Decomposition of the Möbius energy: the Möbius invariance and variational formulae of decomposed energies

（メビウス・エネルギーの分解：分解された
エネルギーのメビウス不変性と変分公式）

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埼玉大学大学院理工学研究科 (博士後期課程)

理工学専攻　(主指導教員　長澤 壯之)

石関　彩

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1 Introduction

What is the most "beautiful knot" which would represent its knot type? We would like to find an optimal embedding of a knot transforming the given knot with preserving its knot type. Considering electrostatic energy of a uniformly charged knot, it will decrease by Coulomb's repulsive force. Based on the energy, O'Hara's energy was defined in 1991 in [18]. In order to find an optimal embedding, we consider a variational problem for this energy. There are some ways to solve this kind of problems, in this study we would like to use a way of gradient flow since the energy will decrease continuously. For that we need first and second variational formulae and L^2-gradient expressions and estimates.

Let $\boldsymbol{f} : \mathbb{R}/\mathcal{L}\mathbb{Z} \ni s \mapsto \boldsymbol{f}(s) \in \mathbb{R}^n$ be a closed curve in $\mathbb{R}^n$ with total length $\mathcal{L}$, where s is an arc-length parameter, $i.e.$, $\|\boldsymbol{f}'(s)\| \equiv 1$. We denote the distance between $\boldsymbol{f}(s_1)$ and $\boldsymbol{f}(s_2)$ along the closed curve by $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$.

O'Hara [18] defined the Möbius energy $\mathcal{E}$ as

$$(1.1) \qquad \mathcal{E}(\boldsymbol{f}) = \text{p.v.} \iint \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \right) ds_1 ds_2,$$

where p.v. $\iint = \lim_{\varepsilon \to +0} \iint_{|s_1 - s_2| > \varepsilon}$ is Cauchy's principal value.

Remark 1.1 This is the original definition of O'Hara. In fact, the integration is not a principal value since the integrand is non-negative. However, many quantities derived from the energy, for example the variational formulae, contain terms each of which is not absolutely integrable. Therefore, when deforming the expression of the energy, we always deal with it as Cauchy's principal value at first, and investigate its absolute integrability later.

Indeed he introduced the energies

$$\mathcal{E}_{(\alpha,p)}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^\alpha} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^\alpha} \right)^p ds_1 ds_2,$$

which are called *O'Hara's* energy. The density contains the negative power of "distance", which implies that a minimizer, if exists, is the "canonical configuration" of knots among the given knot type, even though it makes the analysis hard.

It is easy to see that $\mathcal{E}_{(\alpha,p)}$ is scale-invariant if $\alpha p = 2$, including our energy $\mathcal{E} = \mathcal{E}_{(2,1)}$. In mid-1990's, Freedman-He-Wang [8] showed that $\mathcal{E}$ has the invariance not only under scaling but also under Möbius transformations. Since then, it has been called the *Möbius* energy. Blatt [2] found the proper domain of the energy: $\mathcal{E}(\boldsymbol{f}) < \infty$ if and only if $\boldsymbol{f}$ is bi-Lipschitz and belongs to the fractional Sobolev space $X = H^{3/2} \cap W^{1,\infty}$. See section 2 for the definition of fractional Sobolev spaces. Consequently we may assume the existence of the unit tangent vector $\boldsymbol{\tau}(s) = \boldsymbol{f}'(s)$ almost everywhere. By use of the unit tangent vector field along the curve, the energy may be decomposed into three parts:

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4,$$

where

$$\mathcal{E}_i(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f}) \, ds_1 ds_2,$$

$$\mathscr{M}_1(\boldsymbol{f}) = \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2},$$

$$\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4}$$
$$\times \det \begin{pmatrix} \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix}.$$

This was shown in [11] and we deal with the decomposition theorem in section 3. The first decomposed energy $\mathcal{E}_1$ is an analogue of the Gagliardo semi-norm of $\boldsymbol{\tau}$ in the fractional Sobolev space $H^{1/2}$. This implies the domain of $\mathcal{E}$ is X, as shown by Blatt. The integrand $\mathscr{M}_2$ of second one has the determinant structure, which implies a cancellation of integrand.

In section 4 we focus on the Möbius invariance of the decomposed energy $\mathcal{E}_i$ studied in [11, 13]. As we said before, the energy $\mathcal{E}$ is called the Möbius energy since it is invariant under Möbius transformations; this fact was shown by Freedman-He-Wang [8]. Here 4 we give an alternative proof of this using the decomposition. As a consequence we can show that right circles are only minimizers of $\mathcal{E}_1$ in the class $C^{1,1}$ with minimum value $2\pi^2$. This seems to be related to the fact that the first eigenvalue of the fractional Laplacian $(-\Delta_s)^{\frac{3}{2}}$ is $\left(\dfrac{2\pi}{\mathcal{L}}\right)^3$.

Since the last part "4" of the decomposition is an absolute constant, we can ignore it when considering variational problem. This fact shortens the derivation of variational formulae, and enables us to find their "good" estimates in several functional spaces [12]. Furthermore we find the L^2-gradient of each decomposed energy which contains $(-\Delta_s)^{\frac{3}{2}}$ as the principal term [14]. We deal with the variational formulae in section 5.

After discussing our results with O'Hara, he informed us that our second energy $\mathcal{E}_2$ is the same as the O'Hara-Solanes energy E_{OS} up to multiplication by a constant. The energy E_{OS} was first defined in [21] and they established its Möbius invariance for sufficiently smooth $\boldsymbol{f}$. Note that C^∞ is not dense in X.

Before describing our results, we show known results such as the existence of minimizers or regularity of critical points in section 2. We will state on our results on the decomposition of the energy, the Möbius invariance, and variational formulae of decomposed energies in sections 3–5 respectively. All of them have already published or submitted; Sections 3-4 are based on [11, 13], and section 5 on [12, 14], but some parts of proofs are improved from the original ones.

2 Known results

We summarize the known results on the Möbius energy according to the article [17].

2.1 The existence of minimizers

The existence of minimizers of the Möbius energy has been studied by several mathematicians but it is still a difficult problem. On the other hand, we can easily know the minimum value of this energy by simple calculation. Let S^1 be a right circle (*i.e.*, a circle with the constant curvature) whose center at origin, radius r in the $x_1 x_2$-plane

$$\boldsymbol{f}(s) = r(\cos \tfrac{s}{r}, \sin \tfrac{s}{r}, 0, \cdots, 0).$$

Since

$$\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2 = 2r^2(1 - \cos \tfrac{s_1 - s_2}{r}),$$

and

$$\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2 = (s_1 - s_2)^2 \quad (|s_1 - s_2| \leq \pi r)$$

it is not difficult to see

$$\mathcal{E}(\boldsymbol{f}) = 4.$$

Freedman-He-Wang [8] showed the following.

Theorem 2.1 ([8]) *It holds that $\mathcal{E}(\boldsymbol{f}) \geq 4$. The equality holds if and only of $\boldsymbol{f}$ is the right circle.*

Several proofs of this theorem are known. We will give a new proof in section 3.

Remark 2.1 Abrams-Cantarella-Fu-Ghomi-Howard [1] generalized this fact. Let $\alpha \geq 1$, $p \in (0, 2 + \frac{1}{\alpha})$. Then the right circle is the only global minimizer of $\mathcal{E}_{(\alpha,p)}$.

Now we consider the minimizing problem in each knot type. Since the energy $\mathcal{E}$ is non-negative, there exists a minimizing sequence. However its convergence is not trivial. The scaling-invariance of energy might allows the pull-tight phenomena along the sequence.
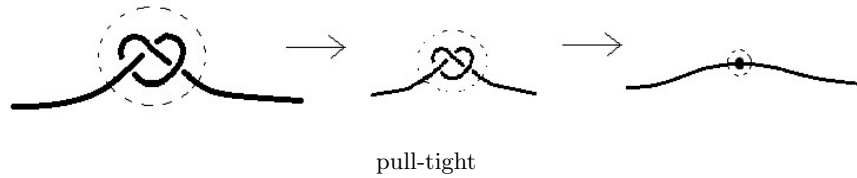
pull-tight

If such a phenomena occurs, the limit knot, if it exists, is not in the same knot type as the sequence. The energy behavior along the pull-tight for $\mathcal{E}_{(\alpha,p)}$ was studied by O'Hara [19].

Theorem 2.2 ([19]) *Let a knot K_ε be a connected sum of a knot K and a small tangle T_ε. The difference of energy $D(\varepsilon) = \mathcal{E}_{(\alpha,p)}(K_\varepsilon) - \mathcal{E}_{(\alpha,p)}(K)$ behaves as follows in a pull-tight process $T_\varepsilon \to \{a\ point\}$:*

- *$D(\varepsilon)$ blows up when $\alpha p > 2$.*

- *$D(\varepsilon)$ converges to a positive constant when $\alpha p = 2$.*

- *$D(\varepsilon)$ vanishes when $\alpha p < 2$.*

We call the cases $\alpha p > 2$, $= 2$, and < 2 respectively *subcritical*, *critical*, and *supercritical*. Pull-tight phenomena is the disappearance of a tangle. The above results implies that a pull-tight may happen in critical and supercritical cases. This shows that the argument of minimizing sequence does not work well. Nevertheless the argument works for the Möbius energy in *prime* knot type. This remarkable result was proven by Freedman-He-Wang [8].

Definition 2.1 Let $n = 3$. A knot is a *composite knot* if it is a connected sum of some non-trivial knots. A *prime knot* is a knot which is not composite.
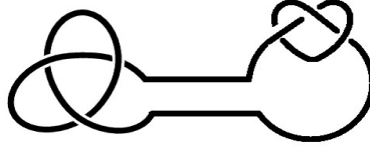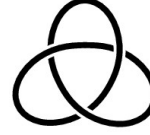
A composite knot   A prime knot

Theorem 2.3 ([8]) *There exists a minimizer of $\mathcal{E}$ for any* prime *knot types.*

The key is how to avoid the pull-tight along the minimizing sequence. If the knot is prime, we can enlarge the tangle by the inversion with respect to a sphere near the shrinking tangle without changing energy level. They passed the limit of minimizing sequence together with, if necessary, enlarging tangle, and showed the limit knot is a minimizer in given knot type. If the knot is composite, two tangles might shrink simultaneously. Hence such a method does not work.

The argument of minimizing sequence works in the subcritical case. Indeed O'Hara [20] showed the following result.

Theorem 2.4 ([20]) *Let $n = 3$. There exists a minimizer (under rescaling) for any knot types if and only if $\alpha p > 2$.*

2.2 The Kusner-Sullivan conjecture

Let $n = 3$, and $[K]$ be a knot type. We denote

$$\mathcal{E}([K]) = \inf_{f \in [K]} \mathcal{E}(\boldsymbol{f}).$$

Note that $\mathcal{E}([K])$ exists, since the energy density is non-negative.

Kusner and Sullivan [15] investigated the energy $\mathcal{E}$ for various knots numerically, and proposed the following conjecture.

Conjecture 2.1 (The Kusner-Sullivan conjecture [15])

1. *There does not exist minimizers of composite knot type.*

2. *Assume that $\boldsymbol{f} \in [K]$ is composite, and it is a connected sum $\boldsymbol{f}_1 \sharp \boldsymbol{f}_2$, $\boldsymbol{f}_i \in [K_i]$ for $i = 1, 2$ (we say $[K] = [K_1] \sharp [K_2]$). It holds that*

$$\mathcal{E}([K]) = \mathcal{E}([K_1]) + \mathcal{E}([K_2]).$$

As far as the author knows, this is still open.

2.3 The bi-Lipschitz continuity

Since the Möbius energy was introduced to determine the "canonical configuration" of knots, we expect that the finiteness of energy suggests some regularity of the curve. Indeed we have the bi-Lipschitz estimate for the curve with finite energy, which means that the curve cannot bend sharply.

Since we use the arc-length parameter, the estimate

$$\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \leq \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$$

is trivial. *That is,* $\boldsymbol{f}$ is Lipschitz continuous with the Lipschitz constant 1. On the other hand, the finiteness of $\mathcal{E}$ implies the converse estimate.

Theorem 2.5 ([20, 2]) *If $\mathcal{E}(\boldsymbol{f}) < M$, then there exists $\lambda = \lambda(M) > 0$ such that*
$$\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \geq \lambda \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)).$$

We call $\boldsymbol{f}$ is *bi-Lipschitz* if $\boldsymbol{f}$ satisfies these both estimates. From this theorem, it is natural to assume that $\boldsymbol{f}$ is bi-Lipschitz when the Möbius energy is finite.

2.4 The regularity of critical points

The bi-Lipschitz continuity holds for all curves with finite energy. The criticality of energy derives more informations about regularity. Several results are known. For example, Freedman-He-Wang [8] and He [9] showed the regularity of local minimizers.

Theorem 2.6 ([8, 9]) *Local minimizers of $\mathcal{E}$ with respect to L^∞ topology are smooth.*

Reiter [23] proved the regularity not only for local minimizers but for critical points.

Theorem 2.7 ([23]) *Any critical points of $\mathcal{E}$ in $W^{2,2}$ are smooth.*

Recently Blatt-Reiter-Shikorra [7] relaxed the assumption of the previous result.

Theorem 2.8 ([7]) *Any critical points of $\mathcal{E}$ with finite energy are smooth.*

The finiteness of energy implies not only bi-Lipschitz continuity but also the integrability of (fractional) derivatives.

Definition 2.2 (Sobolev-Slobodeckij space) *For $j \in \mathbb{N} \cup \{0\}$, and $\alpha \in (0,1)$, $W^{j+\alpha,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ is defined as*

$$W^{j+\alpha,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) = \{\boldsymbol{f} \in W^{j,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n) \,|\, [\boldsymbol{f}^{(j)}]_{\alpha,p} < \infty\},$$

$$[\boldsymbol{f}^{(j)}]_{\alpha,p} = \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{\|\boldsymbol{f}^{(j)}(s_1 + s_2) - \boldsymbol{f}^{(j)}(s_1)\|_{\mathbb{R}^n}^p}{|s_2|^{\alpha p + 1}} \, ds_2 ds_1 \right)^{\frac{1}{p}}$$

with the norm
$$\|\boldsymbol{f}\|_{W^{j+\alpha,p}} = \|\boldsymbol{f}\|_{W^{j,p}} + [\boldsymbol{f}^{(j)}]_{\alpha,p}.$$
When $p = 2$, we denote $W^{j+\alpha,2}$ by $H^{j+\alpha}$.

The following result is due to Blatt [2].

Theorem 2.9 ([2]) *The finiteness $\mathcal{E}(\boldsymbol{f}) < \infty$ implies the bi-Lipschitz continuity of $\boldsymbol{f}$ and $\boldsymbol{f} \in H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. The converse is also true, i.e., if $\boldsymbol{f}$ is bi-Lipschitz and belongs to $H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathcal{E}(\boldsymbol{f})$ is finite.*

This shows the proper domain of $\mathcal{E}$.

Remark 2.2 In fact Blatt [2] considered the general cases. Let $(\alpha, p) \in (0, \infty)^2$ satisfy $\alpha p \geq 2$, $\alpha \geq 1$, $(\alpha - 2)p < 1$. Then $\mathcal{E}_{(\alpha,p)}(\boldsymbol{f})$ is finite if and only if $\boldsymbol{f} \in W^{\frac{(2+p)\alpha - 1}{2\alpha}, 2\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and bi-Lipschitz continuous.

2.5 The gradient flow

Let $\delta_{L^2}\mathcal{E}$ be the L^2-gradient:

$$\langle \delta_{L^2}\mathcal{E}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})} = \left. \frac{d}{d\varepsilon} \mathcal{E}(\boldsymbol{f} + \varepsilon\boldsymbol{\phi}) \right|_{\varepsilon=0}.$$

For the explicit formula of the L^2-gradient, see subsection 5.4 or [14]. Consider the L^2-gradient flow:
$$\partial_t \boldsymbol{f} = -\delta_{L^2}\mathcal{E}(\boldsymbol{f}).$$

The local existence and uniqueness of the L^2-gradient flow was shown by He [9] for smooth initial data, and improved by Blatt [3] for the initial curve in the little Hölder space.

Theorem 2.10 ([9]) *Let the knot $\boldsymbol{f}_0$ be smooth. Then there exists a unique local solution to the L^2-gradient flow with $\boldsymbol{f}(0) = \boldsymbol{f}_0$*

Theorem 2.11 ([3]) *Let $\boldsymbol{f}_0 \in h^{2+\alpha}$, where $h^{2+\alpha}$ is the little Hölder space of order $2 + \alpha$. Then there exists a unique local solution to the L^2-gradient flow with $\boldsymbol{f}(0) = \boldsymbol{f}_0$.*

Blatt [3] also showed the global existence near local minimizers.

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Theorem 2.12 ([3]) *Let $\boldsymbol{f}_*$ be a local minimizer of $\mathcal{E}$ in C^k for some $k \in \{0\} \cup \mathbb{N}$. Assume $\|\boldsymbol{f}(0) - \boldsymbol{f}_*\|_{C^{2+\beta}} \ll 1$. Then the L^2-gradient flow with the initial knot $\boldsymbol{f}(0)$ exists globally in time. $\boldsymbol{f}(t)$ has the limit $\lim_{t \to \infty} \boldsymbol{f}(t) = \boldsymbol{f}_\infty$, and $\boldsymbol{f}_\infty$ is a critical point satisfying $\mathcal{E}(\boldsymbol{f}_*) = \mathcal{E}(\boldsymbol{f}_\infty)$.*

Blatt showed the following *Łojasiewicz-Simon gradient estimate* in his paper. Let $\boldsymbol{f}_*$ be a critical point. Then there exist $\theta \in [0, \frac{1}{2}]$, $\sigma > 0$, $c > 0$ such that $\|\boldsymbol{f} - \boldsymbol{f}_*\|_{H^3} \leq \sigma$ implies

$$|\mathcal{E}(\boldsymbol{f}) - \mathcal{E}(\boldsymbol{f}_*)|^{1-\theta} \leq c\|\delta_{L^2}\mathcal{E}(\boldsymbol{f})\|_{L^2}$$

The assertion of global existence follows from Łojasiewicz's argument. It is still open whether the limit curve $\boldsymbol{f}_\infty$ is the image of some Möbius transformation of $\boldsymbol{f}_*$ or not.

For the case of $n = 2$, then the L^2-gradient flow exists globally in time, and converges to a right circle, *i.e.*, to a global minimizer.

Theorem 2.13 ([4]) *Let $\boldsymbol{f}(0)$ be a planar curve. Then there exists a global solution to the L^2-gradient flow with the initial knot $\boldsymbol{f}(0)$ such that*

$$\boldsymbol{f}(t) \to S^1 \text{(a right circle) as } t \to \infty.$$

Blatt kindly informed the author his result of gradient flow for the subcritical case.

Theorem 2.14 ([5]) *For $\mathcal{E}_{(1,p)}$ with $p > 2$ it holds that*

1. *there exists a global solution of length-constraint-gradient flow for any smooth initial knots,*

2. *the flow converges to a critical point.*

3 The decomposition theorem

In this section we show the decomposition theorem of the Möbius energy, which has already been proved in [11].

At a point where $\boldsymbol{f}$ is differentiable, we denote the unit tangent vector by $\boldsymbol{\tau} = \boldsymbol{f}'$. Similarly $\boldsymbol{\kappa} = \boldsymbol{\tau}'$ stands for the curvature vector at a point where $\boldsymbol{f}$ is twice differentiable.

Theorem 3.1 ([11], Theorem 2.1) *Let $\boldsymbol{f} \in X$ and suppose that there exists a positive constant λ such that $\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \geq \lambda^{-1}\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$. Then the energy $\mathcal{E}(\boldsymbol{f})$ may be decomposed as*

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4,$$

where

$$\mathcal{E}_i(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}_i(\boldsymbol{f}) \, ds_1 ds_2,$$

$$\mathcal{M}_1(\boldsymbol{f}) = \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2},$$

$$\mathcal{M}_2(\boldsymbol{f}) = \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4}$$
$$\times \det \begin{pmatrix} \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix}.$$

Moreover, each $\mathcal{M}_i(\boldsymbol{f})(i = 1, 2)$ is absolutely integrable.

Proof. As we said in the Introduction, to deform the energy density, we first consider the integration in the sense of Cauchy's principal value, and show the absolute integrability later. We differentiate

$$\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2 = \begin{cases} (s_1 - s_2)^2 & (s_1 \leq s_2 \leq s_1 + \frac{\mathcal{L}}{2} \pmod{\mathcal{L}}), \\ (s_1 - s_2 + \mathcal{L})^2 & (s_1 + \frac{\mathcal{L}}{2} \leq s_2 \leq s_1 + \mathcal{L} \pmod{\mathcal{L}}) \end{cases}$$

with respect to s_2. In the sense of distributions,

$$\frac{d}{dx} \log|x| = \mathrm{p.v.} \frac{1}{x},$$

and therefore

$$\frac{\partial}{\partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) = \begin{cases} -\mathrm{p.v.} \frac{1}{s_1 - s_2} & (s_1 \leq s_2 < s_1 + \frac{\mathcal{L}}{2} \pmod{\mathcal{L}}), \\ -\mathrm{p.v.} \frac{1}{s_1 - s_2 + \mathcal{L}} & (s_1 + \frac{\mathcal{L}}{2} < s_2 \leq s_1 + \mathcal{L} \pmod{\mathcal{L}}). \end{cases}$$

Here, the distribution $\mathrm{p.v.} \frac{1}{x}$ is given by

$$\langle \mathrm{p.v.} \frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \to +0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

for $\varphi \in C_0^\infty(\mathbb{R})$ (see [16]). Using the periodicity of $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$, this equals

$$\frac{\partial}{\partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) = \begin{cases} -\mathrm{p.v.} \frac{1}{s_1 - s_2 - \mathcal{L}} & (s_2 + \frac{\mathcal{L}}{2} < s_1 \leq s_2 + \mathcal{L} \pmod{\mathcal{L}}), \\ -\mathrm{p.v.} \frac{1}{s_1 - s_2} & (s_2 \leq s_1 < s_2 + \frac{\mathcal{L}}{2} \pmod{\mathcal{L}}). \end{cases}$$

Regarding this as a distribution of s_1, it is differentiable for $s_1 \neq s_2 + \mathcal{L}/2$ in the weak sense and we obtain

$$\frac{\partial^2}{\partial s_1 \partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) = \begin{cases} \mathrm{p.v.} \frac{1}{(s_1 - s_2 - \mathcal{L})^2} & (s_2 + \frac{\mathcal{L}}{2} < s_1 \leq s_2 + \mathcal{L} \pmod{\mathcal{L}}), \\ \mathrm{p.v.} \frac{1}{(s_1 - s_2)^2} & (s_2 \leq s_1 < s_2 + \frac{\mathcal{L}}{2}) \pmod{\mathcal{L}} \end{cases}$$
$$= \mathrm{p.v.} \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}.$$

As a function of s_2, $\frac{\partial}{\partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$ has a jump discontinuity at $s_2 = s_1 + \mathcal{L}/2$ with gap $-4/\mathcal{L}$; that is,

$$\lim_{s_2 \to s_1 + \frac{\mathcal{L}}{2} + 0} \frac{\partial}{\partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) - \lim_{s_2 \to s_1 + \frac{\mathcal{L}}{2} - 0} \frac{\partial}{\partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) = -\frac{4}{\mathcal{L}}.$$

As functions of s_2, $\frac{\partial^2}{\partial s_1 \partial s_2} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}$ is bounded at $s_2 = s_1 \pm \frac{\mathcal{L}}{2}$, and $\frac{\partial}{\partial s_1} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}$ is continuous at the same points. Therefore we get

$$\int_{\varepsilon < |s_1 - s_2| \leq \frac{\mathcal{L}}{2}} \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \, ds_2$$

$$= \lim_{\delta \to +0} \int_{\varepsilon < |s_1 - s_2| \leq \frac{\mathcal{L}}{2} - \delta} \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \, ds_2$$

$$= \lim_{\delta \to +0} \int_{\varepsilon < |s_1 - s_2| \leq \frac{\mathcal{L}}{2} - \delta} \frac{\partial^2}{\partial s_1 \partial s_2} \log \mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) \, ds_2$$

$$= \lim_{\delta \to +0} \int_{\varepsilon < |s_1 - s_2| \leq \frac{\mathcal{L}}{2} - \delta} \frac{\partial^2}{\partial s_1 \partial s_2} \left(\log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} - \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right) \, ds_2$$

$$= \int_{\varepsilon < |s_1 - s_2| \leq \frac{\mathcal{L}}{2}} \frac{\partial^2}{\partial s_1 \partial s_2} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \, ds_2$$

$$\quad - \lim_{\delta \to +0} \left[\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right]_{s_2 = s_1 - \frac{\mathcal{L}}{2} + \delta}^{s_2 = s_1 - \varepsilon}$$

$$\quad - \lim_{\delta \to +0} \left[\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right]_{s_2 = s_1 + \varepsilon}^{s_2 = s_1 + \frac{\mathcal{L}}{2} - \delta}$$

$$= \int_{\varepsilon < |s_1 - s_2| \leq \frac{\mathcal{L}}{2}} \frac{\partial^2}{\partial s_1 \partial s_2} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \, ds_2$$

$$\quad + \left[\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right]_{s_2 = s_1 - \varepsilon}^{s_2 = s_1 + \varepsilon} - \frac{4}{\mathcal{L}}.$$

