)' \lambda_1 \right\}, \quad f_{2r\theta} = \left\{ -\left(\tilde{\phi}_2^I\right)' \lambda_2 \right\}, \quad f_{ir\theta} = \left\{ -\left(\tilde{\phi}_i^I\right)' \lambda_i \right\} \quad (2.290)$$

Substituting Eqns.(2.284 -2.290) into Eqn.(2.43) yields an expression for the effective stress,

$$\begin{aligned} \sigma_e^2 = \frac{3}{8} \Big\{ & \left[r^{\lambda_0 - 1} f_{0rr} + A_1 r^{\lambda_1 - 1} f_{1rr} + A_2 r^{\lambda_2 - 1} f_{2rr} + \cdots + A_i r^{\lambda_i - 1} f_{irr} \right]^2 \\ & + \left[r^{\lambda_0 - 1} f_{0\theta\theta} + A_1 r^{\lambda_1 - 1} f_{1\theta\theta} + A_2 r^{\lambda_2 - 1} f_{2\theta\theta} + \cdots + A_i r^{\lambda_i - 1} f_{i\theta\theta} \right]^2 \\ & + 8 \left[r^{\lambda_0 - 1} f_{0r\theta} + A_1 r^{\lambda_1 - 1} f_{1r\theta} + A_2 r^{\lambda_2 - 1} f_{2r\theta} + \cdots + A_i r^{\lambda_i - 1} f_{ir\theta} \right]^2 \Big\} \end{aligned} \quad (2.291)$$

Assume,

$$\begin{aligned} & C_1 = r^{\lambda_0 - 1} f_{0rr}, \quad \Delta C_1 = \left(A_1 r^{\lambda_1 - 1} f_{1rr} + A_2 r^{\lambda_2 - 1} f_{2rr} + \cdots + A_i r^{\lambda_i - 1} f_{irr} \right), \\ & C_2 = r^{\lambda_0 - 1} f_{0\theta\theta}, \; \Delta C_2 = \left(A_1 r^{\lambda_1 - 1} f_{1\theta\theta} + A_2 r^{\lambda_2 - 1} f_{2\theta\theta} + \cdots + A_i r^{\lambda_i - 1} f_{i\theta\theta} \right), \\ & C_3 = r^{\lambda_0 - 1} f_{0r\theta}, \; \Delta C_3 = \left(A_1 r^{\lambda_1 - 1} f_{1r\theta} + A_2 r^{\lambda_2 - 1} f_{2r\theta} + \cdots + A_i r^{\lambda_i - 1} f_{ir\theta} \right) \end{aligned} \quad (2.292)$$

Eqn.(2.298) yields,

$$\sigma_e^2 = \frac{3}{8} \left\{ \left[C_1 + \Delta C_1 \right]^2 + \left[C_2 + \Delta C_2 \right]^2 + 8 \left[C_3 + \Delta C_3 \right]^2 \right\} \quad (2.293)$$

Eqn.(2.298) can be written after Taylor expansion as,

$$\sigma_e^{n-1} \approx \left[\frac{3}{8} \left(C_1^{\,2} + C_2^{\,2} + 8C_3^{\,2} \right) \right]^{\frac{n-1}{2}} + \frac{n-1}{2} \left[\frac{3}{8} \left(C_1^{\,2} + C_2^{\,2} + 8C_3^{\,2} \right) \right]^{\frac{n-3}{2}} \times \left\{ \frac{3}{4} \left(C_1 \Delta C_1 + C_2 \Delta C_2 + 8 C_3 \Delta C_3 \right) + \frac{3}{8} \times \left\{ \left(\Delta C_1 \right)^2 + \left(\Delta C_2 \right)^2 + 8 \left(\Delta C_3 \right)^2 \right\} \right\} \quad (2.294)$$

where constant terms can be written as,

$$\left(C_1^{\,2} + C_2^{\,2} + 8C_3^{\,2} \right) = r^{2(\lambda_0 - 1)} \left(f_{0rr}^2 + f_{0\theta\theta}^2 + 8 f_{0r\theta}^2 \right)$$

$$C_1 \Delta C_1 = A_1 \times r^{\lambda_0 - 1} f_{0rr} r^{\lambda_1 - 1} f_{1rr} + A_2 \times r^{\lambda_0 - 1} f_{0rr} r^{\lambda_2 - 1} f_{2rr} + \cdots + A_i \times r^{\lambda_0 - 1} f_{0rr} r^{\lambda_i - 1} f_{irr}$$

$$
\begin{aligned}
C_1\Delta C_1 + C_2\Delta C_2 + 8C_3\Delta C_3 = {} & A_1 \times r^{\lambda_0+\lambda_1-2}\left(f_{0rr}f_{1rr} + f_{0\theta\theta}f_{1\theta\theta} + 8f_{0r\theta}f_{1r\theta}\right) + A_2 \times r^{\lambda_0+\lambda_2-2}\left(f_{0rr}f_{2rr} + f_{0\theta\theta}f_{2\theta\theta} + 8f_{0r\theta}f_{2r\theta}\right) \\
& + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}\left(f_{0rr}f_{irr} + f_{0\theta\theta}f_{i\theta\theta} + 8f_{0r\theta}f_{ir\theta}\right)
\end{aligned}
$$

$$
\left(\Delta C_1\right)^2 = A_1^2 \times \left(r^{\lambda_1-1}f_{1rr}\right)^2 + A_1A_2 \times 2\left(r^{\lambda_1+\lambda_2-2}f_{1rr}f_{2rr}\right) + A_2^2 \times \left(r^{\lambda_2-1}f_{2rr}\right)^2 + \cdots + A_i^2 \times \left(r^{\lambda_i-1}f_{irr}\right)^2 \tag{2.295}
$$

Similarly, $(\Delta C_2)^2$ and $(\Delta C_3)^2$ includes second order of A_1, bi-linear of $A_1 \cdots A_i$ and $A_2 \cdots A_i$ and second order term of $A_2 \cdots A_i$. So, $\left\{(\Delta C_1)^2 + (\Delta C_2)^2 + 8(\Delta C_3)^2\right\}$ also includes second order of A_1, bi-linear of $A_1 \cdots A_i$ and $A_2 \cdots A_i$ and second order term of $A_2 \cdots A_i$. Which means the terms are very small in compared with the initial term. So, here $(\Delta C_1)^2, (\Delta C_2)^2$ and $(\Delta C_3)^2$ order terms can be neglected.

After Taylor series expansion and neglecting higher order terms Eqn. (2.291) yields,

$$
\begin{aligned}
\sigma_e^{n-1} \approx {} & \left[\frac{3}{8}r^{2(\lambda_0-1)}\left(f_{0rr}^2 + f_{0\theta\theta}^2 + 8f_{0r\theta}^2\right)\right]^{\frac{n-1}{2}} + \frac{n-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)}\left(f_{0rr}^2 + f_{0\theta\theta}^2 + 8f_{0r\theta}^2\right)\right]^{\frac{n-3}{2}} \\
& \times \frac{3}{4}\Big(A_1 \times r^{\lambda_0+\lambda_1-2}\left(f_{0rr}f_{1rr} + f_{0\theta\theta}f_{1\theta\theta} + 8f_{0r\theta}f_{1r\theta}\right) + A_2 \times r^{\lambda_0+\lambda_2-2}\left(f_{0rr}f_{2rr} + f_{0\theta\theta}f_{2\theta\theta} + 8f_{0r\theta}f_{2r\theta}\right) \\
& + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}\left(f_{0rr}f_{irr} + f_{0\theta\theta}f_{i\theta\theta} + 8f_{0r\theta}f_{ir\theta}\right)\Big)
\end{aligned} \tag{2.296}
$$

Assume,

$$
\begin{aligned}
& f_0 = \left(f_{0rr}^2 + f_{0\theta\theta}^2 + 8f_{0r\theta}^2\right), \quad f_1 = \left(f_{0rr}f_{1rr} + f_{0\theta\theta}f_{1\theta\theta} + 8f_{0r\theta}f_{1r\theta}\right) \\
& f_2 = \left(f_{0rr}f_{2rr} + f_{0\theta\theta}f_{2\theta\theta} + 8f_{0r\theta}f_{2r\theta}\right), \quad f_i = \left(f_{0rr}f_{irr} + f_{0\theta\theta}f_{i\theta\theta} + 8f_{0r\theta}f_{ir\theta}\right)
\end{aligned} \tag{2.297}
$$

So, expression for the effective stress becomes,

$$
\sigma_e^{n-1} \approx \left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-1}{2}} + \frac{n-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2}f_1 + A_2 \times r^{\lambda_0+\lambda_2-2}f_2 + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}f_i\right) \tag{2.298}
$$

Substituting effective stress term into Eqn.(2.3), strain components have the form in terms of C as,

$$\varepsilon_{rr} \approx \frac{3}{2}\alpha\left(\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-1}{2}}+\frac{n-1}{2}\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-3}{2}}\right.$$
$$\left.\times\frac{3}{4}\left(C_1\Delta C_1+C_2\Delta C_2+8C_3\Delta C_3\right)\right)\times\frac{1}{2}\left\{C_1+\Delta C_1\right\} \tag{2.299}$$

Where,

$$\sigma_e^{n-1} \approx \left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-1}{2}}+\frac{n-1}{2}\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-3}{2}}\times\frac{3}{4}\left(C_1\Delta C_1+C_2\Delta C_2+8C_3\Delta C_3\right) \tag{2.300}$$

$$\varepsilon_{rr} \approx \frac{3}{2}\alpha\left(\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-1}{2}}\times\frac{1}{2}C_1+\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-1}{2}}\times\frac{1}{2}\Delta C_1\right.$$
$$+\frac{n-1}{2}\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-3}{2}}\times\frac{3}{4}\times\frac{1}{2}\left(C_1^{\ 2}\Delta C_1+C_1C_2\Delta C_2+8C_1C_2\Delta C_3\right) \tag{2.301}$$
$$\left.+\frac{n-1}{2}\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-3}{2}}\times\frac{3}{4}\times\frac{1}{2}\left(C_1\left(\Delta C_1\right)^2+C_2\Delta C_1\Delta C_2+8\Delta C_1C_2\Delta C_3\right)\right)$$

Here, $\left(\Delta C_1\right)^2$, $\Delta C_1\Delta C_2$ and $\Delta C_1\Delta C_3$ includes square of small magnitudes. Which means the terms are very small in compared with the initial term so these terms can be neglected.

$$\varepsilon_{rr} \approx \frac{3}{2}\alpha\left(\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-1}{2}}\times\frac{1}{2}C_1+\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-1}{2}}\times\frac{1}{2}\Delta C_1\right.$$
$$\left.+\frac{n-1}{2}\left[\frac{3}{8}\left(C_1^{\ 2}+C_2^{\ 2}+8C_3^{\ 2}\right)\right]^{\frac{n-3}{2}}\times\frac{3}{4}\times\frac{1}{2}\left(C_1^{\ 2}\Delta C_1+C_1C_2\Delta C_2+8C_1C_3\Delta C_2\right)\right) \tag{2.302}$$

In terms of $f_0, f_1, f_2, \cdots, f_i$,Eqn.(2.302) yields,

$$\varepsilon_{rr} \approx \frac{3}{2}\alpha\left(\left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-1}{2}} \times \frac{1}{2}\left\{r^{\lambda_0-1}f_{0rr} + A_1 r^{\lambda_1-1}f_{1rr} + A_2 r^{\lambda_2-1}f_{2rr} + \cdots + A_i r^{\lambda_i-1}f_{irr}\right\}\right.$$

$$+\frac{n-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2}f_1 + A_2 \times r^{\lambda_0+\lambda_2-2}f_2 + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}f_i\right) \times \frac{1}{2}\left\{r^{\lambda_0-1}f_{0rr}\right\}$$

$$+\frac{n-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2}f_1 + A_2 \times r^{\lambda_0+\lambda_2-2}f_2 + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}f_i\right) \times \frac{1}{2}\left\{A_1 r^{\lambda_1-1}f_{1rr}\right\}$$

$$+\frac{n-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2}f_1 + A_2 \times r^{\lambda_0+\lambda_2-2}f_2 + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}f_i\right) \times \frac{1}{2}\left\{A_2 r^{\lambda_2-1}f_{2rr}\right\}$$

$$+\cdots+$$

$$\left.+\frac{n-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2}f_1 + A_2 \times r^{\lambda_0+\lambda_2-2}f_2 + \cdots + A_i \times r^{\lambda_0+\lambda_i-2}f_i\right) \times \frac{1}{2}\left\{A_i r^{\lambda_i-1}f_{irr}\right\}\right) \tag{2.303}$$

Here higher order terms ($(A_i)^2$ order terms) can be neglected and neglecting higher order terms,

$$\varepsilon_{rr} \approx \frac{3}{4}\alpha\left(\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} r^{n(\lambda_0-1)}f_{0rr} + A_1 r^{n\lambda_0-n-\lambda_0+\lambda_1}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{1rr} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0rr}f_1\right)\right\}\right.$$

$$+A_2 r^{n\lambda_0-n-\lambda_0+\lambda_2}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{2rr} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0rr}f_2\right)\right\}$$

$$\left.+\cdots+A_i r^{n\lambda_0-n-\lambda_0+\lambda_i}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{irr} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0rr}f_i\right)\right\}\right) \tag{2.304}$$

Similarly

$$\varepsilon_{\theta\theta} \approx \frac{3}{4}\alpha\left(\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} r^{n(\lambda_0-1)} f_{0\theta\theta} + A_1 r^{n\lambda_0-n-\lambda_0+\lambda_1}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{1\theta\theta} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0\theta\theta}f_1\right)\right\}\right.$$

$$+A_2 r^{n\lambda_0-n-\lambda_0+\lambda_2}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{2\theta\theta} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0\theta\theta}f_2\right)\right\}$$

$$\left.+\cdots+A_i r^{n\lambda_0-n-\lambda_0+\lambda_i}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{i\theta\theta} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0\theta\theta}f_i\right)\right\}\right) \tag{2.305}$$

And

$$\varepsilon_{r\theta} \approx \frac{3}{2}\alpha\left(\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} r^{n(\lambda_0-1)} f_{0r\theta} + A_1 r^{n\lambda_0-n-\lambda_0+\lambda_1}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{1r\theta} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0r\theta}f_1\right)\right\}\right.$$

$$+A_2 r^{n\lambda_0-n-\lambda_0+\lambda_2}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{2r\theta} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0r\theta}f_2\right)\right\}$$

$$\left.+\cdots+A_i r^{n\lambda_0-n-\lambda_0+\lambda_i}\left\{\left[\frac{3}{8}f_0\right]^{\frac{n-1}{2}} f_{ir\theta} + \frac{n-1}{2}\left[\frac{3}{8}f_0\right]^{\frac{n-3}{2}} \times \frac{3}{4}\left(f_{0r\theta}f_i\right)\right\}\right) \tag{2.306}$$

Strain components are in the following form according to the first order terms of $A_1, A_2 \cdots A_i$:

$$\varepsilon_{rr} \approx \varepsilon_{rr0} + \varepsilon_{rr1}\left\{O(A_1)\right\} + \varepsilon_{rr2}\{O(A_2)\} + \cdots\cdots + \varepsilon_{rri}\{O(A_i)\} \tag{2.307}$$

$$\varepsilon_{\theta\theta} \approx \varepsilon_{\theta\theta 0} + \varepsilon_{\theta\theta 1}\left\{O(A_1)\right\} + \varepsilon_{\theta\theta 2}\{O(A_2)\} + \cdots\cdots + \varepsilon_{\theta\theta i}\{O(A_i)\} \tag{2.308}$$

$$\varepsilon_{r\theta} \approx \varepsilon_{r\theta 0} + \varepsilon_{r\theta 1}\left\{O(A_1)\right\} + \varepsilon_{r\theta 2}\{O(A_2)\} + \cdots\cdots + \varepsilon_{r\theta i}\{O(A_i)\} \tag{2.309}$$

In these expressions strain components will have i-th terms with respect to the power of r. $r^{n(\lambda_0-1)}$, $r^{n\lambda_0-n-\lambda_0+\lambda_1}, r^{n\lambda_0-n-\lambda_0+\lambda_2}, \ldots\ldots, r^{n\lambda_0-n-\lambda_0+\lambda_{i-1}}, r^{n\lambda_0-n-\lambda_0+\lambda_i}$

From zero-th order approximation solution it is clear that the terms of $r^{n(\lambda_0-1)}$ satisfy the compatibility condition and from (i-1) th order approximation solution the terms of $r^{n\lambda_0-n-\lambda_0+\lambda_{i-1}}$ also satisfy the compatibility condition. To solve the compatibility condition on the remaining terms, we assume the i-th terms satisfy the conditions independently.

$$\varepsilon_{lj(i)} = \varepsilon_{lj0} + \varepsilon_{lj1}\left\{O(A_1)\right\} + \varepsilon_{lj2}\left\{O(A_2)\right\} + \cdots\cdots + \varepsilon_{lji}\{O(A_i)\} \tag{2.310}$$

Strain components are in the following summation form:

$$\varepsilon^I_{rr(i)} \approx \varepsilon^I_{rr0} + \varepsilon^I_{rr1} + \varepsilon^I_{rr2} + \cdots + \varepsilon^I_{rri} \tag{2.311}$$

$$\varepsilon^I_{\theta\theta(i)} \approx \varepsilon^I_{\theta\theta0} + \varepsilon^I_{\theta\theta1} + \varepsilon^I_{\theta\theta2} + \cdots + \varepsilon^I_{\theta\theta i} \tag{2.312}$$

$$\varepsilon^I_{r\theta(i)} \approx \varepsilon^I_{r\theta0} + \varepsilon^I_{r\theta1} + \varepsilon^I_{r\theta2} + \cdots + \varepsilon^I_{r\theta i} \tag{2.313}$$

Initial part of strain components are given in Eqns.(2.83-2.85) and strain components includes first order term of A_1 are presented in Eqns.(2.155-2.157):

Strain components includes first order term of A_2:

$$\varepsilon^I_{rr2} = A_0^{n-1} A_2 \, r^{n\lambda_0-n-\lambda_0+\lambda_2} \tilde{\varepsilon}^I_{rr2} \tag{2.314}$$

$$\varepsilon^I_{\theta\theta2} = A_0^{n-1} A_2 \, r^{n\lambda_0-n-\lambda_0+\lambda_2} \tilde{\varepsilon}^I_{\theta\theta2} \tag{2.315}$$

$$\varepsilon^I_{r\theta2} = A_0^{n-1} A_2 \, r^{n\lambda_0-n-\lambda_0+\lambda_2} \tilde{\varepsilon}^I_{r\theta2} \tag{2.316}$$

where

$$
\begin{aligned}
\tilde{\varepsilon}_{rr2}^{I} = 2^{-1-n}3^{\frac{n+1}{2}}\alpha \Bigg[& (n-1)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}} \\
& \times\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}\times(-1)\left\{\tilde{\phi}_2^I\left(\lambda_2+1\right)\left(\lambda_2-1\right)-\left(\tilde{\phi}_2^I\right)''\right\}\right. \\
& \quad \left.+4\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right)+\left(\tilde{\phi}_0^I\right)''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times\left\{\left(\tilde{\phi}_2^I\right)'\lambda_2\right\}\right) \\
& \quad +\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times(-1)\left\{\tilde{\phi}_2^I\left(\lambda_2+1\right)\left(\lambda_2-1\right)-\left(\tilde{\phi}_2^I\right)''\right\}\Bigg]
\end{aligned}
\tag{2.317}
$$

$$
\begin{aligned}
\tilde{\varepsilon}_{\theta\theta2}^{I} = 2^{-1-n}3^{\frac{n+1}{2}}\alpha \Bigg[& (n-1)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}} \\
& \times\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}\times\left\{\tilde{\phi}_2^I\left(\lambda_2+1\right)\left(\lambda_2-1\right)-\left(\tilde{\phi}_2^I\right)''\right\}\right. \\
& \quad \left.+4\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right)+\left(\tilde{\phi}_0^I\right)''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times(-1)\left\{\left(\tilde{\phi}_2^I\right)'\lambda_2\right\}\right) \\
& \quad +\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times\left\{\tilde{\phi}_2^I\left(\lambda_2+1\right)\left(\lambda_2-1\right)-\left(\tilde{\phi}_2^I\right)''\right\}\Bigg]
\end{aligned}
\tag{2.318}
$$

$$
\begin{aligned}
\tilde{\varepsilon}_{r\theta2}^{I} = 2^{-n}3^{\frac{n+1}{2}}\alpha \Bigg[& (n-1)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}} \\
& \times\left(\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right)+\tilde{\phi}_0''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times\left\{\tilde{\phi}_2^I\left(\lambda_2+1\right)\left(\lambda_2-1\right)-\left(\tilde{\phi}_2^I\right)''\right\}+4\left\{-\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\times(-1)\left\{\left(\tilde{\phi}_2^I\right)'\lambda_2\right\}\right) \\
& \quad +\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2+\left(\left(\tilde{\phi}_0^I\right)''\right)^2+2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\left(\tilde{\phi}_0^I\right)''\right)\right\}+4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times(-1)\left\{\left(\tilde{\phi}_2^I\right)'\lambda_2\right\}\Bigg]
\end{aligned}
\tag{2.319}
$$

Where, Strain components includes i-th order term of A_i :

$$\varepsilon^I_{rri} = A_0^{n-1} A_i \, r^{n\lambda_0 - n - \lambda_0 + \lambda_i} \, \tilde{\varepsilon}^I_{rri} \tag{2.320}$$

$$\varepsilon^I_{\theta\theta i} = A_0^{n-1} A_i \, r^{n\lambda_0 - n - \lambda_0 + \lambda_i} \, \tilde{\varepsilon}^I_{\theta\theta i} \tag{2.321}$$

$$\varepsilon^I_{r\theta i} = A_0^{n-1} A_i \, r^{n\lambda_0 - n - \lambda_0 + \lambda_i} \, \tilde{\varepsilon}^I_{r\theta i} \tag{2.322}$$

where,

$$\begin{aligned}
\tilde{\varepsilon}^I_{rri} = 2^{-1-n} 3^{\frac{n+1}{2}} \alpha \Bigg[& (n-1) \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I \right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I \right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I \right)' \lambda_0 \right\}^2 \right)^{\frac{n-3}{2}} \\
& \times \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I \right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I \right)'' \right) \right\} \times (-1) \left\{ \tilde{\phi}_i^I \left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\tilde{\phi}_i^I \right)'' \right\} \right. \\
& \left. + 4 \left\{ \tilde{\phi}_0^I \left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I \right)'' \right\} \left\{ \left(\tilde{\phi}_0^I \right)' \lambda_0 \right\} \times \left\{ \left(\tilde{\phi}_i^I \right)' \lambda_i \right\} \right) \\
& + \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I \right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I \right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I \right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \times (-1) \left\{ \tilde{\phi}_i^I \left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\tilde{\phi}_i^I \right)'' \right\} \Bigg]
\end{aligned} \tag{2.323}$$

$$\begin{aligned}
\tilde{\varepsilon}^I_{\theta\theta i} = 2^{-1-n} 3^{\frac{n+1}{2}} \alpha \Bigg[& (n-1) \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I \right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I \right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I \right)' \lambda_0 \right\}^2 \right)^{\frac{n-3}{2}} \\
& \times \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I \right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I \right)'' \right) \right\} \times \left\{ \tilde{\phi}_i^I \left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\tilde{\phi}_i^I \right)'' \right\} \right. \\
& \left. + 4 \left\{ \tilde{\phi}_0^I \left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I \right)'' \right\} \left\{ \left(\tilde{\phi}_0^I \right)' \lambda_0 \right\} \times (-1) \left\{ \left(\tilde{\phi}_i^I \right)' \lambda_i \right\} \right) \\
& + \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I \right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I \right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I \right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \times \left\{ \tilde{\phi}_i^I \left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\tilde{\phi}_i^I \right)'' \right\} \Bigg]
\end{aligned} \tag{2.324}$$

$$
\begin{aligned}
\tilde{\varepsilon}^{I}_{r\theta i} = 2^{-n} 3^{\frac{n+1}{2}} \alpha \Bigg[(n-1) & \left(\left\{ \left(\tilde{\phi}^{I}_{0} \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}^{I}_{0} \right)'' \right)^2 + 2 \left(\tilde{\phi}^{I}_{0} \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}^{I}_{0} \right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}^{I}_{0} \right)' \lambda_0 \right\}^2 \right)^{\frac{n-3}{2}} \\
& \times \left(\left\{ \tilde{\phi}^{I}_{0} \left(1-\lambda_0^2\right) + \tilde{\phi}_0'' \right\} \left\{ \left(\tilde{\phi}^{I}_{0} \right)' \lambda_0 \right\} \times \left\{ \tilde{\phi}^{I}_{i} (\lambda_i + 1)(\lambda_i - 1) - \left(\tilde{\phi}^{I}_{i} \right)'' \right\} + 4 \left\{ - \left(\tilde{\phi}^{I}_{0} \right)' \lambda_0 \right\}^2 \times (-1) \left\{ \left(\tilde{\phi}^{I}_{i} \right)' \lambda_i \right\} \right) \\
& + \left(\left\{ \left(\tilde{\phi}^{I}_{0} \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}^{I}_{0} \right)'' \right)^2 + 2 \left(\tilde{\phi}^{I}_{0} \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}^{I}_{0} \right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}^{I}_{0} \right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \times (-1) \left\{ \left(\tilde{\phi}^{I}_{i} \right)' \lambda_i \right\} \Bigg]
\end{aligned}
\tag{2.325}
$$

Derivatives of strain components with respect to r can be written as,

$$
\frac{\partial \varepsilon^{I}_{rri}}{\partial r} = A_0^{n-1} A_i \, \alpha (n\lambda_0 - n - \lambda_0 + \lambda_i) r^{(n\lambda_0 - n - \lambda_0 + \lambda_i - 1)} \tilde{\varepsilon}^{I}_{rri} \tag{2.326}
$$

$$
\frac{\partial \varepsilon^{I}_{\theta\theta i}}{\partial r} = A_0^{n-1} A_i \, \alpha (n\lambda_0 - n - \lambda_0 + \lambda_i) r^{(n\lambda_0 - n - \lambda_0 + \lambda_i - 1)} \tilde{\varepsilon}^{I}_{\theta\theta i} \tag{2.327}
$$

$$
\frac{\partial \varepsilon^{I}_{r\theta i}}{\partial r} = A_0^{n-1} A_i \, \alpha (n\lambda_0 - n - \lambda_0 + \lambda_i) r^{(n\lambda_0 - n - \lambda_0 + \lambda_i - 1)} \tilde{\varepsilon}^{I}_{r\theta i} \tag{2.328}
$$

$$
\frac{\partial^2 \varepsilon^{I}_{rri}}{\partial r^2} = A_0^{n-1} A_i \, (n\lambda_0 - n - \lambda_0 + \lambda_i) \times \left(n\lambda_0 - n - \lambda_0 + \lambda_i - 1 \right) r^{\left(n\lambda_0 - n - \lambda_0 + \lambda_i - 2\right)} \tilde{\varepsilon}^{I}_{rri} \tag{2.329}
$$

$$
\frac{\partial^2 \varepsilon^{I}_{\theta\theta i}}{\partial r^2} = A_0^{n-1} A_i \, (n\lambda_0 - n - \lambda_0 + \lambda_i) \times \left(n\lambda_0 - n - \lambda_0 + \lambda_i - 1 \right) r^{\left(n\lambda_0 - n - \lambda_0 + \lambda_i - 2\right)} \tilde{\varepsilon}^{I}_{\theta\theta i} \tag{2.330}
$$

$$
\frac{\partial^2 \varepsilon^{I}_{r\theta i}}{\partial r^2} = A_0^{n-1} A_i \, (n\lambda_0 - n - \lambda_0 + \lambda_i) \times \left(n\lambda_0 - n - \lambda_0 + \lambda_i - 1 \right) r^{\left(n\lambda_0 - n - \lambda_0 + \lambda_i - 2\right)} \tilde{\varepsilon}^{I}_{r\theta i} \tag{2.331}
$$

From zero-th order approximation solution it is clear that the terms of $r^{n(\lambda_0 - 1)}$ satisfy the compatibility condition. To solve the compatibility condition on the remaining terms, we assumed the two terms (zero-th order term and first order term of A_i) of strain components satisfy the conditions independently. Finally, initial part of compatibility equation is same as the equation which is satisfied in zero-th order approximation. So, remaining part of compatibility equation should be satisfied independently. Using Eqn. (2.298), the compatibility equation will have three terms with respect to the power of r,

$r^{n(\lambda_0-1)-2}$, $r^{n\lambda_0-n-\lambda_0+\lambda_i-2}$ and $r^{n\lambda_0-3\lambda_0-n+2\lambda_i-1}$. To solve the compatibility condition, we assume that those three terms satisfy the conditions order by order. Here we will neglect the third term. Assuming second term as the incremental part the compatibility equation includes the exponent of r as $r^{n\lambda_0-n-\lambda_0+\lambda_i-2}$. Hence, in the i-th order approximation, compatibility equation becomes in the form of,

$$B\times A_i\frac{d^4\tilde{\phi}_i^I}{d\theta^4}=-C\left(A_i\frac{d^3\tilde{\phi}_i^I}{d\theta^3},\ A_i\frac{d^2\tilde{\phi}_i^I}{d\theta^2},\ A_i\frac{d\tilde{\phi}_i^I}{d\theta},A_i\,\tilde{\phi}_i^I,\cdots,\frac{d^4\tilde{\phi}_0^I}{d\theta^4},\frac{d^3\tilde{\phi}_0^I}{d\theta^3},\frac{d^2\tilde{\phi}_0^I}{d\theta^2},\frac{d\tilde{\phi}_0^I}{d\theta},\ \tilde{\phi}_0^I,\ n,\ \lambda_0,\lambda_1\cdots\lambda_i\right)$$

(2.332)

Equation (2.332) is the fourth-order ordinary differential equation of $\tilde{\phi}_i^I$. Within the i-th order approximation unknowns are $\tilde{\phi}_i^I,\tilde{\phi}_i^{I\prime},\tilde{\phi}_i^{I\prime\prime},\tilde{\phi}_i^{I\prime\prime\prime},A_i$ and λ_i. B and C are derived using Mathematica software and the equation is solved using the Runge-Kutta method.

