Hyperelliptic and trigonal curves among cyclic coverings of the projective line

（射影直線の巡回被覆の中の超楕円曲線とトリゴナル曲線）

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Abstract

Let V be a cyclic covering of the complex projective line with n branch points. We give necessary and sufficient conditions for (1) whether V is hyperelliptic (i.e., has gonality 2) for arbitrary n; (2) whether V is trigonal (i.e., has gonality 3) for $n = 3$.

1 Introduction

Throughout this thesis, all discussions are over the complex number field $\mathbb{C}$. The complex projective line is denoted by $\mathbb{P}^1$. The term *curve* means a complete reduced irreducible (maybe singular) algebraic curve over $\mathbb{C}$. We denote the normalization of C by $\tilde{C}$. The *genus* of a curve C means its geometric genus, i.e., the genus of $\tilde{C}$. A *function* on a smooth curve means a meromorphic function, and a *function* on a singular curve C means a meromorphic function on $\tilde{C}$. In both case, the meromorphic function is assumed to be non-constant unless otherwise stated. An *automorphism* of a smooth curve C means a biholomorphic automorphism. An *automorphism* of a singular curve C means an automorphism of $\tilde{C}$. The trivial automorphism (i.e., the identity map) of a curve C is denoted by id_C. The birational equivalence of curves is denoted by $\sim$. A linear system with dimension d and degree r on a curve is denoted by g^r_d. For a positive integer d, let $\mathbb{Z}_d$ denote the cyclic group of order d. The cardinality of a finite set S is denoted by ${}^{\#}S$.

1.1 Cyclic coverings of the projective line

A curve V is a ***d*-cyclic covering of the projective line** $\mathbb{P}^1$ if the total automorphism group $\mathrm{Aut}(V)$ of V contains a subgroup $\mathbb{Z}_d$ such that the quotient $V/\mathbb{Z}_d$ is rational (i.e., of genus 0). In this thesis, we consider the condition for V having gonality 2 or 3, so V can always be assumed to be non-rational. (See next subsection for the definition of gonality.) Then the natural map $V \to V/\mathbb{Z}_d$ has least 3 branch points. Hence, the curve V has the following plane model:

$$\begin{aligned} V:\ y^d = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\dots(x-\lambda_n)^{a_n}, \quad a_i \not\equiv 0 \pmod{d}, \\ n \geq 3, \quad \gcd(d, a_1, \dots, a_n) = 1, \quad a_1 + \dots + a_n \equiv 0 \pmod{d}. \end{aligned} \tag{1.1}$$

where the branch points $\{\lambda_i \in \mathbb{P}^1\}_{i=1}^n$ are mutually distinct with each other.

If ρ is a projective transformation of $\mathbb{P}^1$, and $\lambda'_i = \rho(\lambda_i)$, then V is obviously birational to the curve $y^d = (x-\lambda'_1)^{a_1}(x-\lambda'_2)^{a_2}\cdots(x-\lambda'_n)^{a_n}$. If some λ_i, say λ_n, is just taken to the infinite point of $\mathbb{P}^1$, then the equation of V is written in the shorter form

$$y^d = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\cdots(x-\lambda_{n-1})^{a_{n-1}}.$$

In this case, the number a_n is recovered as

$$a_n = \min\{a \geq 1 : a_1 + \cdots + a_{n-1} + a \equiv 0 \pmod{d}\}.$$

Moreover, for any six points Q_1, Q_2, Q_3 and Q'_1, Q'_2, Q'_3 on $\mathbb{P}^1$ with $Q_i \neq Q_j$ and $Q'_i \neq Q'_j$, there exists a projective transformation ρ such that $\rho(Q_i) = Q'_i$. Hence, we can take three of the branch points to be arbitrary three distinct points on $\mathbb{P}^1$.

Definition 1. For the curve V given (1.1), we call $(d; a_1, \ldots, a_n)$ the ***type*** of V. And for fixed d and n, two types $(d; a_1, \ldots, a_n)$ and $(d; a'_1, \ldots, a'_n)$ are said to be in the same ***Nielsen class*** if there exists an integer k relatively prime to d and a permutation $\tau \in \mathcal{S}_n$ such that $(a'_1, \ldots, a'_n) \equiv^k_\tau (ka_{\tau(1)}, \ldots, ka_{\tau(n)}) \pmod{d}$. In this case, we write

$$(a'_1, \ldots, a'_n) \equiv^k_\tau (a_1, \ldots, a_n) \pmod{d}.$$

Note that for fixed k and τ, this is not an equivalence relation, but we have $(a_i) \equiv^1_{\mathrm{id}} (a_i) \pmod{d}$; and if $(a'_i) \equiv^k_\tau (a_i) \pmod{d}$ and $(a''_i) \equiv^{k'}_{\tau'} (a'_i) \pmod{d}$, then we have $(a_i) \equiv^l_{\tau^{-1}} (a'_i) \pmod{d}$ and $(a''_i) \equiv^{k'\cdot k}_{\tau'\circ\tau} (a_i) \pmod{d}$, where l is any integer such that $kl \equiv 1 \pmod{d}$.

Proposition 2 (cf. [8], §1.1)**.** *Let V be a curve given in (1.1), $a'_1, \ldots, a'_n$ be positive integers such that $(a'_1, \ldots, a'_n) \equiv^k_\tau (a_1, \ldots, a_n) \pmod{d}$ for some k and τ, and ρ be an arbitrary projective transformation of $\mathbb{P}^1$. Then V is birational to the curve $Y^d = (X-\lambda'_1)^{a'_1}\cdots(X-\lambda'_n)^{a'_n}$ where $\lambda'_i = \sigma(\lambda_{\tau(i)})$.*

Assume additionally that τ is the trivial permutation, ρ is the identity map, and none of the λ_i's is the infinite point, then the birational transformation $\gamma : V \to V'$ can be expressed as $(X, Y) \mapsto \left(x, y^k \prod_{i=1}^n (x-\lambda_i)^{b_i}\right)$ where $b_i = (a'_i - ka_i)/d$.

Example 3. Let $n = 3$ and $(d, a_1, a_2, a_3) = (2a+1, 1, 1, 2a-1)$ for a positive integer a. By taking $k = a$ in Definition 1, we obtain $\gcd(k, d) = 1$ and

$$\{1, 1, 2a-1\} \times a = \{a, a, 2a^2 - a\} \equiv \{a, a, 1\} \pmod{2a+1}.$$

Thus, the curves $y^{2a+1} = x(x-1)$ and $y^{2a+1} = x(x-1)^a$ are birational with each other.

1.2 Gonality of curves

Let C be a curve with genus g. The minimum degree of functions on C is called the ***gonality*** of C and denoted by $\mathrm{Gon}(C)$. We list some equivalent definition as follows:

$$\begin{aligned}\mathrm{Gon}(C) &:= \min\{\deg(f) : \text{function } \varphi \text{ on } C\}\\ &= \min\{\deg(\varphi) : \text{non-constant holomorphic map } \varphi : \tilde{C} \to \mathbb{P}^1\}\\ &= \min\{d : \exists \text{ (base-point-free) linear system } g^1_d \text{ on } \tilde{C}\}.\end{aligned}$$

Brill-Noether theory (cf. [1, IV, §1]) implies that $\mathrm{Gon}(C) \leq (g+3)/2$. We do not have a general method to decide the gonality of curves.

Let C_0 be a plane model of C. Let d be the degree of C_0. Let δ be delta invariant of C_0, and ν be the maximum multiplicity of the points on C_0. By considering the projection from the point with multiplicity ν to a general-position line on $\mathbb{P}^2$, we see that $\mathrm{Gon}(C) = \mathrm{Gon}(C_0) \leq d - \nu$. Some sufficient condition for $\mathrm{Gon}(C) = d - \nu$ have been proved:

- The case where $\nu = 1$, i.e., C_0 is a smooth plane curve, is well-known: We have $\mathrm{Gon}(C) = d-1$, and when $d \geq 3$, every g^1_{d-1} on C_0 is obtained by the projection from a point on C_0 to a line (cf. [7, Theorem 2.3.1]).
- The case where $\nu = 2$, i.e., the only singular points of C_0 are the double points, is partially solved by Coppens and Kato: If there is a positive integer k such that $d \geq 2k+2$ and $\delta < kd - (k+1)^2 + 3$, then $\mathrm{Gon}(C) = d-2$. Moreover, if $d \geq 2k+3$ and $\delta < kd - (k+1)^2 + 2$, then every g^1_{d-2} on C_0 is obtained by the projection from a double point of C_0 to a line (see [3, Theorems 2.1 and 2.3] and [4]).
- The case where $\nu = 3$ was partially solved by Sakai [10].
- The general case was studied by Coppens [2], Ohkouchi and Sakai [9] and so on.

However, on the other hand, even in case $\nu = 2$, there exist curves whose gonality is strictly less then $d - \nu = d - 2$ (see [3] and [4]).

Proposition 4. *Let C and C' be two smooth curves, and φ be a function on C. If there is a non-constant holomorphic map $C \to C'$, then there is a function φ' on C' such that* $\deg(\varphi') = \deg(\varphi)$. *In particular, we have* $\mathrm{Gon}(C') \leq \mathrm{Gon}(C)$.

1.3 Hyperelliptic and trigonal curves

It is obvious that a curve has gonality 1 if and only if $g = 0$, and if and only if the curve is ***rational***. A curve is called ***hyperelliptic*** (resp., ***trigonal***) if its gonality is 2 (resp., 3).

Remark 5. Some authors define hyperelliptic (resp., trigonal) curves with an additional condition $g \geq 2$ (resp., $g \geq 5$) in order to make the linear system g_2^1 (resp., g_3^1) be uniquely determined (see Propositions 6 and 7). We do not use such additional conditions here. In particular, the elliptic curves are regarded as hyperelliptic curves with genus 1.

Let C be a hyperelliptic curve of genus g, and let φ be a holomorphic map $C \to \mathbb{P}^1$ of degree 2. Since every ramification point of φ has ramification index 2, we infer from Riemann-Hurwitz formula that φ has exactly $2g+2$ ramification points. Hence, by a birational transformation, the curve C can be given by the equation

$$Y^2 = (X - \lambda_1) \cdots (X - \lambda_{2g+2}). \tag{1.2}$$

This equation is called the ***Weierstrass normal form*** of C. Moreover, note that the map $\varphi : C \to \mathbb{P}^1$ of degree 2 induces an automorphism σ of C with order 2 such that $C/\langle\sigma\rangle \cong \mathbb{P}^1$, which is called the ***hyperelliptic involution*** of C. The fixed points of σ is just the ramification points of φ. If we write C in the form (1.2), then the hyperelliptic involution is just the map $(x, y) \mapsto (x, -y)$.

Proposition 6 (cf. [5], III.7.3 Theorem, III.7.10 Proposition and III.7.11 Proposition)**.** *Let C be a hyperelliptic curve of genus g. If $g \geq 2$, then we have the following facts:*

(1) *the map $\varphi : C \to \mathbb{P}^1$ of degree 2 is uniquely determined up to the projective transformations of $\mathbb{P}^1$, and hence the hyperelliptic involution σ is also unique;*

(2) *for any function ψ on C, if $\deg(\psi) \leq g$, then $\deg(\psi)$ is even;*

(3) *for any automorphism τ of C, if $\tau \notin \langle\sigma\rangle$, then τ fixes at most 4 points;*

(4) *there are exactly $2g+2$ Weierstrass points on C, which are just the $2g+2$ ramification points of φ.*

Let C be a trigonal curve of genus g. By a birational transformation, we can write express C by an equation with form

$$Y^3 = b(X)Y + c(X), \qquad b(X), c(X) \in \mathbb{C}[X]. \tag{1.3}$$

Proposition 7 (cf. [7], Corollary 2.4.4 and [5], III.8.5 Corollary 4)**.** *Let C be a trigonal curve of genus g. We have the following facts:*

(1) *for any function ψ on C, if $\deg(\psi) \leq (g+1)/2$, then $\deg(\psi)$ is divisible by 3;*

(2) *if $g \geq 5$, then the map $\varphi : C \rightarrow \mathbb{P}^1$ of degree 3 is uniquely determined up to the projective transformations of $\mathbb{P}^1$.*

Proposition 8 (cf. [5], III.7.2 Proposition and III.8.7 Theorem)**.** *A curve of genus 1 or 2 is hyperelliptic; a curve of genus 3 or 4 is either hyperelliptic or trigonal.*

1.4 Main results and remarks

Theorem A. *When $n = 3$, the curve V given in (1.1) is hyperelliptic if and only if $d \geq 3$ and V has the same Nielsen class as one of the following curves:*

(H1) $y^d = x(x-1)$;

(H2) $y^d = x(x-1)^{\frac{d-2}{2}}$ *(d is even and $d \geq 6$).*

Only three of these curves are elliptic: $y^3 = x(x-1)$, $y^4 = x(x-1)$ and $y^6 = x(x-1)^2$.