We integrate this with respect to s_1 and firstly note that

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right]_{s_2 = s_1 - \varepsilon}^{s_2 = s_1 + \varepsilon} \, ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{(\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{s_1 - s_2} \right]_{s_2 = s_1 - \varepsilon}^{s_2 = s_1 + \varepsilon} \, ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{(\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 + \varepsilon)) \cdot \boldsymbol{\tau}(s_1)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 + \varepsilon)\|_{\mathbb{R}^n}^2} - \frac{(\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 - \varepsilon)) \cdot \boldsymbol{\tau}(s_1)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 - \varepsilon)\|_{\mathbb{R}^n}^2} + \frac{2}{\varepsilon} \right\} \, ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{(\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 + \varepsilon)) \cdot \boldsymbol{\tau}(s_1)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 + \varepsilon)\|_{\mathbb{R}^n}^2} - \frac{(\boldsymbol{f}(s_1 + \varepsilon) - \boldsymbol{f}(s_1)) \cdot \boldsymbol{\tau}(s_1 + \varepsilon)}{\|\boldsymbol{f}(s_1 + \varepsilon) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} + \frac{2}{\varepsilon} \right\} \, ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ -\frac{\partial}{\partial s_1} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_1 + \varepsilon)\|_{\mathbb{R}^n} + \frac{2(\boldsymbol{f}(s_1 + \varepsilon) - \boldsymbol{f}(s_1)) \cdot \boldsymbol{\tau}(s_1)}{\|\boldsymbol{f}(s_1 + \varepsilon) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} + \frac{2}{\varepsilon} \right\} \, ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[-2\varepsilon \left(\frac{1}{\|\boldsymbol{f}(s_1 + \varepsilon) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} - \frac{1}{\varepsilon^2} \right) \right.$$

$$\left. + \frac{2}{\|\boldsymbol{f}(s_1 + \varepsilon) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1 + \varepsilon} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_1)) \, ds_3 \right] \, ds_1.$$

We have

$$\left| 2\varepsilon \left(\frac{1}{\|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} - \frac{1}{\varepsilon^2} \right) \right|$$

$$= \frac{2\varepsilon}{\varepsilon^2 \|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1+\varepsilon} \int_{s_1}^{s_1+\varepsilon} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) \, ds_3 ds_4$$

$$= \frac{1}{\varepsilon \|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1+\varepsilon} \int_{s_1}^{s_1+\varepsilon} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4$$

and using estimate $\|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n} \geq \lambda^{-1}\varepsilon$, along with a change in the order of integration, we obtain

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{\varepsilon \|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1+\varepsilon} \int_{s_1}^{s_1+\varepsilon} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4 ds_1$$

$$\leq \frac{\lambda^2}{\varepsilon^3} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_1}^{s_1+\varepsilon} \int_{s_1}^{s_1+\varepsilon} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4 ds_1$$

$$\leq \frac{\lambda^2}{\varepsilon^3} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_4-\varepsilon}^{s_4+\varepsilon} \int_{s_3-\varepsilon}^{s_3} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_1 ds_3 ds_4$$

$$= \frac{\lambda^2}{\varepsilon^2} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_4-\varepsilon}^{s_4+\varepsilon} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4$$

$$\leq \lambda^2 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_4-\varepsilon}^{s_4+\varepsilon} \frac{\|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_4))^2} \, ds_3 ds_4.$$

Since

$$[\boldsymbol{f}']_{H^{1/2}}^2 = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_4))^2} \, ds_3 ds_4$$

is finite, the absolute continuity of integration yields

$$\lim_{\varepsilon \to +0} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{\varepsilon \|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1+\varepsilon} \int_{s_1}^{s_1+\varepsilon} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4 ds_1 = 0.$$

Similarly we have

$$\left| \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{2}{\|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1+\varepsilon} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_1)) \, ds_3 ds_1 \right|$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{\|\boldsymbol{f}(s_1+\varepsilon)-\boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} \int_{s_1}^{s_1+\varepsilon} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 \, ds_3 ds_1$$

$$\leq \lambda^2 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_1}^{s_1+\varepsilon} \frac{\|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_1))^2} \, ds_3 ds_1 \to 0 \quad \text{as} \quad \varepsilon \to +0.$$

Hence we obtain

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right]_{s_2=s_1-\varepsilon}^{s_2=s_1+\varepsilon} ds_1 \to 0 \quad \text{as} \quad \varepsilon \to +0,$$

which leads to the expression

$$\mathcal{E}(\boldsymbol{f}) = \text{p.v.} \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{\partial^2}{\partial s_1 \partial s_2} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \right) ds_1 ds_2 + 4.$$

Manipulating the above log term, we obtain

$$
\mathcal{E}(\boldsymbol{f}) - 4
$$

$$
= \mathrm{p.v.} \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left\{ \frac{1 + \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} \right.
$$

(3.1)
$$
\left. - \frac{2\left\{ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \right\} \left\{ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) \right\}}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4} \right\} ds_1 ds_2.
$$

The density of this integral is expressed as

$$
\frac{1 + \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{2\left\{ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \right\} \left\{ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) \right\}}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4}
$$

$$
= \frac{1 - \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} + \frac{2\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2) \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4}
$$

$$
\quad - \frac{\left\{ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \right\} \left\{ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) \right\}}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4}
$$

$$
= \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}
$$

$$
\quad + \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4} \det \begin{pmatrix} \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)) \cdot \boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix}
$$

$$
= \mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})
$$

and it remains to remove p.v. in front of the double integral in (3.1). To this end it is enough to see

$$
\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}) \geq 0.
$$

We use the notation Δ to mean the difference between values at $s = s_1$ and s_2 of a function $\boldsymbol{v}$ on $\mathbb{R}/\mathcal{L}\mathbb{Z}$:

$$
\Delta s = s_1 - s_2, \quad \Delta \boldsymbol{v} = \boldsymbol{v}(s_1) - \boldsymbol{v}(s_2).
$$

We note that the difference operator Δs is different from Δ_s, which we have defined as the Laplace operator. By using the Lagrange formula, we have

$$
\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\langle \left(\boldsymbol{\tau}(s_1) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right), \left(\boldsymbol{\tau}(s_2) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right) \right\rangle,
$$

where $\wedge$ is the wedge product of vectors, and $\langle \cdot, \cdot \rangle$ is the inner product on $\bigwedge^2 \mathbb{R}^n$. It is easy to see for any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any unit vector $\boldsymbol{e} \in \mathbb{R}^n$ that

$$
\langle (\boldsymbol{x} \wedge \boldsymbol{e}), (\boldsymbol{y} \wedge \boldsymbol{e}) \rangle = (P_e^{\perp} \boldsymbol{x}) \cdot (P_e^{\perp} \boldsymbol{y}),
$$

where

$$
P_e \boldsymbol{x} = (\boldsymbol{x} \cdot \boldsymbol{e})\boldsymbol{e}, \quad P_e^{\perp} \boldsymbol{x} = \boldsymbol{x} - P_e \boldsymbol{x}.
$$

Therefore

$$
\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (P_{Rf}^{\perp} \boldsymbol{\tau}(s_1)) \cdot (P_{Rf}^{\perp} \boldsymbol{\tau}(s_2)),
$$

where

$$
R\boldsymbol{f} = \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}.
$$

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On the other hand, the first density is

$$\mathscr{M}_1(\boldsymbol{f}) = \frac{1}{2}\frac{\|\Delta\boldsymbol{\tau}\|^2_{\mathbb{R}^n}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}.$$

Consequently we obtain

$$
\begin{aligned}
2\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}&(\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}))\\
&= \|\Delta\boldsymbol{\tau}\|^2_{\mathbb{R}^n} + 4(P^\perp_{Rf}\boldsymbol{\tau}(s_1))\cdot(P^\perp_{Rf}\boldsymbol{\tau}(s_2))\\
&= \|P_{Rf}\Delta\boldsymbol{\tau}\|^2_{\mathbb{R}^n} + \|P^\perp_{Rf}\Delta\boldsymbol{\tau}\|^2_{\mathbb{R}^n} + 4(P^\perp_{Rf}\boldsymbol{\tau}(s_1))\cdot(P^\perp_{Rf}\boldsymbol{\tau}(s_2))\\
&= \|P_{Rf}\Delta\boldsymbol{\tau}\|^2_{\mathbb{R}^n} + \|P^\perp_{Rf}(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2))\|^2_{\mathbb{R}^n},
\end{aligned}
$$

which is non-negative and then the non-negativity of $\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})$ is shown.

Finally we show the absolute integrability of each $\mathscr{M}_i(\boldsymbol{f})$. The integrand $\mathscr{M}_1(\boldsymbol{f})$ is non-negative and

$$
\begin{aligned}
\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_1(\boldsymbol{f})\,ds_1 ds_2 &\leq \frac{\lambda^2}{2}\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|^2_{\mathbb{R}^n}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}\,ds_1 ds_2\\
&= \frac{\lambda^2}{2}[\boldsymbol{f}']^2_{H^{1/2}} < \infty,
\end{aligned}
$$

which shows the absolute integrability of $\mathscr{M}_1(\boldsymbol{f})$. Both $\mathscr{M}_1(\boldsymbol{f})$ and $\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})$ are absolutely integrable, hence so is $\mathscr{M}_2(\boldsymbol{f})$. $\qquad\square$

Remark 3.1 The proof of non-negativity of $\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})$ in [11] is improved here.

Remark 3.2 O'Hara kindly informed the author that if $\boldsymbol{f}$ is sufficiently smooth, then (3.1) can be shown by using the cosine formula [15].

As a consequence of the non-negativity of $\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})$, we can give a new proof of Theorem 2.1.

Proof of Theorem 2.1. From the non-negativity, $\mathcal{E}(\boldsymbol{f}) \geq 4$. Furthermore $\mathcal{E}(\boldsymbol{f}) = 4$ holds if and only of $\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}) \equiv 0$. This is also equivalent to

$$P_{Rf}\Delta\boldsymbol{\tau} \equiv \boldsymbol{o}, \quad P^\perp_{Rf}(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)) \equiv \boldsymbol{o}.$$

In particular from the second relation we find a function μ such that

$$(3.2) \qquad \boldsymbol{f}(s_1) - \boldsymbol{f}(s_2) = \mu(s_1, s_2)(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)).$$

Since a minimizer is smooth, so is μ. We differentiate the above relation three

times with respect to s_1 to obtain

(3.3)
$$\boldsymbol{\tau}(s_1) = \frac{\partial \mu}{\partial s_1}(s_1, s_2)\left(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\right) + \mu(s_1, s_2)\boldsymbol{\kappa}(s_1),$$

(3.4)
$$\boldsymbol{\kappa}(s_1) = \frac{\partial^2 \mu}{\partial s_1^2}(s_1, s_2)\left(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\right) + 2\frac{\partial \mu}{\partial s_1}(s_1, s_2)\boldsymbol{\kappa}(s_1) + \mu(s_1, s_2)\boldsymbol{\kappa}'(s_1),$$

(3.5)
$$\boldsymbol{\kappa}'(s_1) = \frac{\partial^3 \mu}{\partial s_1^3}(s_1, s_2)\left(\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\right) + 3\frac{\partial^2 \mu}{\partial s_1^2}(s_1, s_2)\boldsymbol{\kappa}(s_1)$$
$$+ 3\frac{\partial \mu}{\partial s_1}(s_1, s_2)\boldsymbol{\kappa}'(s_1) + \mu(s_1, s_2)\boldsymbol{\kappa}''(s_1).$$

Putting $s_1 = s_2 = s$ in (3.2)–(3.5), we have

$$\mu(s, s) = 0, \quad \frac{\partial \mu}{\partial s_1}(s, s) = \frac{1}{2}, \quad \frac{\partial^2 \mu}{\partial s_1^2}(s, s) = 0,$$

and

(3.6)
$$\boldsymbol{\kappa}'(s) = -4\frac{\partial^3 \mu}{\partial s_1^3}(s, s)\boldsymbol{\tau}(s).$$

Taking the inner product between each side of (3.6) and $\boldsymbol{\kappa}(s)$, we know

$$\left(\|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2\right)' = 0.$$

Therefore $\|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}$ is independent of s, and we write it κ. If $\kappa = 0$, then $\boldsymbol{\tau}'(s) = \boldsymbol{\kappa}(s) = \boldsymbol{o}$, and therefore $\boldsymbol{\tau}(s)$ is a constant vector. It is impossible because $\boldsymbol{f}$ is a closed curve. Consequently $\kappa > 0$. Taking the inner product between each sides of (3.6) and $\boldsymbol{\tau}(s)$, we know

$$-4\frac{\partial^3 \mu}{\partial s_1^3}(s, s) = \boldsymbol{\kappa}'(s) \cdot \boldsymbol{\tau}(s) = -\boldsymbol{\kappa}(s) \cdot \boldsymbol{\kappa}(s) = -\kappa^2.$$

Inserting this into (3.6), we obtain

$$\boldsymbol{\kappa}'(s) + \kappa^2 \boldsymbol{\tau}(s) = \boldsymbol{o}$$

for every $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. Since $\boldsymbol{\tau}(s) = \boldsymbol{f}'(s)$, there exists a constant vector $\boldsymbol{c}$ such that

(3.7)
$$\boldsymbol{\kappa}(s) + \kappa^2 \left(\boldsymbol{f}(s) - \boldsymbol{c}\right) = \boldsymbol{o}.$$

Integrating with respect to s on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, and deviding by $\mathcal{L}$, we find

$$\boldsymbol{c} = \frac{1}{\mathcal{L}}\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \boldsymbol{f}(s)\,ds.$$

We can rewrite (3.7) as the second order differential equation

$$\left(\boldsymbol{f}(s) - \boldsymbol{c}\right)'' + \kappa^2 \left(\boldsymbol{f}(s) - \boldsymbol{c}\right) = \boldsymbol{o}.$$

The solution is

$$\boldsymbol{f}(s) - \boldsymbol{c} = (\boldsymbol{f}(0) - \boldsymbol{c})\cos\kappa s + \frac{\sin\kappa s}{\kappa}\boldsymbol{\tau}(0),$$

that is, $\boldsymbol{f}$ is a right circle with center $\boldsymbol{c}$ and radius κ^{-1}. Since the total length is $\mathcal{L}$, the radius is $\kappa^{-1} = \dfrac{\mathcal{L}}{2\pi}$. $\qquad\qquad\square$

4 The Möbius invariance

In this section we discuss the invariance of $\mathcal{E}_i$ under Möbius transformations. The results and proofs have been published in [11, 13].

The invariance under dilation can be easily shown. In subsection 4.1, we show the invariance of the sum $\mathcal{E}_1 + \mathcal{E}_2$ under the inversion

$$\boldsymbol{f} \mapsto \boldsymbol{p} = \boldsymbol{c} + \frac{r^2\,(\boldsymbol{f} - \boldsymbol{c})}{\|\boldsymbol{f} - \boldsymbol{c}\|_{\mathbb{R}^n}^2}$$

with respect to the sphere with center $\boldsymbol{c}$ and radius r. Note that we assume neither the finiteness of energy $\mathcal{E}(\boldsymbol{f}) < \infty$ nor $\boldsymbol{f}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \ni \boldsymbol{c}$. Indeed we show the invariance of the sum of measures $(\mathscr{M}_1 + \mathscr{M}_2)\,ds_1 ds_2$. This fact was shown by Freedman-He-Wang [8] for $\boldsymbol{f}$ which are parameterized by arc-length, and here we present an alternative proof. We discuss the invariance of each $\mathcal{E}_i$ under the inversion assuming the finiteness of energy and $\boldsymbol{c} \notin \boldsymbol{f}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ in subsection 4.2. We deal with the case $\boldsymbol{c} \in \boldsymbol{f}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, and show the invariance under the assumption $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

4.1 The invariance of the sum of energies

In this section we show the invariance of $(\mathscr{M}_1 + \mathscr{M}_2)\,ds_1 ds_2$ and as a consequence the invariance of the sum of energies follows. First we note

$$\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}) = \frac{1 + \boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2)}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\,(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\,(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}.$$

Even if s is an arc-length parameter for $\boldsymbol{f}$, it is not necessarily so for $\boldsymbol{p}$. Therefore we use a general parameter θ instead of s, and the energy density with respect to $d\theta_1 d\theta_2$ is

$$(\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}))\,\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}$$
$$= \frac{\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n} + \dot{\boldsymbol{f}}(\theta_1)\cdot\dot{\boldsymbol{f}}(\theta_2)}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$
$$+ \frac{1}{2}\left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2\right).$$

Here $\dot{\boldsymbol{f}}$ means the differentiation of $\boldsymbol{f}$ with respect to the general parameter, and similarly for other functions.

Theorem 4.1 ([11], Theorem 3.1) *Let*

$$\boldsymbol{f} \mapsto \boldsymbol{p} = \boldsymbol{c} + \frac{r^2\,(\boldsymbol{f} - \boldsymbol{c})}{\|\boldsymbol{f} - \boldsymbol{c}\|_{\mathbb{R}^n}^2}$$

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be an inversion with respect to the sphere with center $\boldsymbol{c}$ and radius r. Then,

$$(\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}))\,\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}$$

(4.1)
$$- (\mathscr{M}_1(\boldsymbol{p}) + \mathscr{M}_2(\boldsymbol{p}))\,\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n} = 0$$

holds for θ_1 and θ_2 such that

$$\boldsymbol{f}(\theta_1) \neq \boldsymbol{f}(\theta_2), \quad \boldsymbol{f}(\theta_i) \neq \boldsymbol{c} \quad (i = 1,\, 2).$$

Proof. We decompose the difference between the density for $\boldsymbol{f}$ and that for $\boldsymbol{p}$ as follows:

$$(\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}))\,\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}$$

(4.2)
$$- (\mathscr{M}_1(\boldsymbol{p}) + \mathscr{M}_2(\boldsymbol{p}))\,\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n}$$
$$= J_1 + J_2 + J_3,$$

where

$$J_1 = \frac{\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2} - \frac{\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n}}{\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2},$$

$$J_2 = \frac{\dot{\boldsymbol{f}}(\theta_1) \cdot \dot{\boldsymbol{f}}(\theta_2)}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2} - \frac{\dot{\boldsymbol{p}}(\theta_1) \cdot \dot{\boldsymbol{p}}(\theta_2)}{\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2},$$

$$J_3 = \frac{1}{2}\left\{\left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2\right)\right.$$
$$\left. - \left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2\right)\right\}.$$

It holds that

$$\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2$$
$$= \left\|\frac{r^2\,(\boldsymbol{f}(\theta_1) - \boldsymbol{c})}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2} - \frac{r^2\,(\boldsymbol{f}(\theta_2) - \boldsymbol{c})}{\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}\right\|_{\mathbb{R}^n}^2$$
$$= r^4\left\{\frac{1}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2} - \frac{2\,(\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot (\boldsymbol{f}(\theta_2) - \boldsymbol{c})}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2} + \frac{1}{\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}\right\}$$
$$= \frac{r^4\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}.$$

If we define the projection $P_c(\theta)$ and $P_c^\perp(\theta)$ for a vector $\boldsymbol{v}$ by

$$P_c(\theta)\boldsymbol{v} = \left(\boldsymbol{v} \cdot \frac{\boldsymbol{f}(\theta) - \boldsymbol{c}}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}}\right)\frac{\boldsymbol{f}(\theta) - \boldsymbol{c}}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}}, \quad P_c^\perp(\theta) = I - P_c(\theta),$$

then the derivative of $\boldsymbol{p}(\theta)$ may be expressed as

$$\dot{\boldsymbol{p}}(\theta) = r^2\left[\frac{\dot{\boldsymbol{f}}(\theta)}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}^2} - \frac{2\{\dot{\boldsymbol{f}}(\theta) \cdot (\boldsymbol{f}(\theta) - \boldsymbol{c})\}\,(\boldsymbol{f}(\theta) - \boldsymbol{c})}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}^4}\right]$$
$$= \frac{r^2}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}\left(P_c^\perp(\theta) - P_c(\theta)\right)\dot{\boldsymbol{f}}(\theta),$$

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and therefore

$$\|\dot{\boldsymbol{p}}(\theta)\|_{\mathbb{R}^n}^2 = \frac{r^4}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}^4} \left(\|P_c^\perp(\theta)\dot{\boldsymbol{f}}(\theta)\|_{\mathbb{R}^n}^2 + \|P_c(\theta)\dot{\boldsymbol{f}}(\theta)\|_{\mathbb{R}^n}^2 \right)$$

$$= \frac{r^4\|\dot{\boldsymbol{f}}(\theta)\|_{\mathbb{R}^n}^2}{\|\boldsymbol{f}(\theta) - \boldsymbol{c}\|_{\mathbb{R}^n}^4}.$$

Using straightforward considerations,

$$\frac{\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n}}{\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2}$$

$$= \frac{r^2\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2} \frac{r^2\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}}{\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2} \frac{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}{r^4\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$

$$= \frac{\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$

and this demonstrates that $J_1 = 0$. By a similar calculation

$$\frac{\dot{\boldsymbol{p}}(\theta_1) \cdot \dot{\boldsymbol{p}}(\theta_2)}{\|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2} = \frac{\left\{ (I - 2P_c(\theta_1))\,\dot{\boldsymbol{f}}(\theta_1) \right\} \cdot \left\{ (I - 2P_c(\theta_2))\,\dot{\boldsymbol{f}}(\theta_2) \right\}}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$

holds. Observing that

$$(I - 2P_c(\theta_i))\,\dot{\boldsymbol{f}}(\theta_i) = \dot{\boldsymbol{f}}(\theta_i) - 2\left(\frac{\boldsymbol{f}(\theta_i) - \boldsymbol{c}}{\|\boldsymbol{f}(\theta_i) - \boldsymbol{c}\|_{\mathbb{R}^n}} \cdot \dot{\boldsymbol{f}}(\theta_i) \right) \frac{\boldsymbol{f}(\theta_i) - \boldsymbol{c}}{\|\boldsymbol{f}(\theta_i) - \boldsymbol{c}\|_{\mathbb{R}^n}}$$

$$= \dot{\boldsymbol{f}}(\theta_i) - \left(\frac{\partial}{\partial\theta_i} \log \|\boldsymbol{f}(\theta_i) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) (\boldsymbol{f}(\theta_i) - \boldsymbol{c}),$$

we may write

$$J_2 = \frac{1}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$

$$\times \left[\dot{\boldsymbol{f}}(\theta_1) \cdot \dot{\boldsymbol{f}}(\theta_2) - \left\{ \dot{\boldsymbol{f}}(\theta_1) - \left(\frac{\partial}{\partial\theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) (\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \right\} \right.$$

$$\left. \cdot \left\{ \dot{\boldsymbol{f}}(\theta_2) - \left(\frac{\partial}{\partial\theta_1} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) (\boldsymbol{f}(\theta_2) - \boldsymbol{c}) \right\} \right]$$

$$= \frac{1}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2} \left[\left(\frac{\partial}{\partial\theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left\{ (\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_2) \right\} \right.$$

$$+ \left(\frac{\partial}{\partial\theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left\{ (\boldsymbol{f}(\theta_2) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_1) \right\}$$

$$- \left(\frac{\partial}{\partial\theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial\theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$

$$\left. \times \left\{ (\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot (\boldsymbol{f}(\theta_2) - \boldsymbol{c}) \right\} \right].$$

Using

$$(\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_2) = (\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2) + \boldsymbol{f}(\theta_2) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_2)$$

$$= \frac{1}{2} \frac{\partial}{\partial\theta_2} \left(-\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 + \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$

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and

$$(\boldsymbol{f}(\theta_2) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_1) = (\boldsymbol{f}(\theta_2) - \boldsymbol{f}(\theta_1) + \boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_1)$$
$$= \frac{1}{2} \frac{\partial}{\partial \theta_1} \left(-\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 + \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right),$$

we arrive at

$$\frac{(\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_2)}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$
$$= -\frac{1}{2} \frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 + \frac{1}{2} \frac{\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2} \frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2,$$
$$\frac{(\boldsymbol{f}(\theta_2) - \boldsymbol{c}) \cdot \dot{\boldsymbol{f}}(\theta_1)}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$
$$= -\frac{1}{2} \frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 + \frac{1}{2} \frac{\|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2} \frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2$$

upon which our previous expression for J_2 becomes

$$(4.3) \quad J_2 = -\frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 \right)$$
$$- \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$
$$+ \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$
$$\times \frac{\|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 + \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 - 2 (\boldsymbol{f}(\theta_1) - \boldsymbol{c}) \cdot (\boldsymbol{f}(\theta_2) - \boldsymbol{c})}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2}$$
$$= -\frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 \right)$$
$$- \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$
$$+ \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right).$$

Finally,

$$\frac{\partial}{\partial \theta_i} \log \|\boldsymbol{p}(\theta_1) - \boldsymbol{p}(\theta_2)\|_{\mathbb{R}^n}^2 = \frac{\partial}{\partial \theta_i} \left(\log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 - \log \|\boldsymbol{f}(\theta_i) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right),$$

and therefore

$$(4.4) \quad J_3 = \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 \right)$$
$$+ \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$
$$- \frac{1}{2} \left(\frac{\partial}{\partial \theta_1} \log \|\boldsymbol{f}(\theta_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right) \left(\frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(\theta_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \right)$$
$$= -J_2.$$

$\square$

Corollary 4.1 ([11], Corollary 3.1) *Let $\boldsymbol{f} \in W^{1,1}(\mathbb{R}/2\pi\mathbb{Z})$. Then it holds that*

$$\mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + \mathcal{E}_2(\boldsymbol{p}).$$

Remark 4.1 The corollary does not exclude that both sides are infinite.