The expressions for incremental stresses in the i-th order approximation are:

$$\sigma_{rri}^I=A_i\,r^{\lambda_i-1}\tilde{\sigma}_{rri}^I \tag{2.333}$$

$$\sigma_{\theta\theta i}^I=A_i\,r^{\lambda_i-1}\tilde{\sigma}_{\theta\theta i}^I \tag{2.334}$$

$$\sigma_{r\theta i}^I=A_i\,r^{\lambda_i-1}\tilde{\sigma}_{r\theta i}^I \tag{2.335}$$

where,

$$\tilde{\sigma}_{rri}^I=\tilde{\phi}_i^I\left(\lambda_i+1\right)+\left(\tilde{\phi}_i^I\right)'' \tag{2.336}$$

$$\tilde{\sigma}_{\theta\theta i}^I=\tilde{\phi}_i^I\left(\lambda_i+1\right)\lambda_i \tag{2.337}$$

$$\tilde{\sigma}_{r\theta i}^I=-\left(\tilde{\phi}_i^I\right)'\lambda_i \tag{2.338}$$

Displacement fields are expressed in the i-th order approximation as,

$$u_{r(i)}^I=u_{r0}^I+u_{r1}^I+u_{r2}^I+\cdots+u_{ri}^I\approx\int\varepsilon_{rr0}^I dr+\int\varepsilon_{rr1}^I dr+\int\varepsilon_{rr2}^I dr+\cdots+\int\varepsilon_{rri}^I dr \tag{2.339}$$

Initial part of displacement components and initial strain components has a relationship. From (i-1) th order approximation on the interface,

$$u_{r0}^I + u_{r1}^I + u_{r2}^I + \cdots + u_{r\{i-1\}}^I \approx \int \varepsilon_{rr0}^I dr + \int \varepsilon_{rr1}^I dr + \int \varepsilon_{rr2}^I dr + \cdots + \int \varepsilon_{rr\{i-1\}}^I dr \tag{2.340}$$

$$u_{r\{i-1\}}^I = A_0^{n-1}\ r^{(n\lambda_0 - n - \lambda_0 + \lambda_{i-1})+1} \tilde{u}_{r\{i-1\}}^I \tag{2.341}$$

$$u_{ri}^I = A_0^{n-1}\ r^{(n\lambda_0 - n - \lambda_0 + \lambda_i)+1} \tilde{u}_{ri}^I \tag{2.342}$$

where,

$$\begin{aligned}
\tilde{u}_{r\{i-1\}}^I &= \frac{2^{-1-n} 3^{\frac{n+1}{2}}\ \alpha A_{(i-1)}}{(n\lambda_0 - n - \lambda_0 + \lambda_{i-1} + 1)} \\
&\times \Bigg[(n-1) \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I\right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-3}{2}} \\
&\times \Bigg(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I\right)'' \right) \right\} \times (-1) \left\{ \tilde{\phi}_{i-1}^I \left(\lambda_{i-1}+1\right)\left(\lambda_{i-1}-1\right) - \left(\tilde{\phi}_{i-1}^I\right)'' \right\} \\
&\quad + 4 \left\{ \tilde{\phi}_0^I \left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I\right)'' \right\} \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\} \times \left\{ \left(\tilde{\phi}_{i-1}^I\right)' \lambda_{i-1} \right\} \Bigg) \\
&+ \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I\right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \times (-1) \left\{ \tilde{\phi}_{i-1}^I \left(\lambda_{i-1}+1\right)\left(\lambda_{i-1}-1\right) - \left(\tilde{\phi}_{i-1}^I\right)'' \right\} \Bigg]
\end{aligned} \tag{2.343}$$

$$\begin{aligned}
\tilde{u}_{ri}^I &= \frac{2^{-1-n} 3^{\frac{n+1}{2}}\ \alpha A_i}{(n\lambda_0 - n - \lambda_0 + \lambda_i + 1)} \Bigg[(n-1) \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I\right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-3}{2}} \\
&\times \Bigg(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I\right)'' \right) \right\} \times (-1) \left\{ \tilde{\phi}_i^I \left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\tilde{\phi}_i^I\right)'' \right\} \\
&\quad + 4 \left\{ \tilde{\phi}_0^I \left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I\right)'' \right\} \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\} \times \left\{ \left(\tilde{\phi}_i^I\right)' \lambda_i \right\} \Bigg) \\
&+ \left(\left\{ \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right)^2 + \left(\left(\tilde{\phi}_0^I\right)'' \right)^2 + 2 \left(\tilde{\phi}_0^I \left(1-\lambda_0^2\right) \right) \left(\left(\tilde{\phi}_0^I\right)'' \right) \right\} + 4 \left\{ \left(\tilde{\phi}_0^I\right)' \lambda_0 \right\}^2 \right)^{\frac{n-1}{2}} \times (-1) \left\{ \tilde{\phi}_i^I \left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\tilde{\phi}_i^I\right)'' \right\} \Bigg]
\end{aligned} \tag{2.344}$$

Finally, displacement after i-th order approximation:

$$u^I_{r(i)} = u^I_{r0} + u^I_{r1} + u^I_{r2} + \cdots + u^I_{ri} \tag{2.345}$$

On the interface, from zero-th order approximation: $\left(u^I_{r0}\right)_{\theta=0} = 0$

So, $$\left(u^I_{r(i)}\right)_{\theta=0} = u^I_{r1} + u^I_{r2} + \cdots + u^I_{ri} \tag{2.346}$$

And from the (i-1) th order approximation displacement field is known and which is same as the forced displacement field from the elastic material side of the (i-2)-th order approximation. All terms up to the (i-1)-th order approximation is known. Displacement u^I_r at any arbitrary value of θ , and on the interface, is expressed as Eqn.(2.243) and Eqn.(2.346) ,respectively. Where, u^I_{r0} and u^I_{r1} are given in Eqn.(2.89) and Eqn.(2.155). Displacement u^I_{ri} are given in Eqn. (2.342). From strain displacement relation in Eqn.(2.11)substituting $\varepsilon_{r\theta} \approx \varepsilon_{r\theta 0} + \varepsilon_{r\theta 1} + \cdots + \varepsilon_{r\theta i}$ and $u_\theta = u_{\theta 0} + u_{\theta 1} + \cdots + u_{\theta i}$ Strain displacement relation becomes,

$$r\frac{\partial u_{\theta 0}}{\partial r} - u_{\theta 0} + r\frac{\partial u_{\theta 1}}{\partial r} - u_{\theta 1} + \cdots + r\frac{\partial u_{\theta i}}{\partial r} - u_{\theta i} \approx 2r\varepsilon_{r\theta 0} - \frac{\partial u_{r0}}{\partial \theta} + 2r\varepsilon_{r\theta 1} - \frac{\partial u_{r1}}{\partial \theta} + \cdots + 2r\varepsilon_{r\theta i} - \frac{\partial u_{ri}}{\partial \theta} \tag{2.347}$$

Initial part of displacement components and initial part of strain components has a relationship. So incremental part for the i-th order approximation,

$$r\frac{\partial u_{\theta i}}{\partial r} - u_{\theta i} \approx 2r\varepsilon_{r\theta i} - \frac{\partial u_{ri}}{\partial \theta} \tag{2.348}$$

Assume, $u_{\theta i} = Kr^{(n\lambda_0 - n - \lambda_0 + \lambda_i)+1} f(\theta) \therefore \dfrac{\partial u_{\theta i}}{\partial r} = (n\lambda_0 - n - \lambda_0 + \lambda_i + 1)Kr^{(n\lambda_0 - n - \lambda_0 + \lambda_i)+1-1} f(\theta);$

Equation becomes,

$$Kr^{(n\lambda_0 - n - \lambda_0 + \lambda_i)+1} f(\theta)\left\{n\lambda_0 - n - \lambda_0 + \lambda_i\right\} = 2r\varepsilon_{r\theta i} - \frac{\partial u_{ri}}{\partial \theta} \tag{2.349}$$

And we have $u^I_{\theta 0}$ in Eqn.(2.93) and $u^I_{\theta 1}$ in Eqn. (2.157) similarly $u^I_{\theta i}$ can be calculated as,

$$u_{\theta i}^{I} = A_0^{n-1}\ r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{\theta i}^{I} \tag{2.350}$$

where,

$$
\begin{aligned}
\tilde{u}_{\theta i}^{I} &= \frac{2^{1-n}3^{\frac{n+1}{2}}\alpha A_i}{(n\lambda_0-n-\lambda_0+\lambda_i)(n\lambda_0-n-\lambda_0+\lambda_i+1)}\\
&\Bigg\{\Bigg[(n-1)(n\lambda_0-n-\lambda_0+\lambda_i+1)\Bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\Big(\big(\tilde{\phi}_0^I\big)''\Big)\Big\}+4\Big\{\big(\tilde{\phi}_0^I\big)'\lambda_0\Big\}^2\Bigg)^{\frac{n-3}{2}}\\
&\times\Bigg(\Big\{\tilde{\phi}_0^I(1-\lambda_0^2)+\big(\tilde{\phi}_0^I\big)''\Big\}\Big\{\big(\tilde{\phi}_0^I\big)'\lambda_0\Big\}\times\Big\{\tilde{\phi}_i^I(\lambda_i+1)(\lambda_i-1)-\big(\tilde{\phi}_i^I\big)''\Big\}+4\Big\{-\big(\tilde{\phi}_0^I\big)'\lambda_0\Big\}^2\times(-1)\Big\{\big(\tilde{\phi}_i^I\big)'\lambda_i\Big\}\Bigg)\\
&+\Bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\Big(\big(\tilde{\phi}_0^I\big)''\Big)\Big\}+4\Big\{\big(\tilde{\phi}_0^I\big)'\lambda_0\Big\}^2\Bigg)^{\frac{n-1}{2}}\times(-1)\Big\{\big(\tilde{\phi}_0^I\big)'\lambda_i\Big\}\Bigg]\\
&-\Bigg[\frac{(n-3)}{2^3}\Bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\Big(\big(\tilde{\phi}_0^I\big)''\Big)\Big\}+4\Big(\big(\tilde{\phi}_0^I\big)'\lambda_0\Big)^2\Bigg)^{\frac{n-3}{2}-1}\\
&\times\Bigg(-4(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_i^I\big)'\big(\tilde{\phi}_0^I\big)''+n(1-\lambda_0^2)^2\big(\tilde{\phi}_0^I\big)^2\Big((-1+\lambda_i^2)\tilde{\phi}_i^I-\big(\tilde{\phi}_i^I\big)''\Big)+4\Big(\big(\tilde{\phi}_0^I\big)'\lambda_0\Big)^2\Big((-1+\lambda_i^2)\tilde{\phi}_i^I-\big(\tilde{\phi}_i^I\big)''\Big)\\
&+n\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2\Big((-1+\lambda_i^2)\tilde{\phi}_i^I-\big(\tilde{\phi}_i^I\big)''\Big)+2(-1+\lambda_0^2)\tilde{\phi}_0^I\Big(2(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_i^I\big)'+n\big(\tilde{\phi}_0^I\big)''\Big(-(-1+\lambda_i^2)\tilde{\phi}_i^I+\big(\tilde{\phi}_i^I\big)''\Big)\Big)\Bigg)\\
&\times\Bigg(2(1-\lambda_0^2)^2\tilde{\phi}_0^I\big(\tilde{\phi}_0^I\big)'+8\lambda_0^2\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_0^I\big)''-2(-1+\lambda_0^2)\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_0^I\big)''-2(-1+\lambda_0^2)\tilde{\phi}_0^I\big(\tilde{\phi}_0^I\big)^{(3)}+2\big(\tilde{\phi}_0^I\big)''\big(\tilde{\phi}_0^I\big)^{(3)}\Bigg)\Bigg)\\
&-\Bigg(\frac{1}{2^2}\Bigg(\Big\{\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2+\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2+2\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)\Big(\big(\tilde{\phi}_0^I\big)''\Big)\Big\}+4\Big(\big(\tilde{\phi}_0^I\big)'\lambda_0\Big)^2\Bigg)^{\frac{n-3}{2}}\\
&\times\Bigg\{-4(n-1)\lambda_0\lambda_i\tilde{\phi}_i'\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2+2n(1-\lambda_0^2)^2\tilde{\phi}_0^I\big(\tilde{\phi}_0^I\big)'\Big((-1+\lambda_i^2)\tilde{\phi}_i^I-\big(\tilde{\phi}_i^I\big)''+8\lambda_0^2\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_0^I\big)''\Big((-1+\lambda_i^2)\tilde{\phi}_i^I-\big(\tilde{\phi}_i^I\big)''\Big)\\
&-4(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_0^I\big)''\big(\tilde{\phi}_i^I\big)''+2(-1+\lambda_0^2)\big(\tilde{\phi}_0^I\big)'\times\Big\{2(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_0^I\big)'\tilde{\phi}_i'+n\big(\tilde{\phi}_0^I\big)''\Big(-(-1+\lambda_i^2)\tilde{\phi}_i^I+\big(\tilde{\phi}_i^I\big)''\Big)\Big\}\\
&-4(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_i^I\big)'\big(\tilde{\phi}_0^I\big)^{(3)}+2n\big(\tilde{\phi}_0^I\big)''\Big((-1+\lambda_i^2)\tilde{\phi}_i^I-\tilde{\phi}_i''\Big)\big(\tilde{\phi}_0^I\big)^{(3)}+n\big(\tilde{\phi}_0^I(1-\lambda_0^2)\big)^2\Big((-1+\lambda_i^2)\tilde{\phi}_i'-\tilde{\phi}_i^{(3)}\Big)\\
&+4\lambda_0^2\Big(\big(\tilde{\phi}_0^I\big)'\Big)^2\Big((-1+\lambda_i^2)\big(\tilde{\phi}_i^I\big)'-\tilde{\phi}_i^{(3)}\Big)+n\Big(\big(\tilde{\phi}_0^I\big)''\Big)^2\Big((-1+\lambda_i^2)\big(\tilde{\phi}_i^I\big)'-\tilde{\phi}_i^{(3)}\Big)+2(1-\lambda_0^2)^2\tilde{\phi}_0^I\times\Big[2(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_i^I\big)'\big(\tilde{\phi}_0^I\big)''\\
&+2(n-1)\lambda_0\lambda_i\big(\tilde{\phi}_0^I\big)'\big(\tilde{\phi}_i^I\big)''+n\Big(-(-1+\lambda_i^2)\tilde{\phi}_i^I+\big(\tilde{\phi}_i^I\big)''\Big)\big(\tilde{\phi}_0^I\big)^{(3)}+n\big(\tilde{\phi}_0^I\big)''\Big(-(-1+\lambda_i^2)\big(\tilde{\phi}_i^I\big)'+\big(\tilde{\phi}_i^I\big)^{(3)}\Big)\Big]\Bigg\}\Bigg]\Bigg\}
\end{aligned}
\tag{2.351}
$$

Finally, $u_{\theta(i)}^{I} = u_{\theta 0}^{I} + u_{\theta 1}^{I} + u_{\theta 2}^{I} + \cdots + u_{\theta i}^{I}$ (2.352)

From the zero-th order approximation we have: $\left(u_{\theta 0}^{I}\right)_{\theta=0}=0$

On the interface, $\left(u_{\theta(i)}^{I}\right)_{\theta=0}=u_{\theta 1}^{I}+u_{\theta 2}^{I}+\cdots+u_{\theta i}^{I}$ (2.353)

From the (i-1)th order approximation $\left(u_{\theta\{i-1\}}^{I}\right)_{\theta=0}$ is known and the power of r is equal to the power of r from elastic material side of the (i-2) th order approximation , which is used as the forced displacement in the power law material of the (i-1)th order approximation on the interface.

Displacement of Elastic material side after (i-1) th order Approximation which is used as the forced displacement:

$$u_{r(i-1)}^{II}=A_0\, r^{\lambda_0} u_{r0}^{II}+A_1\, r^{\lambda_1} u_{r1}^{II}+\cdots+\cdots+A_{(i-1)}\, r^{\lambda_{i-1}} u_{r\{i-1\}}^{II} \tag{2.354}$$

$$u_{\theta(i-1)}^{II}=A_0\, r^{\lambda_0} u_{\theta 0}^{II}+A_1\, r^{\lambda_1} u_{\theta 1}^{II}+\cdots+\cdots+A_{(i-1)}\, r^{\lambda_{i-1}} u_{\theta\{i-1\}}^{II} \tag{2.355}$$

For the case of Free edges-forced displacement from elastic material side, Boundary conditions are:

$$\begin{matrix} \left(\sigma_{\theta\theta(i)}^{I}\right)_{\theta=\frac{\pi}{2}}=0 & & \left(u_{r(i)}^{I}\right)_{\theta=0}=\left(u_{r(i-1)}^{II}\right)_{\theta=0} \\ & \text{and} & \\ \left(\sigma_{r\theta(i)}^{I}\right)_{\theta=\frac{\pi}{2}}=0 & & \left(u_{\theta(i)}^{I}\right)_{\theta=0}=\left(u_{\theta(i-1)}^{II}\right)_{\theta=0} \end{matrix} \tag{2.356}$$

Last part of boundary equations can be written as,

$$\left(u_{r1}^{I}+u_{r2}^{I}+\cdots+u_{ri}^{I}\right)_{\theta=0}=\left(u_{r0}^{II}+u_{r1}^{II}+\cdots+u_{r\{i-1\}}^{I}\right)_{\theta=0} \qquad (2.357))$$

To satisfy the boundary equation of (i-1) th order approximation, up to (i-1) th terms of both sides are known and is equal. So, all terms are cancelled out up to (i-1) th order and remaining terms should be satisfied on the interface.

On the interface the last part of boundary condition can be described as,

$\left(u_{ri}^{I}\right)_{\theta=0}=\left(u_{r\{i-1\}}^{I}\right)_{\theta=0}$ and similarly $\left(u_{\theta i}^{I}\right)_{\theta=0}=\left(u_{\theta\{i-1\}}^{I}\right)_{\theta=0}$ (2.358)

It can be expressed in detail,

$$A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\tilde{u}_{r1}^{I} + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_2)+1}\tilde{u}_{r2}^{I} + \cdots + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{ri}^{I}$$
$$= A_0\, r^{\lambda_0}\tilde{u}_{r0}^{II} + A_1 r^{\lambda_1}\tilde{u}_{r1}^{II} + \cdots + A_{\{i-1\}} r^{\lambda_{i-1}}\tilde{u}_{r\{i-1\}}^{II} \qquad (2.359)$$

and

$$A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\tilde{u}_{\theta 1}^{I} + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_2)+1}\tilde{u}_{\theta 2}^{I} + \cdots + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{\theta i}^{I}$$
$$= A_0\, r^{\lambda_0}\tilde{u}_{\theta 0}^{II} + A_1 r^{\lambda_1}\tilde{u}_{\theta 1}^{II} + \cdots + A_{\{i-1\}} r^{\lambda_{i-1}}\tilde{u}_{\theta\{i-1\}}^{II} \qquad (2.360)$$

From the (i-1)-th order approximation it is known that,

$$A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\tilde{u}_{r1}^{I} + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_2)+1}\tilde{u}_{r2}^{I} + \cdots + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{r\{i-1\}}^{I}$$
$$= A_0\, r^{\lambda_0}\tilde{u}_{r0}^{II} + A_1 r^{\lambda_1}\tilde{u}_{r1}^{II} + \cdots + A_{\{i-2\}} r^{\lambda_{i-1}}\tilde{u}_{r\{i-2\}}^{II}$$

(2.361)

$$A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_1)+1}\tilde{u}_{\theta 1}^{I} + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_2)+1}\tilde{u}_{\theta 2}^{I} + \cdots + A_0^{n-1}\, r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{\theta\{i-1\}}^{I}$$
$$= A_0\, r^{\lambda_0}\tilde{u}_{\theta 0}^{II} + A_1 r^{\lambda_1}\tilde{u}_{\theta 1}^{II} + \cdots + A_{\{i-2\}} r^{\lambda_{i-1}}\tilde{u}_{\theta\{i-2\}}^{II} \qquad (2.362)$$

So, remaining terms are:

$$A_0^{n-1}\, A_i r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{ri}^{I} = A_{\{i-1\}} r^{\lambda_{i-1}}\tilde{u}_{r\{i-1\}}^{II}\,,\; A_0^{n-1}\, A_i r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\tilde{u}_{\theta i}^{I} = A_{\{i-1\}} r^{\lambda_{i-1}}\tilde{u}_{\theta\{i-1\}}^{II} \qquad (2.363)$$

Applying the boundary condition, remaining terms of boundary equation becomes,

$$\left(u_{ri}^{I}\right)_{\theta=0} = \left(u_{r\{i-1\}}^{II}\right)_{\theta=0} \text{ and } \left(u_{\theta i}^{I}\right)_{\theta=0} = \left(u_{\theta\{i-1\}}^{II}\right)_{\theta=0} \qquad (2.364)$$

Where angular function terms of displacements are presented in Eqns.(2.344,2.351) and

$$\tilde{u}_{r\{i-1\}}^{II} = \frac{E^{I}}{E^{II}}\left\{\frac{\left(1+\nu^{II}\right)}{\lambda_{i-1}}\left[\left(\lambda_{i-1}+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_{i-1}\right)\tilde{\phi}_{i-1}^{II} + \left(1-\nu^{II}\right)\left(\tilde{\phi}_{i-1}^{II}\right)''\right]\right\} \qquad (2.365)$$

$$\tilde{u}_{\theta\{i-1\}}^{II} = -\frac{E^{I}}{E^{II}}\frac{1}{\lambda_{i-1}}\frac{\left(1+\nu^{II}\right)}{\left(\lambda_{i-1}-1\right)}\left[\left\{2\lambda_{i-1}^{2} + \left(\lambda_{i-1}+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_{i-1}\right)\right\}\left(\tilde{\phi}_{i-1}^{II}\right)' + \left(1-\nu^{II}\right)\left(\tilde{\phi}_{i-1}^{II}\right)'''\right]$$

(2.366)

Due to the forced displacement on the interface, displacement of i-th order approximation in the power law material side should be the same as the displacement of (i-1) th order approximation in the elastic material side.

The iterative boundary condition on the interface can be expressed as, $\left(u^{I}_{r(i)}\right)_{\theta=0} = \left(u^{II}_{r(i-1)}\right)_{\theta=0}$, $\left(u^{I}_{\theta(i)}\right)_{\theta=0} = \left(u^{II}_{\theta(i-1)}\right)_{\theta=0}$. (2.367)

Where $\left(u^{I}_{r(i)}\right)_{\theta=0} = \left(u^{I}_{r0} + u^{I}_{r1} + u^{I}_{r2} + \cdots + u^{I}_{r\{i-1\}} + u^{I}_{ri}\right)_{\theta=0}$ (2.368)

$$\left(u^{II}_{r(i-1)}\right)_{\theta=0} = \left(u^{II}_{r0} + u^{II}_{r1} + u^{II}_{r2} + \cdots + u^{II}_{r\{i-1\}}\right)_{\theta=0} \tag{2.369}$$

$$\left(u^{I}_{\theta(i)}\right)_{\theta=0} = \left(u^{I}_{\theta0} + u^{I}_{\theta1} + u^{I}_{\theta2} + \cdots + u^{I}_{\theta\{i-1\}} + u^{I}_{\theta i}\right)_{\theta=0} \tag{2.370}$$

and $\left(u^{II}_{\theta(i-1)}\right)_{\theta=0} = \left(u^{II}_{\theta0} + u^{II}_{\theta1} + u^{II}_{\theta2} + \cdots + u^{II}_{\theta\{i-1\}}\right)_{\theta=0}$ (2.371)

So, $\left(u^{I}_{r0} + u^{I}_{r1} + u^{I}_{r2} + \cdots + u^{I}_{r\{i-1\}} + u^{I}_{ri}\right)_{\theta=0} = \left(u^{II}_{r0} + u^{II}_{r1} + u^{II}_{r2} + \cdots + u^{II}_{r\{i-1\}}\right)_{\theta=0}$ (2.372)

and $\left(u^{I}_{\theta0} + u^{I}_{\theta1} + u^{I}_{\theta2} + \cdots + u^{I}_{\theta\{i-1\}} + u^{I}_{\theta i}\right)_{\theta=0} = \left(u^{II}_{\theta0} + u^{II}_{\theta1} + u^{II}_{\theta2} + \cdots + u^{II}_{\theta\{i-1\}}\right)_{\theta=0}$ (2.373)

From zero-th order approximation: $\left(u^{I}_{r0}\right)_{\theta=0} = 0$ and $\left(u^{I}_{\theta0}\right)_{\theta=0} = 0$

From (i-1) th order approximation:

$$\left(u^{I}_{r0} + u^{I}_{r1} + u^{I}_{r2} + \cdots + u^{I}_{r\{i-1\}}\right)_{\theta=0} = \left(u^{II}_{r0} + u^{II}_{r1} + u^{II}_{r2} + \cdots + u^{II}_{r\{i-2\}}\right)_{\theta=0} \tag{2.374}$$

and $\left(u^{I}_{\theta0} + u^{I}_{\theta1} + u^{I}_{\theta2} + \cdots + u^{I}_{\theta\{i-1\}}\right)_{\theta=0} = \left(u^{II}_{\theta0} + u^{II}_{\theta1} + u^{II}_{\theta2} + \cdots + u^{II}_{\theta\{i-2\}}\right)_{\theta=0}$ (2.375)

So, remaining term $\left(u^{I}_{ri}\right)_{\theta=0} = \left(u^{II}_{r\{i-1\}}\right)_{\theta=0}$ and $\left(u^{I}_{\theta i}\right)_{\theta=0} = \left(u^{II}_{\theta\{i-1\}}\right)_{\theta=0}$ (2.376)

On the interface Power of r should be equal. Where,

$$\lambda_i - \lambda_{i-1} = (1-n)(\lambda_0 - 1) \tag{2.377}$$

From the solution of zero-th order approximation, eigenvalue , λ_0 is known for different *n*. Right hand side of the above equation is known and constant where, $(1-n)$ is negative (-)ve in sign due to the power law hardening material $n>1$, (λ_0-1) also become (-)ve because of $\lambda_0<1$.So R.H. side is always makes a positive constant.

To satisfy the boundary condition in the i-th order approximation the singular exponent would have the form,

$$\lambda_i=\lambda_0+i\times(1-n)(\lambda_0-1)\ . \tag{2.378}$$

It seems the i-th order singularity depends on the hardening exponent n and the zero-th order singularity λ_0.

Equating the power of r in the boundary equation, Radial part is equal in both sides so remaining part means the angular function part should be the same

Equating the power of r, from $\left(u_{ri}^{I}\right)_{\theta=0}=\left(u_{r\{i-1\}}^{II}\right)_{\theta=0}$ and $\left(u_{\theta i}^{I}\right)_{\theta=0}=\left(u_{\theta\{i-1\}}^{II}\right)_{\theta=0}$

On the interface at $\theta=0$, $u_{ri}^{I}=u_{r\{i-1\}}^{II}$ and $u_{\theta i}^{I}=u_{\theta\{i-1\}}^{II}$

$$A_0^{n-1}\ r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\left(\tilde{u}_{ri}^{I}\right)_{\theta=0}=A_{i-1}\,r^{\lambda_{i-1}}\left(\tilde{u}_{r\{i-1\}}^{II}\right)_{\theta=0} \tag{2.379}$$

and

$$A_0^{n-1}\ r^{(n\lambda_0-n-\lambda_0+\lambda_i)+1}\left(\tilde{u}_{\theta i}^{I}\right)_{\theta=0}=A_{i-1}\,r^{\lambda_{i-1}}\left(\tilde{u}_{\theta\{i-1\}}^{II}\right)_{\theta=0} \tag{2.380}$$

Equating the power of r,

$$A_0^{n-1}\left(\tilde{u}_{ri}^{I}\right)_{\theta=0}=A_{i-1}\left(\tilde{u}_{r\{i-1\}}^{II}\right)_{\theta=0}\ \text{ and }\ A_0^{n-1}\left(\tilde{u}_{\theta i}^{I}\right)_{\theta=0}=A_{i-1}\left(\tilde{u}_{\theta\{i-1\}}^{II}\right)_{\theta=0} \tag{2.381}$$

$$A_i=A_0^{1-n}A_{i-1}\times C \tag{2.382}$$

where, $$\frac{\left(\tilde{u}_{r\{i-1\}}^{II}\right)_{\theta=0}}{\left(\tilde{u}_{ri}^{I}\right)_{\theta=0}\Big/A_i}=C\ \text{ and }\ \frac{\left(\tilde{u}_{\theta\{i-1\}}^{II}\right)_{\theta=0}}{\left(\tilde{u}_{\theta i}^{I}\right)_{\theta=0}\Big/A_i}=C \tag{2.383}$$

Compatibility equation becomes in the form of fourth-order ordinary differential equation are given in Eqn. (2.332). Equation (2.332) is the fourth-order ordinary differential equation. For the solution of fourth-order equation of $A_i\tilde{\phi}_i^I$, the equation is reduced into a system of first-order equations: Assume,

$$w1(1) = A_i\tilde{\phi}_i^I, \quad w1(2) = A_i\frac{d\tilde{\phi}_i^I}{d\theta}, \quad w1(3) = A_i\frac{d^2\tilde{\phi}_i^I}{d\theta^2}, \quad w1(4) = A_i\frac{d^3\tilde{\phi}_i^I}{d\theta^3}$$

Where, $w1(1)' = w1(2)$, $w1(2)' = w1(3)$, $w1(3)' = w1(4)$ and $\tilde{\phi}_0, \frac{d\tilde{\phi}_0}{d\theta}, \frac{d^2\tilde{\phi}_0}{d\theta^2}, \frac{d^3\tilde{\phi}_0}{d\theta^3}$ and $\frac{d^4\tilde{\phi}_0}{d\theta^4}$ are known from the solution of 0-th order approximation.

$$\begin{cases} w1(1)' = w1(2) \\ w1(2)' = w1(3), \\ w1(3)' = w1(4), \\ w1(4)' = \dfrac{d^4\tilde{\phi}_i^I}{d\theta^4} \\ = f\left(A_i\dfrac{d^3\tilde{\phi}_i^I}{d\theta^3}, A_i\dfrac{d^2\tilde{\phi}_i^I}{d\theta^2}, A_i\dfrac{d\tilde{\phi}_i^I}{d\theta}, A_i\tilde{\phi}_i^I, ..., \dfrac{d^4\tilde{\phi}_0^I}{d\theta^4}, \dfrac{d^3\tilde{\phi}_0^I}{d\theta^3}, \dfrac{d^2\tilde{\phi}_0^I}{d\theta^2}, \dfrac{d\tilde{\phi}_0^I}{d\theta}, \tilde{\phi}_0^I, n, \lambda_0, \lambda_1 \right) \\ = -\dfrac{C}{B}\left(A_i\dfrac{d^3\tilde{\phi}_i^I}{d\theta^3}, A_i\dfrac{d^2\tilde{\phi}_i^I}{d\theta^2}, A_i\dfrac{d\tilde{\phi}_i^I}{d\theta}, A_i\tilde{\phi}_i^I, ..., \dfrac{d^4\tilde{\phi}_0^I}{d\theta^4}, \dfrac{d^3\tilde{\phi}_0^I}{d\theta^3}, \dfrac{d^2\tilde{\phi}_0^I}{d\theta^2}, \dfrac{d\tilde{\phi}_0^I}{d\theta}, \tilde{\phi}_0^I, n, \lambda_0, \lambda_1 \right) \end{cases}$$

(2.384)

B, C and –C/B equation is derived using Mathematica software. And, therefore, equation can be solved using the Runge-Kutta method.