Theorem B. *When $n = 4$, the curve V given in (1.1) is hyperelliptic if and only if $d \geq 3$ and V has the same Nielsen class as one of the following curves:*

(H3) $y^d = x(x-1)(x-\lambda)^{d-1}$,

(H4) $y^d = x(x-1)^{\frac{d}{2}}(x-\lambda)^{\frac{d}{2}}$ *(d is even and $d \geq 4$),*

(H5) $y^d = x^2(x-1)^{\frac{d}{2}}(x-\lambda)^{\frac{d}{2}}$ *($d \equiv 2 \pmod 4$ and $d \geq 6$),*

where $\lambda \in \mathbb{C}\backslash\{0,1\}$. Only one of these curves is elliptic: $y^2 = x(x-1)(x-\lambda)$.

Theorem C. *When $n \geq 5$, the curve V given in (1.1) is hyperelliptic if and only if d is even and V has the same Nielsen class as one of the following curves:*

(H6) $y^2 = x(x-1)(x-\lambda_3)\ldots(x-\lambda_{n-1})$ *(n is even),*

(H7) $y^d = x(x-1)^{\frac{d}{2}}(x-\lambda_3)^{\frac{d}{2}}\ldots(x-\lambda_{n-1})^{\frac{d}{2}}$ *($d \geq 4$),* (H8) $y^d = x^2(x-1)^{\frac{d}{2}}(x-\lambda_3)^{\frac{d}{2}} \dots (x-\lambda_{n-1})^{\frac{d}{2}}$ *(n is even, $d \equiv 2 \pmod 4$, and $d \geq 6$),*

where the parameters $\lambda_i \in \mathbb{C}\backslash\{0,1\}$ are mutually distinct with each other. None of these curves is elliptic.

Theorem D. *When $n = 3$, the curve V given in (1.1) is trigonal if and only if $d \geq 3$ and V has the same Nielsen class as one of the following curves:*

(T1) $y^d = x(x-1)^2$*;*

(T2) $y^d = x(x-1)^{(d-3)/3}$ *($d \equiv 0 \pmod 3$ and $d \geq 12$);*

(T3) $y^d = x(x-1)^{d/3}$ *($d \equiv 0 \pmod 3$, $d \not\equiv 0 \pmod 9$ and $d \geq 12$).*

Remark 9. By Proposition 2, in every theorems above, the statement "*V has the same Nielsen class as one of the following curves*" can be replaced by "*V is birational to a curve with one of the following forms*".

Remark 10. The hyperelliptic involution is $(x, y) \mapsto (-x+1, y)$ for the curve (H1), is $(x, y) \mapsto (\lambda(x-1)/(x-\lambda), (\lambda^2-\lambda)y/(x-\lambda)^2)$ for the curve (H3), and is $(x, y) \mapsto (x, -y)$ for the other curves in Theorems A, B and C.

Remark 11. The function of degree 3 is y on the curve (T1), and is $y^3/(x-1)$ on the curves (T2) and (T3).

Remark 12. For the same d, the curves (H1) and (H2) are not birational with each other, neither are the curves (H3), (H4) and (H5), and neither are the curves (T1), (T2) and (T3). For the same d and n, the curves (H7) and (H8) are not birational with each other either either.

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2 Preliminaries

2.1 A result on gonality

Let C be a plane curve of degree d. Let $m_1, m_2, \ldots, m_n \geq 2$ be the multiplicities of singularities of C (infinitely near points are included). For a positive integer k, we define (see [9] for the background)

$$r_C(k) := dk - \max\left\{\sum_{i=1}^{n} a_i m_i\right\},$$

where the maximum is taken for non-negative integers $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i^2 = k^2$. Using these notations, we have the following results:

Lemma 13 ([9], Lemma 6)**.** *We have the inequality*

$$r_C(k) \geq k\left(d - \sqrt{\sum_{i=1}^{n} m_i^2}\right).$$

Theorem E. *Let C be a reduced irreducible plane curve of degree d. Let $m_1, \ldots, m_n$ have the same meaning as above. Take a positive integer r such that*

$$d^2 - \sum_{i=1}^{n} m_i^2 > (r+1)^2. \tag{2.1}$$

If $r_C(k) \geq r+1$ for all integers k in the interval

$$1 \leq k \leq \frac{r}{d - \sqrt{\sum_{i=1}^{n} m_i^2}}, \tag{2.2}$$

then there exist no rational functions ψ on C with $\deg(\psi) \leq r$.

Proof. Let $\pi : X \to \mathbb{P}^2$ be the minimal resolution of the singularities of C. Here, we do not require that the total transform of C has normal crossings. Suppose that there is a rational function ψ_0 on C with $\deg(\psi_0) = r_0 \leq r$. According to Serrano's Extension Theorem (see [11, Theorem 3.1]), under the condition (2.1), there exists a meromorphic function Ψ_0 on $\mathbb{P}^2$ which induces ψ_0 such that $\Psi_0 \circ \pi : X \to \mathbb{P}^1$ becomes a holomorphic map. Let k_0 be the degree of Ψ_0. Furthermore, we can obtain non-negative integers $a_1, \ldots, a_n$ such that

$$r_0 = dk_0 - \sum_{i=1}^{n} a_i m_i, \qquad \sum_{i=1}^{n} a_i^2 = k_0^2$$

(see [9, §2] for details). Since $r_0 \geq r_C(k_0)$, by using Lemma 13, we have

$$k_0 \leq \frac{r_0}{d - \sqrt{\sum m_i^2}} \leq \frac{r}{d - \sqrt{\sum m_i^2}}.$$

Now by the hypothesis (2.2), we obtain $r_C(k_0) \geq r + 1 \geq r_0 + 1$, which is absurd. □

Example 14. We show that the gonalities of the following four curves are greater than 3:

$$\begin{aligned} y^{13} &= x(x-1)^3, & y^{14} &= x(x-1)^3, \\ y^{14} &= x(x-1)^5, & y^{31} &= x(x-1)^5. \end{aligned}$$

Firstly, let C be the forth curve $y^{31} = x(x-1)^5$. Using the notations above, we have $d = 31$, $g = 15$, $n = 11$ and $\{m_i\}_{i=1}^n = \{25, 6_4, 5_6\}$. Letting $r = 3$, we have

$$d^2 - \sum a_i^2 = 42 > (r+1)^2 = 16, \qquad r \Big/ \Big(d - \sqrt{\sum m_i^2} \Big) = 4.37 \cdots .$$

An easy computation shows that $(r_C(1), r_C(2), r_C(3), r_C(4)) = (6, 12, 14, 10)$. Therefore, by Theorem E, we obtain $\mathrm{Gon}(C_{31;5}) > 3$. By the same method, we can prove that the gonalities of the first curve $y^{13} = x(x-1)^3$ and the third curve $y^{14} = x(x-1)^5$ are greater than 3. By Proposition 2, the second curve $y^{14} = x(x-1)^3$ is birational to $y^{14} = x(x-1)^5$, so its gonality is also greater than 3.

2.2 Discussion on the curve (1.1)

Now go back to the curve (1.1). By Proposition 2, we can restrict the a_i's in (1.1) to $1 \leq a_i \leq d-1$. That is, we only need to consider the curves

$$\begin{aligned} V:\ & y^d = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\ldots(x-\lambda_n)^{a_n}, \quad 0 < a_i < d, \quad n \geq 3, \\ & \gcd(d, a_1, \ldots, a_n) = 1 \quad \text{and} \quad a_1 + \cdots + a_n \equiv 0 \pmod{d}. \end{aligned} \tag{2.3}$$

Let $c_i = \gcd(d, a_i)$ for $1 \leq i \leq n$. By the conditions $\gcd(d, a_1, \ldots, a_n) = 1$ and $a_1 + \cdots + a_n \equiv 0 \pmod{d}$, we must have

$$\gcd(c_1, \ldots, \widehat{c_i}, \ldots, c_n) = 1, \quad \text{for } 1 \leq i \leq n, \tag{2.4}$$

where $\widehat{c_i}$ means to remove the c_i. It is easy to see that the c_i's are invariant under the transformation given in Proposition 2. Let let P_i be the points

$(\lambda_i, 0)$ on V for $1 \le i \le n$, and $\tilde{V}$ denote the normalization of the curve V. There are c_i points on $\tilde{V}$ lying over P_i, which will be denoted by $P_{i,1}, \ldots, P_{i,c_i}$. Then the Riemann-Hurwitz formula implies that $\tilde{V}$ has genus

$$g = (n-2)d/2 + 1 - \big(c_1 + \cdots + c_n\big)/2. \tag{2.5}$$

Let W denote the set of Weierstrass points of $\tilde{V}$. Note that automorphisms of curves map Weierstrass points to Weierstrass points. Thus, for a fixed i, either all or none of the $P_{i,j}$'s are contained in W. Moreover, we have $^{\#}(W \backslash \{P_{i,j}\text{'s}\}) \equiv 0 \pmod{d}$. Hence, we obtain

$$^{\#}W = \delta_1 c_1 + \cdots + \delta_n c_n + \delta d,$$

where $\delta = {}^{\#}(W \backslash \{P_{i,j}\text{'s}\})/d \ge 0$ and $\delta_i = 1$ (resp., 0) if $\{P_{i,j}\}_{j=1}^{c_i} \subseteq W$ (resp., $\{P_{i,j}\}_{j=1}^{c_i} \cap W = \emptyset$). If V is a hyperelliptic curve, then we infer from Proposition 6 (4) that $^{\#}W = 2g + 2$. Therefore, with the help of (2.5), we conclude that

$$\sum_{i=1}^{n} (\delta_i + 1) c_i = (n - 2 - \delta) d + 4, \qquad \text{hence } \delta \in \{0, 1, \ldots, n-2\}. \tag{2.6}$$

2.3 Other useful facts

Let ξ be a primitive d-root of unity. The map $\sigma : (x, y) \mapsto (x, \xi y)$ is an automorphism the curve V given in (2.3). Thus, the cyclic group $\langle\sigma\rangle$ of order d is a subgroup of the total automorphism group of V. In what follows, when we say the subgroup $\mathbb{Z}_d$ of $\mathrm{Aut}(V)$, we refer to the subgroup $\langle\sigma\rangle$ unless otherwise stated.

Proposition 15 ([6], Theorem 1)**.** *Let V be the curve given in* (2.3)*, and let $n = 3$ and $g \ge 2$. We have* $\mathrm{Aut}(V) = \mathbb{Z}$ *except the cases where there exist positive integer k and permutation τ such that* $\gcd(k, d) = 1$ *and that* $(a_1, a_2, a_3) \equiv_\tau^k (1, b, c) \pmod{d}$ *for the integers $1 \le b, c \le d - 1$ listed in Table 1. In particular, if* $\min\{c_1, c_2, c_3\} \ge 2$*, then we have* $\mathrm{Aut}(V) = \mathbb{Z}_d$*.*

Another fact which will be repeatedly used in our proof is the Castelnuovo-Severi inequality:

Proposition 16 (cf. [1], VIII, C-1)**.** *Let C_1, C_2 and C be curves of genus g_1, g_2 and g, and let $\varphi_1 : C \to C_1$ and $\varphi_2 : C \to C_2$ be holomorphic map of degree d_1 and d_2. If the map $\varphi_1 \times \varphi_2 : C \to C_1 \times C_2$ is birational between C and its image, then we have*

$$g \le (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2.$$

2.4 Proof of Remark 12

As the first application of these preliminaries, we prove Remark 12. Firstly, suppose that d is even and $d \geq 6$. Then the genera of the curves (H1) and (H2) are equal to

$$g(\mathrm{H1}) = d/2 - 1 \qquad g(\mathrm{H2}) = \begin{cases} d/4 & \text{if } d \equiv 0 \pmod 4; \\ d/4 - 1/2 & \text{if } d \equiv 2 \pmod 4. \end{cases}$$

Thus, the curves (H1) and (H2) are not birational with each other. In the same way, for an even integer $d \geq 4$, the genera of (H3) and (H4) are $d - 1$ and $d/2$, respectively, while for an integer $d \geq 6$ with $d \equiv 2 \pmod 4$, the genus of (H5) is $d/2 - 1$. Thus, the curves (H3), (H4) and (H5) are not birational with each other. Similarly, for an even integer $n \geq 6$ and an integer $d \geq 6$ with $d \equiv 2 \pmod 4$, the genera of (H7) and (H8) are $(n-2)d/4$ and $(n-2)d/4 - 1$, respectively. Thus, the curves (H7) and (H8) are not birational with each other.