Proof. First let $\boldsymbol{f} \in H^{3/2}(\mathbb{R}/2\pi\mathbb{Z})$. If the 2-dimensional Lebesgue measure of

$$\{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \,|\, \boldsymbol{f}(\theta_1) = \boldsymbol{f}(\theta_2)\}$$

is positive, then so is that of

$$\{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \,|\, \boldsymbol{p}(\theta_1) = \boldsymbol{p}(\theta_2)\}.$$

Therefore $\mathcal{E}(\boldsymbol{f}) = \mathcal{E}(\boldsymbol{p}) = \infty$ and we have

$$\mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + \mathcal{E}_2(\boldsymbol{p}) = \infty.$$

Thus, we assume that the 2-dimensional Lebesgue measure of

$$\{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \,|\, \boldsymbol{f}(\theta_1) = \boldsymbol{f}(\theta_2)\}$$

is 0, and we claim that the measure of

$$\{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \,|\, \boldsymbol{f}(\theta_1) = \boldsymbol{f}(\theta_2) \text{ or } \boldsymbol{f}(\theta_1) = \boldsymbol{c} \text{ or } \boldsymbol{f}(\theta_2) = \boldsymbol{c}\}$$

is also 0. In order to see this, considering $\boldsymbol{f}$ as a function of s on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, we need to prove that

$$S = \{s \in \mathbb{R}/\mathcal{L}\mathbb{Z} \,|\, \boldsymbol{f}(s) = \boldsymbol{c}\}$$

is a finite set. Arguing by contradiction, we suppose that S is not a finite set and using the compactness of $\mathbb{R}/\mathcal{L}\mathbb{Z}$, there exists a sequence such that $\boldsymbol{f}(s_j) = \boldsymbol{c}$ and $\lim_{j\to\infty} s_j = s_*$. From

$$0 = \|\boldsymbol{f}(s_{j+1}) - \boldsymbol{f}(s_j)\|^2_{\mathbb{R}^n} = \int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \boldsymbol{\tau}(s) \cdot \boldsymbol{\tau}(s') \, ds ds'$$

$$= (s_{j+1} - s_j)^2 - \frac{1}{2} \int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s')\|^2_{\mathbb{R}^n} \, ds ds',$$

it holds that

$$\int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \frac{\|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s')\|^2_{\mathbb{R}^n}}{(s - s')^2} \, ds ds' \geq \int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \frac{\|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s')\|^2_{\mathbb{R}^n}}{(s_{j+1} - s_j)^2} \, ds ds' = 2.$$

However, using that $\boldsymbol{\tau} \in H^{1/2}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and the absolute continuity of integral, this leads to

$$2 \leq \lim_{j\to\infty} \int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \frac{\|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s')\|^2_{\mathbb{R}^n}}{(s - s')^2} \, ds ds' = 0,$$

which is obviously a contradiction. By these arguments we find that (4.1) holds for $\mathscr{L}^2$-a.e. (θ_1, θ_2) and the desired conclusion follows by integrating this.

Finally, we consider the case of $\boldsymbol{f} \notin H^{3/2}(\mathbb{R}/2\pi\mathbb{Z})$ which implies $\mathcal{E}(\boldsymbol{f}) = \infty$, and we will show $\mathcal{E}(\boldsymbol{p}) = \infty$. Again, arguing by contradiction, we suppose that $\mathcal{E}(\boldsymbol{p}) < \infty$ from which we know that $\boldsymbol{p}$ does not have self-intersections. Furthermore $\boldsymbol{p} \in H^{3/2}_{\mathrm{loc}}$ and we remark that $\boldsymbol{p} \in H^{3/2}$ if $\boldsymbol{p}$ does not pass through the point at infinity. If we turn $\boldsymbol{p}$ back by the inversion with respect to the sphere with center $\boldsymbol{c}$ and radius r, then it returns to $\boldsymbol{f}$. Since $\boldsymbol{f}$ does not pass through the point at infinity, $\boldsymbol{p}$ does not pass through $\boldsymbol{c}$. Thereby the 2-dimensional Lebesgue measure of

$$\{(\theta_1, \theta_2) \,|\, \boldsymbol{p}(\theta_1) = \boldsymbol{p}(\theta_2), \text{ or } \boldsymbol{p}(\theta_1) = \boldsymbol{c} \text{ or } \boldsymbol{p}(\theta_2) = \boldsymbol{c}\}$$

is 0. This implies that (4.1) holds for $\mathscr{L}^2$-a.e. (θ_1, θ_2) and integrating this, we get

$$\mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + \mathcal{E}_2(\boldsymbol{p}) = \mathcal{E}(\boldsymbol{p}) - 4 < \infty.$$

However, from this we obtain $\infty = \mathcal{E}(f) = \mathcal{E}_1(f) + \mathcal{E}_2(f) + 4 < \infty$ which is obviously a contradiction. As a conclusion, $\mathcal{E}(p) = \infty$ holds and therefore $\mathcal{E}_1(f) + \mathcal{E}_2(f) = \mathcal{E}_1(p) + \mathcal{E}_2(p) = \infty$ as desired. $\qquad\square$

4.2 The invariance of each energy

We discuss here the invariance of each energy $\mathcal{E}_i$ under the inversion $\boldsymbol{f} \mapsto \boldsymbol{p}$.

Theorem 4.2 ([11], Theorem 3.2) *Assume that the center $\boldsymbol{c}$ of the inversion is not in the image of $\boldsymbol{f}$. We also assume the finiteness of energy $\mathcal{E}(\boldsymbol{f}) < \infty$. Then*

$$\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}), \quad \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_2(\boldsymbol{p})$$

holds.

Proof. In view of Corollary 4.1, it is enough to prove $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p})$. Let J_1 and J_2 be as defined in the proof of Theorem 4.1. It follows that

$$\mathscr{M}_1(\boldsymbol{f})\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n} - \mathscr{M}_1(\boldsymbol{p})\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n} = J_1 - J_2 = -J_2.$$

We need to prove that the integration of this goes to 0. From now on, we use the arc-length variable s_j.

Since we are assuming $\mathcal{E}(\boldsymbol{f}) < \infty$, we know that $\mathscr{M}_1(\boldsymbol{f}) \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$ holds. Since $\mathscr{M}_1(\boldsymbol{p}) \geq 0$, we may write

$$\mathcal{E}_1(\boldsymbol{f}) - \mathcal{E}_1(\boldsymbol{p}) = -\lim_{\varepsilon \to +0} \iint_{|s_1 - s_2| \geq \varepsilon} \frac{J_2}{\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}} \, ds_1 ds_2,$$

where $|s_1 - s_2| \geq \varepsilon$ is in the sense of $\mod \mathcal{L}$. Remarking that

$$\frac{1}{\|\dot{\boldsymbol{f}}(\theta_j)\|_{\mathbb{R}^n}} \frac{\partial}{\partial \theta_j} (\cdots) = \frac{\partial}{\partial s_j} (\cdots),$$

we get

$$- \lim_{\varepsilon \to +0} \iint_{|s_1-s_2|\geq\varepsilon} \frac{J_2}{\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}} ds_1 ds_2$$

$$= \lim_{\varepsilon \to +0} \frac{1}{2} \iint_{|s_1-s_2|\geq\varepsilon} \left\{ \left(\frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2\right) \left(\frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2\right) \right.$$

$$+ \left(\frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2\right) \left(\frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_2)-\boldsymbol{c}\|_{\mathbb{R}^n}^2\right)$$

$$- \left. \left(\frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2\right) \left(\frac{\partial}{\partial \theta_2} \log\|\boldsymbol{f}(s_2)-\boldsymbol{c}\|_{\mathbb{R}^n}^2\right)\right\} ds_1 ds_2$$

$$= \lim_{\varepsilon \to +0} \frac{1}{2} \int_0^{\mathcal{L}} \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 \int_{s_1+\varepsilon}^{s_1+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_2 ds_1$$

$$+ \lim_{\varepsilon \to +0} \frac{1}{2} \int_0^{\mathcal{L}} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_2)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 \int_{s_2+\varepsilon}^{s_2+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_1 ds_2$$

$$- \frac{1}{2} \int_0^{\mathcal{L}} \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 ds_1 \int_0^{\mathcal{L}} \frac{\partial}{\partial \theta_2} \log\|\boldsymbol{f}(s_2)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 ds_2.$$

It holds that

$$\int_{s_1+\varepsilon}^{s_1+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_2 = \log\frac{\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_1-\varepsilon)\|_{\mathbb{R}^n}^2}{\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_1+\varepsilon)\|_{\mathbb{R}^n}^2}.$$

Moreover, $\mathcal{E}(\boldsymbol{f}) < \infty$ implies that $\boldsymbol{f} \in H^{3/2}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and thus

$$\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_1\pm\varepsilon)\|_{\mathbb{R}^n}^2 = \varepsilon^2 + o(\varepsilon^2)$$

uniformly with regard to s_1 as $\varepsilon \to +0$. Therefore

$$\int_{s_1+\varepsilon}^{s_1+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_2 = \log(1+o(1)) = o(1)$$

holds uniformly with regard to s_1. From this and the fact that $\boldsymbol{f}$ does not pass through $\boldsymbol{c}$,

$$\left| \int_0^{\mathcal{L}} \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 \int_{s_1+\varepsilon}^{s_1+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_2 ds_1 \right|$$

$$= o(1) \int_0^{\mathcal{L}} \left| \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 \right| ds_1$$

$$= o(1)$$

holds. Then we arrive at

$$\lim_{\varepsilon \to +0} \int_0^{\mathcal{L}} \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 \int_{s_1+\varepsilon}^{s_1+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_2 ds_1 = 0,$$

and similarly we find that

$$\lim_{\varepsilon \to +0} \int_0^{\mathcal{L}} \frac{\partial}{\partial s_2} \log\|\boldsymbol{f}(s_2)-\boldsymbol{c}\|_{\mathbb{R}^n}^2 \int_{s_2+\varepsilon}^{s_2+\mathcal{L}-\varepsilon} \frac{\partial}{\partial s_1} \log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 ds_1 ds_2 = 0.$$

Finally, since $\boldsymbol{f}$ does not pass through $\boldsymbol{c}$, we have

$$\int_0^{\mathcal{L}} \frac{\partial}{\partial s_1} \log \|\boldsymbol{f}(s_1) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \, ds_1 \int_0^{\mathcal{L}} \frac{\partial}{\partial \theta_2} \log \|\boldsymbol{f}(s_2) - \boldsymbol{c}\|_{\mathbb{R}^n}^2 \, ds_2 = 0$$

from which the desired conclusion $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}(\boldsymbol{p})$ follows. $\qquad\square$

If the center is in the image of $\boldsymbol{f}$, then the above invariance does not hold. Indeed, let $\boldsymbol{f}$ be a right circle, and then $\boldsymbol{p}$ is a line. Therefore,

$$\mathcal{E}_1(\boldsymbol{f}) = 2\pi^2 \neq 0 = \mathcal{E}_1(\boldsymbol{p}).$$

Taking this typical case and Theorem 4.2 into consideration, we expect that

$$\mathcal{E}_1(\boldsymbol{p}) = \mathcal{E}_1(\boldsymbol{f}) - 2\pi^2, \quad \mathcal{E}_2(\boldsymbol{p}) = \mathcal{E}_2(\boldsymbol{f}) + 2\pi^2$$

holds for the case $\boldsymbol{c} \in \boldsymbol{f}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Here we show the above relation under the assumption that $\boldsymbol{f} \in C^{1,1}$ and it has bi-Lipschitz continuity.

Theorem 4.3 ([13], Theorem 1.2) *Assume that $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and that there exists a positive constant λ satisfying $\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \geq \lambda^{-1}\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$. Let $\boldsymbol{c}$ be a point on the curve $\boldsymbol{f}$, and let $\boldsymbol{p}$ be an inversion of $\boldsymbol{f}$ with respect to a sphere whose center is $\boldsymbol{c}$. Then, it follows that*

$$\mathcal{E}_1(\boldsymbol{p}) = \mathcal{E}_1(\boldsymbol{f}) - 2\pi^2, \quad \mathcal{E}_2(\boldsymbol{p}) = \mathcal{E}_2(\boldsymbol{f}) + 2\pi^2.$$

As a corollary, we find that the global minimizers of $\mathcal{E}_1$ in $C^{1,1}$ are right circles.

Corollary 4.2 ([13], Corollary 1.1) *It holds that*

$$\inf\{\mathcal{E}_1(\boldsymbol{f}) \mid \boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}), \text{ bi-Lipschitz}\} = 2\pi^2.$$

The infimum is attained if and only if $\boldsymbol{f}$ is a right circle.

Remark 4.2 Blatt [2] showed that the finiteness of $\mathcal{E}(\boldsymbol{f})$ implies that $\boldsymbol{f} \in X$ and that $\boldsymbol{f}$ has bi-Lipschitz continuity. However, we need $C^{1,1}$ regularity on $\boldsymbol{f}$ for the proof of the main theorem. Extending our theorem for this regularity seems to be an interesting problem. Note that $C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is not dense in X.

Since we need lengthy calculations to prove Theorem 4.3, we first give our strategy and the full details can be found in the next step. The proof of Corollary 4.2 will be given in subsection 4.3. Then let us describe the strategy for proving Theorem 4.3.

Let θ be a general parameter with 2π period. Under our assumption, $\boldsymbol{f}$ has no self-intersections, and therefore $\boldsymbol{f}$ passes through the center $\boldsymbol{c}$ at most once per period. Thus, we may assume that $\boldsymbol{c} = \boldsymbol{f}(0)$. Then, if $\theta \neq 0 \pmod{2\pi}$, it holds that $\boldsymbol{f}(\theta) \neq \boldsymbol{c}$.

From equations (4.2), (4.3), (4.4) and the fact $J_1 = 0$, it holds that

$$(4.5) \quad \mathscr{M}_1(\boldsymbol{f})\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n} - \mathscr{M}_1(\boldsymbol{p})\|\dot{\boldsymbol{p}}(\theta_1)\|_{\mathbb{R}^n}\|\dot{\boldsymbol{p}}(\theta_2)\|_{\mathbb{R}^n} = \frac{1}{2}J(\theta_1, \theta_2),$$

where

$$\frac{1}{2}J(\theta_1,\theta_2) = \left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2\right)$$
$$+\left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(\theta_2)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_2)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)$$
$$-\left(\frac{\partial}{\partial\theta_1}\log\|\boldsymbol{f}(\theta_1)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial\theta_2}\log\|\boldsymbol{f}(\theta_2)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right).$$

Since

$$\frac{\partial}{\partial\theta_i}(\cdots)\,d\theta_i = \frac{\partial}{\partial s_i}(\cdots)\,ds_i,$$

it is easily shown that

$$(4.6) \qquad \iint_{(\mathbb{R}/2\pi\mathbb{Z})^2} J(\theta_1,\theta_2)d\theta_1 d\theta_2 = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{J}(s_1,s_2)\,ds_1 ds_2,$$

where

$$\mathscr{J}(s_1,s_2) = \left(\frac{\partial}{\partial s_1}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial s_2}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2\right)$$
$$+\left(\frac{\partial}{\partial s_1}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial s_2}\log\|\boldsymbol{f}(s_2)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)$$
$$-\left(\frac{\partial}{\partial s_1}\log\|\boldsymbol{f}(s_1)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial s_2}\log\|\boldsymbol{f}(s_2)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right).$$

Note that here, we put $s = 0$ at $\theta = 0$. Also we introduce a function $\mathscr{F}$ to replace the Euclidean distance $\|\cdot\|_{\mathbb{R}^n}$ in $\mathscr{J}$ with the intrinsic distance $\mathscr{D}$:

$$\mathscr{F}(s_1,s_2) = \left(\frac{\partial}{\partial s_1}\log\mathscr{D}_{10}^2\right)\left(\frac{\partial}{\partial s_2}\log\mathscr{D}_{12}^2\right)$$
$$+\left(\frac{\partial}{\partial s_1}\log\mathscr{D}_{12}^2\right)\left(\frac{\partial}{\partial s_2}\log\mathscr{D}_{20}^2\right)$$
$$-\left(\frac{\partial}{\partial s_1}\log\mathscr{D}_{10}^2\right)\left(\frac{\partial}{\partial s_2}\log\mathscr{D}_{20}^2\right),$$

where, for simplicity, we denote

$$\mathscr{D}(\boldsymbol{f}(s_i),\boldsymbol{f}(0)) = \mathscr{D}_{i0}, \quad \mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2)) = \mathscr{D}_{12}.$$

In the following section, we shall show that

$$(4.7) \qquad \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{F}(s_1,s_2)\,ds_1 ds_2 = 4\pi^2$$

and

$$(4.8) \qquad \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (\mathscr{J}(s_1,s_2) - \mathscr{F}(s_1,s_2))\,ds_1 ds_2 = 0.$$

The assertion of Theorem 4.3 easily follows from (4.5)–(4.8).

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We begin by considering (4.7). From

$$\mathscr{D}_{10} = \begin{cases} s_1 & \left(0 \le s_1 \le \dfrac{\mathcal{L}}{2}\right), \\ \mathcal{L} - s_1 & \left(\dfrac{\mathcal{L}}{2} \le s_1 \le \mathcal{L}\right), \end{cases}$$

we have

$$\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2 = \begin{cases} \dfrac{2}{s_1} & \left(0 < s_1 < \dfrac{\mathcal{L}}{2}\right), \\ \dfrac{2}{s_1 - \mathcal{L}} & \left(\dfrac{\mathcal{L}}{2} < s_1 < \mathcal{L}\right). \end{cases}$$

Similarly, we have

$$\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2 = \begin{cases} \dfrac{2}{s_2} & \left(0 < s_2 < \dfrac{\mathcal{L}}{2}\right), \\ \dfrac{2}{s_2 - \mathcal{L}} & \left(\dfrac{\mathcal{L}}{2} < s_2 < \mathcal{L}\right), \end{cases}$$

$$\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 = \frac{2}{s_1 - s_2}, \quad \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 = \frac{2}{s_2 - s_1}$$
$$\text{on} \quad \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, 0 < |s_1 - s_2| < \frac{\mathcal{L}}{2} \right\},$$

$$\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 = \frac{2}{s_1 - s_2 + \mathcal{L}}, \quad \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 = \frac{2}{s_2 - s_1 - \mathcal{L}}$$
$$\text{on} \quad \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, s_2 - s_1 > \frac{\mathcal{L}}{2} \right\},$$

$$\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 = \frac{2}{s_1 - s_2 - \mathcal{L}}, \quad \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 = \frac{2}{s_2 - s_1 + \mathcal{L}}$$
$$\text{on} \quad \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, s_1 - s_2 > \frac{\mathcal{L}}{2} \right\}.$$

Let

$$U_1 = \left(0, \frac{\mathcal{L}}{2}\right)^2, \quad U_2 = \left(\frac{\mathcal{L}}{2}, \mathcal{L}\right)^2$$

$$U_3 = \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, \frac{\mathcal{L}}{2} < s_1 < s_2 + \frac{\mathcal{L}}{2} < \mathcal{L} \right\},$$

$$U_4 = \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, \frac{\mathcal{L}}{2} < s_2 < s_1 + \frac{\mathcal{L}}{2} < \mathcal{L} \right\},$$

$$U_5 = \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, \frac{\mathcal{L}}{2} < s_2 + \frac{\mathcal{L}}{2} < s_1 < \mathcal{L} \right\},$$

$$U_6 = \left\{ (s_1, s_2) \in [0, \mathcal{L}]^2 \,\middle|\, \frac{\mathcal{L}}{2} < s_1 + \frac{\mathcal{L}}{2} < s_2 < \mathcal{L} \right\}$$

(see Figure 1). These sets are disjoint, and

$$\mathscr{L}^2 \left([0, \mathcal{L}]^2 \setminus \bigcup_{k=1}^{6} U_k \right) = 0,$$

Figure 1

where $\mathscr{L}^2$ denotes two dimensional Lebesgue measure. On U_1, it holds that

$$\mathscr{F} = \frac{2}{s_1} \cdot \frac{2}{s_2 - s_1} + \frac{2}{s_1 - s_2} \cdot \frac{2}{s_2} - \frac{2}{s_1} \cdot \frac{2}{s_2} = 0,$$

and similarly, $\mathscr{F} = 0$ on U_2. On U_3,

$$\mathscr{F} = \frac{2}{s_1 - \mathcal{L}} \cdot \frac{2}{s_2 - s_1} + \frac{2}{s_1 - s_2} \cdot \frac{2}{s_2} - \frac{2}{s_1 - \mathcal{L}} \cdot \frac{2}{s_2} = -\frac{4\mathcal{L}}{(s_1 - \mathcal{L})s_2(s_1 - s_2)},$$

and similarly, on U_4,

$$\mathscr{F} = \frac{4\mathcal{L}}{s_1(s_2 - \mathcal{L})(s_1 - s_2)}.$$

It follows that on U_5,

$$\mathscr{F} = \frac{2}{s_1 - \mathcal{L}} \cdot \frac{2}{s_2 - s_1 + \mathcal{L}} + \frac{2}{s_1 - s_2 - \mathcal{L}} \cdot \frac{2}{s_2} - \frac{2}{s_1 - \mathcal{L}} \cdot \frac{2}{s_2} = 0,$$

and on U_6, $\mathscr{F} = 0$.