Applying boundary conditions:

$$\left(\sigma_{\theta\theta(i)}^I\right)_{\theta=\frac{\pi}{2}} = \left(\sigma_{\theta\theta 0}^I\right)_{\theta=\frac{\pi}{2}} + \left(\sigma_{\theta\theta 1}^I\right)_{\theta=\frac{\pi}{2}} + \left(\sigma_{\theta\theta 2}^I\right)_{\theta=\frac{\pi}{2}} + \cdots + \left(\sigma_{\theta\theta i}^I\right)_{\theta=\frac{\pi}{2}} = 0 \tag{2.385}$$

From zero-th order approximation we know $\left(\sigma_{\theta\theta 0}^I\right)_{\theta=\frac{\pi}{2}} = 0$, and from (i-1) th order approximation we know $\left(\sigma_{\theta\theta\{i-1\}}^I\right)_{\theta=\frac{\pi}{2}} = 0$ so $\left(\sigma_{\theta\theta i}^I\right)_{\theta=\frac{\pi}{2}} = 0$

similarly,

$$\left(\sigma^{I}_{r\theta(i)}\right)_{\theta=\frac{\pi}{2}}=\left(\sigma^{I}_{r\theta0}\right)_{\theta=\frac{\pi}{2}}+\left(\sigma^{I}_{r\theta1}\right)_{\theta=\frac{\pi}{2}}+\left(\sigma^{I}_{r\theta2}\right)_{\theta=\frac{\pi}{2}}+\cdots+\left(\sigma^{I}_{r\theta i}\right)_{\theta=\frac{\pi}{2}}=0 \tag{2.386}$$

From zero-th order approximation we know $\left(\sigma^{I}_{r\theta0}\right)_{\theta=\frac{\pi}{2}}=0$, and from (i-1) th order approximation we know $\left(\sigma^{I}_{r\theta\{i-1\}}\right)_{\theta=\frac{\pi}{2}}=0$ so $\left(\sigma^{I}_{r\theta i}\right)_{\theta=\frac{\pi}{2}}=0$ (2.387)

The expressions for incremental stresses in the i-th order approximation are given in Eqns. (2.336-2.338)

Initial conditions at $\theta=\dfrac{\pi}{2}$:

$$\frac{\sigma^{I}_{\theta\theta i}}{r^{\lambda_i-1}}=(\lambda_i+1)\lambda_i\left(A_i\tilde{\phi}^{I}_i\right)=0,\qquad \left(A_i\tilde{\phi}^{I}_i\right)=0,\text{if}\quad \lambda_i\neq-1,\ \lambda_i\neq0 \tag{2.388}$$

And $\dfrac{\sigma^{I}_{r\theta i}}{r^{\lambda_i-1}}=-\lambda_i\left(A_i\tilde{\phi}^{I}_i\right)'=0, or\ \left(A_i\tilde{\phi}^{I}_i\right)'=0,\text{if}\quad \lambda_i\neq0$ (2.389)

We know λ_i from Eqn. (2.378). Unknown are: $\left(A_i\tilde{\phi}^{I}_i\right)''$, $\left(A_i\tilde{\phi}^{I}_i\right)'''$

After integration final conditions, at $\theta=0$, error can be calculated as,

$$\begin{aligned}\text{error1}&=A_{i-1}\tilde{u}^{II}_{r\{i-1\}}\left(\theta=0;\text{elastic}\right)-A_0^{n-1}A_i\tilde{u}^{I}_{ri}\left(\theta=0;\text{power-law}\right)\\ \text{error2}&=A_{i-1}\tilde{u}^{II}_{\theta\{i-1\}}\left(\theta=0;\text{elastic}\right)-A_0^{n-1}A_i\tilde{u}^{I}_{\theta i}\left(\theta=0;\text{power-law}\right)\end{aligned}$$

Assume at $\theta=\dfrac{\pi}{2}$, $A_i\tilde{\phi}^{I}_i=0$, $\left(A_i\tilde{\phi}^{I}_i\right)'=0$, $\left(A_i\tilde{\phi}^{I}_i\right)''$ and $\left(A_i\tilde{\phi}^{I}_i\right)'''$ hence, after i-th integration at $\theta=0$, $\left(A_i\tilde{\phi}^{I}_i\right)=\left(A_i\tilde{\phi}^{I}_i\right)^{(in)}$, $\left(A_i\tilde{\phi}^{I}_i\right)'=\left(\left(A_i\tilde{\phi}^{I}_i\right)'\right)^{(in)}$, $\left(A_i\tilde{\phi}^{I}_i\right)''=\left(\left(A_i\tilde{\phi}^{I}_i\right)''\right)^{(in)}$

and $\left(A_i\tilde{\phi}^{I}_i\right)'''=\left(\left(A_i\tilde{\phi}^{I}_i\right)'''\right)^{(in)}$

$$
\begin{aligned}
\text{error1} = {} & A_{i-1}\tilde{u}^{II}_{r\{i-1\}}\left(\theta = 0;\text{elastic}\right) - \frac{2^{-1-n}3^{\frac{n+1}{2}}A_0^{n-1}}{(n\lambda_0 - n - \lambda_0 + \lambda_i + 1)} \\
& \times\alpha\left[\left(n-1\right)\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\tilde{\phi}_0^I\right)''\right\} + 4\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}^2\right)^{\frac{n-3}{2}}\right. \\
& \times\left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\tilde{\phi}_0^I\right)''\right\}\times\left(-1\right)\left\{\left(\left(A_i\tilde{\phi}_i^I\right)\right)^{(in)}\left(\lambda_i+1\right)\left(\lambda_i-1\right) - \left(\left(A_i\tilde{\phi}_i^I\right)''\right)^{(in)}\right\}\right. \\
& \left. + 4\left\{\tilde{\phi}_0^I\left(1-\lambda_0^2\right) + \left(\tilde{\phi}_0^I\right)''\right\}\left\{\left(\tilde{\phi}_0^I\right)'\lambda_0\right\}\times\left\{\left(\left(A_i\tilde{\phi}_i^I\right)'\right)^{(in)}\lambda_i\right\}\right) \\
& \left. + \left(\left\{\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)^2 + \left(\left(\tilde{\phi}_0^I\right)''\right)^2 + 2\left(\tilde{\phi}_0^I\left(1-\lambda_0^2\right)\right)\left(\tilde{\phi}_0^I\right)''\right\} + 4\left\{\left(\left(A_i\tilde{\phi}_i^I\right)'\right)^{(in)}\lambda_0\right\}^2\right)^{\frac{n-1}{2}}\times\left(-1\right)\left\{\left(A_i\tilde{\phi}_i^I\right)^{(in)}\left(\lambda_i^2-1\right) - \left(\left(A_i\tilde{\phi}_i^I\right)''\right)^{(in)}\right\}\right]
\end{aligned}
\tag{2.390}
$$

$$
\begin{aligned}
&\mathrm{error2} = A_{i-1}\tilde{u}^{II}_{\theta\{i-1\}}\left(\theta=0;\mathrm{elastic}\right) - \frac{2^{1-n}3^{\frac{n+1}{2}}\alpha A_0^{n-1}}{(n\lambda_0-n-\lambda_0+\lambda_i)(n\lambda_0-n-\lambda_0+\lambda_i+1)}\\
&\Bigg\{\Bigg[(n-1)(n\lambda_0-n-\lambda_0+\lambda_i+1)\Bigg(\Bigg\{\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)^2+\Big(\left(\tilde\phi_0^I\right)''\Big)^2+2\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)\Big(\left(\tilde\phi_0^I\right)''\Big)\Bigg\}+4\Big\{\left(\tilde\phi_0^I\right)'\lambda_0\Big\}^2\Bigg)^{\frac{n-3}{2}}\\
&\times\Bigg(\Big\{\tilde\phi_0^I\left(1-\lambda_0^2\right)+\left(\tilde\phi_0^I\right)''\Big\}\Big\{\left(\tilde\phi_0^I\right)'\lambda_0\Big\}\times\Bigg\{\Big(\big(A_i\tilde\phi_i^I\big)\Big)^{(in)}(\lambda_i+1)(\lambda_i-1)-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg\}+4\Big\{-\left(\tilde\phi_0^I\right)'\lambda_0\Big\}^2\times(-1)\Bigg\{\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}\lambda_i\Bigg\}\Bigg)\\
&+\Bigg(\Bigg\{\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)^2+\Big(\left(\tilde\phi_0^I\right)''\Big)^2+2\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)\Big(\left(\tilde\phi_0^I\right)''\Big)\Bigg\}+4\Big\{\left(\tilde\phi_0^I\right)'\lambda_0\Big\}^2\Bigg)^{\frac{n-1}{2}}\times(-1)\Bigg\{\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}\lambda_i\Bigg\}\Bigg]\\
&-\Bigg[\frac{(n-3)}{2^3}\Bigg(\Bigg\{\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)^2+\Big(\left(\tilde\phi_0^I\right)''\Big)^2+2\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)\Big(\left(\tilde\phi_0^I\right)''\Big)\Bigg\}+4\Big(\left(\tilde\phi_0^I\right)'\lambda_0\Big)^2\Bigg)^{\frac{n-3}{2}-1}\\
&\times\Bigg(-4(n-1)\lambda_0\lambda_i\left(\tilde\phi_0^I\right)'\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}\left(\tilde\phi_0^I\right)''+n\left(1-\lambda_0^2\right)^2\left(\tilde\phi_0^I\right)^2\Bigg(\left(-1+\lambda_i^2\right)\big(A_i\tilde\phi_i^I\big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)+4\Big(\left(\tilde\phi_0^I\right)'\lambda_0\Big)^2\Bigg(\left(\lambda_i^2-1\right)\big(A_i\tilde\phi_i^I\big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)\\
&+n\Big(\left(\tilde\phi_0^I\right)''\Big)^2\Bigg(\left(-1+\lambda_i^2\right)\big(A_i\tilde\phi_i^I\big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)+2\left(-1+\lambda_0^2\right)\tilde\phi_0^I\Bigg(2(n-1)\lambda_0\lambda_i\left(\tilde\phi_0^I\right)'\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}+n\left(\tilde\phi_0^I\right)''\Bigg(\left(1-\lambda_i^2\right)\big(A_i\tilde\phi_2^I\big)^{(in)}+\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)\Bigg)\Bigg)\\
&\times\Bigg(2\left(1-\lambda_0^2\right)^2\tilde\phi_0^I\left(\tilde\phi_0^I\right)'+8\lambda_0^2\left(\tilde\phi_0^I\right)'\left(\tilde\phi_0^I\right)''-2\left(\lambda_0^2-1\right)\left(\tilde\phi_0^I\right)'\left(\tilde\phi_0^I\right)''-2\left(\lambda_0^2-1\right)\tilde\phi_0^I\left(\tilde\phi_0^I\right)^{(3)}+2\left(\tilde\phi_0^I\right)''\left(\tilde\phi_0^I\right)^{(3)}\Bigg)\Bigg)\\
&-\Bigg(\frac{1}{2^2}\Bigg(\Bigg\{\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)^2+\Big(\left(\tilde\phi_0^I\right)''\Big)^2+2\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)\Big(\left(\tilde\phi_0^I\right)''\Big)\Bigg\}+4\Big(\left(\tilde\phi_0^I\right)'\lambda_0\Big)^2\Bigg)^{\frac{n-3}{2}}\\
&\times\Bigg\{-4(n-1)\lambda_0\lambda_i\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}\Big(\left(\tilde\phi_0^I\right)''\Big)^2+2n\left(1-\lambda_0^2\right)^2\tilde\phi_0^I\left(\tilde\phi_0^I\right)'\Bigg(\left(\lambda_i^2-1\right)\big(A_i\tilde\phi_i^I\big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}+8\lambda_0^2\left(\tilde\phi_0^I\right)'\left(\tilde\phi_0^I\right)''\Bigg(\left(\lambda_i^2-1\right)\big(A_i\tilde\phi_i^I\big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)\\
&-4(n-1)\lambda_0\lambda_i\left(\tilde\phi_0^I\right)'\left(\tilde\phi_0^I\right)''\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}+2\left(\lambda_0^2-1\right)\left(\tilde\phi_0^I\right)'\times\Bigg\{2(n-1)\lambda_0\lambda_i\left(\tilde\phi_0^I\right)'\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}+n\left(\tilde\phi_0^I\right)''\Bigg(\left(1-\lambda_i^2\right)\big(A_i\tilde\phi_i^I\big)^{(in)}+\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)\Bigg\}\\
&-4(n-1)\lambda_0\lambda_i\left(\tilde\phi_0^I\right)'\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}\left(\tilde\phi_0^I\right)^{(3)}+2n\left(\tilde\phi_0^I\right)''\Bigg(\left(\lambda_i^2-1\right)\big(A_i\tilde\phi_i^I\big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)\left(\tilde\phi_0^I\right)^{(3)}+n\Big(\tilde\phi_0^I\left(1-\lambda_0^2\right)\Big)^2\Bigg(\left(\lambda_i^2-1\right)\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)^{(3)}\Big)^{(in)}\Bigg)\\
&+4\lambda_0^2\Big(\left(\tilde\phi_0^I\right)'\Big)^2\Bigg(\left(-1+\lambda_i^2\right)\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)^{(3)}\Big)^{(in)}\Bigg)+n\Big(\left(\tilde\phi_0^I\right)''\Big)^2\Bigg(\left(-1+\lambda_i^2\right)\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}-\Big(\big(A_i\tilde\phi_i^I\big)^{(3)}\Big)^{(in)}\Bigg)+2\left(1-\lambda_0^2\right)^2\tilde\phi_0^I\\
&\times\Bigg[2(n-1)\lambda_0\lambda_i\big(A_i\tilde\phi_i^I\big)'\left(\tilde\phi_0^I\right)''\\
&+2(n-1)\lambda_0\lambda_i\left(\tilde\phi_0^I\right)'\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}+n\Bigg(\left(1-\lambda_i^2\right)\big(A_i\tilde\phi_i^I\big)^{(in)}+\Big(\big(A_i\tilde\phi_i^I\big)''\Big)^{(in)}\Bigg)\left(\tilde\phi_0^I\right)^{(3)}+n\left(\tilde\phi_0^I\right)''\Bigg(\left(1-\lambda_i^2\right)\Big(\big(A_i\tilde\phi_i^I\big)'\Big)^{(in)}+\Big(\big(A_i\tilde\phi_i^I\big)^{(3)}\Big)^{(in)}\Bigg)\Bigg]\Bigg\}\Bigg]\Bigg\}
\end{aligned}
\tag{2.391}
$$

The error value is calculated by using the equation (2.15) and solution is obtained for the minimum error.

2.6.1.1 Determination of the Stress Intensity Factor A_i

Stress can be expressed as,

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}=A_0\,r^{\lambda_0-1}\left(\tilde{\sigma}_{\theta\theta0}^{I}\right)_{\theta=0}+A_1\,r^{\lambda_1-1}\left(\tilde{\sigma}_{\theta\theta1}^{I}\right)_{\theta=0}+A_2\,r^{\lambda_2-1}\left(\tilde{\sigma}_{\theta\theta2}^{I}\right)_{\theta=0}+\cdots+A_i\,r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^{I}\right)_{\theta=0} \tag{2.392}$$

where, $\left(\tilde{\sigma}_{\theta\theta0}^{I}\right)_{\theta=0}=1$, $\left(\tilde{\sigma}_{\theta\theta1}^{I}\right)_{\theta=0}=1$ and $\left(\tilde{\sigma}_{\theta\theta\{i-1\}}^{I}\right)_{\theta=0}=1$

$$\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}=A_0\,r^{\lambda_0-1}+A_1\,r^{\lambda_1-1}+A_2\,r^{\lambda_2-1}+\cdots+A_i\,r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^{I}\right)_{\theta=0} \tag{2.393}$$

From FEM for joint material (power law hardening/elastic material joint), $\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}$ is known and from zero-th order approximation $A_0\,r^{\lambda_0-1}$ from first order approximation $A_1\,r^{\lambda_1-1}$ and from (i-1)th order approximation $A_{\{i-1\}}\,r^{\lambda_{i-1}-1}$ is also known

$$A_i\,r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^{I}\right)_{\theta=0}=\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}-\left(A_0\,r^{\lambda_0-1}+A_1\,r^{\lambda_1-1}+A_2\,r^{\lambda_2-1}+\cdots+A_{i-1}\,r^{\lambda_{i-1}-1}\right) \tag{2.394}$$

From i-th order approximation after solution of differential equation the numerical value of $\left(A_i\tilde{\phi}_i^{I}\right)$ is known.

we have, $A_i\,r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^{I}\right)_{\theta=0}=\left(A_i\tilde{\phi}_i^{I}\right)_{\theta=0}r^{\lambda_i-1}\left(\lambda_i+1\right)\lambda_i$ (2.395)

where, $\tilde{\sigma}_{\theta\theta i}^{I}=\tilde{\phi}_i^{I}\left(\lambda_i+1\right)\lambda_i$ (2.396)

As the definition of stress intensity factor A_i , $\left(\tilde{\sigma}_{\theta\theta i}^{I}\right)_{\theta=0}=1$ (2.397)

$$\Rightarrow\left(\tilde{\phi}_i^{I}\right)_{\theta=0}=\frac{1}{\left(\lambda_i+1\right)\lambda_i} \tag{2.398}$$

From theory $\left(A_i\tilde{\phi}_i^{I}\right)_{\theta=0}$ is known

$$\Rightarrow A_i=\left(A_i\tilde{\phi}_i^{I}\right)_{\theta=0}\left(\lambda_i+1\right)\lambda_i \tag{2.399}$$

From i-th order approximation after solution of differential equation the numerical value of $\left(A_i\tilde{\phi}_i^I\right)$ and all of its derivatives are known. From FEM, A_i is known numerically so, $\tilde{\phi}_i^I = \left(A_i\tilde{\phi}_i^I\right)\big/A_i$ and similarly all of its derivatives are known. From boundary equation on the interface at $\theta = 0$,are presented in Eqns. (2.381-2.382). For $n > 1$, power of A_0 is smaller than 1 (means negative power shows opposite behavior). From FEM of joint material we can write the following equation,

$$\left(\sigma_{\theta\theta}^I\right)_{\theta=0} = A_0\, r^{\lambda_0-1}\left(\tilde{\sigma}_{\theta\theta 0}^I\right)_{\theta=0} + A_1\, r^{\lambda_1-1}\left(\tilde{\sigma}_{\theta\theta 1}^I\right)_{\theta=0} + A_2\, r^{\lambda_2-1}\left(\tilde{\sigma}_{\theta\theta 2}^I\right)_{\theta=0} + \cdots + A_i\, r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^I\right)_{\theta=0}$$

(2.400)

where, $\left(\tilde{\sigma}_{\theta\theta 0}^I\right)_{\theta=0} = 1$, $\left(\tilde{\sigma}_{\theta\theta 1}^I\right)_{\theta=0} = 1$ and $\left(\tilde{\sigma}_{\theta\theta\{i-1\}}^I\right)_{\theta=0} = 1$

$$\left(\sigma_{\theta\theta}^I\right)_{\theta=0} = A_0\, r^{\lambda_0-1} + A_1\, r^{\lambda_1-1} + A_2\, r^{\lambda_2-1} + \cdots + A_i\, r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^I\right)_{\theta=0} \qquad (2.401)$$

From FEM for joint material (power law hardening/elastic material joint), $\left(\sigma_{\theta\theta}^I\right)_{\theta=0}$ is known and from zero-th order approximation $A_0\, r^{\lambda_0-1}$ from first order approximation $A_1\, r^{\lambda_1-1}$ and from (i-1)th order approximation $A_{\{i-1\}}\, r^{\lambda_{i-1}-1}$ is also known

$$A_i\, r^{\lambda_i-1}\left(\tilde{\sigma}_{\theta\theta i}^I\right)_{\theta=0} = \left(\sigma_{\theta\theta}^I\right)_{\theta=0} - \left(A_0\, r^{\lambda_0-1} + A_1\, r^{\lambda_1-1} + A_2\, r^{\lambda_2-1} + \cdots + A_{i-1}\, r^{\lambda_{i-1}-1}\right) \qquad (2.402)$$

As the definition of stress intensity factor A_i , $\left(\tilde{\sigma}_{\theta\theta i}^I\right)_{\theta=0} = 1$,Eqn. (2.402) yields,

$$A_i\, r^{\lambda_i-1} = \left(\sigma_{\theta\theta}^I\right)_{\theta=0} - \left(A_0\, r^{\lambda_0-1} + A_1\, r^{\lambda_1-1} + A_2\, r^{\lambda_2-1} + \cdots + A_{i-1}\, r^{\lambda_{i-1}-1}\right) \qquad (2.403)$$

Taking logarithmic distribution,

$$\log A_i + (\lambda_i - 1)\log r = \log\left\{\left(\sigma_{\theta\theta}^I\right)_{\theta=0} - \left(A_0\, r^{\lambda_0-1} + A_1\, r^{\lambda_1-1} + A_2\, r^{\lambda_2-1} + \cdots + A_{i-1}\, r^{\lambda_{i-1}-1}\right)\right\}$$

(2.404)

Right hand side have the term should be calculated at first $\left\{\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}-\left(A_0\, r^{\lambda_0-1}+A_1\, r^{\lambda_1-1}+A_2\, r^{\lambda_2-1}+\cdots+A_{i-1}\, r^{\lambda_{i-1}-1}\right)\right\}$ and then logarithmic distribution is calculated. $\left(\sigma_{\theta\theta}^{I}\right)_{\theta=0}$ is numerically known along radial distance r by FEM for joint material and $\left(A_0\, r^{\lambda_0-1}+A_1\, r^{\lambda_1-1}+A_2\, r^{\lambda_2-1}+\cdots+A_{i-1}\, r^{\lambda_{i-1}-1}\right)$ is calculated and subtracted for the same radial distance where A_0 and $\left(\lambda_0-1\right)$ is known from rigid/power law hardening material and A_1, $\left(\lambda_1-1\right)$, A_2, $\left(\lambda_1-1\right)$ and A_{i-1}, $\left(\lambda_{i-1}-1\right)$ is known after first order, second order and(i-1)th order approximation, respectively by FEM.

Once singular exponent, λ_i is known the angular variation of stresses can be computed. To compute the stresses we need to calculate unknown angular functions to satisfy traction on the interface.

2.6.2 Formulation of i^{th} Order Approximation: Constitutive Equations in the Elastic Material Subjected to Traction

For the case of stress free edge and traction on the interface boundary conditions:

$$\begin{matrix} \left(\sigma_{\theta\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}}=0 \\ \left(\sigma_{r\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}}=0 \end{matrix} \quad \text{and} \quad \begin{matrix} \left(\sigma_{\theta\theta i}^{I}\right)_{\theta=0}=\left(\sigma_{\theta\theta i}^{II}\right)_{\theta=0} \\ \left(\sigma_{r\theta i}^{I}\right)_{\theta=0}=\left(\sigma_{r\theta i}^{II}\right)_{\theta=0} \end{matrix} \tag{2.405}$$

Assumed,

$$\phi=\phi_0^{II}+\phi_1^{II}+\phi_2^{II}+\cdots+\phi_i^{II}=A_0\, r^{\lambda_0+1}\tilde{\phi}_0^{II}+A_1\, r^{\lambda_1+1}\tilde{\phi}_1^{II}+A_2\, r^{\lambda_2+1}\tilde{\phi}_2^{II}+\cdots+A_i\, \tilde{\phi}_i^{II} r^{\lambda_i+1} \tag{2.406}$$

So, compatibility equation in terms of stress component becomes,

$$
\begin{aligned}
&\frac{1}{r^3}\left(A_0\left(\lambda_0+1\right)r^{\lambda_0}\tilde{\phi}_0^{II}+A_1\left(\lambda_1+1\right)r^{\lambda_1}\tilde{\phi}_1^{II}+A_2\left(\lambda_2+1\right)r^{\lambda_2}\tilde{\phi}_2^{II}+\cdots+A_i\left(\lambda_i+1\right)r^{\lambda_i}\tilde{\phi}_i^{II}\right)\\
&-\frac{1}{r^2}\left(A_0\left(\lambda_0+1\right)\lambda_0 r^{\lambda_0-1}\tilde{\phi}_0^{II}+A_1\left(\lambda_1+1\right)\lambda_1 r^{\lambda_1-1}\tilde{\phi}_1^{II}+A_2\left(\lambda_2+1\right)\lambda_2 r^{\lambda_2-1}\tilde{\phi}_2^{II}+\cdots+A_i\left(\lambda_i+1\right)\lambda_i r^{\lambda_i-1}\tilde{\phi}_i^{II}\right)\\
&+\frac{4}{r^4}\left(r^{\lambda_0+1}A_0\left(\tilde{\phi}_0^{II}\right)''+r^{\lambda_1+1}A_1\left(\tilde{\phi}_1^{II}\right)''+r^{\lambda_2+1}A_2\left(\tilde{\phi}_2^{II}\right)''+\cdots+r^{\lambda_i+1}A_i\left(\tilde{\phi}_i^{II}\right)''\right)\\
&+\frac{2}{r}\Big(A_0\left(\lambda_0+1\right)\lambda_0\left(\lambda_0-1\right)r^{\lambda_0-2}\tilde{\phi}_0^{II}+A_1\left(\lambda_1+1\right)\lambda_1\left(\lambda_1-1\right)r^{\lambda_1-2}\tilde{\phi}_1^{II}\\
&\qquad+A_2\left(\lambda_2+1\right)\lambda_2\left(\lambda_2-1\right)r^{\lambda_2-2}\tilde{\phi}_2^{II}+\cdots+A_i\left(\lambda_i+1\right)\lambda_i\left(\lambda_i-1\right)r^{\lambda_i-2}\tilde{\phi}_i^{II}\Big)\\
&-\frac{2}{r^3}\left(A_0\left(\lambda_0+1\right)r^{\lambda_0}\left(\tilde{\phi}_0^{II}\right)''+A_1\left(\lambda_1+1\right)r^{\lambda_1}\left(\tilde{\phi}_1^{II}\right)''+A_2\left(\lambda_2+1\right)r^{\lambda_2}\left(\tilde{\phi}_2^{II}\right)''+\cdots+A_i\left(\lambda_i+1\right)r^{\lambda_i}\left(\tilde{\phi}_i^{II}\right)''\right)\\
&+\frac{2}{r^2}\Big(A_0\left(\lambda_0+1\right)\lambda_0 r^{\lambda_0-1}\left(\tilde{\phi}_0^{II}\right)''+A_0\left(\lambda_0+1\right)\lambda_0\left(\lambda_0-1\right)\left(\lambda_0-2\right)r^{\lambda_0-3}\tilde{\phi}_0^{II}\\
&\qquad+A_1\left(\lambda_1+1\right)\lambda_1 r^{\lambda_1-1}\left(\tilde{\phi}_1^{II}\right)''+A_1\left(\lambda_1+1\right)\lambda_1\left(\lambda_1-1\right)\left(\lambda_1-2\right)r^{\lambda_1-3}\tilde{\phi}_1^{II}\big)\\
&\qquad+A_2\left(\lambda_2+1\right)\lambda_2 r^{\lambda_2-1}\left(\tilde{\phi}_2^{II}\right)''+A_2\left(\lambda_2+1\right)\lambda_2\left(\lambda_2-1\right)\left(\lambda_2-2\right)r^{\lambda_2-3}\tilde{\phi}_2^{II}\big)\\
&\qquad+\cdots+\\
&\qquad+A_i\left(\lambda_i+1\right)\lambda_i r^{\lambda_i-1}\left(\tilde{\phi}_i^{II}\right)''+A_i\left(\lambda_i+1\right)\lambda_i\left(\lambda_i-1\right)\left(\lambda_i-2\right)r^{\lambda_i-3}\tilde{\phi}_i^{II}\big)\Big)\\
&+\frac{1}{r^4}\left(A_0\,r^{\lambda_0+1}\left(\tilde{\phi}_0^{II}\right)^{(4)}+r^{\lambda_1+1}A_1\left(\tilde{\phi}_1^{II}\right)^{(4)}+r^{\lambda_2+1}A_2\left(\tilde{\phi}_2^{II}\right)^{(4)}+\cdots+r^{\lambda_i+1}A_i\left(\tilde{\phi}_i^{II}\right)^{(4)}\right)=0
\end{aligned}
\tag{2.407}
$$

From zero-th order approximation solution it is clear that the terms of r^{λ_0-3} and from (i-1) th order solution the terms r^{λ_i-3} satisfy the compatibility condition. Compatibility condition satisfied up to (i-1) th order approximation that includes the term $r^{\lambda_0-3}, r^{\lambda_1-3}, \cdots r^{\lambda_{i-1}-3}$. To solve the compatibility condition, remaining terms satisfy the condition independently. So the compatibility equation for remaining terms,

$$
\frac{\partial^4\tilde{\phi}_i^{II}}{\partial\theta^4}=-\left(1-\lambda_i^2\right)^2\tilde{\phi}_i^{II}-2\left(\lambda_i^2+1\right)\frac{\partial^2\tilde{\phi}_i^{II}}{\partial\theta^2} \tag{2.408}
$$

Equation (2.408) is the fourth-order ordinary differential equation. For the solution of fourth-order equation, the equation is reduced into a system of first-order equations:

Assume, $w1(1)=\tilde{\phi}_i^{II}$, $w1(2)=\dfrac{d\tilde{\phi}_i^{II}}{d\theta}$, $w1(3)=\dfrac{d^2\tilde{\phi}_i^{II}}{d\theta^2}$, $w1(4)=\dfrac{d^3\tilde{\phi}_i^{II}}{d\theta^3}$

Where, $w1(1)' = w1(2)$, $w1(2)' = w1(3)$, $w1(3)' = w1(4)$

$$w1(4)' = \frac{d^4\tilde{\phi}_i^{II}}{d\theta^4} = f\left(\frac{d^3\tilde{\phi}_i^{II}}{d\theta^3}, \frac{d^2\tilde{\phi}_i^{II}}{d\theta^2}, \frac{d\tilde{\phi}_i^{II}}{d\theta}, \tilde{\phi}_i^{II}, \lambda_i\right) \tag{2.409}$$

And, therefore, equation is solved using the Runge-Kutta method.

The resulting expressions for incremental stresses are:

$$\sigma_{rri}^{II} = A_i\, r^{\lambda_i - 1}\tilde{\sigma}_{rri}^{II} \tag{2.410}$$

$$\sigma_{\theta\theta i}^{II} = A_i\, r^{\lambda_i - 1}\tilde{\sigma}_{\theta\theta i}^{II} \tag{2.411}$$

$$\sigma_{r\theta i}^{II} = A_i\, r^{\lambda_i - 1}\tilde{\sigma}_{r\theta i}^{II} \tag{2.412}$$

where,

$$\tilde{\sigma}_{rri}^{II} = (\lambda_i + 1)\tilde{\phi}_i^{II} + \left(\tilde{\phi}_i^{II}\right)'' \tag{2.413}$$

$$\tilde{\sigma}_{\theta\theta i}^{II} = (\lambda_i + 1)\lambda_i\tilde{\phi}_i^{II} \tag{2.414}$$

$$\tilde{\sigma}_{r\theta i}^{II} = -\lambda_i\left(\tilde{\phi}_i^{II}\right)' \tag{2.415}$$

Displacement can be calculated as:

$$u_{r0}^{II} + u_{r1}^{II} + u_{r2}^{II} + \cdots + u_{ri}^{II} = \int \varepsilon_{rr0}^{II} dr + \int \varepsilon_{rr1}^{II} dr + \int \varepsilon_{rr2}^{II} dr + \cdots + \int \varepsilon_{rri}^{II} dr \tag{2.416}$$

Initial part of displacement components and initial strain components has a relationship and given in Eqn. (2.88). From the (i-1) th order approximation,

$$u_{r\{i-1\}}^{II} = A_{i-1}\, r^{\lambda_{i-1}}\tilde{u}_{r\{i-1\}}^{II} \tag{2.417}$$

$$u_{ri}^{II} = A_i r^{\lambda_i}\tilde{u}_{ri}^{II} \tag{2.418}$$

where

$$\tilde{u}^{II}_{r\{i-1\}} = \frac{E^I}{E^{II}} \left\{ \frac{\left(1+\nu^{II}\right)}{\lambda_{i-1}} \left[\left(\lambda_{i-1}+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_{i-1}\right)\tilde{\phi}^{II}_{i-1} + \left(1-\nu^{II}\right)\left(\tilde{\phi}^{II}_{i-1}\right)'' \right] \right\} \tag{2.419}$$

$$\tilde{u}^{II}_{ri} = \frac{E^I}{E^{II}} \left\{ \frac{\left(1+\nu^{II}\right)}{\lambda_i} \left[\left(\lambda_i+1\right)\left(1-\nu^{II}-\nu^{II}\lambda_i\right)\tilde{\phi}^{II}_i + \left(1-\nu^{II}\right)\left(\tilde{\phi}^{II}_i\right)'' \right] \right\} \tag{2.420}$$

Displacement equation of first order approximation of elastic material side can be expressed as,

$$u^{II}_{r(i)} = u^{II}_{r0} + u^{II}_{r1} + u^{II}_{r2} + \cdots + u^{II}_{ri} = A_0 r^{\lambda_0}\tilde{u}^{II}_{r0} + A_1 r^{\lambda_1}\tilde{u}^{II}_{r1} + A_2 r^{\lambda_2}\tilde{u}^{II}_{r2} + \cdots + A_i r^{\lambda_i}\tilde{u}^{II}_{ri} \tag{2.421}$$

And again we know, from Eqn.(2.10) substituting, $\varepsilon^{II}_{\theta\theta(i)} = \varepsilon^{II}_{\theta\theta 0} + \varepsilon^{II}_{\theta\theta 1} + \varepsilon^{II}_{\theta\theta 2} + \cdots + \varepsilon^{II}_{\theta\theta i}$, $u^{II}_{\theta(i)} = u^{II}_{\theta 0} + u^{II}_{\theta 1} + u^{II}_{\theta 2} + \cdots + u^{II}_{\theta i}$ and $u^{II}_{r(i)} = u^{II}_{r0} + u^{II}_{r1} + u^{II}_{r2} + \cdots + u^{II}_{ri}$

Strain displacement relation becomes,

$$\begin{aligned} &u^{II}_{\theta 0} + u^{II}_{\theta 1} + u^{II}_{\theta 2} + \cdots + u^{II}_{\theta i} \\ &= r\int_{-\frac{\pi}{2}}^{0} \varepsilon^{II}_{\theta\theta 0} d\theta - \int_{-\frac{\pi}{2}}^{0} u^{II}_{r0} d\theta + r\int_{-\frac{\pi}{2}}^{0} \varepsilon^{II}_{\theta\theta 1} d\theta - \int_{-\frac{\pi}{2}}^{0} u^{II}_{r1} d\theta + r\int_{-\frac{\pi}{2}}^{0} \varepsilon^{II}_{\theta\theta 2} d\theta - \int_{-\frac{\pi}{2}}^{0} u^{II}_{r2} d\theta + \cdots + r\int_{-\frac{\pi}{2}}^{0} \varepsilon^{II}_{\theta\theta i} d\theta - \int_{-\frac{\pi}{2}}^{0} u^{II}_{ri} d\theta \end{aligned} \tag{2.422}$$

Initial part of displacement components and initial part of strain components has a relationship. So incremental part,

$$u^{II}_{\theta i} = r\int_{-\frac{\pi}{2}}^{0} \varepsilon^{II}_{\theta\theta i} d\theta - \int_{-\frac{\pi}{2}}^{0} u^{II}_{ri} d\theta \tag{2.423}$$

Finally,

$$u^{II}_{\theta(i)} = u^{II}_{\theta 0} + u^{II}_{\theta 1} + u^{II}_{\theta 2} + \cdots + u^{II}_{\theta i} = A_0 r^{\lambda_0}\tilde{u}^{II}_{\theta 0} + A_1 r^{\lambda_1}\tilde{u}^{II}_{\theta 1} + A_2 r^{\lambda_2}\tilde{u}^{II}_{\theta 2} + \cdots + A_i r^{\lambda_i}\tilde{u}^{II}_{\theta i} \tag{2.424}$$

Using another expression of strain Eqn. (2.11), displacement can be calculated as:

$$\left(\varepsilon_{r\theta}^{II}\right)_{\theta=-\frac{\pi}{2}} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_{r(0)}^{II}}{\partial\theta} + C_0 d r^{d-1} - C_0 r^{d-1}\right)_{\theta=-\frac{\pi}{2}} \tag{2.425}$$

Displacement can be calculated From Eqn.(2.11).Rigid body rotation is depends on r and angular function. To overcome the rigid body rotation we have to assume the displacement as a function of r and the angular function term.