Secondly, suppose that d is divisible by 3 and $d \geq 12$. We write $e = d/3$,

	d	b	c	$G = \mathrm{Aut}(C)$	$\|G\|$
A.1	d is odd	1	$d-2$	$\mathbb{Z}_{2n}$	$2n$
A.2	d is even	1	$d-2$	(central $\mathbb{Z}_2$) : D_{2n}	$4n$
B.1	$d \not\equiv 0 \bmod 8,\ d \neq 12$	$b \neq 1$, $b^2 \equiv 1 \bmod d$	$d-b-1$	$\mathbb{Z}_d \ltimes \mathbb{Z}_2$	$2n$
B.2	$d \equiv 0 \bmod 8,\ d \neq 8$	$d/2-2$	$d/2+1$	$(\mathbb{Z}_d \ltimes \mathbb{Z}_2) : \mathbb{Z}_2$	$4n$
B.3	8	2	5	$(\mathbb{Z}_4 \oplus \mathbb{Z}_4) \ltimes S_3$	96
C.1	$d \geq 8,\ d$ is odd, $d \equiv 0 \bmod p$ for $p \equiv 1 \bmod 3$	$b \neq 1$, $\gcd(b,d) = 1$	b^2	$\mathbb{Z}_d \ltimes \mathbb{Z}_3$	$3n$
C.2	7	2	4	$\mathrm{PSL}(2,7)$	168
D.1	12	3	8	(central $\mathbb{Z}_4$) : A_4	48
E.1	8	3	4	$\mathrm{GL}(2,3)$	48
E.2	12	4	7	(central $\mathbb{Z}_3$) : A_4	72
E.3	24	4	19	(central $\mathbb{Z}_6$) : S_4	144

Table 1: Exception cases in Proposition 15

then the genera of the curves (T1), (T2) and (T3) are equal to

$$g(\mathrm{T1}) = \begin{cases} (3e-4)/2 & \text{if } e \text{ is even;} \\ (3e-3)/2 & \text{if } e \text{ is odd;} \end{cases} \quad g(\mathrm{T2}) = \begin{cases} e-1 & \text{if } e \equiv 1 \ (\mathrm{mod} 3); \\ e-2 & \text{if } e \not\equiv 1 \ (\mathrm{mod} 3); \end{cases}$$

$$g(\mathrm{T3}) = \begin{cases} e & \text{if } e \equiv 1 \ (\mathrm{mod} 3); \\ e-1 & \text{if } e \equiv 2 \ (\mathrm{mod} 3). \end{cases}$$

From the distinct genera, we can see that the curves (T1), (T2) and (T3) are not birational with each other possibly except $C_{12;3}$ and $C_{12;4}$. But by Proposition 15, we have $|\operatorname{Aut}(C_{12;2})| = 12$ and $|\operatorname{Aut}(C_{12;4})| = 72$, and hence $C_{12;2} \not\sim C_{12;4}$.

3 Proof of Theorems A, B and C

Lemma 17. *For the curve V given in* (2.3)*, if d is even and $d \geq 4$, then we define the curve:*

$$V_0 : \ y^{d/2} = (x - \lambda_1)^{a_1} \cdots (x - \lambda_n)^{a_n}. \tag{3.1}$$

Then the following conditions are equivalent:

(1) *the curve V_0 is rational;*

(2) $^{\#}\{a_i : a_i = d/2, \ 1 \leq i \leq n\} = n - 2$*;*

(3) *V is birational to the curve* (H2)*,* (H4)*,* (H5)*,* (H7) *or* (H8) *in Theorems A, B and C;*

(4) *one of the following conditions holds:*

(4-1) $\{c_i\}_{i=1}^n = \{1, 1, d/2, \ldots, d/2\}$,
(4-2) $\{c_i\}_{i=1}^n = \{1, 2, d/2, \ldots, d/2\}$,
(4-3) $\{c_i\}_{i=1}^n = \{2, 2, d/2, \ldots, d/2\}$.

Moreover, when (4-2) *holds, we have $d \equiv 2 \pmod 4$, and when* (4-3) *holds, we have $d \equiv 2 \pmod 4$ and n is even. The conditions* (4-1) *and* (4-2) *correspond to the curve* (H2)*,* (H4) *or* (H7)*, and the condition* (4-3) *corresponds to the curve* (H5) *or* (H8)*.*

Proof. It is clear that (1) ⇒ (2) and that (3) ⇒ (1). Note that since $1 \leq a_i \leq d - 1$, we have $c_i = d/2$ if and only if $a_i = d/2$.

(2) ⇒ (4). By Proposition 2, we may assume that $c_3 = \cdots = c_n = d/2$ and $c_1, c_2 \neq d/2$. We infer from (2.4) that $\gcd(c_1, d/2) = \gcd(c_2, d/2) = 1$, so $\{c_1, c_2\} = \{1, 1\}$, $\{1, 2\}$ or $\{2, 2\}$, and if $2 \in \{c_1, c_2\}$, then $d/2$ is odd, i.e., $d \equiv 2 \pmod 4$. Moreover, when $c_1 = c_2 = 2$, both a_1 and a_2 are even, but $d/2$ is odd. Thus, we have $a_1 + a_2 \neq d/2$. Since $a_1 + a_2 + (n-2)d/2 = a_1 + \cdots + a_n \equiv 0 \pmod d$, we see that n is even.

(4-1) or (4-2) ⇒ (3). Similarly, we may assume that $c_1 = 1$ and $a_2 = \cdots = a_{n-1} = d/2$. Hence, there is a positive integer k_0 such that $k_0 a_1 \equiv 1 \pmod d$ and hence $\gcd(k_0, d) = 1$. By taking $k = k_0$ in Definition 1, we can make $a_1 = 1$ and $a_2 = \cdots = a_{n-1} = d/2$. Now by a projective transformation of $\mathbb{P}^1$, we can make $(\lambda_1, \lambda_2, \lambda_3) = (0, \infty, 1)$ when $n = 3$ or $(\lambda_1, \lambda_2, \lambda_n) = (0, 1, \infty)$ when $n \geq 4$, which implies that V is birational to (H2), (H4) or (H7).

(4-3) $\Rightarrow$ (3). Similarly, we may assume that $c_1 = c_n = 2$, $a_2 = \cdots = a_{n-1} = d/2$ and $(\lambda_1, \lambda_2, \lambda_n) = (0, 1, \infty)$. Thus, there is a positive integer k_0 such that $k_0 a_1 \equiv 2 \pmod d$ and $\gcd(k_0, d/2) = 1$. Now taking $k = k_0$ (resp., $k = k_0 + d/2$) when k_0 is odd (resp., when k_0 is even) in Definition 1, we make $a_1 = 2$ and $a_2 = \cdots = a_{n-1} = d/2$, which implies that V is birational to (H5) or (H8). □

3.1 Proof of Theorems A

When $n = 3$, the conditions (2.4), (2.5) and (2.6) become

$$\gcd(c_i, c_j) = 1, \qquad \text{if } i \neq j; \tag{3.2}$$

$$g = (d + 2 - c_1 - c_2 - c_3)/2; \tag{3.3}$$

$$(\delta_1 + 1)c_1 + (\delta_2 + 1)c_2 + (\delta_3 + 1) = (1 - \delta)d + 4, \text{ hence } \delta = 0 \text{ or } 1. \tag{3.4}$$

The "if" part of Theorems A follows from Remark 10. Since all the c_i's are divisors of d, by (3.2), we can write $d = d' c_1 c_2 c_3$ for some integer $d' \geq 1$. Hence, from (3.3), we deduce that $g = 1$ if and only if one of the following three conditions holds: (1) $d = 3$ and $\{c_i\}_{i=1}^3 = \{1, 1, 1\}$; (2) $d = 4$ and $\{c_i\}_{i=1}^3 = \{1, 1, 2\}$; (3) $d = 6$ and $\{c_i\}_{i=1}^3 = \{1, 2, 3\}$. It is easy to check that all of the corresponding curves are birational to the elliptic curves listed in Theorem A, and hence satisfy Theorem A. Therefore, in what follows of this subsection, we may assume that $g \geq 2$.

Lemma 18. *When $n = 3$, if the curve V given in* (1.1) *is hyperelliptic, then* $\min\{c_i\}_{i=1}^3 = 1$.

Proof. Assume that $\min\{c_i\} \geq 2$. Since $g \geq 2$, we infer from the "in particular" part of Proposition 15 that the cyclic group $\mathbb{Z}_d$ is the full automorphism group of V. Then V is hyperelliptic if and only if d is even and the map $(x, y) \to (x, -y)$ is the hyperelliptic involution, that is, the condition (1) of Lemma 17 holds. By our assumption $\min\{c_i\} \geq 2$, this implies the condition (4-3) of Lemma 17. But the condition (4-3) holds only when n is even, a contradiction. □

Now by taking a suitable k in Definition 1 and by using Lemma 18, we may assume that $\min\{a_i\}_{i=1}^3 = 1$, and by a projective transformation of $\mathbb{P}^1$, we may assume that $a_1 = c_1 = 1$ and $a_2 \leq a_3$. That is, it suffices to consider the curves with the following form

$$C_{d;a}: \qquad y^d = x(x-1)^a, \qquad d \geq 2a + 1. \tag{3.5}$$

Lemma 19. *If $d = la$ for some integer l, then we have*

$$\mathrm{Gon}(C_{la;a}) = \begin{cases} a+1 & \text{if } l > a+1, \\ a & \text{if } l = a+1, \\ l & \text{if } l < a+1. \end{cases}$$

Proof. From the equation of the curve, we see that $x = (y^l/(x-1))^a$. Letting $X = y^l/(x-1)$ and $Y = y$, the curve $C_{la;a}$ is birational to the curve $Y^l = X(X^a - 1)$. Then the assertion follows from [10]. □

On the curve $C_{d;a}$, we have $a_2 = a$ and $a_3 = d-a-1$, hence $c_2 = \gcd(d, a)$ and $c_3 = \gcd(d, a+1)$. By Lemma 18, to prove the "only if" part of Theorems A, it suffices to show that

Theorem A (restatement). *If $C_{d;a}$ is hyperelliptic, then $a = 1$ or $d \in \{2a+1, 2a+2\}$.*

Thus, in what follows of this subsection, we assume that *there exists a hyperelliptic curve $C_{d;a}$ with $a \geq 2$ and $d \geq 2a+3$.*

Lemma 20. *If $c_2 \leq 2$ and $c_3 = 1$, then $\delta_3 = 1$ in* (3.4).

Proof. Review the definition of δ_i, we need to show that $P_{3,1}$ is a Weierstrass point of $C_{d;a}$. The divisor of the rational function y on $\tilde{C}_{d;a}$ can be written as

$$(y) = P_{1,1} + (a/c_2) \cdot (P_{2,1} + \cdots + P_{2,c_2}) - (a+1) \cdot P_{3,1}.$$

By (3.3), we have $g \geq (d-2)/2 \geq (2a+1)/2$, so $g \geq a+1$. It follows that the divisor $(y) + g \cdot P_{2,1}$ is effective, and hence $P_{3,1}$ is a Weierstrass point. □

Lemma 21. *We have $\delta = 1$ in* (3.4).

Proof. Review that $d = d'c_1c_2c_3 = d'c_2c_3$ for some integer $d' \geq 1$. Suppose $\delta = 0$. Then since $c_1 = 1$, we infer from (3.4) that

$$d'c_2c_3 = d = \delta_1 + (\delta_2+1)c_2 + (\delta_3+1)c_3 - 3 \leq 2c_2 + 2c_3 - 2. \tag{3.6}$$

Since $d \geq 2a+3 \geq 2c_2+3$ and $d \geq 2(a+1)+1 \geq 2c_3+1$, we have

$$c_2 \geq 2 \qquad \text{and} \qquad c_3 \geq 3. \tag{3.7}$$

Case 1: $d' = 1$, i.e., $d = c_2c_3$. Since $d \geq 2c_2+1$, we have $c_2 \geq 3$. Then (3.6) implies that $(c_2, c_3) = (3,3)$, $(4,3)$ or $(3,4)$. The first pair contradicts (3.2). The other two pairs give $d = 12$ and hence $a \leq 4$, so the only possible corresponding curve is $C_{12;3}$. However, by Lemma 19 we have $\mathrm{Gon}(C_{12;3}) = 3$.