It follows from the above that $\mathscr{F}$ is positive on U_3 and satisfies

$$\iint_{U_3} \mathscr{F} \, ds_1 ds_2 = \int_{\frac{\mathcal{L}}{2}}^{\mathcal{L}} \frac{4\mathcal{L}}{\mathcal{L} - s_1} \int_{s_1 - \frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{ds_2}{s_2(s_1 - s_2)} \, ds_1$$

$$= -\int_{\frac{\mathcal{L}}{2}}^{\mathcal{L}} \frac{8\mathcal{L}}{s_1(\mathcal{L} - s_1)} \log\left(\frac{2s_1}{\mathcal{L}} - 1\right) ds_1.$$

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Making the substitution $u = \dfrac{2s_1}{\mathcal{L}} - 1$, we have

$$\iint_{U_3} \mathscr{F}(s_1, s_2)\, ds_1 ds_2 = -16 \int_0^1 \frac{\log u}{1 - u^2}\, du = 2\pi^2$$

(see [22, 2.6.5.9]). Similarly, it follows that

$$\iint_{U_4} \mathscr{F}(s_1, s_2)\, ds_1 ds_2 = 2\pi^2,$$

and hence we get

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{F}(s_1, s_2)\, ds_1 ds_2 = 4\pi^2.$$

Next, we consider (4.8). It can be directly calculated that

$$
\begin{aligned}
&\mathscr{J}(s_1, s_2) - \mathscr{F}(s_1, s_2) \\
&= \frac{\partial}{\partial s_1}\left(\log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial s_2} \log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2\right)\left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2\right) \\
&\quad + \frac{\partial}{\partial s_1}\left(\log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial s_2} \log \|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2\right)\left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2\right) \\
&\quad - \frac{\partial}{\partial s_1}\left(\log \|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right)\left(\frac{\partial}{\partial s_2} \log \|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2\right)\left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2\right) \\
&= \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right)\left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2\right)\left\{\frac{\partial}{\partial s_2}\left(\log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} - \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right)\right\} \\
&\quad + \left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2\right)\left\{\frac{\partial}{\partial s_1}\left(\log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} - \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\right\} \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2\right)\left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right) \\
&= \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad - \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right) \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2\right)\left\{\frac{\partial}{\partial s_2}\left(\log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} - \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}\right)\right\} \\
&\quad + \left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2\right)\left\{\frac{\partial}{\partial s_1}\left(\log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} - \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2}\right)\right\} \\
&\quad + \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2\right)\left\{\frac{\partial}{\partial s_2}\left(\log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2} - 2\log \frac{\left\|\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right) - \boldsymbol{f}(0)\right\|_{\mathbb{R}^n}^2}{\mathscr{D}\left(\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right), \boldsymbol{f}(0)\right)^2}\right)\right\} \\
&\quad + \left(\frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2\right)\left\{\frac{\partial}{\partial s_1}\left(\log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2} - 2\log \frac{\left\|\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right) - \boldsymbol{f}(0)\right\|_{\mathbb{R}^n}^2}{\mathscr{D}\left(\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right), \boldsymbol{f}(0)\right)^2}\right)\right\} \\
&=: \sum_{j=1}^7 A_j(s_1, s_2),
\end{aligned}
$$

using

$$
\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 = -\frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2
$$

and

$$
\frac{\partial}{\partial s_1} \log \frac{\left\|\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right) - \boldsymbol{f}(0)\right\|_{\mathbb{R}^n}^2}{\mathscr{D}\left(\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right), \boldsymbol{f}(0)\right)^2} = \frac{\partial}{\partial s_2} \log \frac{\left\|\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right) - \boldsymbol{f}(0)\right\|_{\mathbb{R}^n}^2}{\mathscr{D}\left(\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right), \boldsymbol{f}(0)\right)^2}.
$$

The function A_j is of the form

$$
A_j(s_1, s_2) = \frac{\partial}{\partial s_1} A_{j1}(s_1, s_2)\, \frac{\partial}{\partial s_2} A_{j2}(s_1, s_2).
$$

In what follows, we will show the absolute integrability of $\frac{\partial}{\partial s_i} A_{jk}(s_1, s_2)$ with respect to s_i, and

$$
\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\partial}{\partial s_i} A_{ji}(s_1, s_2)\, ds_i = 0.
$$

Then, by virtue of Fubini's theorem, it holds that

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_j(s_1, s_2) ds_1 ds_2 = 0.$$

Lemma 4.1 *Let s_i, $s_j \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. If necessary, we use $s_j + m\mathcal{L}$ ($m \in \mathbb{Z}$) instead of s_j, and then we can assume that $|s_i - s_j| \leq \frac{\mathcal{L}}{2}$. We denote the sign of $s_i - s_j$ when taking s_j as above by $\mathrm{sgn}(s_i - s_j)$. Then,*

$$\frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2}$$
$$= \frac{2\,\mathrm{sgn}(s_i - s_j)}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} \int_{s_j}^{s_i} \int_{s_j}^{s_i} \boldsymbol{\tau}(s_k) \cdot (\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell))\, ds_k ds_\ell.$$

Proof. From the way in which we chose s_i and s_j, we have $\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j)) = |s_i - s_j|$, and therefore it follows that

$$\frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} = \frac{2\boldsymbol{\tau}(s_i) \cdot (\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j))}{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} - \frac{2}{s_i - s_j}$$
$$= \frac{2\{\boldsymbol{\tau}(s_i) \cdot (\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)) - (s_i - s_j)\}}{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}$$
$$\quad + 2(s_i - s_j) \left\{ \frac{1}{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} - \frac{1}{(s_i - s_j)^2} \right\}$$
$$= \frac{2}{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} \int_{s_j}^{s_i} (\boldsymbol{\tau}(s_i) \cdot \boldsymbol{\tau}(s_k) - 1)\, ds_k$$
$$\quad + \frac{2}{(s_i - s_j)\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} \int_{s_j}^{s_i} \int_{s_j}^{s_i} (1 - \boldsymbol{\tau}(s_k) \cdot \boldsymbol{\tau}(s_\ell))\, ds_k ds_\ell$$
$$= \frac{2}{(s_i - s_j)\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} \int_{s_j}^{s_i} \int_{s_j}^{s_i} \boldsymbol{\tau}(s_k) \cdot (\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell))\, ds_k ds_\ell$$
$$= \frac{2\,\mathrm{sgn}(s_i - s_j)}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2} \int_{s_j}^{s_i} \int_{s_j}^{s_i} \boldsymbol{\tau}(s_k) \cdot (\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell))\, ds_k ds_\ell.$$

$\square$

Corollary 4.3 *Assume that $\boldsymbol{f} \in H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n} \geq \lambda^{-1}\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$. Then, we have*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left| \frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \right| ds_i \leq 3\lambda^2 [\boldsymbol{\tau}]_{H^{\frac{1}{2}}}^2.$$

Furthermore, it follows that

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2}\, ds_i = 0.$$

Proof. From

$$\boldsymbol{\tau}(s_k) \cdot (\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell)) = (\boldsymbol{\tau}(s_k) - \boldsymbol{\tau}(s_i)) \cdot (\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell)) + 1 - \boldsymbol{\tau}(s_i) \cdot \boldsymbol{\tau}(s_\ell),$$

we arrive at

$$|\boldsymbol{\tau}(s_k) \cdot (\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell))| \leq \|\boldsymbol{\tau}(s_k) - \boldsymbol{\tau}(s_i)\|_{\mathbb{R}^n} \|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n} + \frac{1}{2}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n}^2$$
$$\leq \frac{1}{2}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_k)\|_{\mathbb{R}^n}^2 + \|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n}^2,$$

and hence by using Lemma 4.1, when $|s_i - s_j| \leq \frac{\mathcal{L}}{2}$,

$$\left| \frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \right|$$
$$\leq \frac{2\lambda^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^3} \int_{s_j}^{s_i} \int_{s_j}^{s_i} \left(\frac{1}{2}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_k)\|_{\mathbb{R}^n}^2 + \|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n}^2 \right) ds_k ds_\ell$$
$$= \frac{3\lambda^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \left| \int_{s_j}^{s_i} \|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_k)\|_{\mathbb{R}^n}^2 ds_k \right|$$
$$\leq 3\lambda^2 \left| \int_{s_j}^{s_i} \frac{\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_k)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_k))^2} ds_k \right|.$$

Thus, we have

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left| \frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \right| ds_i$$
$$= \int_{s_j - \frac{\mathcal{L}}{2}}^{s_j + \frac{\mathcal{L}}{2}} \left| \frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \right| ds_i$$
$$\leq 3\lambda^2 \int_{s_j - \frac{\mathcal{L}}{2}}^{s_j + \frac{\mathcal{L}}{2}} \left| \int_{s_j}^{s_i} \frac{\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s_k)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_k))^2} ds_k \right| ds_i$$
$$\leq 3\lambda^2 [\boldsymbol{\tau}]_{H^{\frac{1}{2}}}^2.$$

For the remaining claim, first observe that

$$0 \leq 1 - \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} = \frac{1}{2\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \int_{s_j}^{s_i} \int_{s_j}^{s_i} \|\boldsymbol{\tau}(s_k) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n}^2 ds_k ds_\ell$$
$$\leq \frac{1}{2} \int_{s_j}^{s_i} \int_{s_j}^{s_i} \frac{\|\boldsymbol{\tau}(s_k) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_k), \boldsymbol{f}(s_\ell))^2} ds_k ds_\ell.$$

Since

$$[\boldsymbol{\tau}]_{H^{\frac{1}{2}}}^2 = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\boldsymbol{\tau}(s_k) - \boldsymbol{\tau}(s_\ell)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_k), \boldsymbol{f}(s_\ell))^2} ds_k ds_\ell < \infty,$$

by using the absolute integrability, we obtain

$$1 - \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} \to 0 \quad (s_i \to s_j).$$

Hence, $\log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2}$ is a continuous function on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, and we obtain

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\partial}{\partial s_i} \log \frac{\|\boldsymbol{f}(s_i) - \boldsymbol{f}(s_j)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_i), \boldsymbol{f}(s_j))^2} ds_i = 0.$$

$\square$

Claim 4.1 *We have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_j(s_1, s_2)\, ds_1 ds_2 = 0$$

for $j = 1, 2$ and 3.

Proof. Applying Fubini's theorem and using Corollary 4.3, we have

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |A_1(s_1, s_2)|\, ds_1 ds_2$$
$$= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left| \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2} \right) \left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} \right) \right| ds_1 ds_2$$
$$\le 9\lambda^4 [\boldsymbol{\tau}]_{H^{\frac{1}{2}}}^4 < \infty,$$

and again using the corollary, we obtain

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_1(s_1, s_2) ds_1 ds_2$$
$$= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2} \right) \left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} \right) ds_1 ds_2$$
$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2} \right) \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} ds_2 \right) ds_1$$
$$= 0.$$

In the same way, we have

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_2(s_1, s_2)\, ds_1 ds_2$$
$$= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} \right) \left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2} \right) ds_1 ds_2$$
$$= 0,$$
$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_3(s_1, s_2)\, ds_1 ds_2$$
$$= -\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{\partial}{\partial s_1} \log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{10}^2} \right) \left(\frac{\partial}{\partial s_2} \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2} \right) ds_1 ds_2$$
$$= 0.$$

$\square$

Next we consider $A_4(s_1, s_2)$ and so we introduce some symbols. We set

$$\mathscr{M}(\boldsymbol{f}) = \frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}$$

and in what follows, as necessary we denote $\boldsymbol{v}(s_i)$ simply by $\boldsymbol{v}_i$. It is not difficult to see that

$$\lim_{(s_1, s_2) \to (s, s)} \frac{\boldsymbol{v}_i' - (R\boldsymbol{f} \cdot \boldsymbol{\tau}_i) R\boldsymbol{v}}{\Delta s} = \frac{(-1)^{i-1}}{2} \boldsymbol{v}''$$

for $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ (see Lemma 5.2). Taking this relation into consideration, we set

$$\tilde{Q}_i \boldsymbol{v} = (-1)^{i-1} 2\{\boldsymbol{v}'_i - (R\boldsymbol{f} \cdot \boldsymbol{\tau}_i) R\boldsymbol{v}\}.$$

Then setting

$$Q\boldsymbol{v} = \Delta \boldsymbol{v}',$$

it follows that

$$\lim_{(s_1,s_2)\to(s,s)} \frac{Q\boldsymbol{v}}{\Delta s} = \lim_{(s_1,s_2)\to(s,s)} \frac{\tilde{Q}_i \boldsymbol{v}}{\Delta s} = \boldsymbol{v}''$$

if $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. The operations Q and $\tilde{Q}_i$ are defined on functions $\boldsymbol{v}$ for which the derivatives $\boldsymbol{v}'$ exist almost everywhere.

Since $Q\boldsymbol{f} = \Delta\boldsymbol{\tau}$ and $\tilde{Q}_i \boldsymbol{f} = (-1)^{i-1}2\{\boldsymbol{\tau}_i - (R\boldsymbol{f} \cdot \boldsymbol{\tau}_i)R\boldsymbol{f}\} = (-1)^{i-1}2P_*^\perp \boldsymbol{\tau}_i$, we have

$$\mathcal{N}_1(\boldsymbol{f}) = \frac{1}{2}\|Q\boldsymbol{f}\|_{\mathbb{R}^n}^2, \quad \mathcal{N}_2(\boldsymbol{f}) = -\frac{1}{2}\tilde{Q}_1\boldsymbol{f} \cdot \tilde{Q}_2\boldsymbol{f}.$$

Lemma 4.2 *Let $Y = H^{\frac{1}{2}} \cap L^\infty$ and $\|\cdot\|_Y = \|\cdot\|_{H^{\frac{1}{2}}} + \|\cdot\|_{L^\infty}$. The followings hold.*

1. *If $\boldsymbol{f}$ is in $C^{1,1}$ and bi-Lipschitz, then $\mathscr{M}(\boldsymbol{f})$ is bounded.*

2. *For $\boldsymbol{v} \in C^{1,1}$,*

$$\left\| \frac{Q\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

3. *Assume that $\boldsymbol{f} \in X$ and that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))|$. Then there exists a positive constant C depending on $\|\boldsymbol{f}'\|_Y$ and λ such that*

$$\left\| \frac{\tilde{Q}_i \boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{v}'\|_Y$$

holds for all $\boldsymbol{v} \in X$.

Proof. 1. First we show the boundedness of $\mathscr{M}$. We may assume that $|s_1 - s_2| \leq \frac{\mathcal{L}}{2}$. We write $\boldsymbol{\kappa}(s) = \boldsymbol{\tau}'(s)$ and note that this exists for almost every s and satisfies $\|\boldsymbol{\kappa}\|_{L^\infty} < \infty$. Using this, it holds that

$$\begin{aligned}
0 \leq \mathscr{M}(\boldsymbol{f}) &= \frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \\
&= \frac{(s_1-s_2)^2 - \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{(s_1-s_2)^2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} \\
&= \frac{1}{(s_1-s_2)^2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1}\int_{s_2}^{s_1} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4))\, ds_3 ds_4 \\
&= \frac{1}{2(s_1-s_2)^2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1}\int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2\, ds_3 ds_4 \\
&= \frac{1}{2(s_1-s_2)^2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} \int_{s_2}^{s_1}\int_{s_2}^{s_1}\int_{s_4}^{s_3}\int_{s_4}^{s_3} \boldsymbol{\kappa}(s_5) \cdot \boldsymbol{\kappa}(s_6)\, ds_5 ds_6 ds_3 ds_4 \\
&\leq C\lambda^2 \|\boldsymbol{\kappa}\|_{L^\infty}^2 < \infty.
\end{aligned}$$

2. Without loss of generality, we may assume that $|s_1 - s_2| \leq \frac{\mathcal{L}}{2}$, and then we use $|\Delta s|$ instead of $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$ for simplicity. The assertions 1–2 are almost trivial. Indeed, it follows immediately that

$$\left\| \frac{Q\boldsymbol{v}}{\Delta s} \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} = \|\boldsymbol{v}'\|_{\mathrm{Lip}} \leq \|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

3. To show this assertion, we decompose $\frac{(-1)^{i-1}}{2}\tilde{Q}_i\boldsymbol{v} = \boldsymbol{v}'_i - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v}$ into

$$\boldsymbol{v}'_i - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v} = \left(\boldsymbol{v}'_i - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) + \left(\frac{\Delta\boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right) + (1 - R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v} = V_1 + V_2 + V_3.$$

We show L^2 estimate for each $V_i/\Delta s$. Estimates on $V_1/\Delta s$. It is easy to see that

$$\frac{1}{\Delta s}\left(\boldsymbol{v}'_i - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) = \frac{1}{(s_1 - s_2)^2}\int_{s_2}^{s_1}(\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s))\,ds$$

and from Hölder's inequality, it follows that

$$\left\| \frac{1}{s_1 - s_2}\int_{s_2}^{s_1}(\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s))\,ds \right\|_{\mathbb{R}^n} \leq \left\{ \frac{1}{s_1 - s_2}\int_{s_2}^{s_1}\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s))\|_{\mathbb{R}^n}^2\,ds \right\}^{\frac{1}{2}}.$$

Changing the order of integration, we have

$$\left\| \frac{1}{\Delta s}\left(\boldsymbol{v}'_i - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)}^2$$

$$\leq \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{s_1-\frac{\mathcal{L}}{2}}^{s_1+\frac{\mathcal{L}}{2}}\frac{1}{(s_1 - s_2)^3}\int_{s_2}^{s_1}\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s)\|_{\mathbb{R}^n}^2\,ds\,ds_2\,ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{s_1-\frac{\mathcal{L}}{2}}^{s_1}\frac{1}{(s_1 - s_2)^3}\int_{s_2}^{s_1}\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s)\|_{\mathbb{R}^n}^2\,ds\,ds_2\,ds_1$$

$$\qquad - \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{s_1}^{s_1+\frac{\mathcal{L}}{2}}\frac{1}{(s_1 - s_2)^3}\int_{s_1}^{s_2}\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s)\|_{\mathbb{R}^n}^2\,ds\,ds_2\,ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{s_1-\frac{\mathcal{L}}{2}}^{s_1}\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s)\|_{\mathbb{R}^n}^2\int_{s_1-\frac{\mathcal{L}}{2}}^{s}\frac{ds_2}{(s_1 - s_2)^3}\,ds\,ds_1$$

$$\qquad - \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{s_1}^{s_1+\frac{\mathcal{L}}{2}}\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s)\|_{\mathbb{R}^n}^2\int_{s}^{s_1+\frac{\mathcal{L}}{2}}\frac{ds_2}{(s_1 - s_2)^3}\,ds\,ds_1$$

$$\leq \frac{1}{2}\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{s_1-\frac{\mathcal{L}}{2}}^{s_1+\frac{\mathcal{L}}{2}}\frac{\|\boldsymbol{v}'(s_i) - \boldsymbol{v}'(s)\|_{\mathbb{R}^n}^2}{(s_i - s)^2}\,ds\,ds_1$$

$$= \frac{1}{2}[\boldsymbol{v}']_{H^{\frac{1}{2}}}^2.$$

Estimates on $V_2/\Delta s$. It follows that

$$\frac{1}{\Delta s}\left(\frac{\Delta\boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right) = \frac{1}{\Delta s}\left\{1 - \left\|\frac{\Delta s}{\Delta\boldsymbol{f}}\right\|_{\mathbb{R}^n}\right\}\frac{\Delta\boldsymbol{v}}{\Delta s}$$

$$= \frac{1}{\Delta s}\left\|\frac{\Delta s}{\Delta\boldsymbol{f}}\right\|_{\mathbb{R}^n}\left\{\left\|\frac{\Delta\boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n} - 1\right\}\frac{\Delta\boldsymbol{v}}{\Delta s}$$

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by the definition of R. From

$$0 \leq 1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2$$

$$= \frac{1}{(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} (1 - \boldsymbol{f}'(s_3) \cdot \boldsymbol{f}'(s_4)) \, ds_3 ds_4$$

$$= \frac{1}{(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) \, ds_3 ds_4$$

$$= \frac{1}{2(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4$$

$$\leq \frac{1}{(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \left(\|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 + \|\boldsymbol{\tau}(s_4) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 \right) ds_3 ds_4$$

$$\leq \frac{2}{s_1 - s_2} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 \, ds$$

and

$$0 \leq 1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2$$

$$= \frac{1}{(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} (1 - \boldsymbol{f}'(s_3) \cdot \boldsymbol{f}'(s_4)) \, ds_3 ds_4$$

$$= \frac{1}{(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) \, ds_3 ds_4$$

$$= \frac{1}{2(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4$$

$$\leq \frac{1}{2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \frac{\|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2}{|s_3 - s_4|^2} \, ds_3 ds_4$$

$$\leq \frac{1}{2} [\boldsymbol{\tau}]_{H^{\frac{1}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})}^2 \leq \frac{1}{2} \|\boldsymbol{f}'\|_Y^2$$

we have that

$$\left| 1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n} \right| = \frac{1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2}{1 + \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}} \leq \|\boldsymbol{f}'\|_Y \left\{ \frac{1}{s_1 - s_2} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 \, ds \right\}^{\frac{1}{2}}$$

and therefore

$$\left\| \frac{1}{\Delta s} \left(\frac{\Delta \boldsymbol{v}}{\Delta s} - R\boldsymbol{v} \right) \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)}^2$$

$$\leq \lambda^2 \|\boldsymbol{f}'\|_Y^2 \|\boldsymbol{v}\|_{\mathrm{Lip}}^2 \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{1}{(s_1 - s_2)^3} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s) - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 \, ds ds_1 ds_2$$

$$\leq \frac{1}{2} \lambda^2 \|\boldsymbol{f}'\|_Y^4 \|\boldsymbol{v}'\|_Y^2.$$

Estimates on $V_3 / \Delta s$.

$$\frac{(1 - R\boldsymbol{f} \cdot \boldsymbol{\tau}_i) R\boldsymbol{v}}{\Delta s} = \frac{1}{2} \frac{\|\boldsymbol{\tau}_i - R\boldsymbol{f}\|_{\mathbb{R}^n}^2}{\Delta s} \left\| \frac{\Delta s}{\Delta \boldsymbol{f}} \right\|_{\mathbb{R}^n} \frac{\Delta \boldsymbol{v}}{\Delta s}.$$

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Since

$$\boldsymbol{\tau}_i - R\boldsymbol{f} = \boldsymbol{f}'_i - \left\|\frac{\Delta s}{\Delta \boldsymbol{f}}\right\|_{\mathbb{R}^n} \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= \boldsymbol{f}'_i - \frac{\Delta \boldsymbol{f}}{\Delta s} + \left(1 - \left\|\frac{\Delta s}{\Delta \boldsymbol{f}}\right\|_{\mathbb{R}^n}\right)\frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= \frac{1}{s_1 - s_2}\int_{s_2}^{s_1}(\boldsymbol{f}'(s_i) - \boldsymbol{f}'(s))\,ds + \left(1 - \left\|\frac{\Delta s}{\Delta \boldsymbol{f}}\right\|_{\mathbb{R}^n}\right)\frac{\Delta \boldsymbol{f}}{\Delta s},$$

we have

$$\|\boldsymbol{\tau}_i - R\boldsymbol{f}\|_{\mathbb{R}^n}^2 \leq \frac{2}{s_1 - s_2}\int_{s_2}^{s_1}\|\boldsymbol{f}'(s_i) - \boldsymbol{f}'(s)\|_{\mathbb{R}^n}^2\,ds + 2\left(1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}\right)^2$$

$$\leq \frac{6}{s_1 - s_2}\int_{s_2}^{s_1}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2\,ds.$$

Hence,

$$\left\|\frac{(1 - R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v}}{\Delta s}\right\|_{\mathbb{R}^n} \leq \frac{3\lambda\|\boldsymbol{v}\|_{\mathrm{Lip}}}{(s_1 - s_2)^2}\left|\int_{s_2}^{s_1}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2\,ds\right|$$

and using $\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n} \leq 2$, we have

$$\left|\int_{s_2}^{s_1}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2\,ds\right|^2 \leq 4(s_1 - s_2)\int_{s_2}^{s_1}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2\,ds.$$

Consequently,

$$\left\|\frac{(1 - R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v}}{\Delta s}\right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)}^2$$

$$\leq 36\lambda^2\|\boldsymbol{v}\|_{\mathrm{Lip}}^2\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{1}{(s_1 - s_2)^3}\int_{s_2}^{s_1}\|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2\,ds\,ds_1\,ds_2$$

$$\leq 18\lambda^2\|\boldsymbol{f}'\|_Y^2\|\boldsymbol{v}'\|_Y^2$$

holds for $\boldsymbol{f} \in X$. From the assertions 2–3, boundedness of $\mathscr{M}_1$ and $\mathscr{M}_2$ are shown. $\qquad\square$

Using the L^∞-estimate, we obtain the following lemma.

Lemma 4.3 *It follows that*

$$|A_4(s_1, s_2)| \leq 4\|\mathscr{M}(\boldsymbol{f}) - \mathscr{M}_1(\boldsymbol{f}) - \mathscr{M}_2(\boldsymbol{f})\|_{L^\infty}$$
$$+ \frac{16}{\mathcal{L}\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(0))}\left(\chi_{U_3}(s_1, s_2) + \chi_{U_4}(s_1, s_2)\right) \quad \mathscr{L}^2\text{-}a.e.$$

Proof. Recall that

$$A_4(s_1, s_2) = \frac{\partial}{\partial s_1}\log\mathscr{D}_{10}^2\frac{\partial}{\partial s_2}A_{42}(s_1, s_2),$$

$$A_{42}(s_1, s_2) = \log\frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} - \log\frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2}.$$

We may assume that $(s_1, s_2) \in (0, \mathcal{L})^2$, and we will obtain estimates for $(s_1, s_2) \in U_1, U_2, U_3, U_4, U_5,$ and U_6, in that order.

First, let $(s_1, s_2) \in U_1$. It holds that

$$\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2 = \frac{2}{s_1},$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial s_2} A_{42}(s_1, s_2) \\
&= \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} - \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 - \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} + \frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2 \\
&= \int_0^{s_1} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} \, ds_3 - \frac{2}{s_2 - s_1} + \frac{2}{s_2} \\
&= \int_0^{s_1} \left[-\frac{2\boldsymbol{\tau}(s_2) \cdot \boldsymbol{\tau}(s_3)}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} + \frac{4\{\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))\} \cdot \{\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))\}}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^4} \right] ds_3 \\
&\quad - \frac{2s_1}{s_2(s_2 - s_1)} \\
&= \int_0^{s_1} \left(-2\mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - 2\mathscr{M}_2(\boldsymbol{f})(s_2, s_3) + \frac{2}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} \right) ds_3 - \frac{2s_1}{s_2(s_2 - s_1)} \\
&= \int_0^{s_1} \bigg\{ -2\mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - 2\mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \\
&\qquad + \frac{2}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} - \frac{2}{(s_2 - s_3)^2} \bigg\} \, ds_3 \\
&= 2 \int_0^{s_1} (\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3)) \, ds_3,
\end{aligned}
$$

and then we have

$$A_4(s_1, s_2) = \frac{4}{s_1} \int_0^{s_1} (\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3)) \, ds_3.$$

From the above, we obtain the claimed estimate on U_1.