Assume, $u_{\theta i}^{II} = k r^{\lambda_i} f(\theta) \therefore \frac{\partial u_{\theta i}^{II}}{\partial r} = \lambda_i k r^{\lambda_i - 1} f(\theta);\ or,\ \ r\frac{\partial u_{\theta i}^{II}}{\partial r} = \lambda_i k r^{\lambda_i} f(\theta)$ (2.426)

So equation becomes, When $\lambda_i \neq 1$,

$$u_{\theta i}^{II} = A_i r^{\lambda_i} \tilde{u}_{\theta i}^{II} \tag{2.427}$$

where,

$$\tilde{u}_{\theta i}^{II} = -\frac{E^I}{E^{II}}\frac{\left(1+\nu^{II}\right)}{\lambda_i\left(\lambda_i - 1\right)}\left[\left\{2\lambda_i^2 + \left(\lambda_i + 1\right)\left(1-\nu^{II} - \nu^{II}\lambda_i\right)\right\}\left(\tilde{\phi}_i^{II}\right)' + \left(1-\nu^{II}\right)\left(\tilde{\phi}_i^{II}\right)'''\right] \tag{2.428}$$

Total displacement yields,

$$u_{\theta(i)}^{II} = A_0 r^{\lambda_0}\tilde{u}_{\theta 0}^{II} + A_1 r^{\lambda_1}\tilde{u}_{\theta 1}^{II} + A_2 r^{\lambda_2}\tilde{u}_{\theta 2}^{II} + \cdots + A_i r^{\lambda_i}\tilde{u}_{\theta i}^{II} \tag{2.429}$$

Boundary Conditions:

$$\begin{matrix} \left(\sigma_{\theta\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0 \\ \left(\sigma_{r\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \left(\sigma_{\theta\theta i}^{I}\right)_{\theta=0} = \left(\sigma_{\theta\theta i}^{II}\right)_{\theta=0} \\ \left(\sigma_{r\theta i}^{I}\right)_{\theta=0} = \left(\sigma_{r\theta i}^{II}\right)_{\theta=0} \end{matrix} \tag{2.430}$$

Applying boundary conditions:

$$\left(\sigma_{\theta\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}} = \left(\sigma_{\theta\theta 0}^{II}\right)_{\theta=-\frac{\pi}{2}} + \left(\sigma_{\theta\theta 1}^{II}\right)_{\theta=-\frac{\pi}{2}} + \left(\sigma_{\theta\theta 2}^{II}\right)_{\theta=-\frac{\pi}{2}} + \cdots + \left(\sigma_{\theta\theta i}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0 \tag{2.431}$$

From zero-th order approximation we know $\left(\sigma_{\theta\theta 0}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0$, and from up to (i-1) th order approximation $\left(\sigma_{\theta\theta(i-1)}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0$ so, $\left(\sigma_{\theta\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0$ and $\left(\sigma_{r\theta(i)}^{II}\right)_{\theta=-\frac{\pi}{2}} = 0$

$$\left(\sigma^{II}_{r\theta(i)}\right)_{\theta=-\frac{\pi}{2}}=\left(\sigma^{II}_{r\theta 0}\right)_{\theta=-\frac{\pi}{2}}+\left(\sigma^{II}_{r\theta 1}\right)_{\theta=-\frac{\pi}{2}}+\left(\sigma^{II}_{r\theta 2}\right)_{\theta=-\frac{\pi}{2}}+\cdots+\left(\sigma^{II}_{r\theta i}\right)_{\theta=-\frac{\pi}{2}}=0 \tag{2.432}$$

From zero-th order approximation we know $\left(\sigma^{II}_{r\theta 0}\right)_{\theta=-\frac{\pi}{2}}=0$, and from up to (i-1) th order approximation $\left(\sigma^{II}_{r\theta(i-1)}\right)_{\theta=-\frac{\pi}{2}}=0$ so, $\left(\sigma^{II}_{r\theta i}\right)_{\theta=-\frac{\pi}{2}}=0$. (2.433)

Initial conditions at $\theta=-\dfrac{\pi}{2}$:

$$\frac{\sigma^{II}_{\theta\theta i}}{r^{\lambda_i-1}}=(\lambda_i+1)\lambda_i\left(A_i\tilde{\phi}^{II}_i\right)=0, \qquad \left(A_i\tilde{\phi}^{II}_i\right)=0, \text{if} \quad \lambda_i\neq-1,\ \lambda_i\neq 0 \tag{2.434}$$

And $\dfrac{\sigma^{II}_{r\theta i}}{r^{\lambda_i-1}}=-\lambda_i\left(A_i\tilde{\phi}^{II}_i\right)'=0, or \ \left(A_i\tilde{\phi}^{II}_i\right)'=0,$ if $\lambda_i\neq 0$ (2.435)

Assume at $\theta=-\dfrac{\pi}{2}$, $A_i\tilde{\phi}^{II}_i=0, \left(A_i\tilde{\phi}^{II}_i\right)'=0, \left(A_i\tilde{\phi}^{II}_i\right)''$ and $\left(A_i\tilde{\phi}^{II}_i\right)'''$ hence, after i-th integration at $\theta=0, \left(A_i\tilde{\phi}^{II}_i\right)=\left(A_i\tilde{\phi}^{II}_i\right)^{(in)}, \left(A_i\tilde{\phi}^{II}_i\right)'=\left(\left(A_i\tilde{\phi}^{II}_i\right)'\right)^{(in)}$,

$\left(A_i\tilde{\phi}^{II}_i\right)''=\left(\left(A_i\tilde{\phi}^{II}_i\right)''\right)^{(in)}$ and $\left(A_i\tilde{\phi}^{II}_i\right)'''=\left(\left(A_i\tilde{\phi}^{II}_i\right)'''\right)^{(in)}$

$$\text{error1}=A_i\tilde{\phi}^{I}_i-\left(A_i\tilde{\phi}^{II}_i\right)^{(in)} \tag{2.436}$$

$$\text{error2}=\left(\left(A_i\tilde{\phi}^{I}_i\right)'\right)-\left(\left(A_i\tilde{\phi}^{II}_i\right)'\right)^{(in)} \tag{2.437}$$

The error value is then calculated by using the equation (2.15).

2.7 Results for the Interface-Edge Problem of Elastic/Power-Law Hardening Materials Joint

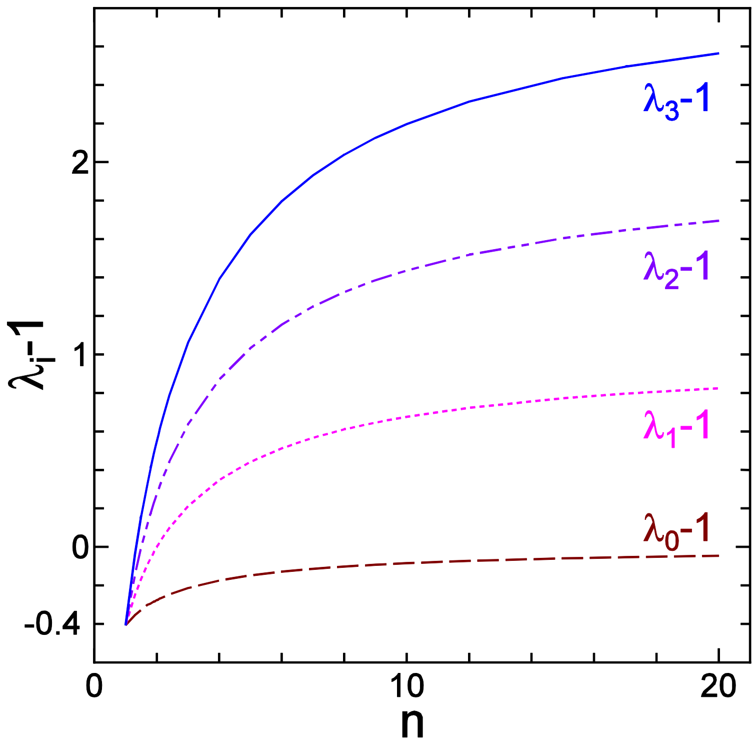


Fig. 2.18: Graphical form of stress singularity $\lambda_i - 1$ with n

Figure 2.18 shows the relation between the order of singularity, $\lambda_i - 1$, and power law hardening exponent, n. As the hardening exponent in the power law hardening material is increased the order of the singularity, $\lambda_i - 1$, tends to increase which means the absolute value of the order of singularity, $|\lambda_i - 1|$, when $\lambda_i - 1 < 0$ tends to decrease to zero. $\lambda_i - 1$ continues increasing when $\lambda_i - 1 > 0$ which means no more singularity is in the incremental term. Two or more singular terms exist for n<2.0, three or more singular terms exist for n<1.50, four or more singular terms exist when n<1.333. In general $(i+1)$ or more singular terms exist in the i-th order approximation for $n < (i+1)/i$, $i = 1, 2, 3 \cdots$.

The magnitude of incremental stress components $|\sigma_{ij(i)}|$ on the interface are decreased with the increase of iteration number i as shown in Fig.2.19 and Fig.2.20

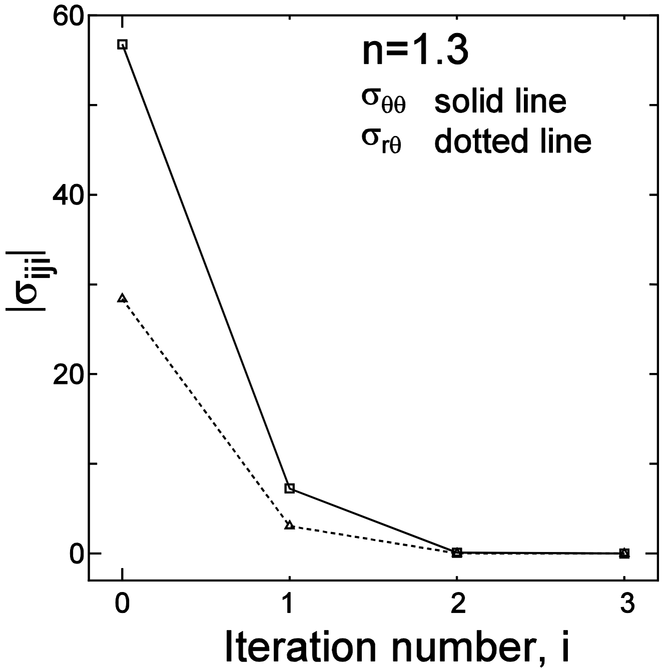


Fig. 2.19: Variation of incremental stresses on the interface with the iteration at $r = 10^{-4}$

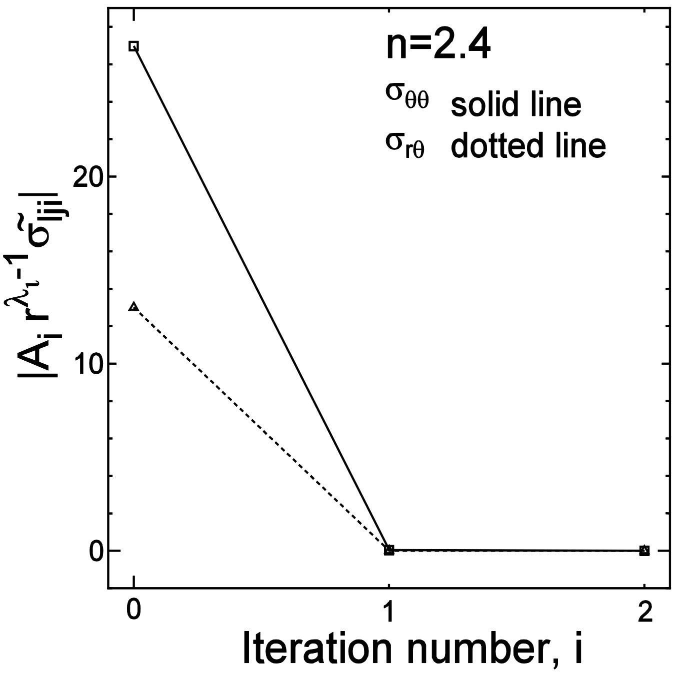


Fig. 2.20: Variation of incremental stresses on the interface with the iteration at $r = 10^{-4}$

for $n = 1.3$ and $n = 2.4$, respectively. The decrement of the incremental stress indicates that the solution is converged iteratively to satisfy the equilibrium condition of traction on the interface. Comparing Fig.2.19 and Fig.2.20 it can be shown that the solution converged rapidly (with minimum iteration number) for $n = 2.4$ than that of $n = 1.3$ because of the no multiple singular terms exist for the case $n = 2.4$.

The magnitude of incremental displacement components $|u_{j(i)}|$ on the interface are shown in Figs. 2.21 and 2.22. The incremental displacements are decreased with the increase of iteration number. The decrement of the incremental displacement means that the solution is converged iteratively to satisfy the continuous condition of displacement on the interface between the power-law and elastic material joint. Due to the existence of only single singular term for $n = 2.4$, the solution converged rapidly (with minimum iteration number) for $n = 2.4$ while $n = 1.3$ converged slowly (with maximum iteration number) that can be shown in Fig.2.22 compared with Fig.2.21.

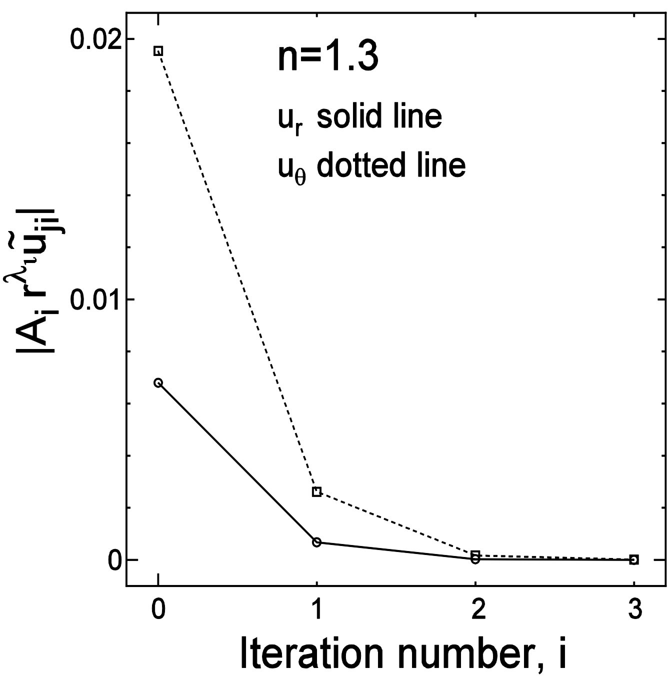


Fig. 2.21: Incremental displacements variation on the interface with the iteration at $r = 10^{-4}$

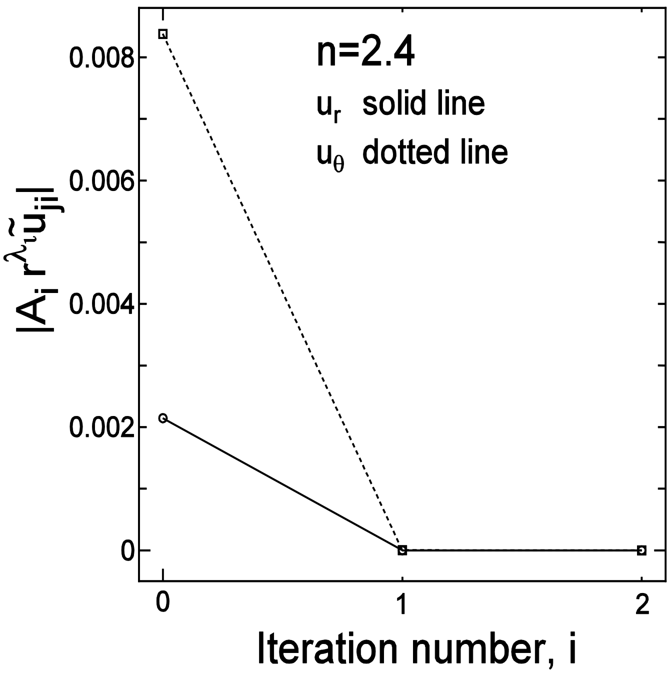


Fig. 2.22: Incremental displacements variation on the interface with the iteration at $r = 10^{-4}$

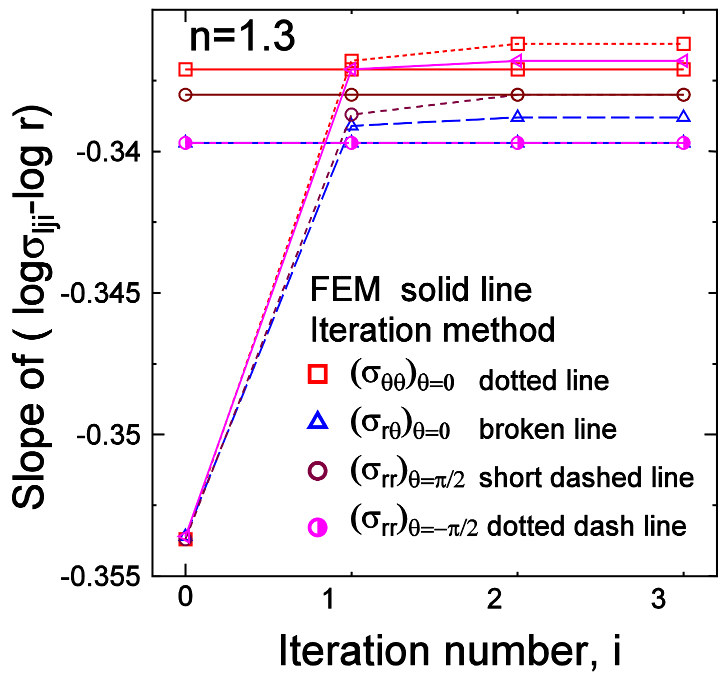


Fig. 2.23: Converging slope of $\log \sigma_{lji} - \log r$ with iteration number at $r = 10^{-4}$

Fig.2.23 shows the variation of slope of $\log\sigma_{ij} - \log r$ plots with the iteration number for $n = 1.3$. The slope of stress fields obtained from higher order approximation are closer to that found in FEM than that of the zero-th order approximation, which indicates that the higher order approximation is important to describe the real field.

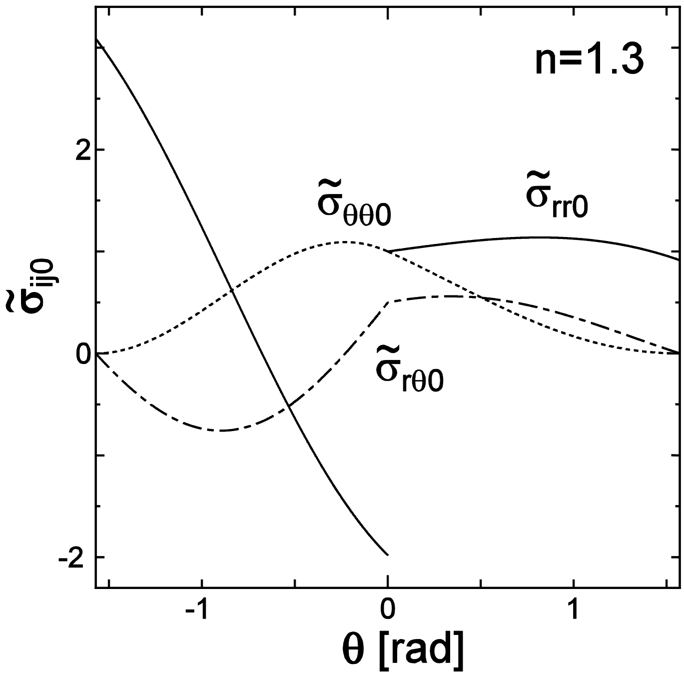


Fig. 2.24: Angular variation of stresses of 0th order approximation near the interface edge of elastic/elastic-plastic materials joint for n=1.3

The angular variations of normalized stresses for power-law hardening material for plane strain condition obtained from theoretical analysis shown in Figures 2.24-2.27 for $n = 1.3$ and Figures 2.28-2.30 for $n = 2.4$. It can be seen that tractions are continuous across the interface, but there is a big jump in the radial stress, $\tilde{\sigma}_{rr}$. The stress $\tilde{\sigma}_{rr}$ on the elastic material side is much greater than that on the power-law hardening material side. This means that the interface is really a stress raiser which results in serious stress concentration. Total stresses and displacements after i-th order approximation are depicted in Figs. 2.31-2.32 and Figs. 2.33-2.34, respectively. Stresses and displacements are continuous on the interface of two dissimilar elastic-plastic and elastic materials joint.

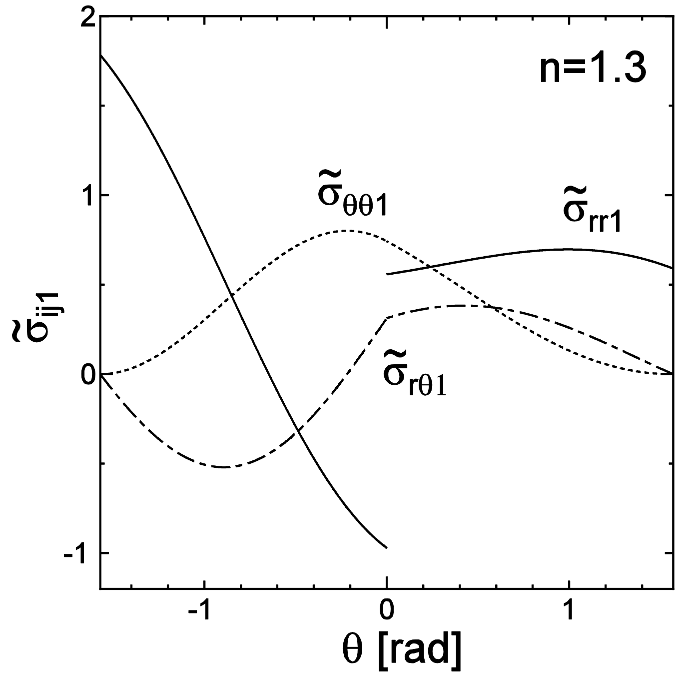


Fig. 2.25: Angular variation of incremental stresses of 1st order approximation near the interface edge of elastic/elastic-plastic materials joint for n=1.3

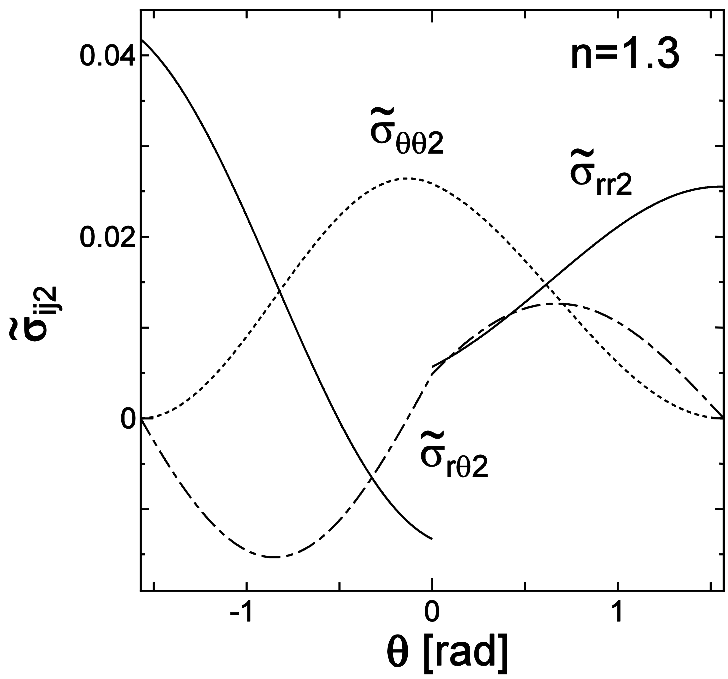


Fig. 2.26: Angular variation of incremental stresses of 2nd order approximation near the interface edge of elastic/elastic-plastic materials joint for n=1.3

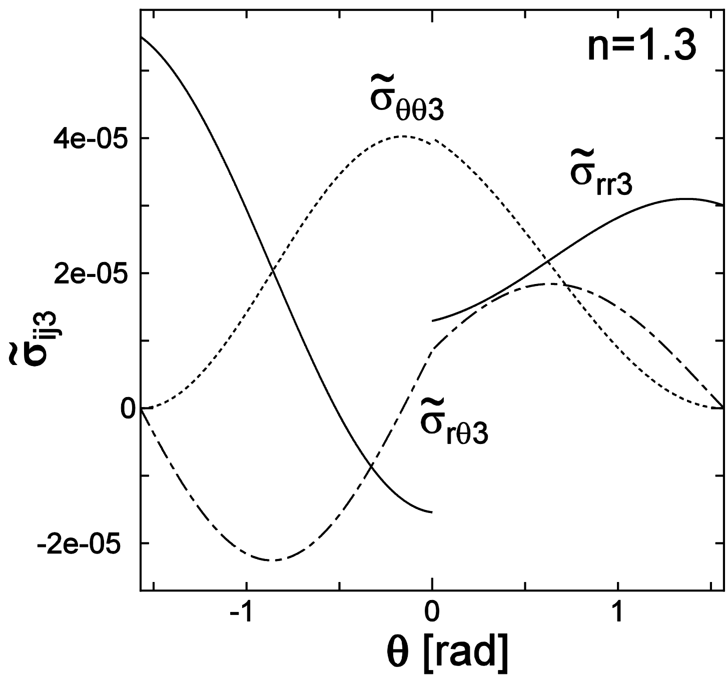


Fig. 2.27: Angular variation of incremental stresses of 3rd order approximation near the interface edge of elastic/elastic-plastic materials joint for n=1.3

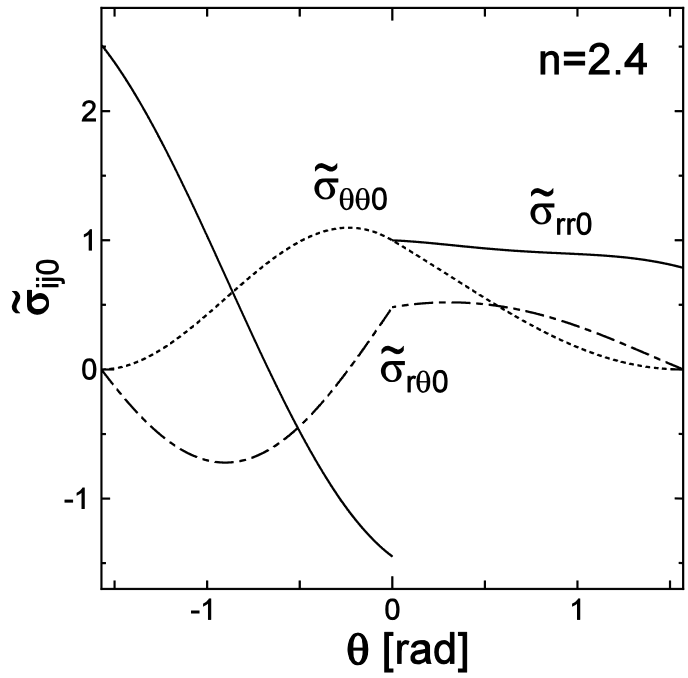


Fig. 2.28: Angular variation of stresses of 0th order approximation near the interface edge of elastic/elastic-plastic materials joint for n=2.4

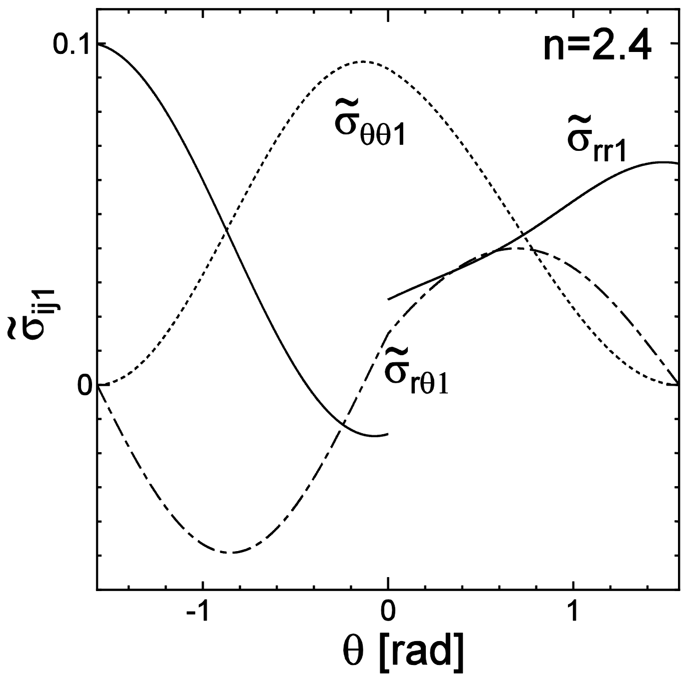


Fig. 2.29: Angular variation of incremental stresses of 1st order approximation near the interface edge of elastic/elastic-plastic materials joint for n=2.4

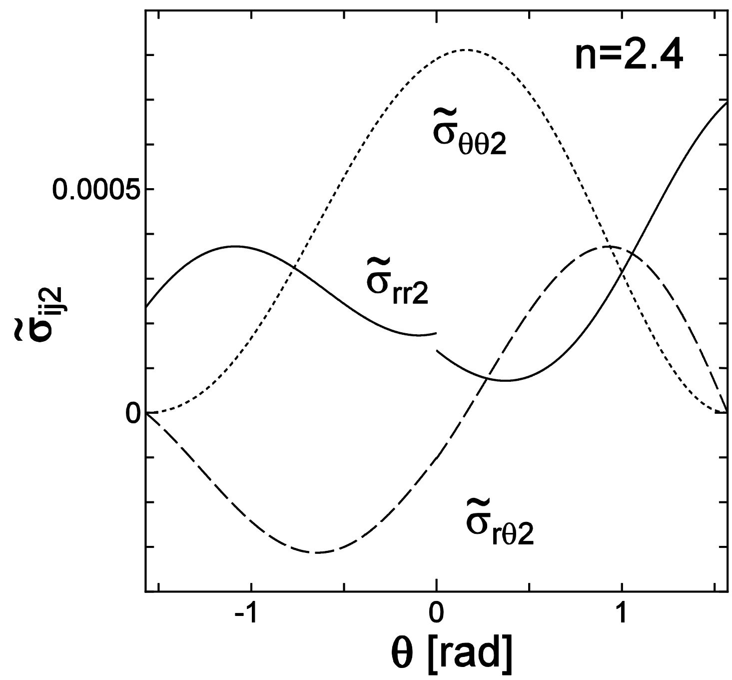


Fig. 2.30: Angular variation of incremental stresses of 2nd order approximation near the interface edge of elastic/elastic-plastic materials joint for n=2.4

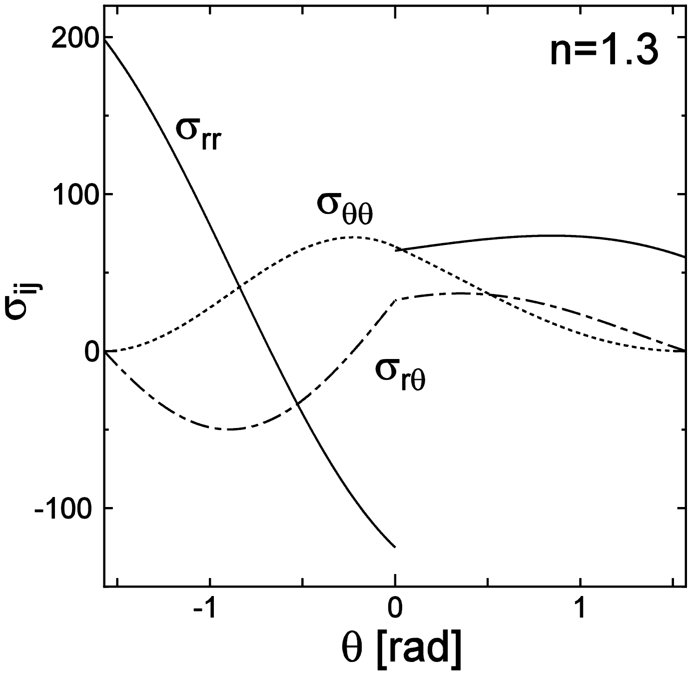


Fig. 2.31: Angular variation of total stresses after i-th order approximation near the interface edge of elastic/elastic-plastic materials joint for n=1.3 at $r = 10^{-4}$.