Case 2: $d' \geq 2$**.** In this case, the inequality (3.6) implies that $d' = 2$ and either c_2 or c_3 is equal to 1, which contradicts (3.7). □

Proof of Theorem A (restatement) for the case $(c_2, c_3) = (1, 1)$**.** We suppose that $c_2 = c_3 = 1$. By (3.3), we have $d = 2g + 1$. We will show that $P_{1,1}$ is a Weierstrass point, then with the help of Lemmata 20 and 21, we have $\delta = \delta_1 = \delta_3 = 1$. However, substituting $\delta = \delta_1 = \delta_3 = 1$ and $c_1 = c_2 = c_3 = 1$ into (2.6), we obtain $\delta_2 = -1$, which contradicts the definition of the δ_i's.

Case 1: a **is even.** Write $a = 2b$ for some $1 \leq b \leq (g-1)/2$, and write φ_1 for the rational function $(x-1)^b/y^g$ on $\tilde{C}_{d;a}$. We have

$$(\varphi_1) = -g \cdot P_{1,1} + b \cdot P_{2,1} + (g - b) \cdot P_{3,1}.$$

We see that $(\varphi_1) + g \cdot P_{1,1} \geq 0$, so $P_{1,1}$ is a Weierstrass point.

Case 2: a **is odd.** Write $a = 2b + 1$ for some $1 \leq b \leq g/2 - 1$, write

$$m_0 = (2g + 1)b/(2b + 1), \qquad m = \big[m_0\big],$$

where [ ] denotes the greatest integer, and write φ_2 for the rational function $(x-1)^b/y^m$. We have

$$(\varphi_2) = -m \cdot P_{1,1} + (2b + 1)(m_0 - m) \cdot P_{2,1} + \big(m - (2b + 1)(m_0 - m)\big) \cdot P_{3,1}.$$

Now we claim that $(\varphi_2) + g \cdot P_{1,1} \geq 0$, i.e., $P_{1,1}$ is a Weierstrass point. Indeed, by the definitions of m_0 and m, it is easy to see that the coefficients of $P_{1,1}$ and $P_{2,1}$ in $(\varphi_2) + g \cdot P_{1,1}$ are non-negative. Moreover, we have $m_0 - m \leq 2b/(2b + 1)$, which implies that

$$(2b+1)(m_0-m) \leq 2b = \frac{(4b + 4)b - 2b}{2b + 1} < \frac{(2g + 1)b - 2b}{2b + 1} = m_0 - \frac{2b}{2b + 1} \leq m.$$

Hence, the coefficient of $P_{3,1}$ is also non-negative. □

Remark 22. Note that if d is a prime number, then we always have $c_2 = c_3 = 1$. Thus, we have proved Theorem A for the case where d is prime. In fact, this special case is implicit in [12, Theorems 1 and 3].

Proof of Theorem A (restatement). We prove it by induction on d. Note that the minimum d we need to consider is 7, a prime number, which has been solved. Now, we assume that $d \geq 8$, $(c_2, c_3) \neq (1, 1)$, and Theorem A holds for any integer d' with $3 \leq d' < d$.

Since $\delta = 1$ (Lemma 21) and $(c_2, c_3) \neq (1,1)$, from (3.4), we deduce that $\delta_1 = \delta_2 = \delta_3 = 0$ and $(c_2, c_3) = (1,2)$ or $(2,1)$. But the second pair of (c_2, c_3) contradicts Lemma 20. Hence, we see that d is even, a is odd, $\gcd(d,a) = 1$ and $\gcd(d, a+1) = 2$. Write $d = 2t$ for some integer t. Since $d \geq 2a+3$ and $a \geq 2$, we have

$$\begin{aligned} t \geq a+2 \geq 4, \qquad & \gcd(t,a) = 1, \\ \gcd(t, a+1) \leq 2, \qquad & t \neq 2a \text{ or } 2a+2. \end{aligned} \tag{3.8}$$

By these conditions, it is easy to see that the following curve is irreducible and not rational

$$C' : y^t = x(x-1)^a.$$

Since there is a surjective map $C_{d;a} \to C'$ of degree 2, we infer from Proposition 4 that $\mathrm{Gon}(C') \leq \mathrm{Gon}(C_{d;a}) = 2$, and hence C' is hyperelliptic.

In order to apply the inductive hypothesis on C', we need to write C' in the form (3.5). In case $t \geq 2a+1$, the condition in (3.5) holds, so C' can be written as $C_{t;a}$ directly. In case $a+2 \leq t \leq 2a-1$, the curve C' is birational to the curve $y^t = x(x-1)^{t-a-1}$ and $t \geq 2(t-a-1)+1$, so we have $C' \sim C_{t;t-a-1}$. That is, either $C_{t;a}$ (with $t \geq 2a+1$) or $C_{t;t-a-1}$ (with $a+2 \leq t \leq 2a-1$) is hyperelliptic. By the inductive hypothesis and (3.8), we obtain $t = a+2$ or $2a+1$, and hence $d = 2a+4$ or $4a+2$. Note that a must be odd and hence $a \geq 3$, because $\gcd(d,a) = 1$.

Since both $C_{d;a}$ and C' are hyperelliptic, the hyperelliptic involution of $C_{d;a}$ is not contained in the subgroup $\mathbb{Z}_d$ of $\mathrm{Aut}(C_{d;a})$. In particular, we obtain $\mathrm{Aut}(C_{d;a}) \neq \mathbb{Z}_d$, so $C_{d;a}$ is in the same Nielsen class with one of the curves listed in Table 1 of Proposition 15. In order to apply Proposition 15, we need to find all of integers b and c such that $1 \leq b, c \leq d-1$ and $(1,a,d) \approx (1,b,c) \pmod{d}$.

In case $d = 4a+2$, we have $\{b,c\} = \{a, 3a+1\}$ or $\{2a-1, 2a+2\}$. The conditions from A.1, A.2, B.2–E.3 are not satisfied obviously, so we only have to check the condition B.1. Suppose $a^2 \equiv 1 \pmod{4a+2}$. Then there is a positive integer k such that $a^2 - 1 = k(4a+2)$, which yields $a = 2k \pm \sqrt{4k^2+2k+1}$. But this is not an integer. Using the same method, we can check that $(3a+1)_1^2 \not\equiv 1$, $(2a-1)_1^2 \not\equiv 1$ and $(2a+2)_1^2 \not\equiv 1 \pmod{4a+2}$.

In case $d = 2a+4$, we have $\{b,c\} = \{a, a+3\}$ or $\{(a+1)/2, (3a+5)/2\}$. The conditions from A.1, A.2, B.2–E.1 and E.3 are not satisfied obviously, and by the same way as above, we can check that the condition B.1 is not satisfied either. Finally, by Lemma 19 we have $\mathrm{Gon}(C_{12;4}) = 3$, which means the condition E.2 is not satisfied. $\square$

3.2 Proof of Theorems B and Theorems C

Lemma 23. *Assume that the curve V in* (2.3) *is hyperelliptic. If $n \geq 4$, $d \geq 3$ and V does not satisfy the equivalent conditions in Lemma 17, then we have $n = 4$ and $c_1 = c_2 = c_3 = c_4 = 1$.*

Proof. **Step 1: to show $\max\{c_i\}_{i=1}^n \leq 4$.** Suppose that $c_i \geq 5$ for some i. Consider the automorphism $\sigma_1 : (x, y) \mapsto (x, \xi^{c_i} y)$, where ξ is the d-th root of unity. It is easy to see that σ_1 fixes the points $P_{i,1}, \ldots, P_{i,c_i}$. Since $c_i \geq 5$ and $\sigma_1 \neq \mathrm{id}_V$, by Proposition 6 (3), we see that σ_1 is just the hyperelliptic involution. Then $c_i = d/2$ and V satisfies the condition (1) in Lemma 17, which contradicts the assumption.

Step 2: to show $\max\{c_i\}_{i=1}^n \leq 3$. Suppose that $c_i = 4$ for some i. Consider the automorphism $\sigma_2 : (x, y) \mapsto (x, \xi^4 y)$. We have $\sigma_2 \neq \mathrm{id}_V$, and if some $c_j = 1$, 2 or 4, then the points $P_{j,1}, \ldots, P_{j,c_j}$ are fixed by σ_2. Hence, if there is a j such that $j \neq i$ and $c_j = 1$, 2 or 4, then σ_2 fixed at least 5, 6 or 8 points, respectively. Similarly as Step 1, this is impossible. Hence, we obtain $c_j = 3$ unless $j = i$, but this contradicts (2.4).

Step 3. Let $n_k = {}^{\#}\{c_i : c_i = k\}$ for $k = 1, 2, 3$. Consider the automorphisms $\sigma_3 : (x, y) \mapsto (x, \xi^2 y)$ and $\sigma_4 : (x, y) \mapsto (x, \xi^3 y)$. When $d \geq 4$, we have $\sigma_3 \neq \mathrm{id}_V \neq \sigma_4$. Using the same method as above, we deduce that $n_1 + 2n_2 \leq 4$ and $n_1 + 3n_3 \leq 4$. Since $n_1 + n_2 + n_3 = n \geq 4$, we conclude that $n_1 = 4$ and $n_2 = n_3 = 0$, which is just what we need. When $d = 3$, we have $c_i = 1$ for all i, and by considering $\sigma_3 \neq \mathrm{id}_V$, we obtain the same result. $\square$

Proof of Theorem C. The "if" part is clear from Remark 10. When $d = 2$, the only curve given in (2.3) is the curve (H6), which it is hyperelliptic obviously and satisfies Theorem C. Hence, it remains to show the "only if" part for $d \geq 3$. By (2.4), we have $c_1 + \cdots + c_n \leq (n-2)d/2 + 2d/3$. Then by (2.5), we obtain $g \geq (3n-10)d/12+1$. Hence, when $n \geq 5$, we have $g \geq 2$. If V is hyperelliptic, then by Lemma 23, the equivalent conditions in Lemma 17 are satisfied. By Proposition 2, we may assume that $a_2 = \cdots = a_{n-1} = d/2$ and $c_1 = 1$. Then by taking a suitable k in Definition 1 (note that since d is even, k must be odd), the assertion follows. $\square$

It remains to consider Theorem B. The "if" part follows from Remark 10. When $n = 4$, by (2.4) and (2.5), we have $g = 1$ iff $2d = c_1 + c_2 + c_3 + c_4$ iff $d = 2$. (The second "iff" comes from (2.4) and the condition $c_i \leq d/2$.) In this case, the only possible curve is $y^2 = x(x-1)(x-\lambda)$, which satisfies our assertion. Thus, in what follows of this subsection, we may assume that $g \geq 2$ and $d \geq 3$.

Now by taking a suitable k in Definition 1 and by using Lemma 23, we may assume that $\min\{a_i\} = 1$, and by a projective transformation of $\mathbb{P}^1$, we may assume that $a_1 = 1$ and $a_2 \leq a_3 \leq a_4$. That is, it suffices to consider the curves with the following form

$$\begin{aligned} C_{d;a,b}(\lambda): \quad & y^d = x(x-1)^a(x-\lambda)^b, \quad 1+a+b \not\equiv 0 \pmod{d}, \\ & d \geq 3, \qquad 1 \leq a \leq b \leq c \leq d-1, \\ & \gcd(d,a) = \gcd(d,b) = \gcd(d,c) = 1, \end{aligned} \tag{3.9}$$

where c is defined to be $\min\{c' \geq 1 : 1+a+b+c' \equiv 0 \pmod{d}\}$. Then by Lemmata 17 and 23, to prove the "only if" part of Theorem B, it suffices to show that

Theorem B (restatement). *If $C_{d;a,b}(\lambda)$ is hyperelliptic, then $a = 1$ and $b = d-1$.*

Lemma 24. *For an even integer $e \geq 4$, we have* $\mathrm{Gon}(C_{2e;e-1,e+1}(\lambda)) = 4$.