Next, let $(s_1, s_2) \in U_2$. In a calculation similar to the above, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial s_2} A_{42}(s_1, s_2) \\
&= \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} - \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 - \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(\mathcal{L}))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(\mathcal{L})\|_{\mathbb{R}^n}^2} + \frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2 \\
&= \int_{\mathcal{L}}^{s_1} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} \, ds_3 - \frac{2}{s_2 - s_1} + \frac{2}{s_2 - \mathcal{L}} \\
&= \int_{\mathcal{L}}^{s_1} \left[-\frac{2\boldsymbol{\tau}(s_2) \cdot \boldsymbol{\tau}(s_3)}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} + \frac{4\{\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))\} \cdot \{\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))\}}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^4} \right] ds_3 \\
&\quad - \frac{2(s_1 - \mathcal{L})}{(s_2 - \mathcal{L})(s_2 - s_1)} \\
&= \int_{\mathcal{L}}^{s_1} \left(-2\mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - 2\mathscr{M}_2(\boldsymbol{f})(s_2, s_3) + \frac{2}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} \right) ds_3 - \frac{2(s_1 - \mathcal{L})}{s_2(s_2 - s_1)} \\
&= \int_{\mathcal{L}}^{s_1} \bigg\{ -2\mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - 2\mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \\
&\qquad + \frac{2}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} - \frac{2}{(s_2 - s_3)^2} \bigg\} \, ds_3 \\
&= 2 \int_{\mathcal{L}}^{s_1} (\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3)) \, ds_3,
\end{aligned}
$$

and hence

$$A_4(s_1, s_2) = \frac{4}{s_1 - \mathcal{L}} \int_{\mathcal{L}}^{s_1} \left(\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \right) ds_3,$$

and we have the claimed estimate on U_2.

Next, we assume that $(s_1, s_2) \in U_3$. From a straightforward computation, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial s_2} A_{42}(s_1, s_2) \\
&= \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} - \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 - \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(\mathcal{L}))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(\mathcal{L})\|_{\mathbb{R}^n}^2} + \frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2 \\
&= \int_{\mathcal{L}}^{s_1} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} ds_3 - \frac{2}{s_2 - s_1} + \frac{2}{s_2}.
\end{aligned}
$$

Here, the range of s_3 is $(s_1, \mathcal{L})$, and note that $s_1 < s_2 + \frac{\mathcal{L}}{2} < \mathcal{L}$ when $(s_1, s_2) \in U_3$. By definition, it holds that

$$
\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2 = \begin{cases} (s_3 - s_2)^2 & \left(s_1 < s_3 < s_2 + \dfrac{\mathcal{L}}{2} \right), \\[2mm] (s_3 - s_2 - \mathcal{L})^2 & \left(s_2 + \dfrac{\mathcal{L}}{2} < s_3 < \mathcal{L} \right), \end{cases}
$$

and then we have

$$
\begin{aligned}
\int_{s_1}^{\mathcal{L}} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} &= \int_{s_1}^{s_2 + \frac{\mathcal{L}}{2}} \frac{ds_3}{(s_3 - s_2)^2} + \int_{s_2 + \frac{\mathcal{L}}{2}}^{\mathcal{L}} \frac{ds_3}{(s_3 - s_2 - \mathcal{L})^2} \\
&= \frac{1}{s_1 - s_2} + \frac{1}{s_2} - \frac{4}{\mathcal{L}}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial s_2} A_{42}(s_1, s_2) \\
&= \int_{\mathcal{L}}^{s_1} \left\{ \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} \right\} ds_3 + \frac{8}{\mathcal{L}} \\
&= 2 \int_{\mathcal{L}}^{s_1} \left(\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \right) ds_3 + \frac{8}{\mathcal{L}}.
\end{aligned}
$$

Combining this with $\dfrac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2 = \dfrac{2}{s_1 - \mathcal{L}}$, we obtain

$$
\begin{aligned}
&A_4(s_1, s_2) \\
&= \frac{4}{s_1 - \mathcal{L}} \int_{\mathcal{L}}^{s_1} \left(\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \right) ds_3 + \frac{16}{\mathcal{L}(s_1 - \mathcal{L})}
\end{aligned}
$$

holds. Thus we have the claimed estimate on U_3.

Suppose that $(s_1, s_2) \in U_4$. It follows that

$$
\begin{aligned}
&\frac{\partial}{\partial s_2} A_{42}(s_1, s_2) \\
&= \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1))}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 - \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} + \frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2 \\
&= \int_{0}^{s_1} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} ds_3 - \frac{2}{s_2 - s_1} + \frac{2}{s_2 - \mathcal{L}}.
\end{aligned}
$$

Now, $0 < s_1 < \frac{\mathcal{L}}{2} < s_2 < s_1 + \frac{\mathcal{L}}{2}$ holds. In particular, it follows that $0 < s_2 - \frac{\mathcal{L}}{2} < s_1$. We decompose the range of s_3, that is, $(0, s_1)$, into $\left(0, s_2 - \frac{\mathcal{L}}{2}\right]$ and $\left[s_2 - \frac{\mathcal{L}}{2}, s_1\right)$. Then, from

$$
\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2 = \begin{cases} (s_3 - s_2 + \mathcal{L})^2 & \left(0, s_2 - \dfrac{\mathcal{L}}{2}\right], \\ (s_3 - s_2)^2 & \left[s_2 - \dfrac{\mathcal{L}}{2}, s_1\right), \end{cases}
$$

we have

$$
\int_0^{s_1} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} = \int_0^{s_2 - \frac{\mathcal{L}}{2}} \frac{ds_3}{(s_3 - s_2 + \mathcal{L})^2} + \int_{s_2 - \frac{\mathcal{L}}{2}}^{s_1} \frac{ds_3}{(s_3 - s_2)^2}
$$
$$
= -\frac{1}{s_2 - \mathcal{L}} + \frac{1}{s_2 - s_1} - \frac{4}{\mathcal{L}},
$$

and thus we obtain

$$
\frac{\partial}{\partial s_2} A_{42}(s_1, s_2)
$$
$$
= \int_0^{s_1} \left\{ \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} \right\} ds_3 - \frac{8}{\mathcal{L}}
$$
$$
= 2 \int_0^{s_1} \left(\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \right) ds_3 - \frac{8}{\mathcal{L}}
$$

which leads to the claimed estimate on U_4.

Next, let $(s_1, s_2) \in U_5$. From the calculations

$$
\frac{\partial}{\partial s_2} A_{42}(s_1, s_2)
$$
$$
= \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_1)\|_{\mathbb{R}^n}^2} - \frac{\partial}{\partial s_2} \log \mathscr{D}_{12}^2 - \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(\mathcal{L}))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(\mathcal{L})\|_{\mathbb{R}^n}^2} + \frac{\partial}{\partial s_2} \log \mathscr{D}_{20}^2
$$
$$
= \int_{\mathcal{L}}^{s_1} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} ds_3 - \frac{2}{s_2 - s_1 + \mathcal{L}} + \frac{2}{s_2}
$$
$$
= \int_{\mathcal{L}}^{s_1} \left\{ \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} \right\} ds_3
$$
$$
= 2 \int_{\mathcal{L}}^{s_1} \left(\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \right) ds_3,
$$

we obtain the claimed estimate on U_5.

Finally, let $(s_1, s_2) \in U_6$. It follows that

$$
\frac{\partial}{\partial s_2} A_{42}(s_1, s_2) = \int_0^{s_1} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{2}{s_2 - s_1 - \mathcal{L}} + \frac{2}{s_2 - \mathcal{L}}.
$$

From $0 < s_1 < \frac{\mathcal{L}}{2}$ and $s_1 + \frac{\mathcal{L}}{2} < s_2 < \mathcal{L}$, when $s_3 \in (0, s_1)$,

$$
\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2 = (s_3 - s_2 + \mathcal{L})^2.
$$

Therefore, we obtain

$$
\int_0^{s_1} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} = \int_0^{s_1} \frac{ds_3}{(s_3 - s_2 + \mathcal{L})^2} = -\frac{1}{s_2 - \mathcal{L}} + \frac{1}{s_2 - s_1 - \mathcal{L}},
$$

and we arrive at

$$\frac{\partial}{\partial s_2} A_{42}(s_1, s_2)$$
$$= \int_0^{s_1} \left\{ \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_2) \cdot (\boldsymbol{f}(s_2) - \boldsymbol{f}(s_3))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(s_2))^2} \right\} ds_3$$
$$= 2 \int_0^{s_1} \left(\mathscr{M}(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_1(\boldsymbol{f})(s_2, s_3) - \mathscr{M}_2(\boldsymbol{f})(s_2, s_3) \right) ds_3$$

which yields the claimed estimate on U_6. □

Claim 4.2 *We have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_j(s_1, s_2) \, ds_1 ds_2 = 0$$

for $j = 4$ and 5.

Proof. As stated before, it is enough to show $A_j \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Since the proof for A_5 is similar to that for A_4, we discuss A_4 only.

We have

$$\iint_{U_3} \frac{ds_1 ds_2}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(0))} = \int_0^{\frac{\mathcal{L}}{2}} \int_{\frac{\mathcal{L}}{2}}^{s_2 + \frac{\mathcal{L}}{2}} \frac{ds_1}{\mathcal{L} - s_1} \, ds_2 = \int_0^{\frac{\mathcal{L}}{2}} \left[-\log(\mathcal{L} - s_1) \right]_{\frac{\mathcal{L}}{2}}^{s_2 + \frac{\mathcal{L}}{2}} \, ds_2$$
$$= \int_0^{\frac{\mathcal{L}}{2}} \log \frac{\frac{\mathcal{L}}{2}}{\frac{\mathcal{L}}{2} - s_2} \, ds_2 = -\frac{\mathcal{L}}{2} \int_0^1 \log t \, dt = \frac{\mathcal{L}}{2},$$

and by similar considerations, we obtain

$$\iint_{U_4} \frac{ds_1 ds_2}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(0))} = \frac{\mathcal{L}}{2}.$$

Thus, $A_4 \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$, and using Fubini's theorem and Corollary 4.3, we arrive at

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_4(s_1, s_2) \, ds_1 ds_2$$
$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{10}^2 \right)$$
$$\times \left\{ \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\partial}{\partial s_2} \left(\log \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{12}^2} - \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2} \right) ds_2 \right\} ds_1$$
$$= 0.$$

□

Claim 4.3 *We have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} A_j(s_1, s_2) \, ds_1 ds_2 = 0$$

for $j = 6$ and 7.

Proof. We discuss A_6 only; the argument for A_7 is quite similar.

Recall that

$$A_6(s_1, s_2) = \frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 \frac{\partial}{\partial s_2} A_{62}(s_1, s_2),$$

$$A_{62}(s_1, s_2) = \log \frac{\|\boldsymbol{f}(s_2) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}{\mathscr{D}_{20}^2} - 2 \log \frac{\left\|\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right) - \boldsymbol{f}(0)\right\|_{\mathbb{R}^n}^2}{\mathscr{D}\left(\boldsymbol{f}\left(\frac{s_1+s_2}{2}\right), \boldsymbol{f}(0)\right)^2}.$$

From direct calculations, we obtain

$$\frac{\partial}{\partial s_2} A_{62}(s_1, s_2) = \left[\frac{2\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} \right]_{\frac{s_1+s_2}{2}}^{s_2} - \left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 \right]_{\frac{s_1+s_2}{2}}^{s_2}$$

$$= \int_{\frac{s_1+s_2}{2}}^{s_2} \frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} \, ds_3 - \left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 \right]_{\frac{s_1+s_2}{2}}^{s_2}$$

and

$$\frac{\partial}{\partial s_3} \frac{2\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}$$

$$= \frac{2\boldsymbol{\kappa}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} + \frac{2\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_3)}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{4|\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))|^2}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^4}$$

$$= \frac{2\boldsymbol{\kappa}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} + \frac{4\left\{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2 - |\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))|^2\right\}}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^4}$$

$$- \frac{2}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2}.$$

Therefore we have

$$A_6(s_1, s_2) = B_1(s_1, s_2) + B_2(s_1, s_2),$$

where

$$B_1(s_1, s_2) = \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 \right) \int_{\frac{s_1+s_2}{2}}^{s_1} \left[\frac{2\boldsymbol{\kappa}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} \right.$$

$$\left. + \frac{4\left\{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2 - |\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))|^2\right\}}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^4} \right] ds_3,$$

$$B_2(s_1, s_2) = \left(\frac{\partial}{\partial s_1} \log \mathscr{D}_{12}^2 \right) \left\{ -\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{2 \, ds_3}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3} \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 \right]_{\frac{s_1+s_2}{2}}^{s_2} \right\}.$$

We consider $B_1(s_1, s_2)$ first. When $0 < s_3 \le \frac{L}{2}$, we have

$$\boldsymbol{\kappa}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0)) = \int_0^{s_3} \boldsymbol{\kappa}(s_3) \cdot \boldsymbol{\tau}(s_4) \, ds_4 = \int_0^{s_3} \boldsymbol{\kappa}(s_3) \cdot (\boldsymbol{\tau}(s_4) - \boldsymbol{\tau}(s_3)) \, ds_4$$

$$= -\int_0^{s_3} \int_{s_4}^{s_3} \boldsymbol{\kappa}(s_3) \cdot \boldsymbol{\kappa}(s_5) \, ds_5 ds_4,$$

and

$$\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2 - |\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))|^2$$

$$= \int_0^{s_3} \int_0^{s_3} \{\boldsymbol{\tau}(s_4) \cdot \boldsymbol{\tau}(s_5) - (\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4))(\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_5))\} \, ds_4 ds_5$$

$$= \int_0^{s_3} \int_0^{s_3} \{(\boldsymbol{\tau}(s_4) \cdot \boldsymbol{\tau}(s_5) - 1) + (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4))$$
$$\qquad\qquad + (\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4))(1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_5))\} \, ds_4 ds_5$$

$$= \frac{1}{2} \int_0^{s_3} \int_0^{s_3} \Big\{ -\|\boldsymbol{\tau}(s_4) - \boldsymbol{\tau}(s_5)\|_{\mathbb{R}^n}^2 + \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2$$
$$\qquad\qquad + (\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_5)\|_{\mathbb{R}^n}^2 \Big\} \, ds_4 ds_5$$

$$= -\frac{1}{2} \int_0^{s_3} \int_0^{s_3} \int_{s_5}^{s_4} \int_{s_5}^{s_4} \boldsymbol{\kappa}(s_6) \cdot \boldsymbol{\kappa}(s_7) \, ds_6 ds_7 ds_4 ds_5$$
$$\quad + \frac{1}{2} \int_0^{s_3} \int_0^{s_3} \int_{s_4}^{s_3} \int_{s_4}^{s_3} \boldsymbol{\kappa}(s_6) \cdot \boldsymbol{\kappa}(s_7) \, ds_6 ds_7 ds_4 ds_5$$
$$\quad + \frac{1}{2} \int_0^{s_3} \int_0^{s_3} \int_{s_5}^{s_5} \int_{s_5}^{s_5} (\boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4))(\boldsymbol{\kappa}(s_6) \cdot \boldsymbol{\kappa}(s_7)) \, ds_6 ds_7 ds_4 ds_5.$$

If $\frac{\mathcal{L}}{2} \leq s_3 < \mathcal{L}$, then the above identity will still hold replacing $\int_0^{s_3}$ with $\int_{s_3}^{\mathcal{L}}$. Therefore, regardless of the location of s_3, we have

$$\left| \frac{2\boldsymbol{\kappa}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} \right| \leq C\lambda^2 \|\boldsymbol{\kappa}\|_{L^\infty}^2,$$

$$\left| \frac{4 \left\{ \|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2 - |\boldsymbol{\tau}(s_3) \cdot (\boldsymbol{f}(s_3) - \boldsymbol{f}(0))|^2 \right\}}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^4} \right| \leq C\lambda^4 \|\boldsymbol{\kappa}\|_{L^\infty}^2,$$

and hence, we have $B_1 \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$.

Next, we consider B_2 and so we define

$$\tilde{U}_3 = \{(s_1, s_2) \,|\, 0 < s_2 < -s_1 + \mathcal{L} < s_1 < \mathcal{L}\},$$

$$\tilde{U}_4 = \{(s_1, s_2) \,|\, -s_1 + \mathcal{L} < s_2 < \frac{\mathcal{L}}{2} < s_1 < \mathcal{L}\},$$

$$\tilde{U}_5 = \{(s_1, s_2) \,|\, 0 < s_1 < \frac{\mathcal{L}}{2} < s_2 < -s_1 + \mathcal{L}\},$$

$$\tilde{U}_6 = \{(s_1, s_2) \,|\, -s_2 + \mathcal{L} < s_1 < \frac{\mathcal{L}}{2} < s_2 < \mathcal{L}\}$$

(see Figure 2).

Note that

$$\mathscr{L}^2 \left([0, \mathcal{L}]^2 \setminus U_1 \cup U_2 \cup \bigcup_{k=3}^{6} \tilde{U}_k \right) = 0.$$

We will show

$$|B_2(s_1, s_2)| \leq 2\|\mathscr{M}(\boldsymbol{f})\|_{L^\infty} + \frac{16}{\mathcal{L}\mathscr{D}_{12}} \left(\chi_{\tilde{U}_4}(s_1, s_2) + \chi_{\tilde{U}_5}(s_1, s_2) \right) \quad \mathscr{L}^2\text{-a.e.},$$

which implies $B_2 \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Let $(s_1, s_2) \in U_1 = (0, \frac{\mathcal{L}}{2})^2$. Then, $\frac{s_1+s_2}{2} \in (0, \frac{\mathcal{L}}{2})$, and therefore,

$$-\left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 \right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{2}{s_2} + \frac{4}{s_1 + s_2}.$$

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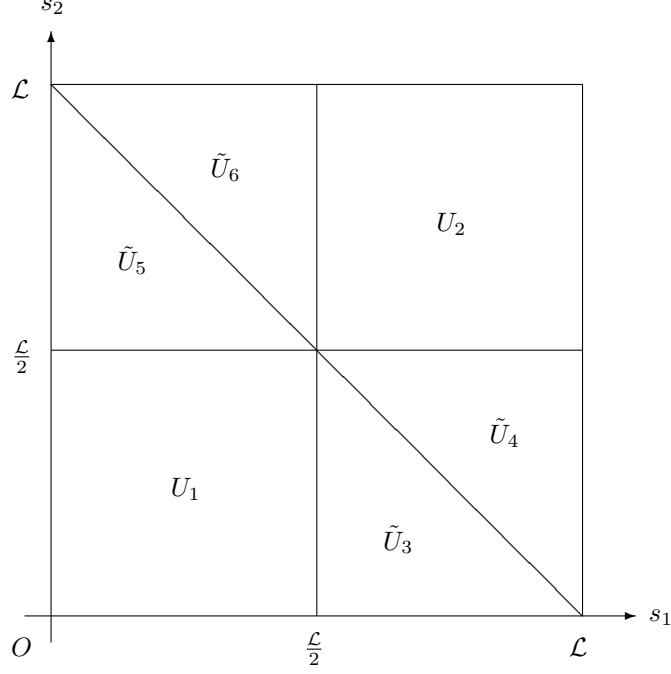
Figure 2

If s_3 is located between $\min\left\{\frac{s_1+s_2}{2}, s_2\right\}$ and $\max\left\{\frac{s_1+s_2}{2}, s_2\right\}$, it follows that $s_3 \in (0, \frac{\mathcal{L}}{2})$, and thus $\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 = s_3^2$. Hence, we obtain

$$\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2} = \int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{s_3^2} = \left[-\frac{1}{s_3}\right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{1}{s_2} + \frac{2}{s_1+s_2},$$

which yields

$$-\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{2\,ds_3}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2}$$
$$= -\int_{\frac{s_1+s_2}{2}}^{s_2} \left(\frac{2}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2}\right) ds_3$$
$$= -2\int_{\frac{s_1+s_2}{2}}^{s_2} \mathscr{M}(\boldsymbol{f})(s_3, 0)\, ds_3.$$

Therefore $|B_2(s_1, s_2)| \leq 2\|\mathscr{M}(\boldsymbol{f})\|_{L^\infty}$ on U_1.

If we assume that $(s_1, s_2) \in U_2 = (\frac{\mathcal{L}}{2}, \mathcal{L})^2$, then $\frac{s_1+s_2}{2} \in (\frac{\mathcal{L}}{2}, \mathcal{L})$. Therefore,

$$-\left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{2}{s_2 - \mathcal{L}} + \frac{2}{\frac{s_1+s_2}{2} - \mathcal{L}}.$$

If s_3 is located between $\min\left\{\frac{s_1+s_2}{2}, s_2\right\}$ and $\max\left\{\frac{s_1+s_2}{2}, s_2\right\}$, it follows that

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$s_3 \in (\frac{\mathcal{L}}{2}, \mathcal{L})$. Therefore, $\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 = (s_3 - \mathcal{L})^2$, and we have

$$\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2} = \int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{(s_3 - \mathcal{L})^2} = -\frac{1}{s_2 - \mathcal{L}} + \frac{1}{\frac{s_1+s_2}{2} - \mathcal{L}},$$

which leads to

$$-\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{2\, ds_3}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2}$$

$$= -\int_{\frac{s_1+s_2}{2}}^{s_2} \left(\frac{2}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2}\right) ds_3$$

$$= -2\int_{\frac{s_1+s_2}{2}}^{s_2} \mathscr{M}(\boldsymbol{f})(s_3, 0)\, ds_3.$$

Next, let set $\tilde{U}_3$ and let $(s_1, s_2) \in \tilde{U}_3$. Then, $\frac{s_1+s_2}{2} \in (\frac{\mathcal{L}}{4}, \frac{\mathcal{L}}{2})$ and thus, we have

$$-\left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{2}{s_2} + \frac{4}{s_1+s_2}.$$

Since $0 < s_2 < \frac{s_1+s_2}{2} < \frac{\mathcal{L}}{2}$, it follows that $\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 = s_3^2$ for $s_3 \in \left(s_2, \frac{s_1+s_2}{2}\right)$, and therefore,

$$\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2} = \int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{s_3^2} = -\frac{1}{s_2} + \frac{2}{s_1+s_2}.$$

Thus, we obtain

$$-\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{2\, ds_3}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2}$$

$$= -\int_{\frac{s_1+s_2}{2}}^{s_2} \left(\frac{2}{\|\boldsymbol{f}(s_3) - \boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2}\right) ds_3$$

$$= -2\int_{\frac{s_1+s_2}{2}}^{s_2} \mathscr{M}(\boldsymbol{f})(s_3, 0)\, ds_3.$$

Assume that $(s_1, s_2) \in \tilde{U}_4$. Then, $\frac{s_1+s_2}{2} \in \left(\frac{\mathcal{L}}{2}, \frac{3}{4}\mathcal{L}\right)$, and therefore, we have

$$-\left[\frac{\partial}{\partial s_3} \log \mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{2}{s_2} + \frac{2}{\frac{s_1+s_2}{2} - \mathcal{L}}.$$

From $0 < s_2 < \frac{\mathcal{L}}{2} < \frac{s_1+s_2}{2} < \frac{3}{4}\mathcal{L}$, for $s_3 \in \left(s_2, \frac{s_1+s_2}{2}\right)$, it holds that

$$\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2 = \begin{cases} s_3^2 & \left(s_2 < s_3 \leq \dfrac{\mathcal{L}}{2}\right), \\[2mm] (s_3 - \mathcal{L})^2 & \left(\dfrac{\mathcal{L}}{2} \leq s_3 < \dfrac{s_1+s_2}{2}\right). \end{cases}$$

Consequently, we have

$$\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3), \boldsymbol{f}(0))^2} = \int_{\frac{s_1+s_2}{2}}^{\frac{\mathcal{L}}{2}} \frac{ds_3}{(s_3 - \mathcal{L})^2} + \int_{\frac{\mathcal{L}}{2}}^{s_2} \frac{ds_3}{s_3^2}$$

$$= -\frac{1}{s_2} + \frac{1}{\frac{s_1+s_2}{2} - \mathcal{L}} + \frac{4}{\mathcal{L}},$$

and therefore, we obtain

$$-\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{2\,ds_3}{\|\boldsymbol{f}(s_3)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3}\log\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2}$$
$$= -\int_{\frac{s_1+s_2}{2}}^{s_2}\left(\frac{2}{\|\boldsymbol{f}(s_3)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2}\right)ds_3 - \frac{8}{\mathcal{L}}$$
$$= -2\int_{\frac{s_1+s_2}{2}}^{s_2}\mathscr{M}(\boldsymbol{f})(s_3,0)\,ds_3 - \frac{8}{\mathcal{L}}.$$

Similarly, let $(s_1,s_2)\in\tilde{U}_5$. Then, $\frac{s_1+s_2}{2}\in\left(\frac{\mathcal{L}}{4},\frac{\mathcal{L}}{2}\right)$, and therefore, we obtain

$$-\left[\frac{\partial}{\partial s_3}\log\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{2}{s_2-\mathcal{L}} + \frac{4}{s_1+s_2}.$$

From $\frac{\mathcal{L}}{4} < \frac{s_1+s_2}{2} < \frac{\mathcal{L}}{2} < s_2$, for $s_3\in\left(\frac{s_1+s_2}{2},s_2\right)$, it follows that

$$\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2 = \begin{cases} s_3^2 & \left(\dfrac{s_1+s_2}{2} < s_3 \le \dfrac{\mathcal{L}}{2}\right), \\[2mm] (s_3-\mathcal{L})^2 & \left(\dfrac{\mathcal{L}}{2} \le s_3 < s_2\right), \end{cases}$$

and thus we obtain

$$\int_{\frac{s_1+s_2}{2}}^{s_2}\frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2} = \int_{\frac{s_1+s_2}{2}}^{\frac{\mathcal{L}}{2}}\frac{ds_3}{s_3^2} + \int_{\frac{\mathcal{L}}{2}}^{s_2}\frac{ds_3}{(s_3-\mathcal{L})^2}$$

$$= -\frac{1}{s_2-\mathcal{L}} + \frac{2}{s_1+s_2} - \frac{4}{\mathcal{L}}.$$

Hence

$$-\int_{\frac{s_1+s_2}{2}}^{s_2}\frac{2\,ds_3}{\|\boldsymbol{f}(s_3)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3}\log\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2}$$
$$= -\int_{\frac{s_1+s_2}{2}}^{s_2}\left(\frac{2}{\|\boldsymbol{f}(s_3)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2}\right)ds_3 + \frac{8}{\mathcal{L}}$$
$$= -2\int_{\frac{s_1+s_2}{2}}^{s_2}\mathscr{M}(\boldsymbol{f})(s_3,0)\,ds_3 + \frac{8}{\mathcal{L}}.$$

To conclude the proof, let $\tilde{U}_6 = \{(s_1,s_2)\,|\,-s_2+\mathcal{L} < s_1 < \frac{\mathcal{L}}{2} < s_2 < \mathcal{L}\}$, and let $(s_1,s_2)\in\tilde{U}_6$. Since $\frac{s_1+s_2}{2}\in\left(\frac{\mathcal{L}}{2},\frac{3}{4}\mathcal{L}\right)$, we have

$$-\left[\frac{\partial}{\partial s_3}\log\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2} = -\frac{2}{s_2-\mathcal{L}} + \frac{2}{\frac{s_1+s_2}{2}-\mathcal{L}}.$$

From $\frac{\mathcal{L}}{2} < \frac{s_1+s_2}{2} < s_2 < \mathcal{L}$, we have $\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2 = (s_3-\mathcal{L})^2$ for $s_3\in\left(\frac{s_1+s_2}{2},s_2\right)$. Therefore, we obtain

$$\int_{\frac{s_1+s_2}{2}}^{s_2}\frac{ds_3}{\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2} = \int_{\frac{s_1+s_2}{2}}^{s_2}\frac{ds_3}{(s_3-\mathcal{L})^2} = -\frac{1}{s_2-\mathcal{L}} + \frac{1}{\frac{s_1+s_2}{2}-\mathcal{L}}$$

and

$$-\int_{\frac{s_1+s_2}{2}}^{s_2} \frac{2\,ds_3}{\|\boldsymbol{f}(s_3)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \left[\frac{\partial}{\partial s_3}\log \mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2\right]_{\frac{s_1+s_2}{2}}^{s_2}$$

$$= -\int_{\frac{s_1+s_2}{2}}^{s_2}\left(\frac{2}{\|\boldsymbol{f}(s_3)-\boldsymbol{f}(0)\|_{\mathbb{R}^n}^2} - \frac{2}{\mathscr{D}(\boldsymbol{f}(s_3),\boldsymbol{f}(0))^2}\right)ds_3$$

$$= -2\int_{\frac{s_1+s_2}{2}}^{s_2} \mathscr{M}(\boldsymbol{f})(s_3,0)\,ds_3.$$

From the above, we obtain the estimate of B_2.