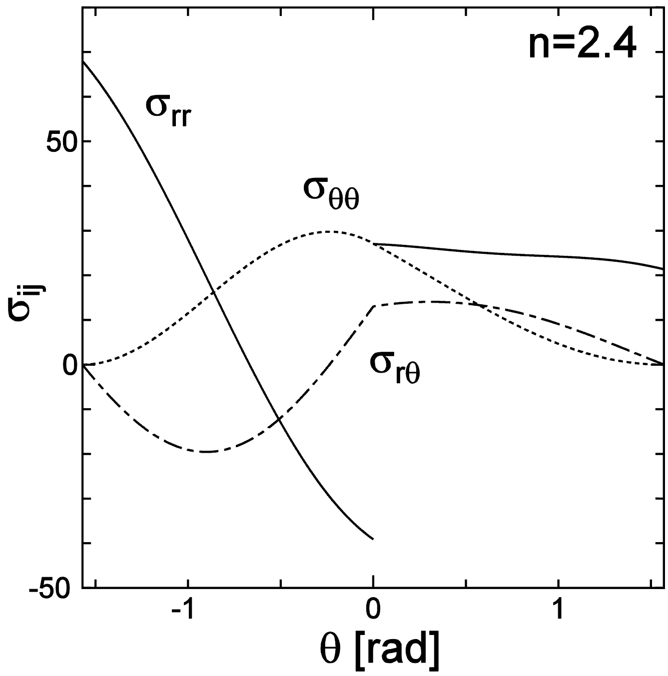


Fig. 2.32: Angular variation of total stresses after i-th order approximation near the interface edge of elastic/elastic-plastic materials joint for n=2.4 at $r = 10^{-4}$.

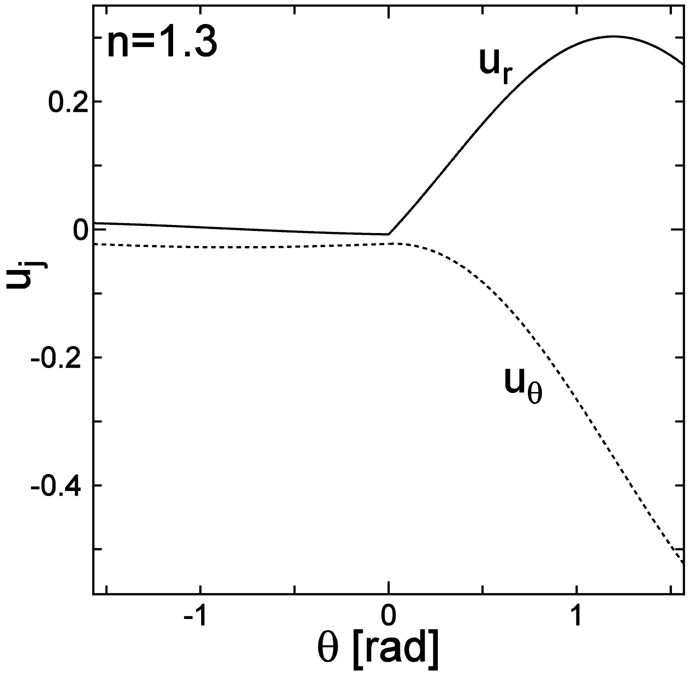


Fig. 2.33: Angular variation of total displacements after i-th order approximation near the interface edge of elastic/elastic-plastic materials joint for n=1.3 at $r = 10^{-4}$.

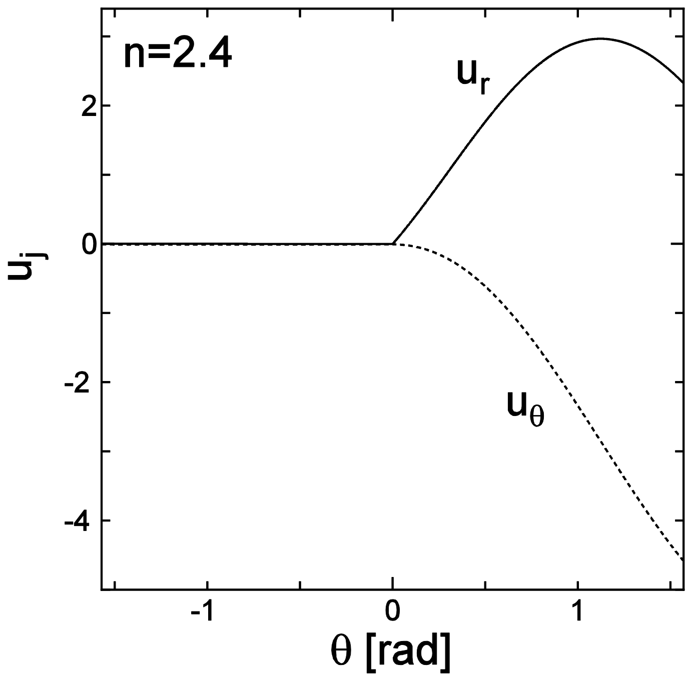


Fig. 2.34: Angular variation of total displacements after i-th order approximation near the interface edge of elastic/elastic-plastic materials joint for n=2.4 at $r = 10^{-4}$.

2.8 Summary of Elastic/Power-Law Hardening Materials Joint

Bonded dissimilar materials in which materials behaves as an elastic and a power-law hardening material were studied and the solution was presented to determine the stress and displacement fields around an interface edge of dissimilar materials joint. An iteration method is proposed for the determination of singular fields around an interface edge of an elastic and a power-law hardening materials joint. In the proposed iteration method, to overcome the problem we have considered at the interface boundary the additional stress fields in the elastic side to satisfy stress continuity, the additional displacement fields in the elastic-plastic side to satisfy displacement continuity, successively. Due to the increase of iteration, the discrepancy of the *r* dependence of the fields along the interface is decreased.

The power of *r* in the stress equation depends on the hardening exponent *n*. As n is increased the absolute value of the i-th order of singularity, $|\lambda_i - 1|$, tends to be decreased to zero when $\lambda_i - 1 < 0$. $(i+1)$ or more singular terms exist in the i-th order approximation for $n < (i+1)/i$. In the zero-th order approximation, the singular exponent λ_0 depends on the hardening exponent n. n ranging from 1(linear stress hardening) to ∞ (perfect plasticity). The dominant singularity as reflected λ_0 is most severe when $n = 1$. The value of λ_0 increased monotonically with the increasing of n .

The angular function of separable form term in the displacement fields in the 0th order approximation should be zero at $\theta = 0$ is a necessary condition for the continuity of displacement on the interface when $r \to 0$ around the interface free edge of elastic/elastic-plastic materials joint. The stress and displacement fields in the elastic-plastic material are controlled by the boundary condition which is the same as the one of an elastic-plastic material on the rigid substrate. The stress fields in the elastic material are also controlled by the index through the equilibrium of force on the interface.

In the power-law hardening material, when n is decreased (for $n=1$ is an incompressible elastic material) the stronger singularity ($\lambda_0 - 1$, goes to more negative) exist in the power-law hardening material due to displacement boundary condition ($r \to 0$, $\tilde{u}_i(\theta=0)=0$) and weaker singularity ($\lambda_0 - 1$, goes to less negative) for larger n. In the elastic/power-law hardening materials joint case the singularity should be weaker with the decreasing value of n because of $n=1$ is an incompressible elastic material but in our approximation is not shows this situation. When regarded as deformation theory materials, the materials just considered each approximate perfect plastic behavior for small values of n for the power-law material. Yet, the calculated values of λ do not agree in this limit. The disparity can be explained as follows. For a power-law material model, elastic strains were neglected because, as stresses grow large for any value of $n>1$, the ratio of elastic strain to plastic strain approaches zero. With an asymptotic expansion of the governing equations in powers of n, it is easily shown that $n \to 1$ as $n \to \infty$.

In the elastic material, singularity $|\lambda - 1|$ depends on Poison's ratio [64]. Singularity, $|\lambda - 1|$ increases with the increase of Poisson's ratio. From the stress-strain relation of power-law hardening material it can be explained that when n is increased; the effective modulus is decreased due to the increasing strain with increasing n for the same stress. The Poisson's ration is fixed, and $\nu = 0.5$ due to the constant volume. Hence, there is no dependency of singularity with the Poisson's ratio.

From 0^{th} order approximation of our present study, the singular exponent for $n=2.4$ is calculated $\lambda_0 = 0.753675 \pm 5\times10^{-6}$. From the previous researches it can be found that for the same value of $n=2.4$, the singular exponent λ_0 is calculated by Duva et al.[65] and Rahman et al.[47] are 0.75 and 0.76, respectively. This shows our results are in a good agreement with available results.

CHAPTER 3

SINGULAR STRESS FIELDS IN DISSIMILAR POWER-LAW HARDENING MATERIALS JOINT

3.1 Introduction

In this chapter, we solved for the singular stress fields at the interface-edge of two dissimilar power-law hardening materials joint where dissimilar materials having different power-law hardening exponents. We have formulated and solved under the plane strain condition. By taking the same wedge angle of two materials, our generic interface-edge model is as butt joint model with the interface-edge of two dissimilar power-law hardening materials.

In [66] an iteration method were presented for the elastic/power-law hardening materials joint to solve the problem on the interface arising due to the dissimilar power of r in the displacement field. In this chapter, using the same iterative method as presented in chapter 2, we have satisfied the boundary conditions to determine the singular fields around an interface edge of two dissimilar power-law hardening materials joint having different hardening exponent.

The study reported in this chapter is an asymptotic analysis for singular stress fields around an interface-edge of dissimilar power-law hardening materials joint under plane-strain condition and J_2 deformation plasticity theory. The emphasis is to establish the effect of constituent material properties and the effect of geometry on the singular stress fields, when the materials behave as:

$$\varepsilon_{ij}^{I} = \frac{3}{2}\alpha^{I}\sigma_{e}^{n_1-1}s_{ij}^{I} \quad \text{(for material I)} \tag{3.1}$$

and

$$\varepsilon_{ij}^{II} = \frac{3}{2}\alpha^{II}\sigma_{e}^{n_2-1}s_{ij}^{II} \quad \text{(for material II)} \tag{3.2}$$

where i and j are used for subscript indicates r,θ. α^{k} and n_k are hardening coefficient and hardening exponent, respectively. σ_e is the effective stress and s_{ij} is the stress deviator.

It should be noted that, there can be material mismatch along the interface arising from either a difference in yield strengths $\sigma_y^I \neq \sigma_y^{II}$ or from a difference in power-law hardening exponents $n_1 \neq n_2$ or both. We first analyze the effect of $n_1 \neq n_2$ by focusing on bimaterial in which only $n_1 \neq n_2$ but $\sigma_y^I = \sigma_y^{II}$. When $n_1 \neq n_2$ then there is the possibility of $n_1 > n_2$ or $n_1 < n_2$. We will assume that the material I is the lower strain hardening material $n_1 > n_2$ in this research. Stress, strain and displacement quantities are normalized by yield stress or corresponding yield strain of the material I. In this research both the hardening exponents, $n_1, n_2 > 1$.

In Section 3.2, we formulate the governing equations for the singular stress field under the plane strain when $n_1 \neq n_2$ but $\sigma_y^I = \sigma_y^{II}$.The solution method is presented in section 3.3. In Section 3.4 the asymptotic analysis of two dissimilar power-law hardening materials joint is given elaborately. Results for the interface-edge problem of two power-law hardening materials joint having different power-law hardening exponents is presented in Section 3.5. This chapter is concluded with summery in Section 3.6.

3.2 Formulation: Two Power-Law Hardening Materials Joint Cases

Consider two power-law hardening materials joint with different hardening exponent n_1 and n_2, as shown in Fig.3.1. We will assume that the material I is the lower strain hardening material in this study.

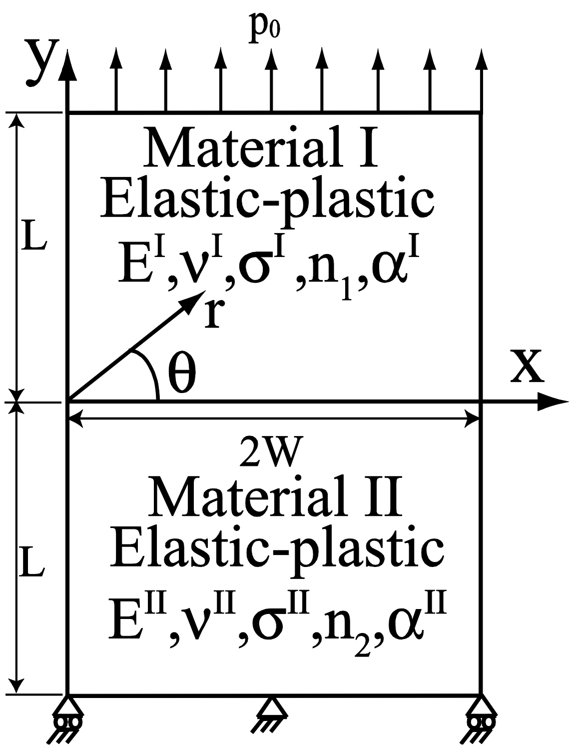


Fig. 3.1: Theoretical model.

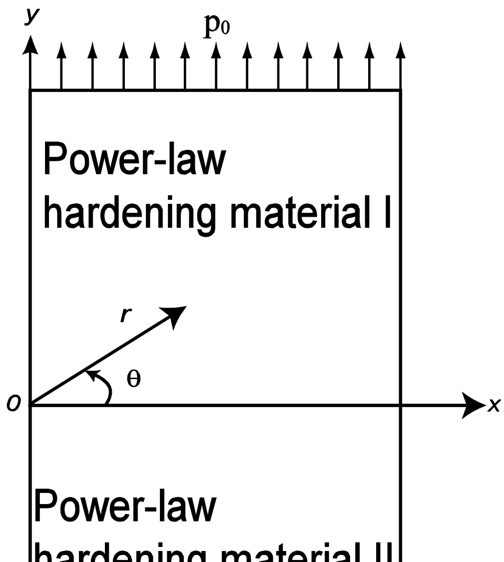


Fig. 3.2: Analysis model geometries.

The boundary condition can be expressed as follows in the polar coordinate system located on the interface edge for power-law hardening materials joint having different hardening exponent.

$$\begin{aligned}
&\sigma^{I}_{\theta\theta}\Big|_{\theta=\frac{\pi}{2}} = \sigma^{II}_{\theta\theta}\Big|_{\theta=-\frac{\pi}{2}} = 0, \qquad && \sigma^{I}_{\theta\theta}\Big|_{\theta=0} = \sigma^{II}_{\theta\theta}\Big|_{\theta=0}, \qquad && u^{I}_{r}\Big|_{\theta=0} = u^{II}_{r}\Big|_{\theta=0}, \\
&\sigma^{I}_{r\theta}\Big|_{\theta=\frac{\pi}{2}} = \sigma^{II}_{r\theta}\Big|_{\theta=-\frac{\pi}{2}} = 0, && \sigma^{I}_{r\theta}\Big|_{\theta=0} = \sigma^{II}_{r\theta}\Big|_{\theta=0}, && u^{I}_{\theta}\Big|_{\theta=0} = u^{II}_{\theta}\Big|_{\theta=0}.
\end{aligned} \tag{3.3}$$

The stresses and displacements of the upper material are referred to with a superscript “I” while those of the lower material, with a superscript “II” as shown in Fig.3.1 and 3.2.

3.3 Solution Method

In the zero-th order approximation, the stress and displacement fields in the power-law hardening material I are assumed to be the same as the ones in the plate jointed to rigid substrate instead of the material II and subjected to the same tensile load.

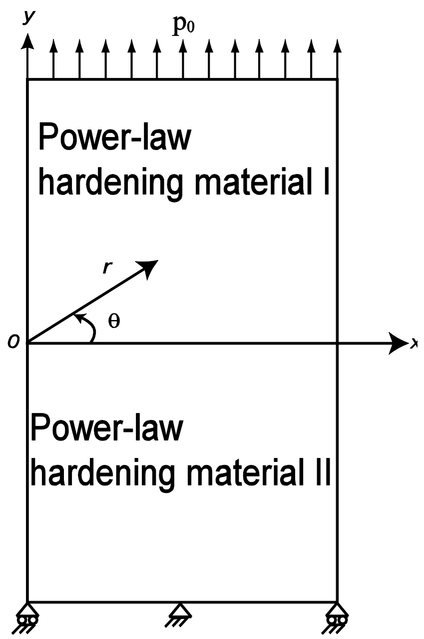


Fig. 3.3: Elastic-plastic/ Elastic-plastic materials joint.

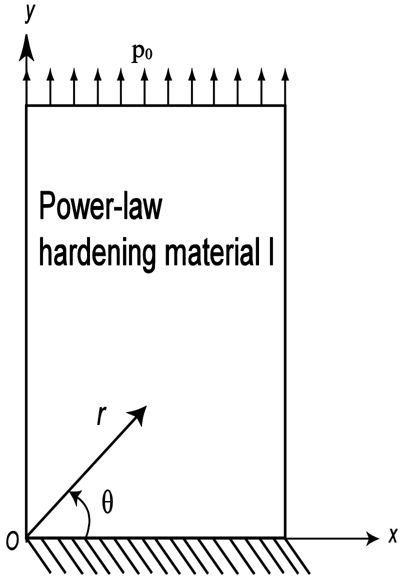


Fig. 3.4: Material I (upper material) bonded to rigid substrate.

The stress fields in the power-law hardening material II can be described by the fields in the power-law hardening material wedge which is subjected to distributed tractions along the one edge. The tractions are the same as the stress distributions on the rigid/material I interface.

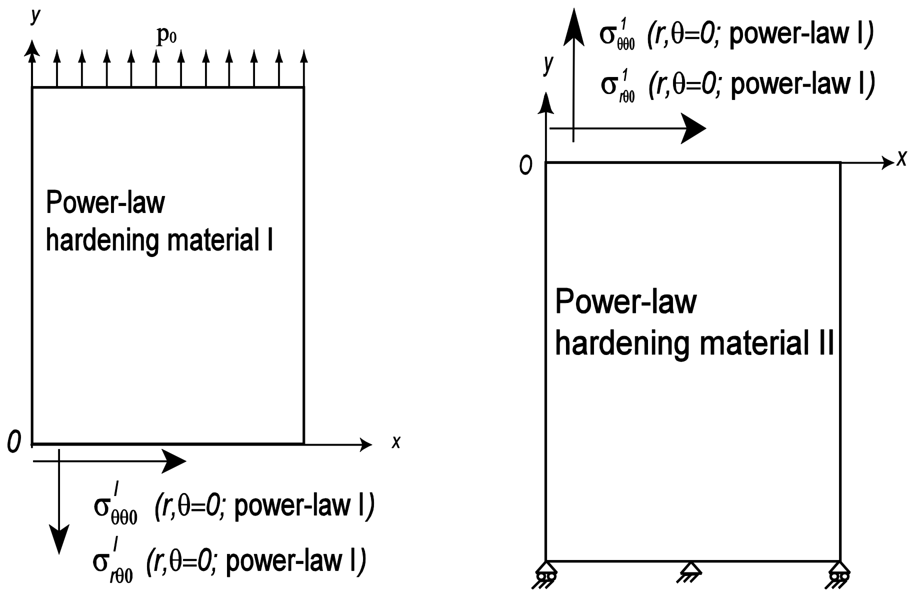


Fig. 3.5: Schematic diagram of applying Traction from upper material to the lower material.(a) Traction from upper material (b) Traction to lower material.

In the first order approximation, the power-law hardening material I having the initial fields of the zero-th order approximation is subjected to a forced displacement which is the same displacement induced along the edge of power-law hardening material II in the zero-th order approximation.

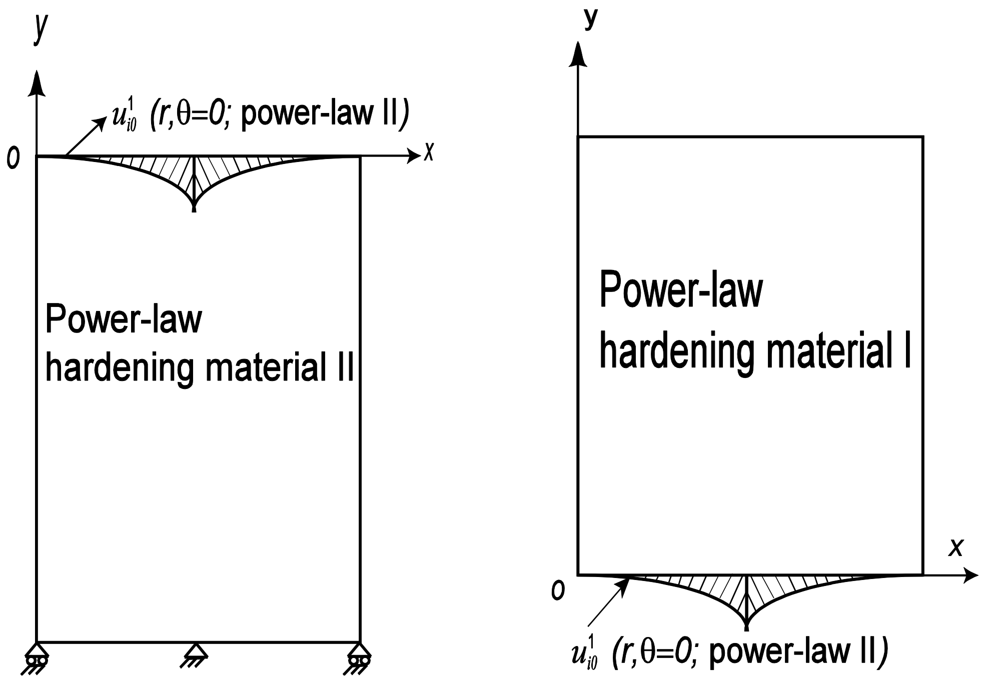


Fig. 3.6: Schematic diagram of applying Forced displacement from lower material to the power-law hardening material I(upper material). (a) Forced displacement from lower material (b) Forced displacement to upper material.

The increase of stress fields in the power-law hardening material II can be described by the fields of the power-law hardening material II having the initial fields of the zero-th order approximation which is subjected to distributed tractions along the one edge. The tractions are the same as the incremental stress distributions on the power-law hardening material I wedge in the first order approximation. The iteration process continues as the approximation goes.

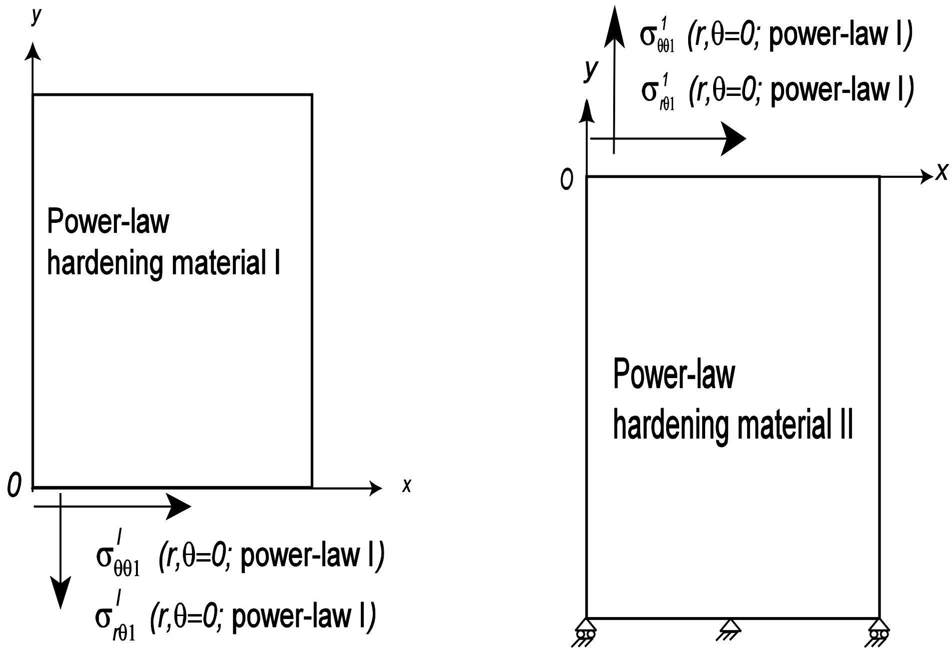


Fig. 3.7: Schematic diagram of applying Traction from power-law hardening material I to the power-law hardening material II (a) Traction from power-law hardening material I (b) Traction to power-law hardening material II.

Stress-strain relations used in the material I and material II, respectively are given in Eqn.(3.1) and Eqn.(3.2),

where i and j are used for subscript indicates r,θ . α^k and n_k are hardening coefficient and hardening exponent, respectively. σ_e is the effective stress and s_{ij} is the stress deviator. We will assume that the material I is the lower strain hardening material $\left(n_1 > n_2\right)$ in this study. Stress, strain and displacement quantities are normalized by yield stress or corresponding yield strain of the material I. In this thesis both the hardening exponents, $n_1, n_2 > 1$.

3.4 Asymptotic Analysis

An asymptotic expansion of the Airy stress function in a separable form is assumed

$$\phi^k(i) = \sum_i A_i r^{\lambda_i+1} \tilde{\phi}_i^k, \qquad i = 0,1,2,...; as \quad r \to 0, \qquad (3.4)$$

where $\lambda_0 < \lambda_1 ....$ and $k = I$ for material I and $k = II$ for material II. $\tilde{\phi}_i^k$ is the angular function of airy stress function in the i-th order of approximation. A_i is a constant which is proportional to the stress intensity factor of i-th order incremental fields. A_i is defined as,

$$\left(\sigma^I_{\theta\theta(0)}\right)_{\theta=0} = \sum_i A_i\, r^{\lambda_i - 1}, \qquad i = 0,1,2,.... \tag{3.5}$$

In the higher order approximation $(i \geq 1)$, nonlinear effective stress term σ_e^{n-1} was expanded by Taylor series expansion method and the first two terms were considered for further calculations.

In the first order approximation before expansion this term is written as,

$$\sigma_e^{n_k - 1} = r^{(\lambda_0 - 1)(n_k - 1)} \left[\frac{3}{8}\left(f_0 + 2A_1\, r^{(\lambda_1 - \lambda_0)} f_1 + A_1^2\, r^{2(\lambda_1 - \lambda_0)} f_2 \right)\right]^{\frac{n_k - 1}{2}}, \tag{3.6}$$

where,

$$f_0 = \left(f_{0rr}^2 + f_{0\theta\theta}^2 + 8 f_{0r\theta}^2 \right),\ f_1 = \left(f_{0rr} f_{1rr} + f_{0\theta\theta} f_{1\theta\theta} + 8 f_{0r\theta} f_{1r\theta} \right),\ f_2 = \left(f_{1rr}^2 + f_{1\theta\theta}^2 + 8 f_{1r\theta}^2 \right),$$

$$f_{0rr} = A_0 \left\{ \tilde{\phi}_0^k (\lambda_0 + 1)(1 - \lambda_0) + \left(\tilde{\phi}_0^k\right)'' \right\},\ f_{0\theta\theta} = -A_0 \left\{ \tilde{\phi}_0^k (\lambda_0 + 1)(1 - \lambda_0) + \left(\tilde{\phi}_0^k\right)'' \right\},\ f_{0r\theta} = \left\{ -A_0 \left(\tilde{\phi}_0^k\right)' \lambda_0 \right\},$$

$$f_{1rr} = \left\{ \tilde{\phi}_1^k (\lambda_1 + 1)(1 - \lambda_1) + \left(\tilde{\phi}_1^k\right)'' \right\},\ f_{1\theta\theta} = -\left\{ \tilde{\phi}_1^k (\lambda_1 + 1)(1 - \lambda_1) + \left(\tilde{\phi}_1^k\right)'' \right\},\ f_{1r\theta} = \left\{ -\left(\tilde{\phi}_1^k\right)' \lambda_1 \right\}. \tag{3.7}$$

Assuming smaller range of r $(r < 1)$ near the interface edge it is reasonable to have the singular exponent of incremental stress λ_1 which is larger than the zero-th order singular exponent λ_0, i.e., $\lambda_0 < \lambda_1$. The order of r of the terms in the part powered by $(n_k - 1)/2$ in Eqn. (3.6) is $0, (\lambda_1 - \lambda_0)$ and $2(\lambda_1 - \lambda_0)$ respectively which means the order of r in the second and third terms are positive in magnitude. Positive power of small r gives the value smaller than 1. Also $A_1 < 1$, $f_1 < 1$ and $f_2 < 1$. The summation of the second and third terms has the smaller magnitude than 1.This satisfies the convergence condition for the Taylor expansion. Assuming the first term as the leading term and remaining terms as the incremental term Taylor expansion is applied to Eqn. (3.6). After expansion and neglecting the higher order term of A_1 the equation becomes,

$$\sigma_e^{n_k-1} \approx \left[\frac{3}{8}r^{2(\lambda_0-1)} f_0\right]^{\frac{n_k-1}{2}} + \frac{n_k-1}{2}\left[\frac{3}{8}r^{2(\lambda_0-1)} f_0\right]^{\frac{n_k-3}{2}} \times \frac{3}{4}\left(A_1 \times r^{\lambda_0+\lambda_1-2} f_1\right). \tag{3.8}$$

Substituting effective stress term from Eqn. (3.8) into Eqn. (3.1) and Eqn. (3.2) , strain components are expressed according to the order of A_1 as:

$$\varepsilon_{ij}^k = \varepsilon_{ij0}^k + \varepsilon_{ij1}^k \left\{O(A_1)\right\} + \varepsilon_{ij2}^k \{O(A_1^2)\} \tag{3.9}$$

In this expression strain components will have three terms with respect to the power of *r*. $r^{n_k(\lambda_0-1)}$, $r^{n_k\lambda_0-n_k-\lambda_0+\lambda_1}$ and $r^{n_k\lambda_0-3\lambda_0-n_k+1+2\lambda_1}$. From zero-th order approximation solution it is clear that the terms of $r^{n_k(\lambda_0-1)}$ satisfy the compatibility condition. To solve the compatibility condition on the remaining terms, we assume the two terms satisfy the conditions independently. At first we will consider it neglecting the third term.

$$\varepsilon_{ij(1)}^k = \varepsilon_{ij0}^k + \varepsilon_{ij1}^k \left\{O(A_1)\right\} \tag{3.10}$$

Strain components are in the following summation form:

$$\varepsilon_{rr(1)}^k \approx \varepsilon_{rr0}^k + \varepsilon_{rr1}^k \tag{3.11}$$

$$\varepsilon_{\theta\theta(1)}^k \approx \varepsilon_{\theta\theta0}^k + \varepsilon_{\theta\theta1}^k \tag{3.12}$$

$$\varepsilon_{r\theta(1)}^k \approx \varepsilon_{r\theta0}^k + \varepsilon_{r\theta1}^k \tag{3.13}$$

Initial part of strain components can be expressed as,

$$\varepsilon_{rr0}^k = A_0^{n_k}\, r^{n_k(\lambda_0-1)}\tilde{\varepsilon}_{rr0}^k \tag{3.14}$$

$$\varepsilon_{\theta\theta0}^k = A_0^{n_k}\, r^{n_k(\lambda_0-1)}\tilde{\varepsilon}_{\theta\theta0}^k \tag{3.15}$$

$$\varepsilon_{r\theta0}^k = A_0^{n_k}\, r^{n_k(\lambda_0-1)}\tilde{\varepsilon}_{r\theta0}^k \tag{3.16}$$

where, the expression of $\tilde{\varepsilon}_{ij0}^{k}$ are given in chapter 2. It is noted that n is replaced by n_k ($n = n_k$) . $n = n_1$ for upper material while $n = n_2$ for lower material part.