Proof. We write C for the curve $C_{2e;e-1,e+1}(\lambda)$, and use the notations in [10]. The data of multiplicities of singularities on C is $[e+1, (e-1)_3, 2_{e-2}]$. We have $\eta = 4(e-1)/(e+1)$. Let $d_C = 2e+1$, $\nu = e+1$, $q = e-4$. Here we write d_C for d in [10], i.e., the degree of C. Then we have

$$h(\eta,\nu,q) = \frac{8e^2-12e+5}{2(e+1)(2e-3)}.$$

Since $\sigma = 3(e-1)/(e+1)$, we find that $d_C/\nu = \sigma - q/\nu$, so $k_0 = 3$. We have

$$f_3(\eta,\nu,q) = \frac{3\sqrt{e^2-1}}{e+1} - \frac{e-1}{e+1}.$$

It follows that

$$\begin{aligned} d_C/\nu - h(\eta,\nu,q) &= \frac{4e-11}{2(e+1)(2e-3)} > 0, \\ d_C/\nu - f_3(\eta,\nu,q) &= \frac{3(e-\sqrt{e^2-1})}{e+1} > 0. \end{aligned}$$

From [10, Theorem 3], we deduce that $\mathrm{Gon}(C) \geq d_C - \nu - q = 4$. Since $\deg(y^2/(x-1)(x-\lambda)) = 4$, we conclude that the gonality of C is just 4. $\square$

Proof of Theorem B (restatement). By (2.5), the curve $C_{d;a,b}(\lambda)$ has genus $d-1$. We divide the proof into three cases as follows:

Case 1: $1+a+b<d$**.** It follows that $c=d-a-b-1$. The rational functions y, $y/(x-1)$ and $y/(x-\lambda)$ have degrees $1+a+b$, $d-a$ and $d-b$, respectively. It is easy to see that none of them is greater than g, and at least one of them is odd, which contradicts Proposition 6 (2).

Case 2: $1+a+b>d$ **and** d **is odd.** It follows that $c=2d-a-b-1$ and the degree of $C_{d;a,b}(\lambda)$ is $a+b+1$. Then the conditions in (3.9) becomes

$$\begin{gathered} 1\le a\le b\le d-1, \qquad a+b\ge d, \qquad a+2b\le 2d-1, \\ a\le d-2, \qquad b\ge (d+1)/2. \end{gathered} \tag{3.10}$$

The rational functions induced by the projections from $(1,0)$ and $(\lambda,0)$ to a line of $\mathbb{P}^2$ have degrees $b+1$ and $a+1$, respectively. By Proposition 6 (2), a must be odd, and either b is odd or $b=d-1$. By (3.10), we have $b=d-1$ if and only if $a=1$, which is just what we need. We consider the remaining cases where a and b are odd and $3\le a\le b\le d-2$.

In case $a\le (d-1)/2$, we have $\deg(y^2/(x-1)(x-\lambda))=2b-d+2$, which is odd and not greater than g. In case $a\ge (d+1)/2$, we have $\deg(y^{d-2}/(x-1)^{a-1}(x-\lambda)^{b-1})=d-2$, which is also odd and not greater than g. Both cases contradict Proposition 6 (2).

Case 3: $1+a+b>d$ **and** d **is even.** Like Case 2, we have $c=2d-a-b-1$, and the conditions in (3.9) becomes

$$\begin{gathered} 1\le a\le b\le d-1, \qquad a+b\ge d, \qquad a+2b\le 2d-1, \\ a\le d-2, \qquad b\ge d/2. \end{gathered} \tag{3.11}$$

We prove Case 3 by induction on d. When $d=4$, by (3.9) and (3.11), we have $(a,b)=(1,3)$, which satisfies our assertion. Now assume that $d\ge 6$ and Theorem B holds for every integer less than d (because of the results of Cases 1 and 2, the inductive hypothesis is based on the whole theorem instead of only on Case 3).

Write $d=2e$ for some $e\ge 3$. There is a natural double covering from $C_{2e;a,b}(\lambda)$ to the following curve:

$$C':\ y^e=x(x-1)^a(x-\lambda)^b.$$

Since d is relatively prime to a, b and c, so is e. Thus, the curve C' is an e-cyclic covering of $\mathbb{P}^1$ with 4 branch points, and its genus is equal to $e-1\ge 2$. Furthermore, since there is a covering map $C_{2e;a,b}(\lambda)\to C'$, we infer from Proposition 4 that $\mathrm{Gon}(C') \leq \mathrm{Gon}(C_{2e;a,b}(\lambda)) = 2$, so C' is also hyperelliptic.

In order to apply the inductive hypothesis on C', we try to find integers $a' \leq b' \leq c'$ and $\mu \in \mathbb{C}$ such that the curve $C_{e;a',b'}(\mu)$ is birational to C' and satisfies the condition of (3.9). By (3.11) and the condition that e is relatively prime to a, b and c, we have $a \in [1, e-1] \cup [e+1, 2e-1]$ and $e+1 \leq b \leq c \leq 2e-1$. In case $a < e$, letting $(X, Y) = (x, y/(x-1))$, the curve C' is birational to the curve $Y^e = X(X-1)^a(X-\lambda)^{b-e}$. Hence, we can take

$$(a', b', c') = \begin{cases} (a, b-e, c-e) & \text{if } a < e \text{ and } a \leq b-e, \\ (b-e, a, c-e) & \text{if } a < e \text{ and } b-e < a < c-e, \\ (b-e, c-e, a) & \text{if } a < e \text{ and } a \geq c-e. \end{cases} \tag{3.12}$$

In case $a > e$, similarly C' is birational to the curve $Y^e = X(X-1)^{a-e}(X-\lambda)^{b-e}$. Hence, we can take

$$(a', b', c') = (a-e, b-e, c-e) \qquad \text{if } a > e. \tag{3.13}$$

Because of the distinct genera, the curve C' cannot be birational to the curve $y^e = x(x-1)^{e/2}(x-\mu)^{e/2}$ or $y^e = x^2(x-1)^{e/2}(x-\mu)^{e/2}$ for any $\mu \in \mathbb{C}\backslash\{0,1\}$. Hence, by the inductive hypothesis, we obtain $(a', b') = (1, e-1)$. Combining it with (3.12) and (3.13), we conclude that $(a, b) = (1, 2e-1)$ or $(e-1, e+1)$. The former pair is just what we need, and the latter pair corresponds to the curve $C_{2e;e-1,e+1}(\lambda)$, where e is even since $\gcd(d, a) = \gcd(2e, e-1) = 1$. By Lemma 24, this curve is not hyperelliptic. □

4 Proof of Theorems D

4.1 Preliminaries

Lemma 25. *For the curve V given in* (2.3)*, if d is divisible by 3 and $d \geq 6$, then we define the curve:*

$$V_0: \ y^{d/3} = (x-\lambda_1)^{a_1} \cdots (x-\lambda_n)^{a_n}. \tag{4.1}$$

Then the following conditions are equivalent:

(1) *the curve V_0 is rational;*

(2) *let $k_1 = {}^{\#}\{a_i : a_i = d/3\}$ and $k_2 = {}^{\#}\{a_i : a_i = 2d/3\}$, then $k_1 + k_2 = n - 2$;*

(3) *one of the following conditions holds:*

(3-1) $\{c_i\}_{i=1}^n = \{1, 1, d/3, \ldots, d/3\}$,
(3-2) $\{c_i\}_{i=1}^n = \{1, 3, d/3, \ldots, d/3\}$,
(3-3) $\{c_i\}_{i=1}^n = \{3, 3, d/3, \ldots, d/3\}$.

Moreover, when (3-2) *holds, we have $d \equiv 3$ or $6 \pmod 9$; and when* (3-3) *holds, we have $d \equiv 3$ or $6 \pmod 9$ and $k_1 + 2k_2$ is divisible by 3.*

Proof. Similarly as Lemma 17. □

Lemma 26. *Let V be the curve given in* (2.3)*. Then V is trigonal and has genus 3 if and only if it has one of the following Nielsen classes:*

- $n = 3$: $(7; 1, 2, 4)$, $(8; 1, 2, 5)$, $(9; 1, 2, 6)$, $(12; 1, 3, 8)$;
- $n = 4$: $(4; 1, 1, 1, 1)$, $(6; 1, 3, 4, 4)$;
- $n = 5$: $(3; 1, 1, 1, 1, 2)$;

And V is trigonal and has genus 4 if and only if it has one of the following Nielsen classes:

- $n = 3$: $(10; 1, 2, 7)$, $(12; 1, 2, 9)$, $(12; 1, 4, 7)$, $(15; 1, 5, 9)$;
- $n = 4$: $(5; 1, 1, 1, 2)$, $(5; 1, 2, 3, 4)$, $(6; 1, 1, 1, 3)$, $(6; 1, 1, 2, 2)$, $(6; 1, 2, 4, 5)$;
- $n = 5$: $(4; 1, 1, 1, 2, 3)$, $(6; 2, 2, 2, 3, 3)$, $(6; 3, 3, 4, 4, 4)$;
- $n = 6$: $(3; 1, 1, 1, 1, 1, 1)$, $(3; 1, 1, 1, 2, 2, 2)$;

Proof. Let $k = {}^{\#}\{a_i : a_i = d/2\}$. By (2.4), we have $k \le n-2$. By Lemma 17, if $k = n-2$, then V is hyperelliptic. Hence, we may assume that $k \le n-3$. When $n = 3$, the c_i's are distinct with each other, so

$$\begin{aligned} g &= d/2 + 1 - (c_1 + c_2 + c_3)/2 \\ &\le d/2 + 1 - (d/3 + d/4 + d/5)/2 = 13d/120 + 1. \end{aligned}$$

Thus, if $g \le 4$, then $d \le 28$. When $n \ge 4$, since $k \le n-3$, we have

$$\begin{aligned} g &= (n-2)d/2 + 1 - (c_1 + \cdots + c_n)/2 \\ &\le (n-2)d/2 + 1 - (d/2 \times (n-3) + d/3 \times 2)/2 = (3n-7)d/12 + 1. \end{aligned}$$

Thus, if $g \le 4$, then $n \le 7$, and when $n = 4$, 5, 6 and 7, we have $d \le 12$, $d \le 6$, $d \le 4$ and $d \le 3$, respectively. That is, there is only finitely many $(d; a_1, \dots, a_n)$'s to check. Then by combining the types in the same Nielsen class, and by excluding those corresponding to hyperelliptic curves, we obtain the assertion. □

Assume the curve V given in (2.3) is trigonal. Let c be a divisor of d, then there is a natural (d/c)-cyclic covering φ from V to the curve

$$V' : \ y^c = (x - \lambda_1)^{a_1}(x - \lambda_2)^{a_2} \dots (x - \lambda_n)^{a_n}. \tag{4.2}$$

If we additionally assume that V' is rational, then there is a map ψ of degree 3 from V to V'.

Lemma 27. *Use the assumptions and notations in the preceding paragraph. If $g \ge 5$ and the map $(\varphi, \psi) : V \to V' \times V'$ is not birational between V and its image, then d is divisible by 3 and V satisfies the equivalent conditions in Lemma 25.*

Proof. Let $s = d/c$, and let $U' \subset V'$ be a connected compact open subset such that $\varphi^{-1}(U')$ is a union of distinguished open subsets $U_1, \dots, U_s$ in V, and that U' does not contain branch points of ψ. For a point $p \in V'$, write $\varphi^{-1}(p) = p_1 + \cdots + p_s$, where $p_i \in U_i$. Since the map (φ, ψ) is not birational, at least two of the p_i's are contained in the same fiber of ψ. Set $U'_{i,j} = \{p \in U' : \psi(p_i) = \psi(p_j)\}$ for $1 \le i < j \le s$, then we have $U' = \bigcup U'_{i,j}$. Hence, by an rearrangement of the indexes if necessary, we may assume that $U'_{1,2}$ is an infinite set.

There is an automorphism $\sigma \in \mathbb{Z}_d \subseteq \mathrm{Aut}(V)$ mapping U_1 to U_2. Since $g \ge 5$, by Proposition 7 (2), the map $V \to V'$ with degree 3 is uniquely determined up to the birational transformations of V'. That is, we have

$\psi \circ \sigma = \rho \circ \psi$ for an birational transformation ρ of V'. Now for any $p_1 \in U_1 \cap \varphi^{-1}(U'_{1,2})$, let $p_2 = U_2 \cap \varphi^{-1}(\varphi(p_1))$. Then by the definition of $U'_{1,2}$, we have $\rho(\psi(p_1)) = \psi(\sigma(p_1)) = \psi(p_2) = \psi(p_1)$. That is, we obtain $\rho = \mathrm{id}$ on the set $\psi(U_1 \cap \varphi^{-1}(U'_{1,2}))$.