Combining this with the previous estimates, we have $A_6(s_1,s_2) \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$, and by applying Fubini's theorem and Corollary 4.3, we can show that the integrated value is 0. □

Combining these claims, we obtain (4.8). Thus, the proof of Theorem 4.3 is completed.

4.3 Global minimizers of $\mathcal{E}_1$

We give here the proof of Corollary 4.2, which says that global minimizers of $\mathcal{E}_1$ in $C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ are only right circles.

Recall that

$$\boldsymbol{f} \mapsto \boldsymbol{p} = \boldsymbol{c} + \frac{r^2(\boldsymbol{f}-\boldsymbol{c})}{\|\boldsymbol{f}-\boldsymbol{c}\|_{\mathbb{R}^n}^2}$$

is an inversion with respect to a sphere with radius r and center $\boldsymbol{c}$ in the image of $\boldsymbol{f}$.

From the main theorem, we have

$$\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + 2\pi^2.$$

It is clear that

$$\mathcal{E}_1(\boldsymbol{p}) \geq 0,$$

and thus we have

$$\mathcal{E}_1(\boldsymbol{f}) \geq 2\pi^2.$$

Of course, $\mathcal{E}_1(\boldsymbol{f}) = 2\pi^2$ implies that $\mathcal{E}_1(\boldsymbol{p}) = 0$, and from this we may deduce that $\mathscr{M}_1(\boldsymbol{p}) \equiv 0$. Therefore, we know that $\boldsymbol{p}'$ is a constant vector, and hence $\boldsymbol{p}$ is a straight line. Finally, we conclude that $\boldsymbol{f}$ is a preimage of the straight line, *i.e.*, a right circle. Conversely, it is easy to see that $\mathcal{E}_1(\boldsymbol{f}) = 2\pi^2$ when $\boldsymbol{f}$ is a right circle. □

5 Variational formulae

In this section, we show that the first and second variations of the Möbius energy components are defined as (multi-)linear functionals on X. Formulae for the first and second variations have been obtained by several authors, e.g., [8, 9]; however, the formulae were not given explicitly except that of the principal term. Direct calculation produces many terms, most of which are not integrable even in the sense of Cauchy's principal value. By combining these terms appropriately,

we find that absolute integrability can be recovered. This was shown by the author [10] for $\boldsymbol{f} \in C^{3+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. This space is narrower than the natural domain X (see [2]).

Using the above decomposition, we can obtain explicit expressions for the variational formulae (Propositions 5.1, 5.2) along with estimates on X and other function spaces.

Setting

$$\mathscr{M}_i(\boldsymbol{f}) = \frac{\mathscr{N}_i(\boldsymbol{f})}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

it is not difficult to see that

$$\mathscr{N}_1(\boldsymbol{f}) = \frac{1}{2}\|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2, \quad \mathscr{N}_2(\boldsymbol{f}) = 2P_*^{\perp}\boldsymbol{\tau}(s_1) \cdot P_*^{\perp}\boldsymbol{\tau}(s_2).$$

Let $\mathscr{F}(\boldsymbol{f})$ be a geometric quantity determined by the closed curve $\boldsymbol{f}$, and let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be functions from $\mathbb{R}/\mathcal{L}\mathbb{Z}$ to $\mathbb{R}^n$. We use δ and δ^2 to mean

$$\delta\mathscr{F}(\boldsymbol{f})[\boldsymbol{\phi}] = \left.\frac{d}{d\varepsilon}\mathscr{F}(\boldsymbol{f} + \varepsilon\boldsymbol{\phi})\right|_{\varepsilon=0},$$

$$\delta^2\mathscr{F}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \left.\frac{d^2}{d\varepsilon_1 d\varepsilon_2}\mathscr{F}(\boldsymbol{f} + \varepsilon_1\boldsymbol{\phi} + \varepsilon_2\boldsymbol{\psi})\right|_{\varepsilon_1=\varepsilon_2=0}.$$

Then the first variation $\mathscr{G}_i$ and the second variation $\mathscr{H}_i$ are given by

$$\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\, ds_1 ds_2 = \delta(\mathscr{M}_i(\boldsymbol{f})\, ds_1 ds_2)[\boldsymbol{\phi}],$$

$$\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\, ds_1 ds_2 = \delta^2(\mathscr{M}_i(\boldsymbol{f})\, ds_1 ds_2)[\boldsymbol{\phi}, \boldsymbol{\psi}].$$

These symbols stand for $\mathscr{N}$umerator, $\mathscr{G}$radient, and $\mathscr{H}$essian, respectively. The main result of the present study is the following:

Theorem 5.1 ([12], Theorem 2) *Assume that there exists a positive constant λ such that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))|$.*

1. *If $\boldsymbol{f}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in X$, then $\mathscr{M}_i(\boldsymbol{f})$, $\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]$, $\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Furthermore, there exists a positive constant C depending on $\|\boldsymbol{f}'\|_Y$ and λ such that*
$$\|\mathscr{M}_i(\boldsymbol{f})\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C,$$
$$\|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{\phi}'\|_Y,$$
$$\|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{\phi}'\|_Y\|\boldsymbol{\psi}'\|_Y.$$

2. *If $\boldsymbol{f}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathscr{M}_i(\boldsymbol{f})$, $\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]$, $\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Furthermore, there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}$, λ and $\mathcal{L}$ such that*
$$\|\mathscr{M}_i(\boldsymbol{f})\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C,$$
$$\|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{\phi}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})},$$
$$\|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{\phi}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}\|\boldsymbol{\psi}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

3. If $\boldsymbol{f}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathscr{M}_i(\boldsymbol{f})$, $\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]$, $\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]$ can be extended on the diagonal set $\{(s_1, s_2)\,|\,s_1 \equiv s_2 \pmod{\mathcal{L}\mathbb{Z}}\}$ such that these functions are continuous everywhere. The limits of sums vanish on the diagonal set, as follows:

$$\lim_{(s_1, s_2) \to (s, s)} (\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})) = 0,$$

$$\lim_{(s_1, s_2) \to (s, s)} (\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}] + \mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}]) = 0,$$

$$\lim_{(s_1, s_2) \to (s, s)} (\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] + \mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]) = 0.$$

Furthermore, there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}$, λ and $\mathcal{L}$ such that

$$\|\mathscr{M}_i(\boldsymbol{f})\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \le C,$$

$$\|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \le C\|\boldsymbol{\phi}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})},$$

$$\|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \le C\|\boldsymbol{\phi}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}\|\boldsymbol{\psi}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

Our strategy is as follows. We express one component of $\mathscr{G}_i$ in terms of $\mathscr{M}_i$ and express three components of $\mathscr{H}_i$ in terms of $\mathscr{M}_i$ and $\mathscr{G}_i$. Then estimates of $\mathscr{G}_i$ are derived from those of $\mathscr{M}_i$ and subsequently we derive estimates of $\mathscr{H}_i$ from those of $\mathscr{M}_i$ and $\mathscr{G}_i$.

We use the operations Q and $\tilde{Q}$ defined in subsection 4.2. Since $Q\boldsymbol{f} = \Delta\boldsymbol{\tau}$ and $\tilde{Q}_i\boldsymbol{f} = (-1)^{i-1}2\{\boldsymbol{\tau}_i - (R\boldsymbol{f} \cdot \boldsymbol{\tau}_i)R\boldsymbol{f}\} = (-1)^{i-1}2P_*^\perp\boldsymbol{\tau}_i$, we have

$$\mathscr{N}_1(\boldsymbol{f}) = \frac{1}{2}\|Q\boldsymbol{f}\|_{\mathbb{R}^n}^2, \quad \mathscr{N}_2(\boldsymbol{f}) = -\frac{1}{2}\tilde{Q}_1\boldsymbol{f} \cdot \tilde{Q}_2\boldsymbol{f}.$$

To show Theorem 5.1, we calculate the variations of Q and $\tilde{Q}_i$, and establish several estimates (see Lemma 5.2). Explicit expressions for $\mathscr{G}_i$ and $\mathscr{H}_i$ are given in subsections 5.1 and 5.2, respectively, and Theorem 5.1 is proved in subsection 5.3.

If $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$, by a formal integration by parts, it seems that the first variation can be extended into $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ as a linear form. Indeed, the principal term of $\delta\mathcal{E}_1(\boldsymbol{f})[\boldsymbol{\phi}]$ is

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{(\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)) \cdot (\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_2))}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}\,ds_1 ds_2,$$

where $\boldsymbol{\tau} = \boldsymbol{f}'$. Using the bi-Lipschitz continuity, we may replace the denominator with $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2$ and then

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{(\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)) \cdot (\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_2))}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}\,ds_1 ds_2 = 2\pi \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (-\Delta_s)^{\frac{1}{4}}\boldsymbol{\tau}\cdot(-\Delta_s)^{\frac{1}{4}}\boldsymbol{\phi}'\,ds.$$

Here, Δ_s is the Laplace operator with respect to the arc-length parameter, and $(-\Delta_s)^\alpha$ is its fractional power. Formally integrating by parts, we obtain

$$2\pi \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (-\Delta_s)^{\frac{3}{2}}\boldsymbol{f} \cdot \boldsymbol{\phi}\,ds.$$

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This seems to apply to $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\phi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. In this paper, we shall justify this not only for the principal term but also for all terms, including $\delta\mathcal{E}_2$.

Here, we define a new operation T_i^k, which we will use to describe the L^2-*gradient* expression of $\delta\mathcal{E}_i$:

$$(5.1) \qquad T_i^k \boldsymbol{f} := \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_i).$$

From now on, $\langle \cdot, \cdot \rangle_{L^2}$ denotes the inner product on $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Similarly $\| \cdot \|_{H^s}$ and $\| \cdot \|_{L^p}$ are respectively the norms on $H^s(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $L^p(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Theorem 5.2 *Let $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ be bi-Lipschitz. We denote the curvature of $\boldsymbol{f}$ by $\boldsymbol{\kappa}$, that is, $\boldsymbol{\kappa} = \boldsymbol{f}''$. Then for $\phi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, it holds that*

$$\delta\mathcal{E}_i(\boldsymbol{f})[\phi] = \langle L_i \boldsymbol{f} + \boldsymbol{N}_i(\boldsymbol{f}), \phi \rangle_{L^2},$$

where

$$L_1 \boldsymbol{f} = 2\pi(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} - 4 \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \mathrm{si}(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \varphi_k + \frac{8}{\mathcal{L}} \Delta_s (\boldsymbol{f} - \check{\boldsymbol{f}}),$$

$$L_2 \boldsymbol{f} = -\frac{4}{3} \pi(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} + \frac{8}{3} \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \mathrm{si}(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \varphi_k + \frac{16}{3\mathcal{L}} \Delta_s \check{\boldsymbol{f}} + \frac{128}{3\mathcal{L}^3} (\boldsymbol{f} - \check{\boldsymbol{f}}),$$

$$\mathrm{si}(t) = -\int_t^\infty \frac{\sin\lambda}{\lambda}\, d\lambda, \quad \varphi_k(s) = \frac{1}{\sqrt{\mathcal{L}}} \exp\left(\frac{2\pi i k s}{\mathcal{L}} \right), \quad \check{\boldsymbol{f}}(s) = \boldsymbol{f}(s + \tfrac{\mathcal{L}}{2}),$$

$$\boldsymbol{N}_1(\boldsymbol{f})(s_1) = -2 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{2}{(\Delta s)^2} (T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \Delta \boldsymbol{\tau} - \mathscr{M}(\boldsymbol{f}) \boldsymbol{\kappa}(s_1) \right\} ds_2$$

$$\qquad\qquad - 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\mathscr{M}_1(\boldsymbol{f})}{\Delta s} T_1^2 \boldsymbol{f} + \frac{1}{\Delta s} \left\{ \mathscr{M}_1(\boldsymbol{f}) - \frac{1}{2} \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}}^2 \right\} \boldsymbol{\tau}(s_1) \right] ds_2,$$

$$\boldsymbol{N}_2(\boldsymbol{f})(s_1) = -4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\{ (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) T_2^0 \boldsymbol{f} + (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) T_1^0 \boldsymbol{f} \right\} ds_2$$

$$\qquad - 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)^3} \left[(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) T_2^0 \boldsymbol{f} + (T_1^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) T_1^4 \boldsymbol{f} \right.$$

$$\qquad\qquad \left. + 2 \left\{ (T_2^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) + 1 \right\} (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) T_1^4 \boldsymbol{f} \right] ds_2$$

$$\qquad - 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)^3} \left[T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) - T_2^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) \right.$$

$$\qquad\qquad + 2 \left\{ (T_2^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) + 1 \right\} (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))$$

$$\qquad\qquad \left. + T_1^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) - \frac{(\Delta s)^2}{6} \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \right] \boldsymbol{\tau}(s_1) ds_2$$

$$\qquad - 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\mathscr{M}_2(\boldsymbol{f})}{\Delta s} T_1^2 \boldsymbol{f} + \frac{1}{\Delta s} \left\{ \mathscr{M}_2(\boldsymbol{f}) + \frac{1}{2} \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \right\} \boldsymbol{\tau}(s_1) \right] ds_2.$$

Furthermore, for $\alpha \in (0, \frac{1}{2})$ it holds that

$$\|\boldsymbol{N}_i(\boldsymbol{f})\|_{L^2} \le C_\alpha (\|\boldsymbol{f}\|_{H^{3-\alpha}}),$$

where $C_\alpha(\|\boldsymbol{f}\|_{H^{3-\alpha}})$ is a constant depending on α and $\|\boldsymbol{f}\|_{H^{3-\alpha}}$.

We call the operator $L_i \boldsymbol{f} + \boldsymbol{N}_i(\boldsymbol{f})$ the L^2-*gradient* of $\delta\mathcal{E}_i$. We will show Theorem 5.2 in subsection 5.4 .

5.1 The first variation

First we summarize several facts regarding the calculation of variations.

Lemma 5.1 *The following first variational formulae hold.*

1. $\delta\boldsymbol{\tau}[\boldsymbol{\phi}] = \boldsymbol{\phi}' - (\boldsymbol{\tau}\cdot\boldsymbol{\phi}')\boldsymbol{\tau}$.

2. $\delta\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}] = 2\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}' - \|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2(\boldsymbol{\tau}(s_1)\cdot\boldsymbol{\phi}'(s_1) + \boldsymbol{\tau}(s_2)\cdot\boldsymbol{\phi}'(s_2))$.

3. $\delta\left(\dfrac{1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right)[\boldsymbol{\phi}] = -\dfrac{2\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}$.

4. $\delta(ds_j)[\boldsymbol{\phi}] = \boldsymbol{\tau}(s_j)\cdot\boldsymbol{\phi}'(s_j)ds_j$.

Proof. Since the parameter s is not an arc-length parameter for $\boldsymbol{f} + \varepsilon\boldsymbol{\phi}$, we use a new parameter θ independent of ε. We denote differentiation with respect to θ by $\dot{}$. From the definition of first variation, we have

$$\delta\boldsymbol{\tau}[\boldsymbol{\phi}] = \frac{d}{d\varepsilon}\frac{\dot{\boldsymbol{f}} + \varepsilon\dot{\boldsymbol{\phi}}}{\|\dot{\boldsymbol{f}} + \varepsilon\dot{\boldsymbol{\phi}}\|_{\mathbb{R}^n}}\bigg|_{\varepsilon=0} = \frac{\dot{\boldsymbol{\phi}}}{\|\dot{\boldsymbol{f}}\|_{\mathbb{R}^n}} - \frac{\dot{\boldsymbol{f}}}{\|\dot{\boldsymbol{f}}\|_{\mathbb{R}^n}^2}\frac{\dot{\boldsymbol{f}}\cdot\dot{\boldsymbol{\phi}}}{\|\dot{\boldsymbol{f}}\|_{\mathbb{R}^n}} = \boldsymbol{\phi}' - (\boldsymbol{\tau}\cdot\boldsymbol{\phi}')\boldsymbol{\tau}.$$

Using the variation $\delta\boldsymbol{\tau}$, we obtain

$$\delta\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}]$$

$$= \delta\left\|\frac{\dot{\boldsymbol{f}}(\theta_1)}{\|\dot{\boldsymbol{f}}(\theta_1)\|_{\mathbb{R}^n}} - \frac{\dot{\boldsymbol{f}}(\theta_2)}{\|\dot{\boldsymbol{f}}(\theta_2)\|_{\mathbb{R}^n}}\right\|^2$$

$$= 2\Delta\boldsymbol{\tau}\cdot\Delta\delta\boldsymbol{\tau}[\boldsymbol{\phi}]$$

$$= 2\Delta\boldsymbol{\tau}\cdot\{\Delta\boldsymbol{\phi}' - (\boldsymbol{\tau}(s_1)\cdot\boldsymbol{\phi}'(s_1))\boldsymbol{\tau}(s_1) + (\boldsymbol{\tau}(s_2)\cdot\boldsymbol{\phi}'(s_2))\boldsymbol{\tau}(s_2)\}$$

$$= 2\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}' - \|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2(\boldsymbol{\tau}(s_1)\cdot\boldsymbol{\phi}'(s_1) + \boldsymbol{\tau}(s_2)\cdot\boldsymbol{\phi}'(s_2)),$$

$$\delta\left(\frac{1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right)[\boldsymbol{\phi}] = -\frac{\delta\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} = -\frac{2\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}$$

and

$$\delta(ds_j)[\boldsymbol{\phi}] = \delta(\|\dot{\boldsymbol{f}}(s_j)\|_{\mathbb{R}^n}d\theta_j)[\boldsymbol{\phi}] = \frac{\dot{\boldsymbol{f}}(s_j)\cdot\dot{\boldsymbol{\phi}}(s_j)}{\|\dot{\boldsymbol{f}}(s_j)\|_{\mathbb{R}^n}}d\theta_j = \boldsymbol{\tau}(s_j)\cdot\boldsymbol{\phi}'(s_j)ds_j.$$

$\square$

Next we obtain an explicit expression for $\mathscr{G}_i$.

Propositon 5.1 ([12], Proposition 1) *The $\mathscr{G}_i$'s can be written as*

$$\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{Q\boldsymbol{f}\cdot Q\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

$$\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}] = -\frac{\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi} + \tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$

in terms of $\mathscr{M}_i$, Q, and $\tilde{Q}_i$.