Strain components contains first order term of A_1:

$$\varepsilon_{rr1}^{k} = A_0^{n_k - 1} A_1 r^{n_k\lambda_0 - n_k - \lambda_0 + \lambda_1} \tilde{\varepsilon}_{rr1}^{k} \tag{3.17}$$

$$\varepsilon_{\theta\theta 1}^{k} = A_0^{n_k - 1} A_1 r^{n_k\lambda_0 - n_k - \lambda_0 + \lambda_1} \tilde{\varepsilon}_{\theta\theta 1}^{k} \tag{3.18}$$

$$\varepsilon_{r\theta 1}^{k} = A_0^{n_k - 1} A_1 r^{n_k\lambda_0 - n_k - \lambda_0 + \lambda_1} \tilde{\varepsilon}_{r\theta 1}^{I} \tag{3.19}$$

where, $\tilde{\varepsilon}_{ij1}^{k}$ are given in chapter 2. It is noted that n is replaced by n_k ($n = n_k$) . $n = n_1$ for upper material while $n = n_2$ for lower material part. From chapter 2 only equations presented for power-law material have been considered.

Strain components contains first order term of A_i:

$$\varepsilon_{rri}^{k} = A_0^{n_k - 1} A_i r^{n_k\lambda_0 - n_k - \lambda_0 + \lambda_i} \tilde{\varepsilon}_{rri}^{k} \tag{3.20}$$

$$\varepsilon_{\theta\theta i}^{k} = A_0^{n_k - 1} A_i r^{n_k\lambda_0 - n_k - \lambda_0 + \lambda_i} \tilde{\varepsilon}_{\theta\theta i}^{k} \tag{3.21}$$

$$\varepsilon_{r\theta i}^{k} = A_0^{n_k - 1} A_i r^{n_k\lambda_0 - n_k - \lambda_0 + \lambda_i} \tilde{\varepsilon}_{r\theta i}^{I} \tag{3.22}$$

where, $\tilde{\varepsilon}_{ij1}^{k}$ are given in chapter 2. It is noted that n is replaced by n_k ($n = n_k$) . $n = n_1$ for upper material while $n = n_2$ for lower material part. From chapter 2 only equations presented for power-law material have been considered.

In the 0^{th} order approximation, displacement field can be expressed as,

$$u_{j0}^{k} = A_0^{n_k} r^{n_k\lambda_0 - n_k + 1} \tilde{u}_{j0}^{k} \tag{3.23}$$

where,

$$u_{r0}^{k} = A_0^{n_k}\, r^{n_k\lambda_0 - n_k + 1}\tilde{u}_{r0}^{k} \tag{3.24}$$

$$u_{\theta 0}^{k} = A_0^{n_k}\, r^{n_k\lambda_0 - n_k + 1}\tilde{u}_{\theta 0}^{k} \tag{3.25}$$

In the 1st order approximation, displacement field can be expressed as,

$$u_{j1}^{k} = A_0^{n_k - 1} A_1\, r^{(n_k\lambda_0 - n_k - \lambda_0 + \lambda_1)+1}\tilde{u}_{j1}^{k} \tag{3.26}$$

where,

$$u_{r1}^{k} = A_0^{n_k - 1} A_1\, r^{(n_k\lambda_0 - n_k - \lambda_0 + \lambda_1)+1}\tilde{u}_{r1}^{k} \tag{3.27}$$

$$u_{\theta 1}^{k} = A_0^{n_k - 1} A_1\, r^{(n_k\lambda_0 - n_k - \lambda_0 + \lambda_1)+1}\tilde{u}_{\theta 1}^{k} \tag{3.28}$$

In the ith order approximation, displacement field can be expressed as,

$$u_{ji}^{k} = A_0^{n_k - 1} A_i\, r^{(n_k\lambda_0 - n_k - \lambda_0 + \lambda_i)+1}\tilde{u}_{ji}^{k} \tag{3.29}$$

where,

$$u_{ri}^{k} = A_0^{n_k - 1} A_i\, r^{(n_k\lambda_0 - n_k - \lambda_0 + \lambda_i)+1}\tilde{u}_{ri}^{k} \tag{3.30}$$

$$u_{\theta i}^{k} = A_0^{n_k - 1} A_i\, r^{(n_k\lambda_0 - n_k - \lambda_0 + \lambda_i)+1}\tilde{u}_{\theta i}^{k} \tag{3.31}$$

where, angular function of displacement $\tilde{u}_{ji}^{k}$ are given in chapter 2. It is noted that n is replaced by n_k ($n = n_k$) . $n = n_1$ for upper material while $n = n_2$ for lower material part. From chapter 2 only equations presented for power-law material have been considered.

The resulting expressions for incremental stresses are:

$$\sigma_{rri}^{k} = A_i\, r^{\lambda_i - 1}\tilde{\sigma}_{rri}^{k} \tag{3.32}$$

$$\sigma_{\theta\theta i}^{k} = A_i\, r^{\lambda_i - 1}\tilde{\sigma}_{\theta\theta i}^{k} \tag{3.33}$$

$$\sigma_{r\theta i}^{k} = A_i\, r^{\lambda_i - 1}\tilde{\sigma}_{r\theta i}^{k} \tag{3.34}$$

where angular function of stresses are,

$$\tilde{\sigma}_{rri}^{k} = \tilde{\phi}_i^k \left(\lambda_i + 1\right) + \left(\tilde{\phi}_i^k\right)'' \tag{3.35}$$

$$\tilde{\sigma}_{\theta\theta i}^{k} = \tilde{\phi}_i^k \left(\lambda_i + 1\right)\lambda_i \tag{3.36}$$

$$\tilde{\sigma}_{r\theta i}^{k} = -\left(\tilde{\phi}_i^k\right)' \lambda_i \tag{3.37}$$

From zero-th order approximation solution it is clear that the terms of $r^{n_k(\lambda_0-1)}$ satisfy the compatibility condition. To solve the compatibility condition on the remaining terms, we assumed the two terms (zero-th order term and first order term of A_1) of strain components satisfy the conditions independently. Finally, initial part of compatibility equation is same as the equation which is satisfied in zero-th order approximation. So, remaining part of compatibility equation should be satisfied independently. Using Eqn. (3.8), the compatibility equation will have three terms with respect to the power of r , $r^{n_k(\lambda_0-1)-2}$, $r^{n_k\lambda_0-n_k-\lambda_0+\lambda_1-2}$ and $r^{n_k\lambda_0-3\lambda_0-n_k+2\lambda_1-1}$. To solve the compatibility condition, we assume that those three terms satisfy the conditions order by order. Here we will neglect the third term. Assuming second term as the incremental part the compatibility equation includes the exponent of r as $r^{n_k\lambda_0-n_k-\lambda_0+\lambda_1-2}$. Hence, in the first order approximation, compatibility equation becomes in the form of,

$$B^k \times \frac{d^4\tilde{\phi}_1^k}{d\theta^4} = -C^k\left(\frac{d^3\tilde{\phi}_1^k}{d\theta^3}, \frac{d^2\tilde{\phi}_1^k}{d\theta^2}, \frac{d\tilde{\phi}_1^k}{d\theta}, \tilde{\phi}_1^k, \frac{d^4\tilde{\phi}_0^k}{d\theta^4}, \frac{d^3\tilde{\phi}_0^k}{d\theta^3}, \frac{d^2\tilde{\phi}_0^k}{d\theta^2}, \frac{d\tilde{\phi}_0^k}{d\theta}, \tilde{\phi}_0^k, n_k, \lambda_0, \lambda_1, A_1\right). \tag{3.38}$$

where, B^k and C^k are derived using Mathematica software. Equation (3.45) is the fourth-order ordinary differential equation of $\tilde{\phi}_1^k$. The governing differential equation and boundary conditions define an eigenvalue problem. Within the first order approximation unknowns are $\tilde{\phi}_1^I, \tilde{\phi}_1^{I\prime}, \tilde{\phi}_1^{I\prime\prime}, \tilde{\phi}_1^{I\prime\prime\prime}, A_1$ and λ_1. B and C are derived using Mathematica software. A fourth-order Runge-Kutta method and the shooting method were used to solve the problem.

To apply traction boundary condition on the interface at $\theta = 0$ can be expressed as,

$$\sigma_{\theta\theta(0)}^{II} = A_0\, r^{\lambda_0-1}\left(\lambda_0+1\right)\lambda_0\tilde{\phi}_0^{II} = A_0\, r^{\lambda_0-1}\left(\lambda_0+1\right)\lambda_0\tilde{\phi}_0^{I}, \tag{3.39}$$

$$\sigma^{II}_{r\theta(0)} = -A_0\, r^{\lambda_0 - 1}\lambda_0 \frac{d\tilde{\phi}^{II}_0}{d\theta} = -A_0\, r^{\lambda_0 - 1}\lambda_0 \frac{d\tilde{\phi}^{I}_0}{d\theta}. \tag{3.40}$$

Finally, governing differential equation is solved to satisfy the traction boundary condition on the interface.

To apply forced displacement from the second material side to the 1st material side. The iterative boundary condition on the interface can be expressed as,

$$\left(u^{I}_{r(i)}\right)_{\theta=0} = \left(u^{II}_{r(i-1)}\right)_{\theta=0},\ \left(u^{I}_{\theta(i)}\right)_{\theta=0} = \left(u^{II}_{\theta(i-1)}\right)_{\theta=0}. \tag{3.41}$$

When $i = 1$, the boundary equation can be written as,

$$\left(u^{I}_{r(1)}\right)_{\theta=0} = \left(u^{II}_{r(0)}\right)_{\theta=0},\ \left(u^{I}_{\theta(1)}\right)_{\theta=0} = \left(u^{II}_{\theta(0)}\right)_{\theta=0}, \tag{3.42}$$

where $u^{II}_{r(0)}$ and $u^{II}_{\theta(0)}$ are the displacements of zero-th order approximation in the power-law hardening material I side, $u^{I}_{r(1)}$ and $u^{I}_{\theta(1)}$ are the incremental displacements of the first order approximation in the power-law hardening material II side. To derive the expressions for displacement, small deformation strain-displacement relations, presented in Chapter 2, Eq.(2.9)-Eq.(2.11), have been used. Strain components are derived using Eq.(3.1) and Eq.(3.2) for power-law hardening material I and power-law hardening material II, respectively. From Eq.(2.9) the strain component, ε_{rr}, is integrated by r to derive the expression of displacement u_r and the expression of u_θ is also derived from $\varepsilon_{r\theta}$ using Eq.(2.11). Within the first order approximation in the power-law hardening material I side we have,

$$u^{I}_{r} = u^{I}_{r(0)} + u^{I}_{r(1)},\, u^{I}_{\theta} = u^{I}_{\theta(0)} + u^{I}_{\theta(1)}. \tag{3.43}$$

From the zero-th order approximation we have,

$$\left(u^{I}_{r(0)}\right)_{\theta=0} = 0,\, \left(u^{I}_{\theta(0)}\right)_{\theta=0} = 0. \tag{3.44}$$

Displacement functions of power-law hardening material II side within zero-th order approximation which is used as the forced displacement on the power-law hardening material I side in the first order approximation are,

$$u_{r0}^{II} = A_0^{n_2}\, r^{n_2\lambda_0 - n_2 + 1}\tilde{u}_{r0}^{II},\ u_{\theta 0}^{II} = A_0^{n_2}\, r^{n_2\lambda_0 - n_2 + 1}\tilde{u}_{\theta 0}^{II}. \tag{3.45}$$

To satisfy the displacement continuity condition on the interface order by order the boundary equation can be expressed as,

$$A_0^{n-1}\, r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1}\left(\tilde{u}_{r(1)}^{I}\right)_{\theta=0} = A_0^{n_2}\, r^{n_2\lambda_0 - n_2 + 1}\tilde{u}_{r0}^{II}, \tag{3.46}$$

and

$$A_0^{n-1}\, r^{(n\lambda_0 - n - \lambda_0 + \lambda_1)+1}\left(\tilde{u}_{\theta(1)}^{I}\right)_{\theta=0} = A_0^{n_2}\, r^{n_2\lambda_0 - n_2 + 1}\tilde{u}_{\theta 0}^{II}, \tag{3.47}$$

where $\tilde{u}_{r(i)}^{k}$ and $\tilde{u}_{\theta(i)}^{k}$ are the angular function terms of displacement component, $i = 0$ and $1, k = I$ and II . $\tilde{u}_{r(1)}^{I}$ is a function of $\tilde{\phi}_1^{I}, \tilde{\phi}_1^{I\prime}, \tilde{\phi}_1^{I\prime\prime}, \tilde{\phi}_1^{I\prime\prime\prime}, A_1$ and λ_1. and λ_1 .

In the zero-th order approximation, from the solution of differential equation of the power-law hardening material I the singular exponent λ_0 is calculated for hardening exponent, n_1 . Displacement of material II at $\theta = 0$ within the zero-th order approximation is applied as the forced displacement to the power-law material I at $\theta = 0$ in the first order approximation.

The iterative boundary condition can be expressed on the interface as,

$$\left(u_{r(i)}^{I}\right)_{\theta=0} = \left(u_{r(i-1)}^{II}\right)_{\theta=0},\ \left(u_{\theta(i)}^{I}\right)_{\theta=0} = \left(u_{\theta(i-1)}^{II}\right)_{\theta=0}, \tag{3.48}$$

where $\tilde{u}_{r(i)}^{k}$ and $\tilde{u}_{\theta(i)}^{k}$ are the angular function terms of displacement component, $i = 0, 1, 2, 3 \cdots$.

In the first order approximation,

$$A_0^{n_1-1}\, r^{(n_1\lambda_0 - n_1 - \lambda_0 + \lambda_1)+1}\left(\tilde{u}_{r(1)}^{I}\right)_{\theta=0} = A_0^{n_2-1}\, r^{(n_2\lambda_0 - n_2)+1}\left(\tilde{u}_{r(0)}^{II}\right)_{\theta=0}, \tag{3.49}$$

and

$$A_0^{n_1-1}\, r^{(n_1\lambda_0 - n_1 - \lambda_0 + \lambda_1)+1}\left(\tilde{u}_{\theta(1)}^{I}\right)_{\theta=0} = A_0^{n_2-1}\, r^{(n_2\lambda_0 - n_2)+1}\left(\tilde{u}_{\theta(0)}^{II}\right)_{\theta=0}. \tag{3.50}$$

where $u^{II}_{r(0)}$ and $u^{II}_{\theta(0)}$ are the displacements of zero-th order approximation in the power-law material II, $u^{I}_{r(1)}$ and $u^{I}_{\theta(1)}$ are the incremental displacements of the first order approximation in the power-law material I. $\tilde{u}^{k}_{r(1)}$ is a function of $\tilde{\phi}^{k}_{1}, \tilde{\phi}^{k\prime}_{1}, \tilde{\phi}^{k\prime\prime}_{1}, \tilde{\phi}^{k\prime\prime\prime}_{1}, A_1$ and λ_1. To satisfy the boundary condition on the interface the power of r in Eqn. (3.49) and (3.50) should be equal. Equating the power of r we have,

$$\lambda_1 - 1 = (\lambda_0 - 1)\{1 - (n_1 - n_2)\}. \tag{3.51}$$

Equation (3.51) relates the first order singularity, $\lambda_1 - 1$, with the zero-th order singularity, $\lambda_0 - 1$ and power-law hardening exponents, n_1 and n_2. When $(n_1 - n_2) \geq 1$ then $\lambda_1 - 1 \geq 0$ which means there is no first order singular term. When $(n_1 - n_2) < 1$ then $\lambda_1 - 1 < 0$ which means two or more singular terms exist. $n_2 = n_1 - 1$ gives a critical material combination on the existence of higher order singularity. Similarly, to satisfy the boundary condition in the higher order approximation the singular exponent would have the form,

$$\lambda_i - 1 = n_2^i \times \lambda_0 - \left(\sum_{l=0}^{l=i-1} n_2^l\right) \times \{(n_1 - 1)\lambda_0 - (n_1 - n_2)\} - 1. \tag{3.52}$$

It seems the i-th order singularity depends on the hardening exponents n_1, n_2 and the zero-th order singularity λ_0. To calculate the singular exponent when the material II is an incompressible elastic material we can assume $n_2 = 1$. The obtained singular exponent is as,

$$\lambda_i - 1 = (\lambda_0 - 1) \times \{1 - i \times (n_1 - 1)\} \tag{3.53}$$

Equation (3.53) is exactly the same as the expression determined for the elastic-plastic/elastic materials joint case in chapter 2.

The Mechanical properties of power-law hardening materials are given in Table 3.1. E is Young's modulus, ν is the Poisson's ratio.

Table 3.1: Mechanical Properties of jointed materials for dissimilar power-law hardening materials joint

Properties	E[GPa]	ν	σ_y [MPa]	n	α	p_0 [MPa]
Material I	108	0.33	30	1-20	10.1	130
Material II	108	0.33	30	1-20	10.1	130

3.5 Results for the Power-Law / Power-Law Materials Joint

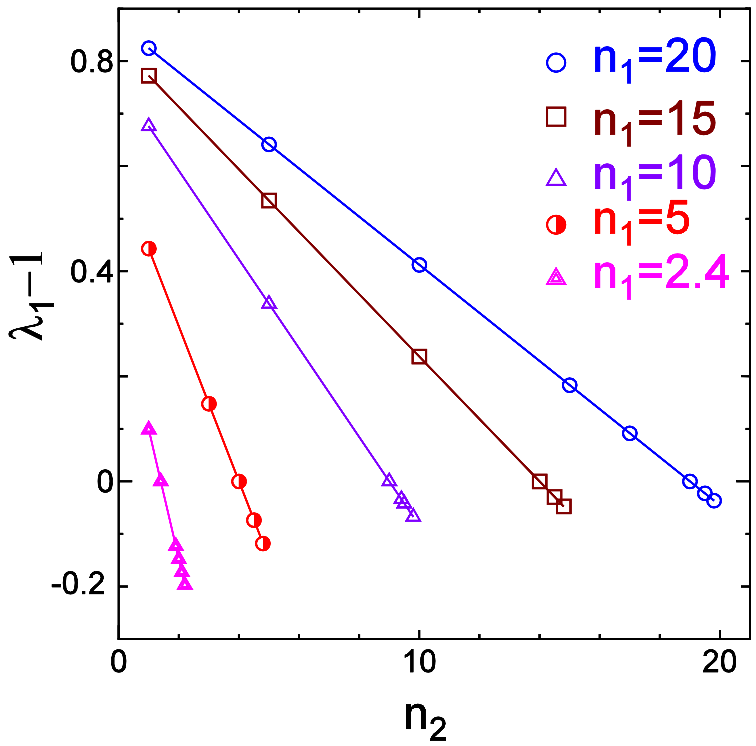


Fig. 3.8: Variation of stress singularity $\lambda_1 - 1$ with n_2 for different n_1

Figure 3.8 shows the variation of the first order singularity, $\lambda_1 - 1$ with n_2 when n_1 is constant upto the terminated point $n_2 = n_1$. The absolute value of the first order singularity, $|\lambda_1 - 1|$, when $\lambda_1 - 1 > 0$ tends to decrease to zero and when $\lambda_1 - 1 < 0$, $|\lambda_1 - 1|$ is increased with the increase of n_2. The gradient of line in Fig.3.8 is larger for smaller n_1 than that of larger n_1.

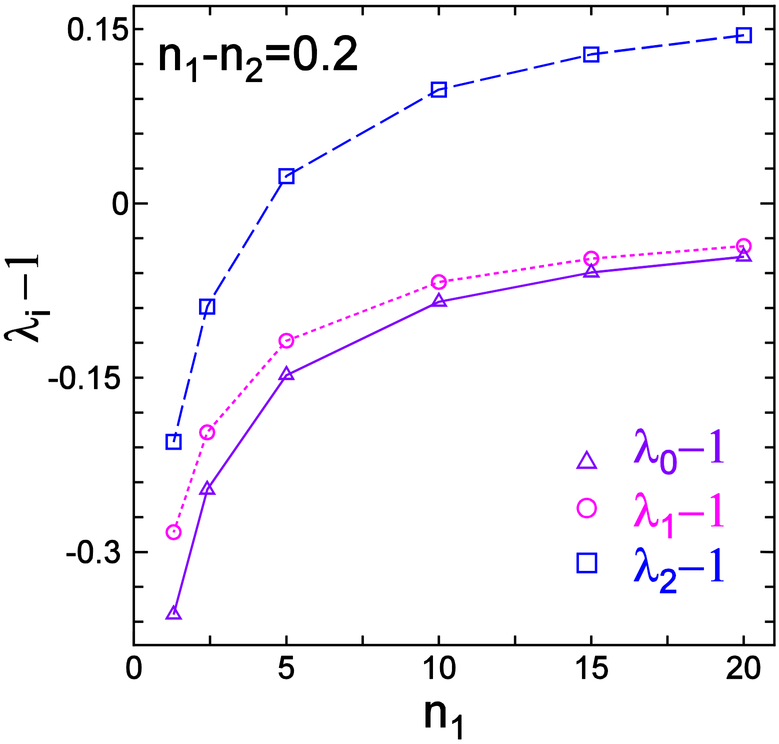


Fig. 3.9: Variation of stress singularity $\lambda_i - 1$ with n_1 for same difference of n_1 and n_2

Figure 3.9 shows the variation of i-th order singularity, $\lambda_i - 1$, with n_1 for the same difference between hardening exponents, n_1 and n_2 where $|n_1 - n_2| = 0.2$. The absolute value of i-th order singularity, $|\lambda_i - 1|$, is increased which means stronger singularity exists when the hardening exponent, n_1 is small. The qualitative tendency of Fig.3.9 is almost the same as the i-th order singularity presented in [66] for elastic-plastic/elastic materials joint.

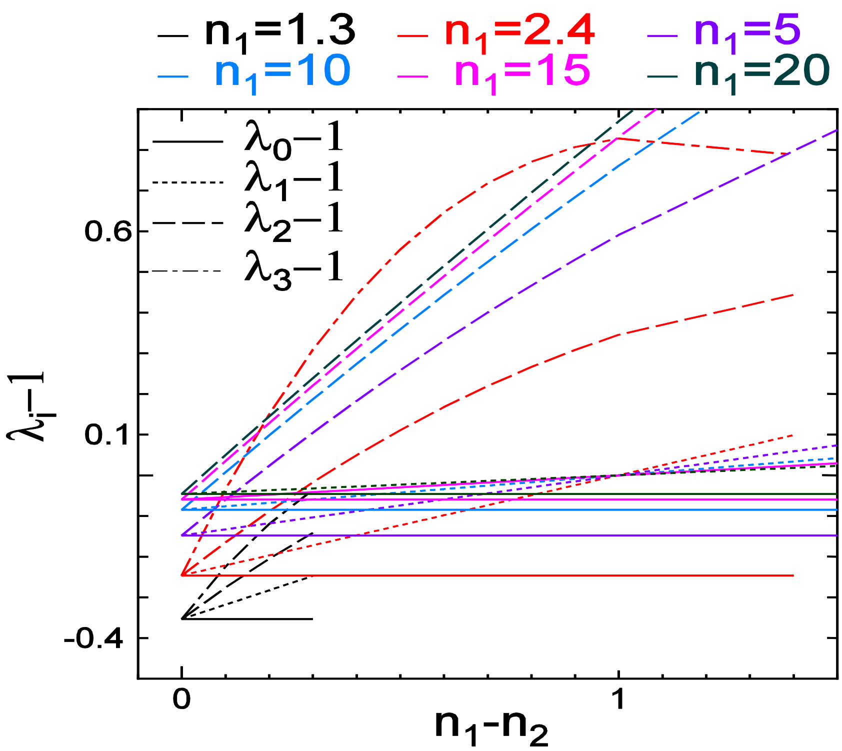


Fig. 3.10: Variation of stress singularity $\lambda_i - 1$ with difference of n_1 and n_2 for different n_1

The variation of i-th order singularity, $\lambda_i - 1$, with difference between hardening exponents, n_1 and n_2 for the different values of n_1 is shown in Fig. 3.10. The absolute value of i-th order singularity, $|\lambda_i - 1|$, is increased when the difference of hardening exponents, $n_1 - n_2$ is small which means stronger singularity exists. More singular terms exist for smaller difference of n_1 and n_2 which means higher hardening material (harder material with smaller n) includes more singular terms in compared with lower hardening material (softer material with larger n). The magnitude of first order singular exponent is increased with decreasing the difference of n_1 and n_2 and for $(n_1 - n_2) < 1$, $\lambda_1 = 1$ .One singularity exist for any combination of n_1 and n_2. Two or more singularity exists only when $(n_1 - n_2) < 1$.

3.6 Summary of Power-Law Hardening/Power-Law Hardening Materials Joint

Asymptotic solution for the interface-edge problem of power-law hardening materials having different power-law hardening exponent is presented in this chapter to determine the stress and displacement fields around the interface edge of jointed materials. An iteration method is proposed for the determination of singular fields around an interface edge of two dissimilar power-law hardening materials joint. Our analyses show the order of stress singularity has a dependency with the combination of hardening exponents. Multiple stress singular terms exist for $\left(n_1 - n_2\right) < 1$ in the higher order approximation.

If the hardening exponents of bonded power-law hardening materials are very close, then the material with smaller hardening exponent can be assumed as a very hard material within the plastic strain range under which the small deformation condition is satisfied, the assuming method of power-law hardening material I bonded with rigid substrate instead of power-law hardening material II will lose its physical meaning. In such a case, we cannot regard one of the composed materials as a rigid.

In this analysis in the zero-th order approximation we make is that, since the power-law hardening materials with different power-law hardening exponents behaves as if the more stress hardening material is attached to a rigid material, to calculate the zero-th order singularity in the $n_1 > n_2$ case. In this manner we will be able to get the singular exponent, λ_0, and the angular variation of the stresses $\tilde{\sigma}_{ij}$ in the more stress hardening material, but not the correct angular variations of stresses and displacements on the less stress hardening material (material II). The reason is that Material II (less stress hardening material) behaves as rigid only as far as its influence on Material I(more stress hardening material) is concerned. Material II is actually not rigid and will have displacements will be influenced by the more stress hardening material (material I). Still as far as the solution in the more stress hardening material is concerned our presented formulation of two different power-law hardening materials joint is sufficient.

For power-law hardening materials having different power-law hardening material joint case, a separable form solution can be obtained by observing that in the limit $r \to 0$, $\tilde{u}_i\left(\theta = 0\right) = 0$ (see section 3.4). This condition is restrictive, yet its validity has been suggested and supported by interpreting the finite element results of Shih and Asaro [67].

For the joint of two materials of same power-law hardening exponent, one can assume the zero-th order singular exponent as unity because of there is no singularity exists if the materials properties are same. In that case, applying this iteration method gives the same singular exponent in the higher order approximation as zero-th order. It points to the fact that when n_1 and n_2 are close in magnitude, the region where the $n_1 \neq n_2$ solution is acceptable is more restrictive. The requirement of power-law hardening exponents should be different and not very close is a restriction on application to real problem of our approximation method.

CHAPTER 4

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

4.1 General conclusions

In this work, we have analyzed the singular stress fields around an interface edge of dissimilar materials joint. We have formulated the problem for plane strain condition. We proposed an iteration method for the determination of singular fields around the interface edge of dissimilar materials joint where the interface free-edge geometry has been considered. We developed our formulations for general bimaterial interface-edge geometry. In our formulation butt joint model has been considered to result in specific bimaterial geometries of interest to engineering applications. For the two dissimilar materials joint, we provided solutions for interface-edge of an elastic and a power-law hardening materials joint and two dissimilar power-law hardening materials joint having different power-law hardening exponent. We have correlated our local dominant singularity results obtained from asymptotic analysis to global full-field finite element results to obtain the nondimensional generalized stress intensity factors for the plane strain free-edge interface of two dissimilar materials butt joint. The essential features of our results are given based on the joint materials.

4.1.1 Conclusions on Elastic-plastic/Elastic materials joint

Bonded dissimilar materials in which materials behaves as an elastic and a power-law hardening material were studied and the solution was presented to determine the stress and displacement fields around an interface edge of dissimilar materials joint. An iteration method is proposed for the determination of singular fields around an interface edge of an elastic and a power-law hardening materials joint. In the proposed iteration method, to overcome the problem we have considered at the interface boundary the additional stress fields in the elastic side to satisfy stress continuity, the additional displacement fields in the elastic-plastic side to satisfy displacement continuity, successively.

Both the balance of force and the continuity of displacements are satisfied on the interface iteratively. These continuity of the displacement field and the continuity of the stress field are important features of the stress strain fields near the interface edge of two dissimilar elastic/elastic-plastic materials joint.

Due to the increase of iteration, the discrepancy of the *r* dependence of the fields along the interface is decreased.

In the power-law hardening material, when n is decreased (for $n=1$ is an incompressible elastic material) the stronger singularity ($(\lambda_0 - 1)$ goes to more negative) exist in the power-law hardening material due to displacement boundary condition ($r \to 0$, $\tilde{u}_i(\theta = 0) = 0$) and weaker singularity ($(\lambda_0 - 1)$ goes to less negative) for larger n. In the elastic/power-law hardening materials joint case the singularity should be weaker with the decreasing value of n because of $n=1$ is an incompressible elastic material but in our approximation is not shows this situation. When regarded as deformation theory materials, the materials just considered each approximate perfect plastic behavior for small values of n for the power-law material. Yet, the calculated values of λ do not agree in this limit. The disparity can be explained as follows. For a power-law material model, elastic strains were neglected because, as stresses grow large for any value of $n > 1$, the ratio of elastic strain to plastic strain approaches zero. With an asymptotic expansion of the governing equations in powers of n, it is easily shown that $\lambda_0 \to 1$ as $n \to \infty$.

The power of *r* in the stress equation depends on the hardening exponent *n*. As n is increased the absolute value of the i-th order of singularity, $|\lambda_i - 1|$, tends to be decreased to zero when $\lambda_i - 1 < 0$. $(i+1)$ or more singular terms exist in the i-th order approximation for $n < (i+1)/i$.

For the determination of singular exponent, an explicit equation is presented where the i-th order singularity, λ_i depends on the hardening exponent n and the zero-th order singularity λ_0.

The angular function of separable form term in the displacement fields in the 0^{th} order approximation should be zero at $\theta = 0$ is a necessary condition for the continuity of displacement on the interface when $r \to 0$ around the interface free edge of elastic/elastic-plastic materials joint. The stress and displacement fields in the elastic-plastic material are controlled by the boundary condition which is the same as the one of an elastic-plastic material on the rigid substrate. The stress fields in the elastic material are also controlled by the index through the equilibrium of force on the interface.

The stress fields are compared with the fields of joint material results of FEM where the stress fields are slightly increased with the increase of iteration of asymptotic analysis which is the superposed stress of 0^{th} order and the higher order stresses. Due to the slight increase of stresses in the higher order approximation our theoretical stresses are slightly far from the FEM joint results where 0^{th} order fields are close to that FEM results. This might be due to the effect of regular stress by remote tensile loading.

4.1.2 Conclusions on Power-Law Hardening/ Power-Law Hardening materials joint

Asymptotic solution for the interface-edge problem of power-law hardening materials having different power-law hardening exponent is presented in this thesis to determine the stress and displacement fields around the interface edge of jointed materials. Our analyses show the order of stress singularity has a dependency with the combination of hardening exponents. Multiple stress singular terms exist for $\left(n_1 - n_2\right) < 1$ in the higher order approximation. For the determination of singular exponent, an explicit equation is also been presented where the i-th order singularity, λ_i depends on the combination of hardening exponents, n_1, n_2 and the zero-th order singularity λ_0.

Both the balance of force and the continuity of displacements are satisfied on the interface iteratively. This continuity of the displacement field and the continuity of the stress field are important features of the stress strain fields near the interface edge of two dissimilar elastic-plastic materials joint.