Since φ is 1-1 from $U_1 \cap \varphi^{-1}(U'_{1,2})$ onto $U'_{1,2}$, and since ψ is a finite map, the set $\psi(U_1 \cap \varphi^{-1}(U'_{1,2}))$ is also infinite. Then since we conclude that $\rho = \mathrm{id}$, i.e., $\psi \circ \sigma = \psi$ on the whole V. This means, on a fiber of φ, the number of points in $\varphi^{-1}(p)$ is divisible by $\mathrm{ord}(\sigma)$. Then $\deg(\psi)$ is divisible by $\mathrm{ord}(\sigma)$. Since $\deg(\psi)$ is equal to 3, and since σ is not order 1 (because $\sigma(U_1) = U_2$), we must have $\mathrm{ord}(\sigma) = \deg(\psi) = 3$. That is, the 3 points of $\psi^{-1}(p)$ are in the same fiber of φ, and σ acts as a permutation on these 3 points, so ψ is just the natural map $V \to V/\langle\sigma\rangle$, i.e., the natural map from V to V_0 given in (4.1). Therefore, the curve V_0 is rational. □

Lemma 28. *Assume that the curve V in* (2.3) *is trigonal, $g \geq 5$, $d \equiv 0 \pmod 3$ and $d \geq 6$. Let $c = \max\{c_i\}_{i=1}^n$. If $\min\{c_i\}_{i=1}^n \neq 1$ and the curve V' given in* (4.2) *is rational, then V satisfies the equivalent conditions in Lemma 25.*

Proof. Since V' is rational, we may assume that $a_3 \equiv \cdots \equiv a_n \equiv 0 \pmod c$. It follows that $c_3 \equiv \cdots \equiv c_n \equiv 0 \pmod c$. Since $c_i \leq c$, we obtain $c_3 = \cdots = c_n = c$. If $c = d/2$, then by Lemma 17, the curve V is hyperelliptic, which contradicts our assumption; and if $c = d/3$, then this is what we need. Hence, we may assume that $c \leq d/4$. By Lemma 27, the map $(\varphi, \psi) : V \to V' \times V'$ is birational. Thus, we infer from Proposition 16 that $g \leq 2(d/c - 1)$.

By (2.4), we obtain $\gcd(c_1, c) = 1$ and $\gcd(c_2, c) = 1$. In particular, we have $c \geq 3$. Let $e = \gcd(c_1, c_2)$, $c'_1 = c_1/e$ and $c'_2 = c_2/e$. Then we can write $d = d'cc'_1c'_2e$ for some positive integer d', and hence $g = (n-2)d'cc'_1c'_2e/2 + 1 - e(c'_1 + c'_2)/2 - (n-2)c/2$. We may assume that $c_1 \leq c_2$ and hence $c'_1 \leq c'_2$. Substituting these into the inequality $g \leq 2(d/c - 1)$, we obtain

$$\begin{aligned}(n-2)c &\leq \frac{4d'c'_1c'_2e + (c'_1 + c'_2)e - 6}{d'c'_1c'_2e - 1} \\ &= 4 + \frac{(c'_1 + c'_2)e - 2}{d'c'_1c'_2e - 1} =: r(d', c'_1, c'_2, e)\end{aligned} \tag{4.3}$$

Since $c \geq 3$ and $r(d', c'_1, c'_2, e) \leq r(1, c'_1, c'_2, e) = 6$, we must have $n \leq 4$ and $c \leq 6$.

Case 1: $n = 3$**.** In this case, by (2.4), we have $\gcd(c_1, c_2) = 1$. Combining with the conditions on c_1, c_2 and c above, we deduce that $e = 1$,

$c'_1 = c_1 = 2$, $c'_2 = c_2 = 3$ and $c = 5$. However, we infer from (4.3) that $6d' \leq 4$, which is impossible since $d' \geq 1$.

Case 2: $n = 4$**.** It follows that $c = 3$ and $c_1 = c_2 = 2$, then $e = 2$ and $c'_1 = c'_2 = 1$. However, we infer from (4.3) that $8d' \leq 5$, which is also impossible. □

4.2 Proof of Theorems D

When $n = 3$, review the facts (3.2), (3.3), (3.4) and $d = d'c_1c_2c_3$ for some integer $d' \geq 1$.

Lemma 29. *When $n = 3$, if the curve V given in* (1.1) *is trigonal, then* $\min\{c_i\}_{i=1}^3 = 1$.

Proof. Suppose to the converse that $\min\{c_i\} \geq 2$. By Proposition 2 and (3.2), we may assume that $2 \leq c_1 < c_2 < c_3$. It follows that $c_2 \geq 3$ and $c_3 \geq 5$, then we have $d \geq c_1c_2c_3 \geq 6c_3 \geq 2(c_1 + c_2 + c_3) + 6$. By (3.3), we obtain $g \geq (c_1 + c_2 + c_3)/2 + 4 \geq 9$. Since $(c_1, c_2) \neq (3, 3)$, this contradicts Lemma 25. □

Lemma 30. *When $n = 3$ and $\min\{c_i\}_{i=1}^3 = 1$, we have*

(1) *if $\max\{c_i\}_{i=1}^3 = d/2$, then $V \sim C_{2a+2;a}$ for an integer a;*

(2) *if $\max\{c_i\}_{i=1}^3 = d/3$, then $V \sim C_{3a;a}$ or $V \sim C_{3a+3;a}$ for an integer a;*

(3) *if $\max\{c_i\}_{i=1}^3 \leq d/4$, $g \geq 5$ and V is trigonal, then $\{c_i\} = \{1, 2, 3\}$, $\{1, 3, 4\}$ or $\{1, 1, u\}$, where $u \in \{1, 2, 3, 4\}$.*

Proof. We may assume that $c_1 = 1$ and $c_2 \leq c_3$.

(1) and (2). (1) follows from Lemma 17. For (2), since $c_3 = d/3$, we have $a_3 = d/3$ or $2d/3$. Write $d = 3a$ for an integer a. Then for any integer k such that $ka_1 \equiv c_1 = 1 \pmod{3a}$, we have $ka_3 \equiv a$ or $2a \pmod{d}$. The former case corresponds the curve $C_{3a;a}$, and the latter case corresponds the curve $C_{3a;a-1}$.

(3). Review the definition of V_1 in the proof of Lemma 27, and consider the natural map $\varphi_1 : V \to V_1$ of degree $d'c_2$ as well as the map $\psi_1 : V \to V_1$ of degree 3. Since $c_3 \leq d/4$, we infer from Lemma 27 that $(\varphi_1, \psi_1) : V \to V_1 \times V_1$ is birational between V to its image. Thus, by Proposition 16, we obtain $g \leq 2(d'c_2 - 1)$, that is,

$$d'c_2(c_3 - 4) \leq c_2 + c_3 - 5. \tag{4.4}$$

Case 1: $c_2 = 1$**.** It follows that $d = d'c_3 \geq 4c_3$, i.e., $d' \geq 4$, then (4.4) yields $c_3 \leq 4$.

Case 2: $c_2 \geq 2$ **and** $d' = 1$**.** It follows that $d = c_2c_3 \geq 4c_3$, i.e., $c_2 \geq 4$, then by (4.4), we obtain $(c_2, c_3) = (4, 5)$ and hence $d = 20$. It is easy to see that all of the possible corresponding curves are birational to $C_{20;4}$. But by Lemma 19, we have $\mathrm{Gon}(C_{20;4}) = 4$.

Case 3: $c_2 \geq 2$ **and** $d' \geq 2$**.** By (3.2) and (4.4), we have $(c_2, c_3) = (2, 3)$ or $(3, 4)$. $\square$

Similarly as the proof of Theorem A, the "if" part of Theorem D follows from Remark 11, and by Lemma 29, to prove the "only if" part of Theorem D, it suffices to show that

Theorem D (restatement). *If $C_{d;a}$ is trigonal, then $a = 2$ or $d \in \{2a + 3, 3a, 3a + 1, 3a + 2, 3a + 3\}$.*

In Theorem A (restatement), we have seen that if $C_{d;a}$ is hyperelliptic, then $a = 1$ or $d \in \{2a + 1, 2a + 2\}$. Thus, in what follows of this subsection, we assume that *there exists a trigonal curve $C_{d;a}$ with $a \geq 3$ and $d \in [2a + 4, 3a - 1] \cup [3a + 4, \infty)$.* (Note that $[2a + 4, 3a - 1]$ is empty unless $a \geq 5$.)

It is easy to see that for the (d, a)'s under our assumption, the minimum d satisfying our assumption is 13, and neither $\gcd(d, a)$ nor $\gcd(d, a + 1)$ can be equal to $d/2$ or $d/3$. When $d = 13$, we have $c_2 = c_3 = 1$ and hence $g = 6$. when $d \geq 14$, we have $\max\{c_1, c_2\} \leq d/4$ and $\min\{c_1, c_2\} \leq d/5$, so $g \geq 11d/40 + 1/4 > 4$. Thus, we have $g \geq 5$. Therefore, the curve $C_{d;a}$ must satisfy the condition (3) of Lemma 30. Hence, we may assume additionally that

$$\{\gcd(d, a), \gcd(d, a + 1)\} = \{1, u\}, \{2, 3\} \text{ or } \{3, 4\}, \quad u \in \{1, 2, 3, 4\}. \quad (4.5)$$

In particular, by (3.3), we have

$$g \geq d/2 - 3 =: g_0, \qquad \text{hence} \quad (g + 1)/2 \geq (g_0 + 1)/2 = d/4 - 1. \quad (4.6)$$

We write $\varphi_{\xi,\zeta}$ for the function $y^\xi/(x-1)^\zeta$ on $\tilde{C}_{d;a}$. Then the following lemma is obvious.

Lemma 31. *For two integers $\xi > 0$ and $\zeta \geq 0$, we have*

$$\deg(\varphi_{\xi,\zeta}) = \begin{cases} \xi(a+1) - \zeta d & \text{if } \ \xi a - \zeta d \geq 0 \ \text{ and } \ \zeta d - \xi(a+1) \leq 0, \\ \xi & \text{if } \ \xi a - \zeta d \leq 0 \ \text{ and } \ \zeta d - \xi(a+1) \leq 0, \\ \zeta d - \xi a & \text{if } \ \xi a - \zeta d \leq 0 \ \text{ and } \ \zeta d - \xi(a+1) \geq 0. \end{cases}$$

Lemma 32. *We have* $\mathrm{Gon}(C_{d;3}) \neq 3$ *for* $d \geq 13$*, and* $\mathrm{Gon}(C_{d;4}) \neq 3$ *for* $d \geq 16$*.*

Proof. **Case 1:** $a = 3$ **and** $g \leq 6$**.** By (2.5), we have $13 \leq d \leq 16$. We have shown that the gonalities of $C_{13;3}$ and $C_{14;3}$ are greater than 3 in Example 14. From Proposition 2, we deduce that $C_{16;3} \sim C_{16;4}$. Then by Lemma 19, we obtain $\mathrm{Gon}(C_{d;3}) \neq 3$ for $d = 15$ and 16.

Case 2: $a = 3$ **and** $g \geq 7$**.** Suppose that $\mathrm{Gon}(C_{d;3}) = 3$. Since $\deg(\varphi_{1,0}) = 4$, there exist functions of degrees both 3 and 4, which contradicts [7, Theorem 2.4.3].

Case 3: $a = 4$ **and** $g \leq 8$**.** By (2.5), we have $d \in \{16, 17, 18, 20\}$. From Proposition 2, we deduce that $C_{17;4} \sim C_{17;3}$. Then by Lemma 19 and the result for $a = 3$ above, we obtain $\mathrm{Gon}(C_{d;4}) \neq 3$ for $d = 16$, 17 and 20. On the curve $C_{18;4}$, since $\deg(\varphi_{4,1}) = 4$ and $g = 8$, we infer from [7, Theorem 2.4.3] that its gonality is 4.