Proof. From Lemma 5.1,

$$\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\,ds_1 ds_2 = \delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}]\,ds_1 ds_2 + \mathscr{M}_i(\boldsymbol{f})\,\delta(ds_1 ds_2)[\boldsymbol{\phi}]$$

(5.2)
$$= \left\{\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}] + \mathscr{M}_i(\boldsymbol{f})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\phi}_1' + \boldsymbol{\tau}_2 \cdot \boldsymbol{\phi}_2')\right\}ds_1 ds_2,$$

i.e.,

$$\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}] + \mathscr{M}_i(\boldsymbol{f})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\phi}_1' + \boldsymbol{\tau}_2 \cdot \boldsymbol{\phi}_2').$$

Using Lemma 5.1, we have

$$\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{\delta\mathscr{N}_i(\boldsymbol{f})[\boldsymbol{\phi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} + \mathscr{N}_i(\boldsymbol{f})\,\delta\left(\frac{1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right)[\boldsymbol{\phi}]$$

(5.3)
$$= \frac{\delta\mathscr{N}_i(\boldsymbol{f})[\boldsymbol{\phi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

To calculate $\delta\mathscr{N}_i(\boldsymbol{f})[\boldsymbol{\phi}]$, using $Q\boldsymbol{f} = \Delta\boldsymbol{\tau}$, it follows that

$$\delta\left(\|Q\boldsymbol{f}\|^2\right)[\boldsymbol{\phi}] = 2\Delta\boldsymbol{\tau}\cdot\delta(\Delta\boldsymbol{\tau})[\boldsymbol{\phi}]$$

$$= 2\Delta\boldsymbol{\tau}\cdot\Delta P^\perp\boldsymbol{\tau}$$

$$= 2\Delta\boldsymbol{\tau}\cdot\left\{\Delta\boldsymbol{\phi}' - (\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')\boldsymbol{\tau}_1 + (\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\boldsymbol{\tau}_2\right\}$$

$$= 2\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}' - 2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')(1 - \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_1) + 2(\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 - 1)$$

$$= 2\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}' - (\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\|\Delta\boldsymbol{\tau}\|^2$$

$$= 2Q\boldsymbol{f}\cdot Q\boldsymbol{\phi} - 2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\mathscr{N}_1(\boldsymbol{f}).$$

Therefore, we obtain

(5.4)
$$\delta\mathscr{N}_1(\boldsymbol{f})[\boldsymbol{\phi}] = Q\boldsymbol{f}\cdot Q\boldsymbol{\phi} - (\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\mathscr{N}_1(\boldsymbol{f}).$$

It follows from the definition of $\tilde{Q}_i$ and Lemma 5.1 that

$$\delta(\tilde{Q}_i\boldsymbol{f})[\boldsymbol{\phi}] = (-1)^{i-1}2\delta\{\boldsymbol{\tau}_i - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{f}\}[\boldsymbol{\phi}]$$

$$= (-1)^{i-1}2\left[\boldsymbol{\phi}_i' - (\boldsymbol{\phi}_i'\cdot\boldsymbol{\tau}_i)\boldsymbol{\tau}_i - \{P_*^\perp R\boldsymbol{\phi}\cdot\boldsymbol{\tau}_i + R\boldsymbol{f}\cdot(\boldsymbol{\phi}_i' - (\boldsymbol{\phi}_i'\cdot\boldsymbol{\tau}_i)\boldsymbol{\tau}_i)\}R\boldsymbol{f}\right.$$

$$\left. - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)P_*^\perp R\boldsymbol{\phi}\right]$$

$$= (-1)^{i-1}2\left[\boldsymbol{\phi}_i' - (\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\boldsymbol{\tau}_i\right.$$

$$- \{(R\boldsymbol{\phi}\cdot P_*^\perp\boldsymbol{\tau}_i) + (R\boldsymbol{f}\cdot\boldsymbol{\phi}_i') + (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\}R\boldsymbol{f}$$

$$\left. - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)\{R\boldsymbol{\phi} - (R\boldsymbol{f}\cdot R\boldsymbol{\phi})R\boldsymbol{f}\}\right]$$

$$= (-1)^{i-1}2\left\{\boldsymbol{\phi}_i' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{\phi} - (\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\{\boldsymbol{\tau}_i - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{f}\}\right.$$

$$\left. - \left((R\boldsymbol{\phi}\cdot P_*^\perp\boldsymbol{\tau}_i) + \left[R\boldsymbol{f}\cdot\{\boldsymbol{\phi}_i' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{\phi}\}\right]\right)R\boldsymbol{f}\right\}$$

$$= \tilde{Q}_i\boldsymbol{\phi} - (\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\tilde{Q}_i\boldsymbol{f} - \{(R\boldsymbol{\phi}\cdot\tilde{Q}_i\boldsymbol{f}) + (R\boldsymbol{f}\cdot\tilde{Q}_i\boldsymbol{\phi})\}R\boldsymbol{f}.$$

We take the inner product of this and $\tilde{Q}_j\boldsymbol{f} = (-1)^{i-1}2P_*^\perp\boldsymbol{\tau}_j$. Since $R\boldsymbol{f}\cdot P_*^\perp\boldsymbol{\tau}_j = 0$ for $i \neq j$, we obtain

$$\delta(\tilde{Q}_i\boldsymbol{f})[\boldsymbol{\phi}]\cdot\tilde{Q}_j\boldsymbol{f} = (\tilde{Q}_i\boldsymbol{\phi}\cdot\tilde{Q}_j\boldsymbol{f}) - (\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')(\tilde{Q}_i\boldsymbol{f}\cdot\tilde{Q}_j\boldsymbol{f})$$

$$= (\tilde{Q}_i\boldsymbol{\phi}\cdot\tilde{Q}_j\boldsymbol{f}) + 2(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\mathscr{N}_2(\boldsymbol{f}).$$

This shows that

(5.5)
$$\delta\mathscr{N}_2(\boldsymbol{f})[\boldsymbol{\phi}] = -\frac{1}{2}(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi} + \tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi}) - (\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\mathscr{N}_2(\boldsymbol{f})$$

and substituting (5.3)–(5.5) into (5.2), we obtain the assertion. $\qquad\square$

5.2 The second variation

Define the differential operator $\hat{R}$ and the operation S by

$$\hat{R}\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v}_1' + \boldsymbol{v}_2'),$$

and

$$S(\boldsymbol{v}, \boldsymbol{w}) = \hat{R}\boldsymbol{v} \cdot Q\boldsymbol{w} + Q\boldsymbol{v} \cdot \hat{R}\boldsymbol{w},$$

respectively. It is easy to see that $S(\boldsymbol{v}, \boldsymbol{w}) = \Delta(\boldsymbol{v}' \cdot \boldsymbol{w}')$, however, we choose to use the above expression, which is slightly difficult to understand, in order to make a comparison with the operation $\tilde{S}_i$ defined as follows:

$$\tilde{S}_i(\boldsymbol{v}, \boldsymbol{w}) = R\boldsymbol{v} \cdot \tilde{Q}_i\boldsymbol{w} + \tilde{Q}_i\boldsymbol{v} \cdot R\boldsymbol{w}.$$

The operations S and $\tilde{S}_i$ appear in the expressions for $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

Propositon 5.2 ([12], Proposition 2) *The $\mathscr{H}_i$'s can be written as*

$$\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \frac{Q\boldsymbol{\phi} \cdot Q\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{S(\boldsymbol{f}, \boldsymbol{\phi})S(\boldsymbol{f}, \boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$- \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f} \cdot \Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f} \cdot \Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{\phi} \cdot \Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

$$\mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = -\frac{\tilde{Q}_1\boldsymbol{\phi} \cdot \tilde{Q}_2\boldsymbol{\psi} + \tilde{Q}_2\boldsymbol{\phi} \cdot \tilde{Q}_1\boldsymbol{\psi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$+ \frac{\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\phi})\tilde{S}_2(\boldsymbol{f}, \boldsymbol{\psi}) + \tilde{S}_2(\boldsymbol{f}, \boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$- \frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f} \cdot \Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f} \cdot \Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{\phi} \cdot \Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$

in terms of $\mathscr{M}_i$, $\mathscr{G}_i$ Q, $\tilde{Q}_i$, S, and $\tilde{S}_i$.

Proof. Defining $\mathscr{P}_i$ by

$$\mathscr{P}_1(\boldsymbol{f})[\boldsymbol{\phi}] = Q\boldsymbol{f} \cdot Q\boldsymbol{\phi},$$

$$\mathscr{P}_2(\boldsymbol{f})[\boldsymbol{\phi}] = -\frac{1}{2}(\tilde{Q}_1\boldsymbol{f} \cdot \tilde{Q}_2\boldsymbol{\phi} + \tilde{Q}_2\boldsymbol{f} \cdot \tilde{Q}_1\boldsymbol{\phi}),$$

we have

$$\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f} \cdot \Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

We can show that

$$\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \delta(\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}] + \mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}](\boldsymbol{\tau}_1 \cdot \boldsymbol{\psi}_1' + \boldsymbol{\tau}_2 \cdot \boldsymbol{\psi}_s')$$

in a similar manner to the proof of Proposition 5.1. It follows from Lemma 5.1 that

$$\delta(\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}] = \frac{\delta(\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}$$
$$- \frac{2\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{\psi}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$+ \frac{4\mathscr{M}_i(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\boldsymbol{\phi})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}$$
$$= \frac{\delta(\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$- \frac{2\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

Substituting

$$\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\psi}] = \mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}] - \mathscr{M}_i(\boldsymbol{f})(\boldsymbol{\tau}_1\cdot\boldsymbol{\psi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\psi}_2')$$

into the above relation, we obtain

$$\delta(\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}] = \frac{\delta(\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$- \frac{2\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$+ \frac{2\mathscr{M}_i(\boldsymbol{f})(\boldsymbol{\tau}_1\cdot\boldsymbol{\psi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\psi}_2')\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$- \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

and hence

$$\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{\delta(\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}]}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$- \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$

(5.6)
$$+ \left\{ \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} + \mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}] \right\} (\boldsymbol{\tau}_1\cdot\boldsymbol{\psi}_1' + \boldsymbol{\tau}_2\cdot\boldsymbol{\psi}_2').$$

To calculate $\delta(\mathscr{P}_i(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}]$, since $\delta\boldsymbol{\phi}'[\boldsymbol{\psi}] = -(\boldsymbol{\tau}\cdot\boldsymbol{\psi})\boldsymbol{\phi}'$ by Lemma 5.1, we have

$\delta(Q\phi)[\psi] = -\Delta\{(\tau \cdot \psi)\phi'\}$. Therefore, the following holds:

$$\delta(\mathscr{P}_1(\boldsymbol{f})[\phi])[\psi]$$
$$= \delta(Q\boldsymbol{f})[\psi] \cdot Q\phi + Q\boldsymbol{f} \cdot \delta(Q\phi)[\psi]$$
$$= \Delta P^\perp \psi' \cdot Q\phi - Q\boldsymbol{f} \cdot \Delta\{(\tau \cdot \psi')\phi'\}$$
$$= \Delta\psi' \cdot Q\phi - \Delta\{(\tau \cdot \psi')\tau\} \cdot \Delta\phi' - \Delta\tau \cdot \Delta\{(\tau \cdot \psi')\phi'\}$$
$$= Q\psi \cdot Q\phi - \left[(\tau_1 \cdot \psi'_1)\Delta\tau + \{\Delta(\tau \cdot \psi')\}\tau_2\right] \cdot \Delta\phi'$$
$$\qquad - \Delta\tau \cdot \left[\{\Delta(\tau \cdot \psi')\}\phi'_1 + (\tau_2 \cdot \psi'_2)\Delta\phi'\right]$$
$$= Q\phi \cdot Q\psi - (\Delta\tau \cdot \Delta\phi')(\tau_1 \cdot \psi'_1 + \tau_2 \cdot \psi'_2)$$
$$\qquad - \{\Delta(\tau \cdot \psi')\}(\tau_2 \cdot \Delta\phi' + \Delta\tau \cdot \phi'_1)$$
$$= Q\phi \cdot Q\psi - \mathscr{P}_1(\boldsymbol{f})[\phi](\tau_1 \cdot \psi'_1 + \tau_2 \cdot \psi'_2) - \{\Delta(\tau \cdot \phi')\}\{\Delta(\tau \cdot \psi')\}$$
$$(5.7) \qquad = Q\phi \cdot Q\psi - \mathscr{P}_1(\boldsymbol{f})[\phi](\tau_1 \cdot \psi'_1 + \tau_2 \cdot \psi'_2) - S(\boldsymbol{f}, \phi)S(\boldsymbol{f}, \psi).$$

From

$$\delta\tilde{Q}_i(\phi)[\psi]$$
$$= (-1)^{i-1}2\delta\{\phi'_i - (R\boldsymbol{f} \cdot \tau_i)R\phi\}[\psi]$$
$$= (-1)^{i-1}2\left(\delta\phi'_i[\psi]\right.$$
$$\qquad -[\{\delta(R\boldsymbol{f})[\psi] \cdot \tau_i\} + R\boldsymbol{f} \cdot \delta\tau_i[\psi]]R\phi - (R\boldsymbol{f} \cdot \tau_i)\delta(R\phi)[\psi])$$
$$= (-1)^{i-1}2\left[-(\tau_i \cdot \psi'_i)\phi'_i\right.$$
$$\qquad -\{P^\perp_* R\psi \cdot \tau_i + R\boldsymbol{f} \cdot (\psi'_i - (\psi'_i \cdot \tau_i)\tau_i)\}R\phi + (R\boldsymbol{f} \cdot \tau_i)(R\boldsymbol{f} \cdot R\psi)R\phi]$$
$$= (-1)^{i-1}2\left[-(\tau_i \cdot \psi'_i)\phi'_i\right.$$
$$\qquad -\{P^\perp_* \tau_i \cdot R\psi + R\boldsymbol{f} \cdot \{\psi'_i - (\psi'_i \cdot \tau_i)\tau_i\} - (R\boldsymbol{f} \cdot \tau_i)(R\boldsymbol{f} \cdot R\psi)\}R\phi]$$
$$= (-1)^{i-1}2\left[-(\tau_i \cdot \psi'_i)\phi'_i\right.$$
$$\qquad -\{(P^\perp_* \tau_i \cdot R\psi) + (R\boldsymbol{f} \cdot \psi'_i)$$
$$\qquad - (R\boldsymbol{f} \cdot \tau_i)(\tau_i \cdot \psi'_i) - (R\boldsymbol{f} \cdot \tau_i)(R\boldsymbol{f} \cdot R\psi)\}R\phi]$$
$$= (-1)^{i-1}2\left[-(\tau_i \cdot \psi'_i)\{\phi'_i - (R\boldsymbol{f} \cdot \tau_i)R\phi\}\right.$$
$$\qquad -\{(P^\perp_* \tau_i \cdot R\psi) + (R\boldsymbol{f} \cdot \psi'_i) - (R\boldsymbol{f} \cdot \tau_i)(R\boldsymbol{f} \cdot R\psi)\}R\phi]$$
$$= (-1)^{i-1}2\left[-(\tau_i \cdot \psi'_i)\{\phi'_i - (R\boldsymbol{f} \cdot \tau_i)R\phi\}\right.$$
$$\qquad - \left((P^\perp_* \tau_i \cdot R\psi) + [R\boldsymbol{f} \cdot \{\psi'_i - (R\boldsymbol{f} \cdot \tau_i)R\}\psi]\right)R\phi]$$
$$= -(\tau_i \cdot \psi'_i)\tilde{Q}_i\phi - \{(\tilde{Q}_i\boldsymbol{f} \cdot R\psi) + (R\boldsymbol{f} \cdot \tilde{Q}_i\psi)\}R\phi$$
$$= -(\tau_i \cdot \psi'_i)\tilde{Q}_i\phi - \tilde{S}_i(\boldsymbol{f}, \psi)R\phi,$$

it follows that

$$\tilde{Q}_i\boldsymbol{f} \cdot \delta(\tilde{Q}_j\phi)[\psi] = -(\tau_j \cdot \psi'_j)(\tilde{Q}_i\boldsymbol{f} \cdot \tilde{Q}_j\phi) - \tilde{S}_j(\boldsymbol{f}, \psi)(\tilde{Q}_i\boldsymbol{f} \cdot R\phi).$$

We have already calculated $\delta(\tilde{Q}_i\boldsymbol{f})$ in the proof of Proposition 5.1. Using $\tilde{S}_i$, we have

$$\delta(\tilde{Q}_i\boldsymbol{f})[\psi] = \tilde{Q}_i\psi - (\tau_i \cdot \psi'_i)\tilde{Q}_i\boldsymbol{f} - \tilde{S}_i(\boldsymbol{f}, \psi)R\boldsymbol{f}$$

and therefore

$$\delta(\tilde{Q}_i\boldsymbol{f})[\psi] \cdot \tilde{Q}_j\phi = (\tilde{Q}_i\psi \cdot \tilde{Q}_j\phi) - (\tau_i \cdot \psi'_i)(\tilde{Q}_i\boldsymbol{f} \cdot \tilde{Q}_j\phi) - \tilde{S}_i(\boldsymbol{f}, \psi)(R\boldsymbol{f} \cdot \tilde{Q}_j\phi).$$

When $i \neq j$, it follows that

$$
\begin{aligned}
&\delta(\tilde{Q}_i \boldsymbol{f} \cdot \tilde{Q}_j \boldsymbol{\phi})[\boldsymbol{\psi}] \\
&= (\tilde{Q}_i \boldsymbol{\psi} \cdot \tilde{Q}_j \boldsymbol{\phi}) - \tilde{S}_j(\boldsymbol{f}, \boldsymbol{\psi})(\tilde{Q}_i \boldsymbol{f} \cdot R\boldsymbol{\phi}) - \tilde{S}_i(\boldsymbol{f}, \boldsymbol{\psi})(R\boldsymbol{f} \cdot \tilde{Q}_j \boldsymbol{\phi}) \\
&\quad - (\boldsymbol{\tau}_1 \cdot \boldsymbol{\psi}_1' + \boldsymbol{\tau}_2 \cdot \boldsymbol{\psi}_2')(\tilde{Q}_i \boldsymbol{f} \cdot \tilde{Q}_j \boldsymbol{\phi}).
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
&\delta(\mathscr{P}_2(\boldsymbol{f})[\boldsymbol{\phi}])[\boldsymbol{\psi}] \\
&= -\frac{1}{2}(\tilde{Q}_1 \boldsymbol{\phi} \cdot \tilde{Q}_2 \boldsymbol{\psi} + \tilde{Q}_2 \boldsymbol{\phi} \cdot \tilde{Q}_1 \boldsymbol{\psi}) \\
&\qquad + \frac{1}{2}\tilde{S}_2(\boldsymbol{f}, \boldsymbol{\psi})(\tilde{Q}_1 \boldsymbol{f} \cdot R\boldsymbol{\phi}) + \frac{1}{2}\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\psi})(R\boldsymbol{f} \cdot \tilde{Q}_2 \boldsymbol{\phi}) \\
&\qquad + \frac{1}{2}\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\psi})(\tilde{Q}_2 \boldsymbol{f} \cdot R\boldsymbol{\phi}) + \frac{1}{2}\tilde{S}_2(\boldsymbol{f}, \boldsymbol{\psi})(R\boldsymbol{f} \cdot \tilde{Q}_1 \boldsymbol{\phi}) \\
&\qquad + \frac{1}{2}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\psi}_1' + \boldsymbol{\tau}_2 \cdot \boldsymbol{\psi}_2')(\tilde{Q}_1 \boldsymbol{f} \cdot \tilde{Q}_2 \boldsymbol{\phi} + \tilde{Q}_2 \boldsymbol{f} \cdot \tilde{Q}_1 \boldsymbol{\phi}) \\
&= -\frac{1}{2}\{(\tilde{Q}_1 \boldsymbol{\phi} \cdot \tilde{Q}_2 \boldsymbol{\psi}) + (\tilde{Q}_2 \boldsymbol{\phi} \cdot \tilde{Q}_1 \boldsymbol{\psi})\} \\
&\qquad + \frac{1}{2}\{\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\phi})\tilde{S}_2(\boldsymbol{f}, \boldsymbol{\psi}) + \tilde{S}_2(\boldsymbol{f}, \boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\psi})\} \\
&\qquad - \mathscr{P}_2(\boldsymbol{f})[\boldsymbol{\phi}](\boldsymbol{\tau}_1 \cdot \boldsymbol{\psi}_1' + \boldsymbol{\tau}_2 \cdot \boldsymbol{\psi}_2').
\end{aligned}
\tag{5.8}
$$

Substituting (5.7) and (5.8) into (5.6), we obtain the assertion. $\qquad\square$

5.3 Estimates as multi-linear functional

The following lemma is key in order to obtain the desired estimates on $\mathscr{M}_i$, $\mathscr{G}_i$ and $\mathscr{H}_i$.

Lemma 5.2 *1. For $\boldsymbol{v} \in X$, the following estimate holds:*

$$
\left\| \frac{Q\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \|\boldsymbol{v}'\|_Y.
$$

2. Assume $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. If we set $Q\boldsymbol{v}|_{s=s_1=s_2} = \boldsymbol{v}''$, then $Q\boldsymbol{v}$ is continuous everywhere and

$$
\left\| \frac{Q\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}.
$$

3. Assume that $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))|$. Then there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}$, λ, and $\mathcal{L}$ such that

$$
\left\| \frac{\tilde{Q}_i \boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}
$$

holds for all $\boldsymbol{v} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

4. *Assume that $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and that $\boldsymbol{f}$ has no self-intersections. For $\boldsymbol{v} \in$ $C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, $\tilde{Q}_i\boldsymbol{v}$ is continuous everywhere by setting $\tilde{Q}_i\boldsymbol{v}\big|_{s=s_1=s_2} = \boldsymbol{v}''$. If we further assume that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))|$, then there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}$, λ, and $\mathcal{L}$ such that*

$$\left\| \frac{\tilde{Q}_i\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq C\|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

holds for all $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Proof. Without loss of generality, we may assume that $|s_1 - s_2| \leq \frac{\mathcal{L}}{2}$, and then we use $|\Delta s|$ instead of $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$ for simplicity. The assertions 1–2 are almost trivial. Indeed, it follows immediately that

$$\left\| \frac{Q\boldsymbol{v}}{\Delta s} \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} = [\boldsymbol{v}']_{H^{\frac{1}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})} \leq \|\boldsymbol{v}'\|_Y.$$

If $\boldsymbol{v}$ is in the class C^2, then we have

$$\lim_{(s_1,s_2)\to(s,s)} \frac{Q\boldsymbol{v}}{\Delta s} = \boldsymbol{v}''(s),$$

and

$$\left\| \frac{Q\boldsymbol{v}}{\Delta s} \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} = \max_{|\Delta s| \leq \frac{\mathcal{L}}{2}} \left\| \frac{1}{s_1 - s_2} \int_{s_2}^{s_1} \boldsymbol{v}''(s)\,ds \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

To show the assertions 3–4, recall that we decompose $\frac{(-1)^{i-1}}{2}\tilde{Q}_i\boldsymbol{v} = \boldsymbol{v}_i' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v}$ into

$$\boldsymbol{v}_i' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v} = \left(\boldsymbol{v}_i' - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) + \left(\frac{\Delta\boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right) + (1 - R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v} = V_1 + V_2 + V_3.$$

We show L^∞ and C^0 estimates for each $V_i/\Delta s$. Estimates on $V_1/\Delta s$. From the inequality shown in Lemma 4.2 3, it follows that

$$\left\| \frac{1}{\Delta s}\left(\boldsymbol{v}_i' - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)}^2 \leq \sup_{|\Delta s| \leq \frac{\mathcal{L}}{2}} \frac{1}{(s_1 - s_2)^2} \left| \int_{s_2}^{s_1} \|\boldsymbol{v}'\|_{\text{Lip}}|s_i - s|\,ds \right|$$

$$\leq \frac{1}{2}\|\boldsymbol{v}'\|_{\text{Lip}} \leq \frac{1}{2}\|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

If $\boldsymbol{v}$ is in the class C^2, then we have

$$\lim_{(s_1,s_2)\to(s,s)} \frac{1}{\Delta s}\left(\boldsymbol{v}_i' - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) = \frac{(-1)^{i-1}}{2}\boldsymbol{v}''(s)$$

and

$$\left\| \frac{1}{\Delta s}\left(\boldsymbol{v}_i' - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)}^2 \leq \max_{|\Delta s| \leq \frac{\mathcal{L}}{2}} \left\| \frac{1}{(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_i}^{s} \boldsymbol{v}''(\sigma)\,d\sigma ds \right\|_{\mathbb{R}^n}$$

$$\leq \frac{1}{2}\|\boldsymbol{v}''\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})} \leq \frac{1}{2}\|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

Estimates on $V_2/\Delta s$. If $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\boldsymbol{\tau}$ has Lipschitz continuity. Consequently,

$$\left|1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}\right| \leq \frac{1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2}{1 + \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}} \leq 1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2$$

$$= \frac{1}{2(s_1 - s_2)^2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4$$

$$\leq \frac{\|\boldsymbol{\tau}\|_{\text{Lip}}^2 |\Delta s|^2}{12} \leq \frac{\mathcal{L}\|\boldsymbol{\tau}\|_{\text{Lip}}^2 |\Delta s|}{24}$$

and this implies

$$\left\|\frac{1}{\Delta s}\left(\frac{\Delta \boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right)\right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \frac{\lambda \mathcal{L}\|\boldsymbol{f}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}^2 \|\boldsymbol{v}'\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})}}{24}.$$

Similarly, if $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then

$$\left|1 - \left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}\right| \leq \frac{\|\boldsymbol{\tau}'\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})}^2 |\Delta s|^2}{12},$$

$$\lim_{(s_1, s_2) \to (s,s)} \frac{1}{\Delta s}\left(\frac{\Delta \boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right) = \boldsymbol{o},$$

and

$$\left\|\frac{1}{\Delta s}\left(\frac{\Delta \boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right)\right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \frac{\lambda \mathcal{L}\|\boldsymbol{f}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}^2 \|\boldsymbol{v}'\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})}}{24}$$

hold.