For the joint of two materials of same power-law hardening exponent, one can assume the zero-th order singular exponent as unity because of there is no singularity exists if the materials properties are same. In that case, applying this iteration method gives the same singular exponent in the higher order approximation as zero-th order. When material property of two materials tends closer each other, this approximation gives more singularity exists in compared with the large difference of material properties. The requirement of power law hardening exponents should be different and not very close is a restriction on application to real problem of our approximation method.

4.2 Future work

The thesis is a first effort to address the singular stress fields near the interface edge of dissimilar elastic/power-law hardening materials joint and power-law hardening/power-law hardening materials joint. The formulation presented, the iteration methods developed and the sample problem solved provide useful and insightful information for the local mechanics environment dominant at interface free-edges where damage and fracture frequently initiate. Further work needs to be performed to gain additional understanding on the mechanic relevant to interfacial fracture in this plastically deforming bimaterial joint. As a direct extension of our work on dissimilar elastic/power-law hardening materials joint and power-law hardening/power-law hardening materials joint, one now can solve for stress fields of more interfacial geometries by different wedge angle (other than the geometries we have solved like butt joint). Finally, a parallel study to verify experimentally the relevance of this singular stress fields to interfacial fracture should be conducted.

However, when using this expansion in the higher order approximation for comparison to finite element results of jointed material, the higher order terms (more than one term) appears to diverge from the finite element data when the higher order term is added. This suggests that for any values of n a separable solution for the higher order terms may not exist. The regular stress term should be considered for the determination of full field where an eigen expansion with the complex solution is necessary to accomplish the quantitative results of this analysis. A complete analysis and discussion should be reported in future work.

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APPENDICES

APPENDIX A

Calculation of Effective Stress term

Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{A.1}$$

Deviatoric stress components can be calculated in the summation form as,

$$\begin{aligned} s_{ij}s_{ij} &= s_{rr}s_{rr} + s_{r\theta}s_{r\theta} + s_{rz}s_{rz} + s_{\theta r}s_{\theta r} + s_{\theta\theta}s_{\theta\theta} + s_{\theta z}s_{\theta z} + s_{zr}s_{zr} + s_{z\theta}s_{z\theta} + s_{zz}s_{zz} \\ &= s_{rr}^2 + s_{\theta\theta}^2 + s_{zz}^2 + s_{r\theta}^2 + s_{rz}^2 + s_{\theta r}^2 + s_{\theta z}^2 + s_{zr}^2 + s_{z\theta}^2 \end{aligned} \tag{A.2}$$

where,

$$s_{rr} = \sigma_{rr} - \frac{1}{3}\sigma_{kk} = \sigma_{rr} - \frac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = \frac{1}{3}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{zz}) \tag{A.3}$$

$$s_{\theta\theta} = \sigma_{\theta\theta} - \frac{1}{3}\sigma_{kk} = \sigma_{\theta\theta} - \frac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = \frac{1}{3}(2\sigma_{\theta\theta} - \sigma_{rr} - \sigma_{zz}) \tag{A.4}$$

$$s_{zz} = \sigma_{zz} - \frac{1}{3}\sigma_{kk} = \sigma_{zz} - \frac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = \frac{1}{3}(2\sigma_{zz} - \sigma_{\theta\theta} - \sigma_{rr}) \tag{A.5}$$

$$s_{r\theta} = \sigma_{r\theta}, \quad s_{\theta z} = \sigma_{\theta z}, \quad s_{zr} = \sigma_{zr}, \quad s_{rz} = \sigma_{rz}, \quad s_{\theta r} = \sigma_{\theta r}, \quad s_{z\theta} = \sigma_{z\theta} \tag{A.6}$$

We have effective stress term,

$$\sigma_e^2 = \frac{3}{2}s_{ij}s_{ij} \tag{A.7}$$

Substituting Eqn. (A.2-A.6) into Eqn.(A.7), Eqn.(A.8) can be written as,

$$\begin{aligned} \sigma_e^2 &= s_{rr}^2 + s_{\theta\theta}^2 + s_{zz}^2 + s_{r\theta}^2 + s_{rz}^2 + s_{\theta r}^2 + s_{\theta z}^2 + s_{zr}^2 + s_{z\theta}^2 \\ &= \frac{3}{2}\left(\begin{array}{l} \left(\frac{1}{3}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{zz})\right)^2 + \left(\frac{1}{3}(2\sigma_{\theta\theta} - \sigma_{rr} - \sigma_{zz})\right)^2 + \left(\frac{1}{3}(2\sigma_{zz} - \sigma_{\theta\theta} - \sigma_{rr})\right)^2 \\ + \sigma_{r\theta}^2 + \sigma_{\theta z}^2 + \sigma_{zr}^2 + \sigma_{rz}^2 + \sigma_{\theta r}^2 + \sigma_{z\theta}^2 \end{array} \right) \\ &= \frac{1}{6}\left\{ 6\left(\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{zz}^2\right) - 6\left(\sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}\sigma_{zz} + \sigma_{zz}\sigma_{rr}\right) + \frac{3}{2}\left(\sigma_{r\theta}^2 + \sigma_{\theta z}^2 + \sigma_{zr}^2 + \sigma_{rz}^2 + \sigma_{\theta r}^2 + \sigma_{z\theta}^2\right) \right\} \\ &= \left(\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{zz}^2\right) - \left(\sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}\sigma_{zz} + \sigma_{zz}\sigma_{rr}\right) + 3\left(\sigma_{r\theta}^2 + \sigma_{rz}^2 + \sigma_{\theta z}^2\right) \end{aligned} \tag{A.8}$$

Applying plane strain condition,

$$\varepsilon_{zz} = 0,\ \sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right), \quad \sigma_{rz} = \sigma_{\theta z} = \sigma_{zr} = \sigma_{z\theta} = 0 \tag{A.9}$$

$$\begin{aligned}
\sigma_e^2 &= \left(\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{zz}^2\right) - \left(\sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}\sigma_{zz} + \sigma_{zz}\sigma_{rr}\right) + 3\left(\sigma_{r\theta}^2 + \sigma_{rz}^2 + \sigma_{\theta z}^2\right) \\
&= \sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \left\{\nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right\}^2 - \left(\sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}\left\{\nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right\} + \left\{\nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right\}\sigma_{rr}\right) + 3\sigma_{r\theta}^2 \\
&= \sigma_{rr}^2 + \sigma_{\theta\theta}^2 + 2\sigma_{rr}\sigma_{\theta\theta} + \nu^2\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - \left(3\sigma_{rr}\sigma_{\theta\theta} + \left\{\nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right\}\right) + 3\sigma_{r\theta}^2 \\
&= \left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 + \nu^2\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - \left(3\sigma_{rr}\sigma_{\theta\theta} + \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2\right) + 3\sigma_{r\theta}^2 \\
&= \left(\nu^2 - \nu + 1\right)\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2
\end{aligned} \tag{A.10}$$

$$\sigma_e^2 = \left(\nu^2 - \nu + 1\right)\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2 \tag{A.11}$$

For plastic material due to the no volume change, $\nu = \dfrac{1}{2}$

In plane strain, Eqn.(A.11) can be rewritten as,

$$\sigma_e^2 = \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1\right)\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2 \tag{A.12}$$

Finally,

$$\begin{aligned}
\sigma_e^2 &= \frac{3}{4}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2 \\
&= \frac{3}{4}\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2 + \frac{3}{4}\times 4\sigma_{rr}\sigma_{\theta\theta} - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2 \\
&= \frac{3}{4}\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2 + 3\sigma_{r\theta}^2
\end{aligned} \tag{A.12}$$

APPENDIX B

Displacement boundary conditions (0th order approximation)

At interface the displacements are continuous

$$u_{i0}^{I} = u_{i0}^{II} \tag{B.1}$$

or,

$$A_0^{n_1}\, r^{n_1\lambda_0 - n_1 + 1}\tilde{u}_{i0}^{I} = A_0^{n_2}\, r^{n_2\lambda_0 - n_2 + 1}\tilde{u}_{i0}^{II} \tag{B.2}$$

hence,

$$\frac{r^{n_1(\lambda_0-1)+1}}{r^{n_2(\lambda_0-1)+1}} = \frac{A_0^{n_2}\tilde{u}_{i0}^{II}}{A_0^{n_1}\tilde{u}_{i0}^{I}} \tag{B.3}$$

$$r^{n_1(\lambda_0-1)-n_2(\lambda_0-1)} = \frac{A_0^{n_2}\tilde{u}_{i0}^{II}}{A_0^{n_1}\tilde{u}_{i0}^{I}} \tag{B.4}$$

$$r^{(n_2-n_1)(\lambda_0-1)} = \frac{A_0^{n_1}\tilde{u}_{i0}^{I}}{A_0^{n_2}\tilde{u}_{i0}^{II}} \tag{B.5}$$

From Eqn. (B.5), when $n_1 \neq n_2$. As the material I is assumed to be less hardening material (lets assume $n_1 > n_2$), we get, $n_2 - n_1 < 0$. Existing a stress singularity gives $\lambda_0 - 1 < 0$, therefore the exponent of r in Eqn. (B.5), $(n_2 - n_1)(\lambda_0 - 1)$ is positive. So as $r \to 0$, the right hand side of Eqn. (B.5) approaches 0. This implies that in the near field region

$$\tilde{u}_{r0}^{I} = 0 \tag{B.6}$$

$$\tilde{u}_{\theta 0}^{I} = 0 \tag{B.7}$$

This boundary condition for $n_1 \neq n_2$ $(n_1 > n_2)$ has two immediate effects. Relative to the more stress hardening material (material I with n_1), we see that the boundary conditions are identical as if the material I is attached to a perfectly rigid material at the interface. This implies that the λ_0 does not depend on the properties of the less stress hardening material (material II with n_2). The solution can first be obtained for the more stress hardening material (material I with n_1) and once λ_0 and the stress distributions in the material I is known, the response of the less stress hardening material (material II with n_2) can be calculated.

If material II is elastic, n_2 is set to be 1. Where the stress hardening exponent of power-law hardening material is larger than $1\left(n_1 > n_2\right)$ in that case we see that the boundary conditions are identical as if the material I is attached to a perfectly rigid material at the interface.

APPENDIX C

Numerical Shooting Method

Material I (0^{th} order approximation)

Since our governing equations are a pair of fourth order differential equations, on free surface (at $\theta = \frac{\pi}{2}$) we need to prescribe the stress function and its first three derivatives. That is at $\theta = \frac{\pi}{2}$ we need to prescribe $\left\{ \tilde{\phi}_0^I, \left(\tilde{\phi}_0^I\right)', \left(\tilde{\phi}_0^I\right)'', \left(\tilde{\phi}_0^I\right)''' \right\}$. From the boundary conditions we know that at the free-surface $\tilde{\phi}_0^I$ and $\left(\tilde{\phi}_0^I\right)'$ are zero. Since this is an eigen-problem we can arbitrarily assign $\left(\tilde{\phi}_0^I\right)'' = 1.0$. We guess the initial values of the other derivatives of $\tilde{\phi}_0^I$ and λ_0 .With these guessed values we shoot from the stress free-edge to the interface. To satisfy the boundary conditions at the interface we update the initial guessed values and we calculate the error. We use an automatic step size method to accommodate the rapid localized changes in the stress function and its derivatives and minimize truncation error. The solution is finally obtained when the error is minimum.

For exact solution, the assumed value of $\left(\tilde{\phi}_0^I\right)''$ is updated to satisfy $\left(\tilde{\sigma}_{\theta\theta}\right)_{\theta=0} = 1$ and solution is done as above mentioned method.

Material II (0^{th} order approximation)

In the second material side, since our governing equations are a pair of fourth order differential equations, on free surface (at $\theta = -\frac{\pi}{2}$) we need to prescribe the stress function and its first three derivatives. That is at $\theta = -\frac{\pi}{2}$ we need to prescribe

$\left\{\tilde{\phi}_0^{II}, \left(\tilde{\phi}_0^{II}\right)', \left(\tilde{\phi}_0^{II}\right)'', \left(\tilde{\phi}_0^{II}\right)'''\right\}$. From the boundary conditions we know that at the free-surface $\tilde{\phi}_0^{II}$ and $\left(\tilde{\phi}_0^{II}\right)'$ are zero. Since singular exponent, λ_0 is known from the solution of material I side the angular variation of stresses can be computed. To compute the stresses we need to calculate unknown angular functions to satisfy traction on the interface. We guess the initial values of the other derivatives of $\tilde{\phi}_0^{II}$ .With these guessed values we shoot from the stress free-edge to the interface. To satisfy the boundary conditions at the interface we update the initial guessed values and we calculate the error. We use an automatic step size method to accommodate the rapid localized changes in the stress function and its derivatives and minimize truncation error. The solution is finally obtained for the minimum error.

Material I and Material II (i^th order approximation)

In the i-th order approximation the singular exponent, λ_i is known from the forced displacement boundary condition. Since our governing equations are a pair of fourth order differential equations, on each free surface (at $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$) we need to prescribe the stress function and its first three derivatives. That is at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ we need to prescribe $\left\{\tilde{\phi}_i^k, \left(\tilde{\phi}_i^k\right)', \left(\tilde{\phi}_i^k\right)'', \left(\tilde{\phi}_i^k\right)'''\right\}$ where $k = I$ for material I and $k = II$ for material II. From the boundary conditions we know that at the free-surface $\tilde{\phi}_i^k$ and $\left(\tilde{\phi}_i^k\right)'$ are zero. We guess the initial values of the other derivatives of $\tilde{\phi}_i^k$ or $\left\{\left(\tilde{\phi}_i^k\right)'', \left(\tilde{\phi}_i^k\right)'''\right\}$. With these guessed values we shoot from the stress free-edge to the interface. To satisfy the boundary conditions at the interface (forced displacement boundary condition for the solution of material I side and traction boundary condition for the solution of material II side) we update the initial guessed values and we calculate the error. The solution is finally obtained when for the minimum error.

Different terms in the compatibility equations of the first order approximation

$$\begin{aligned}
\frac{\frac{\partial^2 (r\,\epsilon_{\theta\theta 1}(r,\theta))}{\partial r^2}}{r} = {} & 2^{-n-1}\, 3^{\frac{n+1}{2}}\, r^{-n+(n-1)\lambda_0+\lambda_1-2}\, \alpha\, A_0^{n-1}\, ((n-1)(\lambda_0-1)+\lambda_1) \\
& (-n+(n-1)\lambda_0+\lambda_1)\Big((\lambda_0^2-1)^2\, \hat{\phi}_0(\theta)^2 - 2(\lambda_0^2-1)\, \hat{\phi}_0''(\theta)\, \hat{\phi}_0(\theta) + 4\lambda_0^2\, \hat{\phi}_0'(\theta)^2 + \hat{\phi}_0''(\theta)^2\Big)^{\frac{n-3}{2}} \\
& \Big(n(\lambda_0^2-1)^2 \big((\lambda_1^2-1) A_1 \hat{\phi}_1(\theta) - A_1 \hat{\phi}_1''(\theta)\big)\hat{\phi}_0(\theta)^2 + {} \\
& \quad 2(\lambda_0^2-1)\big(2(n-1)\lambda_0\lambda_1\, \hat{\phi}_0'(\theta) A_1 \hat{\phi}_1'(\theta) + n\,\hat{\phi}_0''(\theta)\big(A_1\hat{\phi}_1''(\theta) - (\lambda_1^2-1) A_1 \hat{\phi}_1(\theta)\big)\big)\hat{\phi}_0(\theta) - 4(n-1)\lambda_0\lambda_1\, \hat{\phi}_0'(\theta) \\
& \quad A_1 \hat{\phi}_1'(\theta)\, \hat{\phi}_0''(\theta) + 4\lambda_0^2\, \hat{\phi}_0'(\theta)^2 \big((\lambda_1^2-1) A_1\hat{\phi}_1(\theta) - A_1\hat{\phi}_1''(\theta)\big) + n\,\hat{\phi}_0''(\theta)^2\big((\lambda_1^2-1) A_1\hat{\phi}_1(\theta) - A_1\hat{\phi}_1''(\theta)\big)\Big)
\end{aligned}$$

$$\begin{aligned}
\frac{\frac{\partial^2 \epsilon_{rr1}(r,\theta)}{\partial \theta^2}}{r^2} = {} & 2^{-n-1}\, 3^{\frac{n+1}{2}}\, r^{-n+(n-1)\lambda_0+\lambda_1-2}\, \alpha\, A_0^{n-1}\, \Big((\lambda_0^2-1)^2\, \hat{\phi}_0(\theta)^2 - 2(\lambda_0^2-1)\, \hat{\phi}_0''(\theta)\, \hat{\phi}_0(\theta) + 4\lambda_0^2\, \hat{\phi}_0'(\theta)^2 + \hat{\phi}_0''(\theta)^2\Big)^{\frac{n-7}{2}} \\
& \Big(-48(n-1)\,\hat{\phi}_0'(\theta)^6 \big((\lambda_1^2-1) A_1\hat{\phi}_1(\theta) - A_1\hat{\phi}_1''(\theta)\big)\lambda_0^8 - {} \\
& \quad 16(n-1)\lambda_1\,\hat{\phi}_0'(\theta)^5 \big((5n-3)\,\hat{\phi}_1'(\theta)\,\hat{\phi}_0''(\theta) + 8\,\hat{\phi}_0'(\theta) A_1\hat{\phi}_1''(\theta)\big)\lambda_0^7 + {} \\
& \quad 8\,\hat{\phi}_0'(\theta)^4 \Big(4\big((-2\lambda_1^2-3n+5) A_1\hat{\phi}_1''(\theta) + 2 A_1 \hat{\phi}_1^{(4)}(\theta)\big)\hat{\phi}_0'(\theta)^2 + 4(n-1) \\
& \qquad \Big(\hat{\phi}_0''(\theta)\big(A_1\hat{\phi}_1^{(3)}(\theta) - (\lambda_1^2-1) A_1\hat{\phi}_1'(\theta)\big) - A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta) - (n-8)(n-1)\,\hat{\phi}_0''(\theta)^2 A_1\hat{\phi}_1''(\theta) + {} \\
& \quad (n-1)(\lambda_1^2-1) A_1\hat{\phi}_1(\theta)\Big((n-8)\,\hat{\phi}_0''(\theta)^2 + 4\,\hat{\phi}_0'(\theta)\big(3\,\hat{\phi}_0'(\theta) + \hat{\phi}_0^{(3)}(\theta)\big)\Big)\Big)\lambda_0^6 + 4(n-1)\lambda_1\,\hat{\phi}_0'(\theta)^3 \\
& \quad \Big(32 A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0'(\theta)^3 + 8\Big(4 A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta) + 2\,\hat{\phi}_0''(\theta) A_1\hat{\phi}_1^{(3)}(\theta) + A_1\hat{\phi}_1'(\theta)\big((n+3)\,\hat{\phi}_0''(\theta) + 2\,\hat{\phi}_0^{(4)}(\theta)\big)\Big) \\
& \qquad \hat{\phi}_0'(\theta)^2 + 8\,\hat{\phi}_0''(\theta)\Big((3n-7)\,\hat{\phi}_0''(\theta) A_1\hat{\phi}_1''(\theta) + 3(n-1) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta) + {} \\
& \quad (n(9n-41)+18) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0''(\theta)^3\Big)\lambda_0^5 + \hat{\phi}_0'(\theta)^2 \\
& \quad \Big(48(n-1) A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0'(\theta)^4 + 96(n-1)\Big(A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta) + \hat{\phi}_0''(\theta)\big(A_1\hat{\phi}_1^{(3)}(\theta) - (\lambda_1^2-1) A_1\hat{\phi}_1'(\theta)\big)\Big)\hat{\phi}_0'(\theta)^3 + {} \\
& \quad 16\Big(\Big((-(n+2)\lambda_1^2 + n(3n-11)+11) A_1\hat{\phi}_1''(\theta) + (n+2) A_1\hat{\phi}_1^{(4)}(\theta)\Big)\hat{\phi}_0''(\theta)^2 + {} \\
& \qquad 3(n-1)\Big(2\,\hat{\phi}_0^{(3)}(\theta)\big(A_1\hat{\phi}_1^{(3)}(\theta) - (\lambda_1^2-1) A_1\hat{\phi}_1'(\theta)\big) + A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(4)}(\theta)\Big)\hat{\phi}_0''(\theta) + {} \\
& \qquad 3(n-1) A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta)^2\Big)\hat{\phi}_0'(\theta)^2 + 8(n-1)\,\hat{\phi}_0''(\theta)^2 \\
& \qquad \Big(-(3n-7)(\lambda_1^2-1) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0''(\theta) + (3n-7) A_1\hat{\phi}_1^{(3)}(\theta)\,\hat{\phi}_0''(\theta) + 2(4n-13) A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta) + {} \\
& \quad (n-1)(n(9n-62)+116)\,\hat{\phi}_0''(\theta)^4 A_1\hat{\phi}_1''(\theta) - (n-1)(\lambda_1-1)(\lambda_1+1) A_1\hat{\phi}_1(\theta) \\
& \qquad \Big(48\,\hat{\phi}_0'(\theta)^4 + 96\,\hat{\phi}_0^{(3)}(\theta)\,\hat{\phi}_0'(\theta)^3 + 48\Big(\hat{\phi}_0^{(3)}(\theta)^2 + \hat{\phi}_0''(\theta)\big((n-3)\,\hat{\phi}_0''(\theta) + \hat{\phi}_0^{(4)}(\theta)\big)\Big)\hat{\phi}_0'(\theta)^2 + {} \\
& \qquad 16(4n-13)\,\hat{\phi}_0''(\theta)^2\,\hat{\phi}_0^{(3)}(\theta)\,\hat{\phi}_0'(\theta) + (n(9n-62)+116)\,\hat{\phi}_0''(\theta)^4\Big)\Big)\lambda_0^4 + 4(n-1)\lambda_1\,\hat{\phi}_0'(\theta)\,\hat{\phi}_0''(\theta) \\
& \quad \Big(12(n-3) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0'(\theta)^4 + 8\Big((n-1)\,\hat{\phi}_0''(\theta) A_1\hat{\phi}_1''(\theta) + 3(n-3) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)^3 + {} \\
& \quad 2\Big(4\,\hat{\phi}_0''(\theta)\Big((n-1) A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta) + \hat{\phi}_0''(\theta) A_1\hat{\phi}_1^{(3)}(\theta)\Big) + {} \\
& \qquad A_1\hat{\phi}_1'(\theta)\Big((n(3n-17)+36)\,\hat{\phi}_0''(\theta)^2 + 2(n-1)\,\hat{\phi}_0^{(4)}(\theta)\,\hat{\phi}_0''(\theta) + 6(n-3)\,\hat{\phi}_0^{(3)}(\theta)^2\Big)\Big)\hat{\phi}_0'(\theta)^2 + {} \\
& \quad 2\,\hat{\phi}_0''(\theta)^2\Big((3n-2)\,\hat{\phi}_0''(\theta) A_1\hat{\phi}_1''(\theta) + (n(3n-17)+36) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta) + {} \\
& \quad 3(3n-10) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0''(\theta)^4\Big)\lambda_0^3 + \hat{\phi}_0''(\theta)^2\Big(24(n-2)(n-1) A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0'(\theta)^4 - {} \\
& \quad 8(n-1)\Big((n+3)(\lambda_1^2-1) A_1\hat{\phi}_1'(\theta)\,\hat{\phi}_0''(\theta) - (n+3) A_1\hat{\phi}_1^{(3)}(\theta)\,\hat{\phi}_0''(\theta) - 6(n-2) A_1\hat{\phi}_1''(\theta)\,\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)^3 + {} \\
& \quad 2\Big(\Big((-2(2n+1)\lambda_1^2 + n(n(3n-19)+50) - 28) A_1\hat{\phi}_1''(\theta) + 2(2n+1) A_1\hat{\phi}_1^{(4)}(\theta)\Big)\hat{\phi}_0''(\theta)^2 + {}
\end{aligned}$$

$$2(n-1)(n+3)\Big(2\hat{\phi}_0^{(3)}(\theta)\Big(A_1\hat{\phi}_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\Big)+A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)\Big)\hat{\phi}_0''(\theta)+$$

$$12(n-2)(n-1)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2\Big)\hat{\phi}_0'(\theta)^2+2(n-1)\hat{\phi}_0''(\theta)^2\Big(-(3n-8)(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)+$$

$$(3n-8)A_1\hat{\phi}_1^{(3)}(\theta)\hat{\phi}_0''(\theta)+(n-2)(3n-10)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)+$$

$$(n-1)(3n-8)\hat{\phi}_0''(\theta)^4A_1\hat{\phi}_1''(\theta)-(n-1)(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big(24(n-2)\hat{\phi}_0'(\theta)^4+48(n-2)\hat{\phi}_0^{(3)}(\theta)\hat{\phi}_0'(\theta)^3+$$

$$2\Big((n(3n-16)+30)\hat{\phi}_0''(\theta)^2+2(n+3)\hat{\phi}_0^{(4)}(\theta)\hat{\phi}_0''(\theta)+12(n-2)\hat{\phi}_0^{(3)}(\theta)^2\Big)\hat{\phi}_0'(\theta)^2+$$

$$2(n-2)(3n-10)\hat{\phi}_0''(\theta)^2\hat{\phi}_0^{(3)}(\theta)\hat{\phi}_0'(\theta)+(3n-8)\hat{\phi}_0''(\theta)^4\Big)\Big)\lambda_0^2+4(n-1)\lambda_1\hat{\phi}_0''(\theta)^3$$

$$\Big((n-3)(n-2)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0'(\theta)^3+2(n-2)\Big(\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(n-3)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)^2+$$

$$\Big(\hat{\phi}_0''(\theta)\Big(2(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+\hat{\phi}_0''(\theta)A_1\hat{\phi}_1^{(3)}(\theta)\Big)+$$

$$(n-2)A_1\hat{\phi}_1'(\theta)\Big((n-3)\hat{\phi}_0^{(3)}(\theta)^2+\hat{\phi}_0''(\theta)\Big(3\hat{\phi}_0''(\theta)+\hat{\phi}_0^{(4)}(\theta)\Big)\Big)\Big)\hat{\phi}_0'(\theta)+$$

$$\hat{\phi}_0''(\theta)^2\Big(2\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(2n-3)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\Big)\lambda_0-$$

$$n(\lambda_0^2-1)^6\hat{\phi}_0(\theta)^6\Big((\lambda_1^2-1)A_1\hat{\phi}_1''(\theta)-A_1\hat{\phi}_1^{(4)}(\theta)\Big)-(\lambda_0^2-1)^5\hat{\phi}_0(\theta)^5$$

$$\Big(-\lambda_0^2\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)n^2+\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)n^2-2\lambda_1^2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)n^2+2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)n^2+$$

$$2\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta)n^2+A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)n^2+\lambda_0^2\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)n-6\lambda_1^2\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)n+$$

$$8\lambda_0\lambda_1\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)n+5\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)n+2\lambda_1^2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)n+4\lambda_0\lambda_1A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)n-$$

$$2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)n-2\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta)n+(n-1)(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big((\lambda_0^2-1)\hat{\phi}_0''(\theta)-\hat{\phi}_0^{(4)}(\theta)\Big)n-$$

$$A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)n+6\hat{\phi}_0''(\theta)A_1\hat{\phi}_1^{(4)}(\theta)n-8\lambda_0\lambda_1\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)-4\lambda_0\lambda_1A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)+$$

$$2(n-1)\hat{\phi}_0'(\theta)\Big(n(\lambda_0^2-1)(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)+(-n\lambda_0^2+2\lambda_1\lambda_0+n)A_1\hat{\phi}_1^{(3)}(\theta)\Big)\Big)+$$

$$n\hat{\phi}_0''(\theta)^4\Big(-\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)\lambda_1^2-2n\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)\lambda_1^2+2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)\lambda_1^2+$$

$$n^2A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2-3nA_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2+2A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2+$$

$$(n-2)(n-1)\hat{\phi}_0'(\theta)^2A_1\hat{\phi}_1''(\theta)+n\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)+2nA_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)-$$

$$2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)+2n\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta)-2\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta)+$$

$$2(n-1)\hat{\phi}_0'(\theta)\Big((n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+\hat{\phi}_0''(\theta)\Big(A_1\hat{\phi}_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\Big)\Big)+$$

$$n\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)-\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)-$$

$$(n-1)(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big((n-2)\Big(\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\Big)^2+\hat{\phi}_0''(\theta)\Big(\hat{\phi}_0''(\theta)+\hat{\phi}_0^{(4)}(\theta)\Big)\Big)+\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1^{(4)}(\theta)\Big)-$$

$$(\lambda_0^2-1)^4\hat{\phi}_0(\theta)^4\Big(-A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2n^3+3A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2n^2+\lambda_0^2\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)n^2-$$

$$5\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)n^2+10\lambda_1^2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)n^2-8\lambda_0\lambda_1A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)n^2-$$

$$10A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)n^2-10\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta)n^2-5\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)n^2-$$

$$2A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2n+7\lambda_0^2\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)n+15\lambda_1^2\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)n-$$

$$40\lambda_0\lambda_1\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)n-10\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)n-10\lambda_1^2A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)n+$$

$$4\lambda_0\lambda_1A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)n+10A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)n+$$

$$10\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta)n+5\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)n-15\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1^{(4)}(\theta)n-$$

$$8\lambda_0^2\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)+40\lambda_0\lambda_1\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)+4\lambda_0\lambda_1A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)+$$

$$(n-1)(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big((n-2)n(\lambda_0^2-1)^2\hat{\phi}_0'(\theta)^2-2(n-2)((n-2)\lambda_0^2-n)\hat{\phi}_0^{(3)}(\theta)\hat{\phi}_0'(\theta)-$$

$$((n+8)\lambda_0^2-5n)\hat{\phi}_0''(\theta)^2+(n-2)n\hat{\phi}_0^{(3)}(\theta)^2+5n\hat{\phi}_0''(\theta)\hat{\phi}_0^{(4)}(\theta)\Big)+$$

$$2(n-1)\hat{\phi}_0'(\theta)\Big((n-2)(\lambda_0((n-2)\lambda_0-4\lambda_1)-n)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+$$

$$(\lambda_0((n+8)\lambda_0-10\lambda_1)-5n)\hat{\phi}_0''(\theta)A_1\hat{\phi}_1^{(3)}(\theta)+A_1\hat{\phi}_1'(\theta)$$