Case 4: $a = 4$ **and** $g \geq 9$**.** Since $\deg(\varphi_{1,0}) = 5$, the proof is similar as Case 2. □

Lemma 33. *We have* $\mathrm{Gon}(C_{d;5}) \neq 3$ *for* $d = 14$ *or* $d \geq 19$*.*

Proof. By Example 14, we have $\mathrm{Gon}(C_{14;5}) \neq 3$ From Proposition 2, we deduce that $C_{19;5} \sim C_{19;3}$, and by Lemma 32, we have $\mathrm{Gon}(C_{19;5}) \neq 3$. Now suppose that $\mathrm{Gon}(C_{d;5}) = 3$ for some $d \geq 20$.

By (4.5), we only need to consider the d's which are divisible by neither 5 nor 6, that is, $\gcd(d, 5) = 1$ and $\gcd(d, 6) \leq 3$. By (2.5), we have $(g+1) \geq (d-1)/4$. For an integer $k \in [d/6, d/5]$ with $k \not\equiv 0 \pmod 3$, we have $\deg(\varphi_{k,1}) = k \leq (g+1)/2$. Hence, by Proposition 7 (1), such k does not exist.

In case $d \geq 60$, since $d/5 - d/6 \geq 2$, there are at least two successive integers between $d/6$ and $d/5$, then at least one of them is not divisible 3, a contradiction.

In case $20 \leq d \leq 59$, we can check the d's one by one, and there are only 5 of them such that the k having the property above does not exist, which are 31, 32, 33, 34 and 49. We have shown that $\mathrm{Gon}(C_{31;5}) \neq 3$ in Example 14. For the remaining four curves, we have (1) $\deg(\varphi_{5,1}) = 7$, $g = 15$ on $C_{32;5}$; (2) $\deg(\varphi_{5,1}) = 8$, $g = 15$ on $C_{33;5}$; (3) $\deg(\varphi_{7,1}) = 8$, $g = 16$ on $C_{34;5}$; (4) $\deg(\varphi_{10,1}) = 11$, $g = 24$ on $C_{49;5}$. Thus, by Proposition 7 (1), none of these curves is trigonal. □

By Lemmata 32 and 33, we have dealt with the case $a \leq 5$, so in what follows of this subsection, we assume that $a \geq 6$.

Lemma 34. *If* $\mathrm{Gon}(C_{d;a}) = 3$*, then* $d \le 4a + 7$*.*

Proof. Assume that $\mathrm{Gon}(C_{d;a}) = 3$ for some pair of integers (d, a) such that

$$a \ge 6 \qquad \text{and} \qquad d \ge 4a + 8. \tag{4.7}$$

By Lemma 31, we have $\deg(\varphi_{1,0}) = a + 1 \le (g_0 + 1)/2$. Thus, from Proposition 7 (2), we deduce that

$$a \equiv 2 \pmod 3, \qquad \text{hence } a \ge 8 \text{ and } d \ge 40. \tag{4.8}$$

Under the condition of (4.7), for a positive integer k, we have

$$\deg(\varphi_{k,1}) = \begin{cases} d - ka & \le (g_0 + 1)/2 \quad \text{if } r_1 \le k \le r_2, \\ k & \le (g_0 + 1)/2 \quad \text{if } r_2 \le k \le r_3, \\ ka - d + k & \le (g_0 + 1)/2 \quad \text{if } r_3 \le k \le r_4, \end{cases} \tag{4.9}$$

where g_0 is defined in (4.6) and

$$r_1 = \frac{3d + 4}{4a}, \quad r_2 = \frac{d}{a + 1}, \quad r_3 = \frac{d}{a}, \quad r_4 = \frac{5d - 4}{4a + 4}.$$

Note that the reason for $k \le (g_0 + 1)/2$ in the second line of (4.9) is as follow: Otherwise we have $(g_0 + 1)/2 < k \le r_3$. But by (4.7), this yields $d < 4a/(a - 4) \le 8$, which contradicts (4.8). By Proposition 7 (2), the degree of $\varphi_{k,1}$ in any case is divisible by 3.

Case 1: $d \equiv 0 \pmod 3$**.** By (4.8), it follows that $k \in [r_1, r_3]$ only if k is divisible by 3. In particular, we have $r_3 - r_1 < 2$, which yields

$$d \le 8a + 3. \tag{4.10}$$

By (4.7), we have $r_3 > 4$. Since $4 \notin [r_1, r_3]$, we obtain $r_1 > 4$, which yields

$$d \ge (16a - 2)/3. \tag{4.11}$$

By (4.11), we have $r_3 > 5$. Since $5 \notin [r_1, r_3]$, we obtain $r_1 > 5$, which yields

$$d \ge (20a - 1)/3. \tag{4.12}$$

Now we can prove the inequality

$$d \le 6a + 10. \tag{4.13}$$

Suppose that $d \ge 6a + 11$. By (4.12), for a positive integer k, we have

$$\deg(\varphi_{6k+1,k}) = \begin{cases} 6k + 1 & \le (g_0 + 1)/2 \quad \text{if } s_1 \le k \le s_2, \\ kd - 6ka - a & \le (g_0 + 1)/2 \quad \text{if } s_2 \le k \le s_3, \end{cases} \tag{4.14}$$

where g_0 is defined in (4.6) and

$$s_1 = \frac{a}{d-6a}, \quad s_2 = \frac{a+1}{d-6a-6}, \quad s_3 = \frac{d+4a-4}{4d-24a}.$$

Note that the reason for $6k+1 \leq (g_0+1)/2$ in the first line of (4.14) is as follow: Otherwise we have $(g_0+1)/2 < 6k+1 \leq 6s_2+1$. But this yields

$$6a+11 \leq d < 3a+7+\sqrt{9a^2+18a+25},$$

which does not hold for $a \geq 8$. Since neither degree in (4.14) is divisible by 3, we infer from Proposition 7 (2) that there does not exists any integer $k \in [s_1, s_3]$. In particular, we have $s_3 - s_1 < 1$. Under the supposition $d \geq 6a+11$, this yields $d \geq 8a-1$, and by (4.10) and (4.11), we obtain $d = 8a-1$ or $8a+2$. However, one checks that $1 \in [s_1, s_3]$ for either d, a contradiction. Hence, we obtain (4.13).

From (4.7), (4.8), (4.10)–(4.12), (4.13) and the condition of Case 1, we deduce that $(d,a) = (54,8), (57,8), (75,11)$ or $(93,14)$. Note that the first pair does not satisfy the condition (3) of (4.5). And for the other pairs, we have (1) $g = 27$ and $\deg(\varphi_{7,1}) = 7$ on $C_{57;8}$; (2) $g = 36$ on $C_{75;11}$, (3) $g = 45$ on $C_{93;14}$, and $\deg(\varphi_{13,2}) = 13$ on the later two curves. Thus, by Proposition 7 (2), none of these curves is trigonal.

Case 2: $d \equiv 1 \pmod 3$**.** By (4.8), it follows that $k \in [r_1, r_2]$ (resp., $k \in [r_2, r_3]$) only if $k \equiv 2 \pmod 3$ (resp., $k \equiv 0 \pmod 3$), and there dose not exist an integer $k \in [r_3, r_4]$. In particular, we have $r_4 - r_3 < 1$, which yields

$$d < (4a^2+8a)/(a-4). \tag{4.15}$$

By (4.7), we have $r_4 > 5$. Since $5 \notin [r_2, r_4]$, we obtain $r_2 > 5$, which yields

$$d \geq 5a+6. \tag{4.16}$$

By (4.16), we have $r_4 > 6$. Since $6 \notin [r_3, r_4]$, we obtain $r_3 > 6$, which yields

$$d \geq 6a+1. \tag{4.17}$$

Moreover, since $6 \notin [r_1, r_2]$, i.e., $r_1 > 6$ or $r_2 < 6$, we deduce that

$$d \leq 6a+5 \qquad \text{or} \qquad d \geq 8a-1. \tag{4.18}$$

Similarly, since $7, 10 \notin [r_1, r_4]$, we deduce that

$$\begin{aligned} &d \leq (28a+31)/5 \qquad &&\text{or} \qquad d \geq (28a-2)/3, \\ &d \leq 8a+8 \qquad &&\text{or} \qquad d \geq (40a-2)/3. \end{aligned} \tag{4.19}$$

From (4.7), (4.8), (4.15)–(4.19) and the condition of Case 2, we deduce that $(d, a) = (49, 8)$ or $(67, 11)$. However, since $C_{49;8} \sim C_{49;5}$ and $C_{67;11} \sim C_{67;5}$, by Lemma 33, neither of these curves is trigonal.

Case 3: $d \equiv 2 \pmod 3$**.** By (4.8), it follows that $k \in [r_1, r_2]$ (resp., $k \in [r_2, r_3]$) only if $k \equiv 1 \pmod 3$ (resp., $k \equiv 0 \pmod 3$), and there dose not exist an integer $k \in [r_3, r_4]$. Hence, the condition (4.15) in Case 2 still hold. By (4.7), we have $r_4 > 5$. Since $5 \notin [r_1, r_4]$, we obtain $r_1 > 5$, which yields

$$d \geq (20a - 1)/3. \tag{4.20}$$

By (4.20), we have $r_4 > 7$. Since $7 \notin [r_2, r_4]$, we obtain $r_2 > 5$, which yields

$$d \geq 7a + 8. \tag{4.21}$$

By (4.20), we have $r_4 > 8$. Since $8 \notin [r_1, r_4]$, we obtain $r_1 > 8$. However, with the help of (4.15), this yields

$$(32a - 1)/3 \leq d < (4a^2 + 8a)/(a - 4). \tag{4.22}$$

which does not hold for $a \geq 8$. □

Lemma 35. *If* $\mathrm{Gon}(C_{d;a}) = 3$*, then* $d \leq 3a - 1$.

Proof. Suppose that the assertion does not hold. By Lemma 34, we assume that $\mathrm{Gon}(C_{d;a}) = 3$ for some pair of integers (d, a) such that

$$a \geq 6 \qquad \text{and} \qquad 3a + 4 \leq d \leq 4a + 7. \tag{4.23}$$

If $d = 4a$ or $4a+4$, then either $c_1 = \gcd(d, a) = a \geq 6$ or $c_2 = \gcd(d, a+1) = a+1 \geq 7$, which contradicts (4.5). Now we will show that if $d \in [4a+1, 4a+3] \cup [4a+5, 4a+7]$, then there is a contradiction to Proposition 7 (2). Indeed, if $4a+1 \leq d \leq 4a+3$, then $\deg(\varphi_{4,1}) = 4$ but $g \geq 2a - 2 \geq 10$. If $d = 4a+5$ (resp. $4a+7$), then $\deg(\varphi_{4,1}) = 5$ (resp. 7) but $g \geq 2a \geq 12$ (resp. $g \geq 2a + 1 \geq 13$). If $d = 4a + 6$, then we have $g \geq 2a$. In case $a \not\equiv 1 \pmod 3$, we have $\deg(\varphi_{5,1}) = a - 1$. In case $a = 7$ (resp. 10), we have $\deg(\varphi_{1,0}) = 8$ (resp. 11) but $g = 16$ (resp. 22). In case $a \equiv 1 \pmod 3$ and $a \geq 13$, we have $\deg(\varphi_{9,2}) = a - 3$. To sum up, we obtain

$$d \leq 4a - 1. \tag{4.24}$$

By (4.24), we have $\deg(\varphi_{3,1}) = d - 3a$. When $d \leq 4a - 2$, we see that $\deg(\varphi_{3,1}) \leq (g_0 + 1)/2$. When $d = 4a - 1$, since $c_1 = \gcd(d, a) = 1$, by (4.5), we obtain $c_2 \leq 4$ and hence $g \geq 2a-2$, so we still have $\deg(\varphi_{3,1}) \leq (g+1)/2$. Thus, we infer from Proposition 7 (2) that

$$d \equiv 0 \pmod 3, \qquad \text{hence } d \geq 3a+6 \text{ and } a \geq 7. \tag{4.25}$$

Suppose that $d = 3a+6$. For an integer $k \in [a/6, (a-1)/4]$, we have $\deg(\varphi_{3k+1,k}) = 3k+1$. Hence, by Proposition 7 (2), such k does not exist. For $a \geq 15$, since $(a-1)/4 - a/6 \geq 1$, this is impossible. For $7 \leq a \leq 14$, we can check the a's one by one, and only $a = 7$ and 8 satisfy the non-existence of k. We have (1) $\deg(\varphi_{4,1}) = 5$ and $g = 13$ on $C_{27;7}$; and (2) $\deg(\varphi_{7,2}) = 7$ and $g = 13$ on $C_{30;8}$. Thus, neither of these curves is trigonal, and hence we obtain

$$d \geq 3a+9 \text{ and } a \geq 10. \tag{4.26}$$

By (4.25) and (4.26), for a positive integer k, we have

$$\deg(\varphi_{3k+1,k}) = \begin{cases} a+1-(d-3a-3)k \leq \frac{g_0+1}{2} & \text{if } t_1 \leq k \leq t_2, \\ 3k+1 \leq \frac{g_0+1}{2} & \text{if } t_2 \leq k \leq t_3, \\ (d-3a)k-a \leq \frac{g_0+1}{2} & \text{if } t_3 \leq k \leq t_4, \end{cases} \tag{4.27}$$

where g_0 is defined in (4.6) and

$$t_1 = \frac{4a-d+8}{4d-12a-12}, \quad t_2 = \frac{a}{d-3a}, \quad t_3 = \frac{a+1}{d-3a-3}, \quad t_4 = \frac{d+4a-4}{4d-12a}.$$

Note that the reason for $3k+1 \leq (g_0+1)/2$ in the second line of (4.27) is as follow: Otherwise we have $(g_0+1)/2 < 2k+1 \leq 3t_3+1$. But by (4.26), this yields

$$3a+9 \leq d < (3a+11)/2 + \sqrt{9a^2+18a+73}/2,$$

which does not holds for $a \geq 10$. By Proposition 7 (2), the degree of $\varphi_{3k+1,k}$ in any case is divisible by 3.