$$
\begin{aligned}
&((n+8)\lambda_0^2+6(n-2)(\lambda_0^2-1)\lambda_1\lambda_0-((n+8)\lambda_0^2-5n)\lambda_1^2-5n)\hat\phi_0''(\theta)-2(n-2)\lambda_0\lambda_1\hat\phi_0^{(4)}(\theta))) - \\
&\hat\phi_0'(\theta)^2(4(2n+1)A_1\hat\phi_1^{(4)}(\theta)\lambda_0^2+((n-2)(n-1)n+\lambda_0(-4(2n+1)\lambda_0\lambda_1^2-8(n-2)(n-1) \\
&(\lambda_0^2-1)\lambda_1+\lambda_0((n-2)(n-1)n\lambda_0^2-2n((n-3)n-2)+4)))A_1\hat\phi_1''(\theta))) - \\
&(\lambda_0^2-1)^2\hat\phi_0(\theta)^2(24(n-2)(n-1)\hat\phi_0'(\theta)^4((\lambda_1^2-1)\hat\phi_1(\theta)-\hat\phi_1''(\theta))\lambda_0^6+4(n-1)\lambda_1\hat\phi_0'(\theta)^3 \\
&((n(5n-29)+66)\hat\phi_1'(\theta)\hat\phi_0''(\theta)+8(n-1)\hat\phi_0'(\theta)A_1\hat\phi_1''(\theta))\lambda_0^5 - \\
&2\hat\phi_0'(\theta)^2(8((n+2)A_1\hat\phi_1^{(4)}(\theta)-((n+2)\lambda_1^2+n(3n-10)+4)A_1\hat\phi_1''(\theta))\hat\phi_0'(\theta)^2+4(n-1) \\
&(-(n-13)(\lambda_1^2-1)A_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)+(n-13)A_1\hat\phi_1^{(3)}(\theta)\hat\phi_0''(\theta)-2(2n-5)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)) \\
&\hat\phi_0'(\theta)-(n-1)(n^2+2)\hat\phi_0''(\theta)^2A_1\hat\phi_1''(\theta)+ \\
&(n-1)(\lambda_1^2-1)A_1\hat\phi_1(\theta)(24(n-2)\hat\phi_0'(\theta)^2+8(2n-5)\hat\phi_0^{(3)}(\theta)\hat\phi_0'(\theta)+(n^2+2)\hat\phi_0''(\theta)^2))\lambda_0^4- \\
&8(n-1)\lambda_1\hat\phi_0'(\theta)(4(n-1)A_1\hat\phi_1''(\theta)\hat\phi_0'(\theta)^3+(4(n-1)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)+12\hat\phi_0''(\theta)A_1\hat\phi_1^{(3)}(\theta)+ \\
&A_1\hat\phi_1'(\theta)(((n-7)n+24)\hat\phi_0''(\theta)+2(n-1)\hat\phi_0^{(4)}(\theta)))\hat\phi_0'(\theta)^2+\hat\phi_0''(\theta) \\
&(6n\hat\phi_0''(\theta)A_1\hat\phi_1''(\theta)+((n-3)n+20)A_1\hat\phi_1'(\theta)\hat\phi_0^{(3)}(\theta))\hat\phi_0'(\theta)+9(n-4)A_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)^3)\lambda_0^3+ \\
&2(-12(n-2)(n-1)A_1\hat\phi_1''(\theta)\hat\phi_0'(\theta)^4+12(n-1)((n+3)(\lambda_1^2-1)A_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)- \\
&(n+3)A_1\hat\phi_1^{(3)}(\theta)\hat\phi_0''(\theta)-2(n-2)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta))\hat\phi_0'(\theta)^3- \\
&6(((-2(2n+1)\lambda_1^2+n((n-8)n+26)-13)A_1\hat\phi_1''(\theta)+2(2n+1)A_1\hat\phi_1^{(4)}(\theta))\hat\phi_0''(\theta)^2+ \\
&(n-1)(n+3)(2\hat\phi_0^{(3)}(\theta)(A_1\hat\phi_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat\phi_1'(\theta))+A_1\hat\phi_1''(\theta)\hat\phi_0^{(4)}(\theta))\hat\phi_0''(\theta)+2(n-2) \\
&(n-1)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)^2)\hat\phi_0'(\theta)^2-2(n-1)\hat\phi_0''(\theta)^2(-(7n-24)(\lambda_1^2-1)A_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)+ \\
&(7n-24)A_1\hat\phi_1^{(3)}(\theta)\hat\phi_0''(\theta)+3(n-4)(n-2)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta))\hat\phi_0'(\theta)-(n-1)(7n-24) \\
&\hat\phi_0''(\theta)^4A_1\hat\phi_1''(\theta)+(n-1)(\lambda_1^2-1)A_1\hat\phi_1(\theta)(12(n-2)\hat\phi_0'(\theta)^4+24(n-2)\hat\phi_0^{(3)}(\theta)\hat\phi_0'(\theta)^3+ \\
&6(((n-7)n+15)\hat\phi_0''(\theta)^2+(n+3)\hat\phi_0^{(4)}(\theta)\hat\phi_0''(\theta)+2(n-2)\hat\phi_0^{(3)}(\theta)^2)\hat\phi_0'(\theta)^2+ \\
&6(n-4)(n-2)\hat\phi_0''(\theta)^2\hat\phi_0^{(3)}(\theta)\hat\phi_0'(\theta)+(7n-24)\hat\phi_0''(\theta)^4))\lambda_0^2-4(n-1)\lambda_1\hat\phi_0''(\theta) \\
&(3(n-3)(n-2)A_1\hat\phi_1'(\theta)\hat\phi_0'(\theta)^3+6(n-2)(2\hat\phi_0''(\theta)A_1\hat\phi_1''(\theta)+(n-3)A_1\hat\phi_1'(\theta)\hat\phi_0^{(3)}(\theta))\hat\phi_0'(\theta)^2+ \\
&(2\hat\phi_0''(\theta)(6(n-2)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)+5\hat\phi_0''(\theta)A_1\hat\phi_1^{(3)}(\theta))+ \\
&3(n-2)A_1\hat\phi_1'(\theta)((n-3)\hat\phi_0^{(3)}(\theta)^2+2\hat\phi_0''(\theta)(3\hat\phi_0''(\theta)+\hat\phi_0^{(4)}(\theta))))\hat\phi_0'(\theta)+ \\
&2\hat\phi_0''(\theta)^2(10\hat\phi_0''(\theta)A_1\hat\phi_1''(\theta)+(6n-7)A_1\hat\phi_1'(\theta)\hat\phi_0^{(3)}(\theta)))\lambda_0+ \\
&n\hat\phi_0''(\theta)^2(15\hat\phi_0''(\theta)^2A_1\hat\phi_1''(\theta)\lambda_1^2+20nA_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)\hat\phi_0^{(3)}(\theta)\lambda_1^2-20A_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)\hat\phi_0^{(3)}(\theta)\lambda_1^2- \\
&6n^2A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)^2+18nA_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)^2-12A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)^2-6(n-2)(n-1) \\
&\hat\phi_0'(\theta)^2A_1\hat\phi_1''(\theta)-10n\hat\phi_0''(\theta)^2\hat\phi_1''(\theta)-5\hat\phi_0''(\theta)^2A_1\hat\phi_1''(\theta)-20nA_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)\hat\phi_0^{(3)}(\theta)+ \\
&20A_1\hat\phi_1'(\theta)\hat\phi_0''(\theta)\hat\phi_0^{(3)}(\theta)-20n\hat\phi_0''(\theta)\hat\phi_0^{(3)}(\theta)A_1\hat\phi_1^{(3)}(\theta)+20\hat\phi_0''(\theta)\hat\phi_0^{(3)}(\theta)A_1\hat\phi_1^{(3)}(\theta)- \\
&4(n-1)\hat\phi_0'(\theta)(3(n-2)A_1\hat\phi_1''(\theta)\hat\phi_0^{(3)}(\theta)+5\hat\phi_0''(\theta)(A_1\hat\phi_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat\phi_1'(\theta)))- \\
&10n\hat\phi_0''(\theta)A_1\hat\phi_1''(\theta)\hat\phi_0^{(4)}(\theta)+10\hat\phi_0''(\theta)A_1\hat\phi_1''(\theta)\hat\phi_0^{(4)}(\theta)+2(n-1)(\lambda_1^2-1)A_1\hat\phi_1(\theta) \\
&(3(n-2)(\hat\phi_0'(\theta)+\hat\phi_0^{(3)}(\theta))^2+5\hat\phi_0''(\theta)(\hat\phi_0''(\theta)+\hat\phi_0^{(4)}(\theta)))-15\hat\phi_0''(\theta)^2A_1\hat\phi_1^{(4)}(\theta)))- \\
&2(\lambda_0^2-1)^3\hat\phi_0(\theta)^3(2(n-3)(n-2)(n-1)\lambda_1\hat\phi_0'(\theta)^3A_1\hat\phi_1'(\theta)\lambda_0^5+2(n-1)\hat\phi_0'(\theta)^2 \\
&(((n-7)n+15)\hat\phi_0''(\theta)((\lambda_1^2-1)A_1\hat\phi_1(\theta)-A_1\hat\phi_1''(\theta))+2(n+3)\hat\phi_0'(\theta)((\lambda_1^2-1)A_1\hat\phi_1'(\theta)-A_1\hat\phi_1^{(3)}(\theta))) \\
&\lambda_0^4-4(n-1)\lambda_1\hat\phi_0'(\theta)(((n-3)(n-2)A_1\hat\phi_1'(\theta)-4A_1\hat\phi_1^{(3)}(\theta))\hat\phi_0'(\theta)^2+
\end{aligned}
$$

$$\Big(((n-7)\,n+8)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)-4\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)\Big)\,\hat{\phi}_0{}'(\theta)+6\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)^2\Big)\,\lambda_0^3+$$
$$\Big(4\,(n-1)\,(n+3)\,\Big(A_1\,\hat{\phi}_1{}^{(3)}(\theta)-(\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)\Big)\,\hat{\phi}_0{}'(\theta)^3+$$
$$2\,\Big((n-1)\,(n+3)\,\Big(2\,\hat{\phi}_0{}^{(3)}(\theta)\,\Big(A_1\,\hat{\phi}_1{}^{(3)}(\theta)-(\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)\Big)+A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(4)}(\theta)\Big)+$$
$$(n-1)\,(\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)\,\Big(5\,(n-3)\,\hat{\phi}_0{}''(\theta)-(n+3)\,\hat{\phi}_0{}^{(4)}(\theta)\Big)+$$
$$\hat{\phi}_0{}''(\theta)\,\Big(4\,(2\,n+1)\,A_1\,\hat{\phi}_1{}^{(4)}(\theta)-((8\,n+4)\,\lambda_1^2+n\,(5\,n-28)+11)\,A_1\,\hat{\phi}_1{}''(\theta)\Big)\Big)\,\hat{\phi}_0{}'(\theta)^2+$$
$$2\,(n-1)\,\hat{\phi}_0{}''(\theta)\,\Big(-(3\,n-16)\,(\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)+(3\,n-16)\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)\,\hat{\phi}_0{}''(\theta)+$$
$$2\,(n-2)\,\Big((\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)-A_1\,\hat{\phi}_1{}''(\theta)\Big)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)+$$
$$(n-1)\,(3\,n-16)\,\hat{\phi}_0{}''(\theta)^3\,\Big(A_1\,\hat{\phi}_1{}''(\theta)-(\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)\Big)\Big)\,\lambda_0^2+2\,(n-1)\,\lambda_1$$
$$\Big((n-3)\,(n-2)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}'(\theta)^3+2\,(n-2)\,\Big(4\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)+(n-3)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)^2+$$
$$\Big(2\,\hat{\phi}_0{}''(\theta)\,\Big(4\,(n-2)\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)+5\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)\Big)+$$
$$(n-2)\,A_1\,\hat{\phi}_1{}'(\theta)\,\Big((n-3)\,\hat{\phi}_0{}^{(3)}(\theta)^2+4\,\hat{\phi}_0{}''(\theta)\,\Big(3\,\hat{\phi}_0{}''(\theta)+\hat{\phi}_0{}^{(4)}(\theta)\Big)\Big)\Big)\,\hat{\phi}_0{}'(\theta)+$$
$$2\,\hat{\phi}_0{}''(\theta)^2\,\Big(10\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)+(4\,n-3)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\Big)\,\lambda_0+$$
$$n\,\hat{\phi}_0{}''(\theta)\,\Big(-10\,\hat{\phi}_0{}''(\theta)^2\,A_1\,\hat{\phi}_1{}''(\theta)\,\lambda_1^2-10\,n\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\,\lambda_1^2+10\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\,\lambda_1^2+$$
$$2\,n^2\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)^2-6\,n\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)^2+4\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)^2+2\,(n-2)\,(n-1)\,\hat{\phi}_0{}'(\theta)^2$$
$$A_1\,\hat{\phi}_1{}''(\theta)+5\,n\,\hat{\phi}_0{}''(\theta)^2\,A_1\hat{\phi}_1{}''(\theta)+5\,\hat{\phi}_0{}''(\theta)^2\,A_1\,\hat{\phi}_1{}''(\theta)+10\,n\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)-$$
$$10\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)+10\,n\,\hat{\phi}_0{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)-10\,\hat{\phi}_0{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)+$$
$$2\,(n-1)\,\hat{\phi}_0{}'(\theta)\,\Big(2\,(n-2)\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)+5\,\hat{\phi}_0{}''(\theta)\,\Big(A_1\,\hat{\phi}_1{}^{(3)}(\theta)-(\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)\Big)\Big)+$$
$$5\,n\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(4)}(\theta)-5\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(4)}(\theta)-(n-1)\,(\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)$$
$$\Big(2\,(n-2)\,\Big(\hat{\phi}_0{}'(\theta)+\hat{\phi}_0{}^{(3)}(\theta)\Big)^2+5\,\hat{\phi}_0{}''(\theta)\,\Big(\hat{\phi}_0{}''(\theta)+\hat{\phi}_0{}^{(4)}(\theta)\Big)\Big)+10\,\hat{\phi}_0{}''(\theta)^2\,A_1\,\hat{\phi}_1{}^{(4)}(\theta)\Big)\Big)-$$
$$(\lambda_0^2-1)\,\hat{\phi}_0(\theta)\,\Big(48\,(n-3)\,(n-1)\,\lambda_1\,\hat{\phi}_0{}'(\theta)^5\,A_1\,\hat{\phi}_1{}'(\theta)\,\lambda_0^7+48\,(n-1)\,\hat{\phi}_0{}'(\theta)^4$$
$$\Big((n-3)\,\hat{\phi}_0{}''(\theta)\,\Big((\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)-A_1\,\hat{\phi}_1{}''(\theta)\Big)+2\,\hat{\phi}_0{}'(\theta)\,\Big((\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)-A_1\,\hat{\phi}_1{}^{(3)}(\theta)\Big)\Big)\,\lambda_0^6+$$
$$4\,(n-1)\,\lambda_1\,\hat{\phi}_0{}'(\theta)^3\,\Big(-8\,\Big(3\,(n-3)\,A_1\,\hat{\phi}_1{}'(\theta)-2\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)^2+$$
$$8\,\Big(2\,(n-3)\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)-(n-5)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)+(n\,(3\,n-7)-54)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)^2\Big)$$
$$\lambda_0^5-4\,\hat{\phi}_0{}'(\theta)^2\,\Big(24\,(n-1)\,\Big((\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)-A_1\,\hat{\phi}_1{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)^3-$$
$$4\,\Big(3\,(n-1)\,(\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)\,\Big(\hat{\phi}_0{}''(\theta)-\hat{\phi}_0{}^{(4)}(\theta)\Big)+3\,(n-1)\,\Big(2\,\hat{\phi}_0{}^{(3)}(\theta)\,\Big(A_1\,\hat{\phi}_1{}^{(3)}(\theta)-(\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)\Big)+$$
$$A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(4)}(\theta)\Big)+\hat{\phi}_0{}''(\theta)\,\Big(2\,(n+2)\,A_1\,\hat{\phi}_1{}^{(4)}(\theta)-(2\,(n+2)\,\lambda_1^2+n-7)\,A_1\,\hat{\phi}_1{}''(\theta)\Big)\Big)\,\hat{\phi}_0{}'(\theta)^2+$$
$$2\,(n-1)\,\hat{\phi}_0{}''(\theta)\,\Big((5\,n-17)\,(\lambda_1^2-1)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)-(5\,n-17)\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)\,\hat{\phi}_0{}''(\theta)+$$
$$4\,(n-4)\,\Big((\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)-A_1\,\hat{\phi}_1{}''(\theta)\Big)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)+$$
$$(n-1)\,(n\,(3\,n-23)+43)\,\hat{\phi}_0{}''(\theta)^3\,\Big((\lambda_1^2-1)\,A_1\,\hat{\phi}_1(\theta)-A_1\,\hat{\phi}_1{}''(\theta)\Big)\Big)\,\lambda_0^4+8\,(n-1)\,\lambda_1\,\hat{\phi}_0{}'(\theta)$$
$$\Big(6\,(n-3)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}'(\theta)^4+4\,\Big(2\,(n-1)\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)+3\,(n-3)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)^3+$$
$$\Big(4\,\hat{\phi}_0{}''(\theta)\,\Big(2\,(n-1)\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)+3\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}^{(3)}(\theta)\Big)+$$
$$A_1\,\hat{\phi}_1{}'(\theta)\,\Big((n\,(5\,n-29)+66)\,\hat{\phi}_0{}''(\theta)^2+4\,(n-1)\,\hat{\phi}_0{}^{(4)}(\theta)\,\hat{\phi}_0{}''(\theta)+6\,(n-3)\,\hat{\phi}_0{}^{(3)}(\theta)^2\Big)\Big)\,\hat{\phi}_0{}'(\theta)^2+$$
$$\hat{\phi}_0{}''(\theta)^2\,\Big(4\,(2\,n-1)\,\hat{\phi}_0{}''(\theta)\,A_1\,\hat{\phi}_1{}''(\theta)+(n\,(5\,n-27)+64)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}^{(3)}(\theta)\Big)\,\hat{\phi}_0{}'(\theta)+$$
$$6\,(2\,n-7)\,A_1\,\hat{\phi}_1{}'(\theta)\,\hat{\phi}_0{}''(\theta)^4\Big)\,\lambda_0^3+\hat{\phi}_0{}''(\theta)\,\Big(48\,(n-2)\,(n-1)\,A_1\,\hat{\phi}_1{}''(\theta)\,\hat{\phi}_0{}'(\theta)^4-24\,(n-1)$$

$$\Big((n+3)(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta) - (n+3)A_1\hat{\phi}_1^{(3)}(\theta)\hat{\phi}_0''(\theta) - 4(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)^3 +$$

$$4\Big(\Big(\big(-4(2n+1)\lambda_1^2 + n(n(4n-27)+76)-41\big)A_1\hat{\phi}_1''(\theta) + 4(2n+1)A_1\hat{\phi}_1^{(4)}(\theta)\Big)\hat{\phi}_0''(\theta)^2 +$$

$$3(n-1)(n+3)\Big(2\hat{\phi}_0^{(3)}(\theta)\Big(A_1\hat{\phi}_1^{(3)}(\theta) - (\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\Big) + A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta)\Big)\hat{\phi}_0''(\theta) +$$

$$12(n-2)(n-1)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2\Big)\hat{\phi}_0'(\theta)^2 + 2(n-1)\hat{\phi}_0''(\theta)^2\Big(-(11n-32)(\lambda_1^2-1)A_1$$

$$\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta) + (11n-32)A_1\hat{\phi}_1^{(3)}(\theta)\hat{\phi}_0''(\theta) + 4(n-2)(2n-7)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta) +$$

$$(n-1)(11n-32)\hat{\phi}_0''(\theta)^4 A_1\hat{\phi}_1''(\theta) - (n-1)(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big(48(n-2)\hat{\phi}_0'(\theta)^4 + 96(n-2)\hat{\phi}_0^{(3)}(\theta)$$

$$\hat{\phi}_0'(\theta)^3 + 4\Big((n(4n-23)+45)\hat{\phi}_0''(\theta)^2 + 3(n+3)\hat{\phi}_0^{(4)}(\theta)\hat{\phi}_0''(\theta) + 12(n-2)\hat{\phi}_0^{(3)}(\theta)^2\Big)\hat{\phi}_0'(\theta)^2 +$$

$$8(n-2)(2n-7)\hat{\phi}_0''(\theta)^2\hat{\phi}_0^{(3)}(\theta)\hat{\phi}_0'(\theta) + (11n-32)\hat{\phi}_0''(\theta)^4\Big)\Big)\lambda_0^2 + 4(n-1)\lambda_1\hat{\phi}_0''(\theta)^2$$

$$\Big(3(n-3)(n-2)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0'(\theta)^3 + 2(n-2)\Big(4\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta) + 3(n-3)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)^2 +$$

$$\Big(\hat{\phi}_0''(\theta)\Big(8(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta) + 5\hat{\phi}_0''(\theta)A_1\hat{\phi}_1^{(3)}(\theta)\Big) +$$

$$(n-2)A_1\hat{\phi}_1'(\theta)\Big(3(n-3)\hat{\phi}_0^{(3)}(\theta)^2 + 4\hat{\phi}_0''(\theta)\Big(3\hat{\phi}_0''(\theta) + \hat{\phi}_0^{(4)}(\theta)\Big)\Big)\Big)\hat{\phi}_0'(\theta) +$$

$$\hat{\phi}_0''(\theta)^2\Big(10\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta) + (8n-11)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\Big)\lambda_0 +$$

$$n\hat{\phi}_0''(\theta)^3\Big(-6\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)\lambda_1^2 - 10nA_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)\lambda_1^2 + 10A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)\lambda_1^2 +$$

$$4n^2A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2 - 12nA_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2 + 8A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)^2 + 4(n-2)(n-1)$$

$$\hat{\phi}_0'(\theta)^2A_1\hat{\phi}_1''(\theta) + 5n\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta) + \hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta) + 10nA_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta) -$$

$$10A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta) + 10n\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta) - 10\hat{\phi}_0''(\theta)\hat{\phi}_0^{(3)}(\theta)A_1\hat{\phi}_1^{(3)}(\theta) +$$

$$2(n-1)\hat{\phi}_0'(\theta)\Big(4(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta) + 5\hat{\phi}_0''(\theta)\Big(A_1\hat{\phi}_1^{(3)}(\theta) - (\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\Big)\Big) +$$

$$5n\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta) - 5\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(4)}(\theta) - (n-1)(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)$$

$$\Big(4(n-2)\Big(\hat{\phi}_0'(\theta) + \hat{\phi}_0^{(3)}(\theta)\Big)^2 + 5\hat{\phi}_0''(\theta)\Big(\hat{\phi}_0''(\theta) + \hat{\phi}_0^{(4)}(\theta)\Big)\Big) + 6\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1^{(4)}(\theta)\Big)\Big)\Big)$$

$$-\frac{\frac{\partial \epsilon_{\mathrm{rr}1}(r,\theta)}{\partial r}}{r} = 2^{-n-1}\,3^{\frac{n+1}{2}}\,r^{-n+(n-1)\lambda_0+\lambda_1-2}\,\alpha\,A_0^{n-1}\,(-n+(n-1)\lambda_0+\lambda_1)$$

$$\Big((\lambda_0^2-1)^2\hat{\phi}_0(\theta)^2 - 2(\lambda_0^2-1)\hat{\phi}_0''(\theta)\hat{\phi}_0(\theta) + 4\lambda_0^2\hat{\phi}_0'(\theta)^2 + \hat{\phi}_0''(\theta)^2\Big)^{\frac{n-3}{2}}$$

$$\Big(n(\lambda_0^2-1)^2\Big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta) - A_1\hat{\phi}_1''(\theta)\Big)\hat{\phi}_0(\theta)^2 +$$

$$2(\lambda_0^2-1)\Big(2(n-1)\lambda_0\lambda_1\hat{\phi}_0'(\theta)A_1\hat{\phi}_1'(\theta) + n\hat{\phi}_0''(\theta)\Big(A_1\hat{\phi}_1''(\theta) - (\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big)\Big)\hat{\phi}_0(\theta) - 4(n-1)\lambda_0\lambda_1\hat{\phi}_0'(\theta)$$

$$A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta) + 4\lambda_0^2\hat{\phi}_0'(\theta)^2\Big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta) - A_1\hat{\phi}_1''(\theta)\Big) + n\hat{\phi}_0''(\theta)^2\Big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta) - A_1\hat{\phi}_1''(\theta)\Big)\Big)$$

$$-\frac{2\,\frac{\partial\left(\frac{\partial \epsilon_{\mathrm{r}\theta 1}(r,\theta)}{\partial \theta}\,r\right)}{\partial r}}{r^2} = 2^{1-n}\,3^{\frac{n+1}{2}}\,r^{-n+(n-1)\lambda_0+\lambda_1-2}\,\alpha\,A_0^{n-1}\,((n-1)(\lambda_0-1)+\lambda_1)$$

$$\Big((\lambda_0^2-1)^2\hat{\phi}_0(\theta)^2 - 2(\lambda_0^2-1)\hat{\phi}_0''(\theta)\hat{\phi}_0(\theta) + 4\lambda_0^2\hat{\phi}_0'(\theta)^2 + \hat{\phi}_0''(\theta)^2\Big)^{\frac{n-5}{2}}$$

$$\Big(\lambda_1\hat{\phi}_0(\theta)^3\Big((n-1)\hat{\phi}_0'(\theta)A_1\hat{\phi}_1'(\theta) + \hat{\phi}_0(\theta)A_1\hat{\phi}_1''(\theta)\Big)\lambda_0^8 +$$

$$(n-1)\hat{\phi}_0(\theta)^2\Big(-(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0'(\theta)^2 + (\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big((n-2)\hat{\phi}_0'(\theta)^2 + \hat{\phi}_0(\theta)\hat{\phi}_0''(\theta)\Big) +$$

$$\hat{\phi}_0(\theta)\Big(\hat{\phi}_0'(\theta)\Big((\lambda_1^2-1)A_1\hat{\phi}_1'(\theta) - A_1\hat{\phi}_1^{(3)}(\theta)\Big) - \hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\Big)\Big)\lambda_0^7 +$$

$$\lambda_1 \hat{\phi}_0(\theta)\Big(-4 A_1 \hat{\phi}_1''(\theta)\hat{\phi}_0(\theta)^3 + \Big(-4\hat{\phi}_0''(\theta) A_1 \hat{\phi}_1''(\theta) - (n-1) A_1 \hat{\phi}_1'(\theta)\Big(4\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\Big)\Big)\hat{\phi}_0(\theta)^2 +$$

$$\hat{\phi}_0'(\theta)\big(9(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)+4(n+1)\hat{\phi}_0'(\theta)A_1\hat{\phi}_1''(\theta)\big)\hat{\phi}_0(\theta)+4(n-2)(n-1)\hat{\phi}_0'(\theta)^3 A_1\hat{\phi}_1'(\theta)\Big)$$

$$\lambda_0^6 - (n-1)\Big(4\big(A_1\hat{\phi}_1''(\theta) - (\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\big)\hat{\phi}_0'(\theta)^4 +$$

$$2\hat{\phi}_0(\theta)\Big((n-2)\hat{\phi}_0''(\theta)\big(A_1\hat{\phi}_1''(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\big)+\hat{\phi}_0'(\theta)\big(2A_1\hat{\phi}_1^{(3)}(\theta)-2(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\big)\Big)\hat{\phi}_0'(\theta)^2 +$$

$$3\hat{\phi}_0(\theta)^3\Big(\hat{\phi}_0''(\theta)\big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta)-A_1\hat{\phi}_1''(\theta)\big)+\hat{\phi}_0'(\theta)\big((\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)-A_1\hat{\phi}_1^{(3)}(\theta)\big)\Big)+$$

$$\hat{\phi}_0(\theta)^2\Big(-3(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0'(\theta)^2-\Big((n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+3\hat{\phi}_0''(\theta)\big(A_1\hat{\phi}_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\big)\Big)$$

$$\hat{\phi}_0'(\theta)-3\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)+(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big(3\hat{\phi}_0''(\theta)^2+(n-2)\hat{\phi}_0'(\theta)\big(3\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\big)\Big)\Big)\Big)\lambda_0^5+$$

$$\lambda_1\Big(6A_1\hat{\phi}_1''(\theta)\hat{\phi}_0(\theta)^4+3\Big(4\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(n-1)A_1\hat{\phi}_1'(\theta)\big(2\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\big)\Big)\hat{\phi}_0(\theta)^3+$$

$$\Big(-8(n+1)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0'(\theta)^2-15(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0'(\theta)+3\hat{\phi}_0''(\theta)$$

$$\Big(2\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\Big)\hat{\phi}_0(\theta)^2-\hat{\phi}_0'(\theta)\Big(8(n-2)(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0'(\theta)^2+$$

$$4\Big(2(n+1)\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(n-2)(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)+21(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)^2\Big)$$

$$\hat{\phi}_0(\theta)+4\hat{\phi}_0'(\theta)^3\big((n-1)(3n+2)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)+4n\hat{\phi}_0'(\theta)A_1\hat{\phi}_1''(\theta)\big)\Big)\lambda_0^4+$$

$$(n-1)\Big(3\Big(\hat{\phi}_0''(\theta)\big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta)-A_1\hat{\phi}_1''(\theta)\big)+\hat{\phi}_0'(\theta)\big((\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)-A_1\hat{\phi}_1^{(3)}(\theta)\big)\Big)\hat{\phi}_0(\theta)^3+$$

$$\Big(-3(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0'(\theta)^2+\Big(6\hat{\phi}_0''(\theta)\big((\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)-A_1\hat{\phi}_1^{(3)}(\theta)\big)-2(n-2)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)$$

$$\hat{\phi}_0'(\theta)-6\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)+(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big(6\hat{\phi}_0''(\theta)^2+(n-2)\hat{\phi}_0'(\theta)\big(3\hat{\phi}_0'(\theta)+2\hat{\phi}_0^{(3)}(\theta)\big)\Big)\Big)$$

$$\hat{\phi}_0(\theta)^2+\Big(4\big(A_1\hat{\phi}_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\big)\hat{\phi}_0'(\theta)^3+\hat{\phi}_0''(\theta)$$

$$\Big(3(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)-3A_1\hat{\phi}_1^{(3)}(\theta)\hat{\phi}_0''(\theta)+2(n-2)\big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta)-A_1\hat{\phi}_1''(\theta)\big)\hat{\phi}_0^{(3)}(\theta)\Big)$$

$$\hat{\phi}_0'(\theta)+3\hat{\phi}_0''(\theta)^3\big((\lambda_1^2-1)A_1\hat{\phi}_1(\theta)-A_1\hat{\phi}_1''(\theta)\big)\Big)\hat{\phi}_0(\theta)+$$

$$\hat{\phi}_0'(\theta)^2\Big(4A_1\hat{\phi}_1''(\theta)\hat{\phi}_0'(\theta)^2+4\Big(A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+\hat{\phi}_0''(\theta)\big(A_1\hat{\phi}_1^{(3)}(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)\big)\Big)\hat{\phi}_0'(\theta)+$$

$$3(n-2)\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)-(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big(3(n-2)\hat{\phi}_0''(\theta)^2+4\hat{\phi}_0'(\theta)\big(\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\big)\Big)\Big)\Big)\lambda_0^3-$$

$$\lambda_1\big(\hat{\phi}_0(\theta)+\hat{\phi}_0''(\theta)\big)\Big(4A_1\hat{\phi}_1''(\theta)\hat{\phi}_0(\theta)^3+\Big(8\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(n-1)A_1\hat{\phi}_1'(\theta)\big(4\hat{\phi}_0'(\theta)+3\hat{\phi}_0^{(3)}(\theta)\big)\Big)\hat{\phi}_0(\theta)^2+$$

$$\Big(-4(n+1)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0'(\theta)^2-7(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\hat{\phi}_0'(\theta)+\hat{\phi}_0''(\theta)$$

$$\Big(4\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+3(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\Big)\hat{\phi}_0(\theta)-\hat{\phi}_0'(\theta)\Big(4(n-2)(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0'(\theta)^2+$$

$$4\Big((n+1)\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)+(n-2)(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0^{(3)}(\theta)\Big)\hat{\phi}_0'(\theta)+11(n-1)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)^2\Big)\Big)$$

$$\lambda_0^2-(n-1)\big(\hat{\phi}_0(\theta)+\hat{\phi}_0''(\theta)\big)^2\Big(\hat{\phi}_0'(\theta)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)\lambda_1^2-\hat{\phi}_0'(\theta)A_1\hat{\phi}_1'(\theta)\hat{\phi}_0''(\theta)-n\hat{\phi}_0'(\theta)^2A_1\hat{\phi}_1''(\theta)+$$

$$2\hat{\phi}_0'(\theta)^2A_1\hat{\phi}_1''(\theta)-\hat{\phi}_0''(\theta)^2A_1\hat{\phi}_1''(\theta)-n\hat{\phi}_0'(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+2\hat{\phi}_0'(\theta)A_1\hat{\phi}_1''(\theta)\hat{\phi}_0^{(3)}(\theta)+$$

$$(\lambda_1^2-1)A_1\hat{\phi}_1(\theta)\Big(\hat{\phi}_0''(\theta)\big(\hat{\phi}_0(\theta)+\hat{\phi}_0''(\theta)\big)+(n-2)\hat{\phi}_0'(\theta)\big(\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\big)\Big)+$$

$$\hat{\phi}_0(\theta)\Big(\hat{\phi}_0'(\theta)\big((\lambda_1^2-1)A_1\hat{\phi}_1'(\theta)-A_1\hat{\phi}_1^{(3)}(\theta)\big)-\hat{\phi}_0''(\theta)A_1\hat{\phi}_1''(\theta)\Big)-\hat{\phi}_0'(\theta)\hat{\phi}_0''(\theta)A_1\hat{\phi}_1^{(3)}(\theta)\Big)\lambda_0+$$

$$\lambda_1\big(\hat{\phi}_0(\theta)+\hat{\phi}_0''(\theta)\big)^3\Big(\big(\hat{\phi}_0(\theta)+\hat{\phi}_0''(\theta)\big)A_1\hat{\phi}_1''(\theta)+(n-1)A_1\hat{\phi}_1'(\theta)\big(\hat{\phi}_0'(\theta)+\hat{\phi}_0^{(3)}(\theta)\big)\Big)\Big)$$