In case $a \equiv 2 \pmod 3$, by (4.26), there does not exits integer $k \in [t_2, t_4]$. In particular, we have $t_4 - t_2 < 1$, which yields $d \leq 3a-1$ or $d \geq 4a-1$. By the condition $(d,a) \equiv (0,2) \pmod 3$, we obtain $d \leq 3a-3$ or $d \geq 4a+1$, which contradicts (4.24) and (4.26).

In case $a \not\equiv 2 \pmod 3$, by (4.26), there does not exits integer $k \in [t_1, t_3]$. In particular, we have $t_3 - t_1 < 1$. But this yields $d \leq 3a+2$ or $d \geq 4a+3$, which also contradicts (4.24) and (4.26). □

Lemma 36. *If* $\mathrm{Gon}(C_{d;a}) = 3$, *then* $d \geq 3a-4$.

Proof. Suppose that the assertion does not hold. Then we assume that $\mathrm{Gon}(C_{d;a}) = 3$ for an integer d such that

$$2a + 4 \le d \le 3a - 5. \tag{4.28}$$

Case 1: $2a + 4 \le d \le 5a/2 - 1/2$. It follows that $\deg(\varphi) = d - 2a \le (g_0 + 1)/2$, then we obtain

$$d \equiv 2a \pmod 3, \qquad \text{hence } d \ge 2a + 6 \text{ and } a \ge 13. \tag{4.29}$$

Suppose $d = 2a + 6$. By (4.30), for an integer $k \in [a/6, (a-1)/4]$, we have $\deg(\varphi_{2k+1,k}) = 2k + 1 \le (g + 1)/2$. By Proposition 7 (2), there does not exist an integer $k \in [a/6, (a-1)/4]$ such that $k \not\equiv 1 \pmod 3$. For $a \ge 27$, this is impossible since $(a-1)/4 - a/6 \ge 2$. For $13 \le a \le 26$, this is possible only for $a = 19$ and 20. However, since $C_{44;19} \sim C_{44;7}$ and $C_{46;20} \sim C_{46;10}$, by Lemma 35, neither of these curves is trigonal. Therefore, we deduce that

$$d \ge 2a + 9 \qquad \text{and} \qquad a \ge 19. \tag{4.30}$$

By (4.30) and the condition of Case 1, for a positive integer k, we have

$$\deg(\varphi_{2k+1,k}) = \begin{cases} a + 1 - (d - 2a - 2)k \le \frac{g_0+1}{2} & \text{if } u_1 \le k \le u_2, \\ 2k + 1 \le \frac{g_0+1}{2} & \text{if } u_2 \le k \le u_3, \\ (d - 2a)k - a \le \frac{g_0+1}{2} & \text{if } u_3 \le k \le u_4, \end{cases} \tag{4.31}$$

where g_0 is defined in (4.6) and

$$u_1 = \frac{4a - d + 8}{4d - 8a - 8}, \quad u_2 = \frac{a}{d - 2a}, \quad u_3 = \frac{a + 1}{d - 2a - 2}, \quad u_4 = \frac{4a + d - 4}{4d - 8a}.$$

Note that the reason for $2k + 1 \le (g_0 + 1)/2$ in the second line of (4.31) is as follow: Otherwise we have $(g_0 + 1)/2 < 2k + 1 \le 2u_3 + 1$. But under the condition (4.30), this yields $2a + 9 \le d < a + 5 + \sqrt{a^2 + 2a + 17}$, which does not holds for $a \ge 19$. By Proposition 7 (2), all degrees in (4.31) are divisible by 3.

Case 1.1: $a \equiv 0 \pmod 3$. By (4.29), it follows that $k \in [u_1, u_3]$ only if $k \equiv 1 \pmod 3$. In particular, we have $u_3 - u_1 < 2$, which yields

$$d \ge (16a + 13)/7. \tag{4.32}$$

By (4.32), we have $u_1 < 2$. Since $2 \notin [u_1, u_3]$, we obtain $u_3 < 2$. But this yields $d \ge 5a/2 + 3$, which contradicts the condition of Case 1.

Case 1.2: $a \equiv 1 \pmod 3$**.** By (4.30), it follows that $k \in [u_1, u_2]$ (resp., $k \in [u_2, u_3]$) only if $k \equiv 2 \pmod 3$ (resp., $k \equiv 1 \pmod 3$), and there dose not exist an integer $k \in [u_3, u_4]$. In particular, we have $u_4 - u_1 < 2$. Under the condition (4.30), this yields

$$d > \left(14a + 1 + \sqrt{4a^2 - 68a + 49}\right)/6 \geq 8a/3 - 4. \tag{4.33}$$

By (4.33), we have $u_1 < 3$. Since $3 \notin [u_1, u_4]$, we obtain $u_4 < 3$. But this yields $d \geq (28a - 3)/11$, which contradicts the condition of Case 1.

Case 1.3: $a \equiv 2 \pmod 3$**.** By (4.30), it follows that $k \in [u_1, u_2]$ (resp., $k \in [u_2, u_3]$) only if $k \equiv 0 \pmod 3$ (resp., $k \equiv 1 \pmod 3$), and there dose not exist an integer $k \in [u_3, u_4]$. Thus, we obtain $u_4 - u_2 < 2$, which yields

$$d \geq (16a - 3)/7. \tag{4.34}$$

By (4.34), we have $u_1 < 5$. Since $5 \notin [u_1, u_4]$, we obtain $u_4 < 5$, which yields

$$d \geq (44a - 3)/19. \tag{4.35}$$

By the condition of Case 1, we have $u_4 > 2$. Since $2 \notin [u_1, u_4]$, we obtain $u_1 > 2$, which yields

$$d \leq (20a + 23)/9. \tag{4.36}$$

By (4.30), (4.34)—(4.36) and the conditions of Case 1 and Case 1.3, the only possible pair (d, a) is $(67, 29)$. Since $\deg(\varphi_{5,2}) = 16$ and $g = 33$ on $C_{67;29}$, this curve is not trigonal.

Case 2: $5a/2 \leq d \leq 3a - 5$**.** It follows that $a \geq 10$. Suppose that $d \leq (5a+5)/2$. Then we have $\deg(\varphi_{5,2}) = 5 \leq (g_0+1)/2$, which contradicts Proposition 7 (2). Thus, we obtain

$$d \geq 5a/2 + 3. \tag{4.37}$$

By (4.37), we have $\deg(\varphi_{3,1}) = 3a - d + 3 \leq (g_0 + 1)/2$, so we obtain

$$d \equiv 0 \pmod 3, \qquad \text{hence } d \leq 3a - 6 \text{ and } a \geq 18. \tag{4.38}$$

By (4.37) and (4.38), for an integer k, we have

$$\deg(\varphi_{3k-1,k}) = \begin{cases} a - (3a - d)k & \leq \frac{g_0+1}{2} \quad \text{if } v_1 \leq k \leq v_2, \\ 3k - 1 & \leq \frac{g_0+1}{2} \quad \text{if } v_2 \leq k \leq v_3, \\ (3a - d + 3)k - a - 1 & \leq \frac{g_0+1}{2} \quad \text{if } v_3 \leq k \leq v_4, \end{cases} \tag{4.39}$$

where g_0 is defined in (4.6) and

$$v_1 = \frac{4a-d+4}{12a-4d}, \quad v_2 = \frac{a+1}{3a+3-d}, \quad v_3 = \frac{a}{3a-d}, \quad v_4 = \frac{4a+d}{12a+12-4d}.$$

Note that the reason for $3k-1 \le (g_0+1)/2$ in the second line of (4.39) is as follow: Otherwise we have $(g_0+1)/2 < 3k-1 \le 3v_3-1$. But under the condition (4.37) and (4.38), this yields $\left(3a+\sqrt{9a^2-48a}\right)/2 < d \le 3a-6$, which does not holds for $a \ge 18$. By Proposition 7 (2), all degrees in (4.39) are divisible by 3.

In case $a \equiv 0 \pmod 3$, by (4.38), there does not exist an integer $k \in [v_2, v_4]$. Thus, we obtain $v_4 - v_2 < 1$, but this yields $d \le 12a/5+3$, which contradict (4.37).

In case $a \not\equiv 0 \pmod 3$, by (4.38), there does not exist an integer $k \in [v_1, v_3]$. Thus, we obtain $v_3 - v_1 < 1$, but this yields $d \ge (12a+3)/5$, which also contradict (4.37). □

Proof of Theorem D (restatement). It remains to show that $\mathrm{Gon}(C_{d;a}) \neq 3$ for $a \ge 6$ and $\max\{2a+4, 3a-4\} \le d \le 3a-1$. Assume to the contrary that $\mathrm{Gon}(C_{d;a}) = 3$ for such (d,a)'s.

In case $d = 3a-1$, since $\deg(\varphi_{3,1}) = 4$, we must have $4 > (g_0+1)/2$, which yields $a = 6$. Since $C_{17;6} \sim C_{17;3}$, by Lemma 32, it is not trigonal.

In case $d = 3a-2$, since $\deg(\varphi_{3,1}) = 5$, we must have $5 > (g_0+1)/2$, which yields $6 \le a \le 8$. We have (1) $\deg(\varphi_{2,1}) = 4$, $g = 7$ on $C_{16;6}$; (2) $\deg(\varphi_{2,1}) = 5$, $g = 9$ on $C_{19;7}$; (3) $\deg(\varphi_{3,1}) = 5$, $g = 10$ on $C_{22;8}$. Thus, none of these curves is trigonal.

In case $d = 3a-3$, we have $a \ge 7$. For an integer $k \in [(a+1)/6, (a-1)/4]$, we have $\deg(\varphi_{3k-1,k}) = 3k-1 \le (g_0+1)/2$. By Proposition 7 (2), there does not exist an integer $k \in [(a+1)/6, (a-1)/4]$. For $a \ge 17$, this is impossible since $(a-1)/4 - (a+1)/6 \ge 1$. For $7 \le a \le 16$, this is possible only for $a = 7$, 8 and 12. We have (1) $\deg(\varphi_{2,1}) = 4$, $g = 8$ on $C_{18;7}$; (2) $\deg(\varphi_{2,1}) = 5$, $g = 9$ on $C_{21;8}$; (3) $\deg(\varphi_{8,3}) = 8$, $g = 15$ on $C_{33;12}$. Thus, none of these curves is trigonal.

In case $d = 3a-4$, we have $a \ge 8$. Since $\deg(\varphi_{3,1}) = 7$, we must have $7 > (g_0+1)/2$, which yields $8 \le a \le 11$. We have (1) $\deg(\varphi_{2,1}) = 4$, $g = 8$ on $C_{20;8}$; (2) $C_{23;9} \sim C_{23;4}$; (3) $C_{26;10} \sim C_{26;7}$; (3) $C_{29;11} \sim C_{29;8}$. Thus, by Proposition 7 (2), Lemma 32 and 35, none of these curves is trigonal. □

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