Estimates on $V_3/\Delta s$. When $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, using the estimate

$$\left|\int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2 \, ds\right| \leq \frac{1}{3}\|\boldsymbol{\tau}\|_{\text{Lip}} |\Delta s|^3 \leq \frac{\mathcal{L}}{6}\|\boldsymbol{\tau}\|_{\text{Lip}} |\Delta s|^2,$$

we obtain

$$\left\|\frac{(1 - R\boldsymbol{f} \cdot \boldsymbol{\tau}_i)R\boldsymbol{v}}{\Delta s}\right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \frac{1}{2}\mathcal{L}\lambda \|\boldsymbol{\tau}\|_{\text{Lip}} \|\boldsymbol{v}\|_{\text{Lip}}$$

$$\leq \frac{1}{2}\mathcal{L}\lambda \|\boldsymbol{f}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})} \|\boldsymbol{v}'\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

Similarly, if $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then

$$\left|\int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_i) - \boldsymbol{\tau}(s)\|_{\mathbb{R}^n}^2 \, ds\right| \leq \frac{\mathcal{L}}{6}\|\boldsymbol{\tau}'\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})} |\Delta s|^2,$$

$$\lim_{(s_1, s_2) \to (s,s)} \frac{(1 - R\boldsymbol{f} \cdot \boldsymbol{\tau}_i)R\boldsymbol{v}}{\Delta s} = \boldsymbol{o},$$

and

$$\left\|\frac{(1 - R\boldsymbol{f} \cdot \boldsymbol{\tau}_i)R\boldsymbol{v}}{\Delta s}\right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leq \frac{1}{2}\mathcal{L}\lambda \|\boldsymbol{f}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})} \|\boldsymbol{v}'\|_{C^0(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

hold. $\qquad\square$

We are now in position to prove Theorem 5.1. Let $\bar{Q}$ be Q or $\tilde{Q}_i$. Then we can write

$$|\mathscr{M}_i(\boldsymbol{f})| \le \frac{\lambda^2}{2} \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n},$$

for both $i = 1, 2$. Similarly, from Proposition 5.1, it follows that

$$
\begin{aligned}
|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]| &\le \lambda^2 \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\quad + 2\lambda|\mathscr{M}_i(\boldsymbol{f})| \left\| \frac{\Delta\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\le \lambda^2 \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + 2\lambda|\mathscr{M}_i(\boldsymbol{f})|\|\boldsymbol{\phi}\|_{\mathrm{Lip}}.
\end{aligned}
$$

Let $\bar{R}$ be $\hat{R}$ or R, and let $\bar{S}$ be S or $\tilde{S}_i$. Then the definition of these operations yields

$$
\begin{aligned}
&\left| \frac{\bar{S}(\boldsymbol{v}, \boldsymbol{w})}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right| \\
&\le \|\bar{R}\boldsymbol{v}\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{w}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\bar{R}\boldsymbol{w}\|_{\mathbb{R}^n} \\
&\le \lambda \left(\|\boldsymbol{v}\|_{\mathrm{Lip}} \left\| \frac{\bar{Q}\boldsymbol{w}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\boldsymbol{w}\|_{\mathrm{Lip}} \right).
\end{aligned}
$$

Therefore, Proposition 5.2 implies

$$
\begin{aligned}
&|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]| \\
&\le \lambda^2 \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{\psi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\quad + \lambda^4 \left(\|\boldsymbol{f}\|_{\mathrm{Lip}} \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\boldsymbol{\phi}\|_{\mathrm{Lip}} \right) \\
&\quad \times \left(\|\boldsymbol{f}\|_{\mathrm{Lip}} \left\| \frac{\bar{Q}\boldsymbol{\psi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\boldsymbol{\psi}\|_{\mathrm{Lip}} \right) \\
&\quad + 2\lambda^2|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]|\|\boldsymbol{\psi}\|_{\mathrm{Lip}} + 2\lambda^2|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]|\|\boldsymbol{\phi}\|_{\mathrm{Lip}} \\
&\quad + 2\lambda^2|\mathscr{M}_i(\boldsymbol{f})|\|\boldsymbol{\phi}\|_{\mathrm{Lip}}\|\boldsymbol{\psi}\|_{\mathrm{Lip}}.
\end{aligned}
$$

Consequently, the estimates in Theorem 5.1 are easily derived from Lemma 4.2 and Lemma 5.2. If $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then Lemma 5.2 yields

$$\lim_{(s_1, s_2) \to (s,s)} (\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})) = \frac{1}{2}\|\boldsymbol{f}''(s)\|_{\mathbb{R}^n}^2 - \frac{1}{2}\|\boldsymbol{f}''(s)\|_{\mathbb{R}^n}^2 = 0.$$

Similarly, we can show that both the limits of $\mathscr{G}_1(\boldsymbol{f}) + \mathscr{G}_2(\boldsymbol{f})$ and $\mathscr{H}_1(\boldsymbol{f}) + \mathscr{H}_2(\boldsymbol{f})$ vanish. $\qquad\square$

5.4 L^2-gradient expressions

Here we give the proof of the Theorem 5.2. First we decompose $\delta\mathcal{E}_1(\boldsymbol{f})[\boldsymbol{\phi}]$ into the linear and nonlinear pars with respect to $\boldsymbol{f}$ in subsection 5.4.1 and then we deal with the parts in subsections 5.4.2 and 5.4.3 respectively.

5.4.1 Preliminaries

Since $C^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is dense both in $H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, we may assume $\boldsymbol{f}$ and $\boldsymbol{\phi}$ are sufficiently smooth. From Proposition 5.1, the first variation $\delta\mathcal{E}_i(\cdot)[\cdot]$ is expressed as

$$
\delta\mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\, ds_1 ds_2
$$
$$
= \sum_{j=1}^2 \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{ij}(\boldsymbol{f},\boldsymbol{\phi})(s_1,s_2)\, ds_1 ds_2,
$$

where

$$
G_{i1}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{Q_{i1}\boldsymbol{f}\cdot Q_{i2}\boldsymbol{\phi} + Q_{i2}\boldsymbol{f}\cdot Q_{i1}\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},
$$
$$
G_{i2}(\boldsymbol{f},\boldsymbol{\phi}) = -\frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},
$$

$$
Q_1\boldsymbol{v} = \Delta\boldsymbol{v}' = \boldsymbol{v}'(s_1) - \boldsymbol{v}'(s_2), \quad Q_{11} = Q_{12} = Q_1, \quad Q_{2j}\boldsymbol{v} = 2\{\boldsymbol{v}'(s_j) - (R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_j))R\boldsymbol{v}\}.
$$

We decompose these operations Q_{ij} as $Q_{ij}\boldsymbol{v} = \tilde{Q}_{ij}\boldsymbol{v} + \bar{Q}_{ij}\boldsymbol{v}$, where

$$
\tilde{Q}_{1j} = Q_{1j} = Q_1, \quad \bar{Q}_{1j} = 0, \quad \tilde{Q}_{2j}\boldsymbol{v} = 2\left(\boldsymbol{v}'(s_j) - \frac{\Delta\boldsymbol{v}}{\Delta s}\right),
$$
$$
\bar{Q}_{2j}\boldsymbol{v} = 2\left\{\frac{\Delta\boldsymbol{v}}{\Delta s} - (R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_j))R\boldsymbol{v}\right\} = 2\left\{1 - (R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_j))\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right\}\frac{\Delta\boldsymbol{v}}{\Delta s}.
$$

Then, using these operations, we have

$$
G_{i1}(\boldsymbol{f},\boldsymbol{\phi}) = \sum_{k=1}^3 G_{i1k}(\boldsymbol{f},\boldsymbol{\phi}),
$$
$$
G_{i11}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{\tilde{Q}_{i1}\boldsymbol{f}\cdot\tilde{Q}_{i2}\boldsymbol{\phi} + \tilde{Q}_{i2}\boldsymbol{f}\cdot\tilde{Q}_{i1}\boldsymbol{\phi}}{2(\Delta s)^2},
$$
$$
G_{i12}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{1}{2}\mathscr{M}(\boldsymbol{f})(\tilde{Q}_{i1}\boldsymbol{f}\cdot\tilde{Q}_{i2}\boldsymbol{\phi} + \tilde{Q}_{i2}\boldsymbol{f}\cdot\tilde{Q}_{i1}\boldsymbol{\phi}),
$$
$$
G_{i13}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{\tilde{Q}_{i1}\boldsymbol{f}\cdot\bar{Q}_{i2}\boldsymbol{\phi} + \bar{Q}_{i1}\boldsymbol{f}\cdot\tilde{Q}_{i2}\boldsymbol{\phi} + \bar{Q}_{i1}\boldsymbol{f}\cdot\bar{Q}_{i2}\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}
$$
$$
+ \frac{\tilde{Q}_{i2}\boldsymbol{f}\cdot\bar{Q}_{i1}\boldsymbol{\phi} + \bar{Q}_{i2}\boldsymbol{f}\cdot\tilde{Q}_{i1}\boldsymbol{\phi} + \bar{Q}_{i2}\boldsymbol{f}\cdot\bar{Q}_{i1}\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.
$$

G_{i11} is linear with respect to $\boldsymbol{f}$; however, G_{i12}, G_{i13}, and G_{i2} are not. We would like to write

$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{i11}(\boldsymbol{f},\boldsymbol{\phi})\, ds_1 ds_2 = \langle L_i\boldsymbol{f}, \boldsymbol{\phi}\rangle_{L^2},
$$
$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (G_{i12}(\boldsymbol{f},\boldsymbol{\phi}) + G_{i13}(\boldsymbol{f},\boldsymbol{\phi}) + G_{i2}(\boldsymbol{f},\boldsymbol{\phi}))\, ds_1 ds_2 = \langle \boldsymbol{N}_i(\boldsymbol{f}), \boldsymbol{\phi}\rangle_{L^2},
$$

where L_i and $\boldsymbol{N}_i$, respectively, are linear and nonlinear operators from $H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ to $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, and we would like to estimate them.

5.4.2 The linear part

We let

$$\varphi_k(s) = \frac{1}{\sqrt{\mathcal{L}}} \exp\left(\frac{2\pi i k s}{\mathcal{L}}\right)$$

for $k \in \mathbb{Z}$, where $\{\varphi_k\}$ is a complete orthonormal basis that consists of eigenfunctions of the Laplacian on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ and

$$\boldsymbol{f} = \sum_{k \in \mathbb{Z}} \varphi_k \boldsymbol{a}_k, \quad \boldsymbol{\phi} = \sum_{k \in \mathbb{Z}} \varphi_k \boldsymbol{b}_k$$

are the associated Fourier series of $\boldsymbol{f}$ and $\boldsymbol{\phi}$.

Lemma 5.3 *Assume that $\boldsymbol{f}, \boldsymbol{\phi} \in C^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then, it holds that*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{111}(\boldsymbol{f}, \boldsymbol{\phi})\, ds_1 ds_2 = \sum_{k \in \mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^3 \left\{\int_{|\lambda| \le |k\pi|} \frac{2(1 - \cos \lambda)}{\lambda^2}\, d\lambda\right\} \langle \boldsymbol{a}_k, \boldsymbol{b}_k \rangle_{\mathbb{C}^n}.$$

Proof. From

$$\boldsymbol{\tau} = \sum_{k \in \mathbb{Z}} \varphi_k' \boldsymbol{a}_k = \sum_{k \in \mathbb{Z}} \frac{2\pi i k}{\mathcal{L}} \varphi_k \boldsymbol{a}_k,$$

we have

$$\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_1 + h) = \sum_{k \in \mathbb{Z}} \frac{2\pi i k}{\mathcal{L}}(\varphi_k(s_1) - \varphi(s_1 + h))\boldsymbol{a}_k$$

$$= \sum_{k \in \mathbb{Z}} \frac{2\pi i k}{\mathcal{L}} \left\{1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}}\right)\right\} \varphi_k(s_1)\boldsymbol{a}_k,$$

and similarly, we have

$$\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_1 + h) = \sum_{k \in \mathbb{Z}} \frac{2\pi i k}{\mathcal{L}} \left\{1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}}\right)\right\} \varphi_k(s_1)\boldsymbol{b}_k.$$

If we let $s_2 = s_1 + h$, then we obtain

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{111}(\boldsymbol{f}, \boldsymbol{\phi})\, ds_1 ds_2 = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{Q_{11}\boldsymbol{f} \cdot Q_{11}\boldsymbol{\phi}}{(\Delta s)^2} ds_1 ds_2$$

$$= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\Delta\boldsymbol{\tau} \cdot \Delta\boldsymbol{\phi}'}{(\Delta s)^2} ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{(\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_1 + h)) \cdot (\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_1 + h))}{h^2} dh ds_1$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{1}{h^2} \sum_{k,m \in \mathbb{Z}} \frac{2\pi i k}{\mathcal{L}} \frac{2\pi i m}{\mathcal{L}} \left\{1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}}\right)\right\} \left\{1 - \exp\left(\frac{2\pi i m h}{\mathcal{L}}\right)\right\}$$

$$\times \varphi_k(s_1)\varphi_m(s_1)\boldsymbol{a}_k \cdot \boldsymbol{b}_m dh ds_1$$

$$= \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{1}{h^2} \sum_{k \in \mathbb{Z}} \left(\frac{2\pi k}{\mathcal{L}}\right)^2 \left|1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}}\right)\right|^2 \boldsymbol{a}_k \cdot \boldsymbol{b}_{-k} dh$$

$$= 2 \sum_{k \in \mathbb{Z}} \left(\frac{2\pi k}{\mathcal{L}}\right)^2 \left\{\int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{1 - \cos\left(\frac{2\pi k h}{\mathcal{L}}\right)}{h^2} dh\right\} \boldsymbol{a}_k \cdot \boldsymbol{b}_{-k},$$

where $\cdot$ indicates the sum of products of the components. From $\boldsymbol{b}_{-k} = \bar{\boldsymbol{b}}_k$, we have
$$\boldsymbol{a}_k \cdot \boldsymbol{b}_{-k} = \boldsymbol{a}_k \cdot \bar{\boldsymbol{b}}_k = \langle \boldsymbol{a}_k, \boldsymbol{b}_k \rangle_{\mathbb{C}^n}.$$

Letting $\frac{2\pi kh}{\mathcal{L}} = \lambda$, the above integration with respect to h is written as

$$2 \sum_{k \in \mathbb{Z}} \left(\frac{2\pi k}{\mathcal{L}} \right)^2 \left\{ \int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{1 - \cos\left(\frac{2\pi kh}{\mathcal{L}} \right)}{h^2} dh \right\} \boldsymbol{a}_k \cdot \boldsymbol{b}_{-k}$$

$$= 2 \sum_{k \in \mathbb{Z}} \left(\frac{2\pi k}{\mathcal{L}} \right)^3 \left(\int_{-\pi k}^{\pi k} \frac{1 - \cos\lambda}{\lambda^2} d\lambda \right) \langle \boldsymbol{a}_k, \boldsymbol{b}_k \rangle_{\mathbb{C}^n}$$

$$= 2 \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \left(\int_{-\pi|k|}^{\pi|k|} \frac{1 - \cos\lambda}{\lambda^2} d\lambda \right) \langle \boldsymbol{a}_k, \boldsymbol{b}_k \rangle_{\mathbb{C}^n}.$$

$\square$

Lemma 5.4 *If $\boldsymbol{f}, \phi \in C^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{211}(\boldsymbol{f}, \phi) \, ds_1 ds_2$$

$$= \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \left[\int_{-\pi|k|}^{\pi|k|} \frac{4\{\lambda^2 \cos\lambda - 2\lambda \sin\lambda + 2(1 - \cos\lambda)\}}{\lambda^4} d\lambda \right] \langle \boldsymbol{a}_k, \boldsymbol{b}_k \rangle_{\mathbb{C}^n}$$

holds.

Proof. For
$$\boldsymbol{v} = \sum_{k \in \mathbb{Z}} \varphi_k \boldsymbol{c}_k,$$

we have

$$\tilde{Q}_{21} \boldsymbol{v}(s_1, s_1 + h) = 2 \left(\boldsymbol{v}'(s_1) - \frac{\boldsymbol{v}(s_1) - \boldsymbol{v}(s_1 + h)}{-h} \right)$$

$$= 2 \sum_{k \in \mathbb{Z}} \left(\varphi_k'(s_1) - \frac{\varphi_k(s_1) - \varphi_k(s_1 + h)}{-h} \right) \boldsymbol{c}_k$$

$$= 2 \sum_{k \in \mathbb{Z}} \left[\frac{2\pi i k}{\mathcal{L}} + \frac{1}{h} \left\{ 1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}} \right) \right\} \right] \varphi_k(s_1) \boldsymbol{c}_k$$

and

$$\tilde{Q}_{22} \boldsymbol{v}(s_1, s_1 + h) = 2 \left(\boldsymbol{v}'(s_1 + h) - \frac{\boldsymbol{v}(s_1) - \boldsymbol{v}(s_1 + h)}{-h} \right)$$

$$= 2 \sum_{k \in \mathbb{Z}} \left(\varphi_k'(s_1 + h) - \frac{\varphi_k(s_1) - \varphi_k(s_1 + h)}{-h} \right) \boldsymbol{c}_k$$

$$= 2 \sum_{k \in \mathbb{Z}} \left[\frac{2\pi i k}{\mathcal{L}} \exp\left(\frac{2\pi i k h}{\mathcal{L}} \right) + \frac{1}{h} \left\{ 1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}} \right) \right\} \right] \varphi_k(s_1) \boldsymbol{c}_k.$$

Then, setting

$$q_{1k}(h) = \frac{2\pi i k}{\mathcal{L}} + \frac{1}{h}\left\{1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}}\right)\right\}$$

and

$$q_{2k}(h) = \frac{2\pi i k}{\mathcal{L}}\exp\left(\frac{2\pi i k h}{\mathcal{L}}\right) + \frac{1}{h}\left\{1 - \exp\left(\frac{2\pi i k h}{\mathcal{L}}\right)\right\},$$

we obtain

$$\tilde{Q}_{21}\boldsymbol{v}(s_1, s_1 + h) = 2\sum_{k\in\mathbb{Z}} q_{1k}(h)\varphi_k(s_1)\boldsymbol{c}_k,$$

$$\tilde{Q}_{22}\boldsymbol{v}(s_1, s_1 + h) = 2\sum_{k\in\mathbb{Z}} q_{2k}(h)\varphi_k(s_1)\boldsymbol{c}_k.$$

Therefore, we obtain

$$\tilde{Q}_{21}\boldsymbol{f}\cdot\tilde{Q}_{22}\boldsymbol{\phi} + \tilde{Q}_{22}\boldsymbol{f}\cdot\tilde{Q}_{21}\boldsymbol{\phi}$$
$$= 4\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(q_{1k}(h)q_{2m}(h) + q_{2k}(h)q_{1m}(h))\varphi_k(s_1)\varphi_m(s_1)\boldsymbol{a}_k\cdot\boldsymbol{b}_m$$

and

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{211}(\boldsymbol{f}, \boldsymbol{\phi})\,ds_1 ds_2$$
$$= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\tilde{Q}_{21}\boldsymbol{f}\cdot\tilde{Q}_{22}\boldsymbol{\phi} + \tilde{Q}_{22}\boldsymbol{f}\cdot\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2}\,ds_1 ds_2$$
$$= 2\int_{-\mathcal{L}/2}^{\mathcal{L}/2}\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\frac{1}{h^2}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(q_{1k}(h)q_{2m}(h) + q_{2k}(h)q_{1m}(h))\varphi_k(s_1)\varphi_m(s_1)\boldsymbol{a}_k\cdot\boldsymbol{b}_m\,ds_1 dh$$
$$= 2\sum_{k\in\mathbb{Z}}\left\{\int_{-\mathcal{L}/2}^{\mathcal{L}/2}\frac{1}{h^2}(q_{1k}(h)q_{2,-k}(h) + q_{2k}(h)q_{1,-k}(h))dh\right\}\langle\boldsymbol{a}_k, \boldsymbol{b}_k\rangle_{\mathbb{C}^n}.$$

Here, we let $\lambda = \frac{2\pi k h}{\mathcal{L}}$, and then we get

$$q_{1k}(h)q_{2,-k}(h) + q_{2k}(h)q_{1,-k}(h) = \frac{1}{h^2}\left\{2\lambda^2\cos\lambda - 4\lambda\sin\lambda - 4(\cos\lambda - 1)\right\},$$

which leads to the expression

$$2\sum_{k\in\mathbb{Z}}\int_{-\mathcal{L}/2}^{\mathcal{L}/2}\frac{1}{h^2}(q_{1k}(h)q_{2,-k}(h) + q_{2k}(h)q_{1,-k}(h))dh$$
$$= 2\sum_{k\in\mathbb{Z}}\int_{-\pi k}^{\pi k}\left(\frac{2\pi k}{\mathcal{L}}\right)^3\frac{\{2\lambda^2\cos\lambda - 4\lambda\sin\lambda + 4(1 - \cos\lambda)\}}{\lambda^4}d\lambda$$
$$= \sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^3\int_{-\pi|k|}^{\pi|k|}\frac{4\{\lambda^2\cos\lambda - 2\lambda\sin\lambda + 2(1 - \cos\lambda)\}}{\lambda^4}d\lambda.$$

$\square$

Recall that we define the integral sine function si by

$$\text{si}(t) = -\int_t^\infty \frac{\sin\lambda}{\lambda}\, d\lambda$$

and for a function u on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, we define $\check{u}$ by

$$\check{u}(s) = u\left(s + \frac{\mathcal{L}}{2}\right).$$

We note that we will also use the latter notation for vector-valued functions.

Propositon 5.3 *For $\boldsymbol{f}, \boldsymbol{\phi} \in C^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$,*

(5.9)
$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{i11}(\boldsymbol{f}, \boldsymbol{\phi})\, ds_1 ds_2 = \langle L_i \boldsymbol{f}, \boldsymbol{\phi} \rangle_{L^2}$$

holds, where

$$L_1 \boldsymbol{f} = 2\pi(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} - 4 \sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^3 \text{si}(|k\pi|)\langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k$$

$$+ \frac{8}{\mathcal{L}} \Delta_s \left(\boldsymbol{f} - \check{\boldsymbol{f}}\right),$$

$$L_2 \boldsymbol{f} = -\frac{4}{3}\pi(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} + \frac{8}{3} \sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^3 \text{si}(|k\pi|)\langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k$$

$$+ \frac{16}{3\mathcal{L}} \Delta_s \check{\boldsymbol{f}} + \frac{128}{3\mathcal{L}^3} \left(\boldsymbol{f} - \check{\boldsymbol{f}}\right).$$

In particular, the right-hand side of (5.9) can be defined for $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\boldsymbol{\phi} \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Proof. From the above computation, we obtain

$$L_i \boldsymbol{f} = \sum_{k\in\mathbb{Z}} \left|\frac{2\pi k}{\mathcal{L}}\right|^3 \left(\int_{|\lambda|\le|k\pi|} z_i(\lambda)\, d\lambda\right) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k,$$

where

$$z_1(\lambda) = \frac{2(1-\cos\lambda)}{\lambda^2}, \quad z_2(\lambda) = \frac{4\{\lambda^2\cos\lambda - 2\lambda\sin\lambda + 2(1-\cos\lambda)\}}{\lambda^4}.$$

These functions satisfy

$$z_i(\lambda) = \mathcal{O}(\lambda^{-2}) \quad (\lambda \to \pm\infty),$$

and they can be continuously extended at $\lambda = 0$. In particular, $z_i \in L^1(\mathbb{R})$. These are obviously even functions, and we can easily see that

$$z_i(\lambda) = a_i \frac{\sin\lambda}{\lambda} + b_i\, (\lambda z_i(\lambda))',$$

where

$$a_1 = 2, \quad b_1 = -1, \quad a_2 = -\frac{4}{3}, \quad b_2 = -\frac{1}{3}.$$

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Therefore,

$$
\int_{|\lambda| \leq |k\pi|} z_i(\lambda)\, d\lambda = a_i \int_{|\lambda| \leq |k\pi|} \frac{\sin \lambda}{\lambda}\, d\lambda + b_i\, [\lambda z_i(\lambda)]_{-|k\pi|}^{|k\pi|}
$$

$$
= a_i \left(\pi - \int_{|\lambda| \geq |k\pi|} \frac{\sin \lambda}{\lambda}\, d\lambda \right) + b_i\, \{ |k\pi| z_i(|k\pi|) - (-|k\pi| z_i(-|k\pi|)) \}
$$

$$
= a_i\, (\pi + 2\mathrm{si}(|k\pi|)) + 2 b_i |k\pi| z_i(|k\pi|),
$$

and consequently, we obtain

$$
(5.10) \qquad L_i \boldsymbol{f} = a_i \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 (\pi + 2\mathrm{si}(|k\pi|)) \, \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k
$$

$$
+ 2 b_i \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 |k\pi| z_i(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k.
$$

From the definition of z_i,

$$
|k\pi| z_1(|k\pi|) = \frac{2\{ 1 - \cos(|k\pi|) \}}{|k\pi|} = \frac{2\{ 1 - (-1)^k \}}{|k\pi|},
$$

$$
|k\pi| z_2(|k\pi|) = \frac{4[-|k\pi|^2 \cos(|k\pi|) + 2|k\pi| \sin(|k\pi|) - 2\{ 1 - \cos(|k\pi|) \}]}{|k\pi|^3}
$$

$$
= \frac{4[-(-1)^k |k\pi|^2 - 2\{ 1 - (-1)^k \}]}{|k\pi|^3},
$$

and from the definition of the Fourier series, it follows that

$$
\sum_{k \in \mathbb{Z}} \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k = \boldsymbol{f},
$$

$$
\sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^2 \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k = -\Delta_s \boldsymbol{f}.
$$

Since

$$
\langle \check{\boldsymbol{f}}, \varphi_k \rangle_{L^2} = \langle \boldsymbol{f}, \check{\varphi}_k \rangle_{L^2} = (-1)^k \langle \boldsymbol{f}, \varphi_k \rangle_{L^2},
$$

we obtain

$$
\sum_{k \in \mathbb{Z}} (-1)^k \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k = \check{\boldsymbol{f}},
$$

$$
\sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^2 (-1)^k \langle \boldsymbol{f}, \varphi_k \rangle_{L^2}\, \varphi_k = -\Delta_s \check{\boldsymbol{f}},
$$

and therefore

$$\sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 |k\pi| z_1(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k$$

$$= \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \frac{2\{1 - (-1)^k\}}{|k\pi|} \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k$$

$$= \frac{4}{\mathcal{L}} \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^2 \{1 - (-1)^k\} \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k$$

$$= -\frac{4}{\mathcal{L}} \Delta_s \left(\boldsymbol{f} - \check{\boldsymbol{f}} \right),$$

and

$$\sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 |k\pi| z_2(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k$$

$$= \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \frac{4[-(-1)^k |k\pi|^2 - 2\{1 - (-1)^k\}]}{|k\pi|^3} \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k$$

$$= 4 \sum_{k \in \mathbb{Z}} \left[-\frac{2}{\mathcal{L}} \left| \frac{2\pi k}{\mathcal{L}} \right|^2 (-1)^k - \frac{16}{\mathcal{L}^3} \{1 - (-1)^k\} \right] \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k$$

$$= -\frac{8}{\mathcal{L}} \Delta_s \check{\boldsymbol{f}} - \frac{64}{\mathcal{L}^3} \left(\boldsymbol{f} - \check{\boldsymbol{f}} \right).$$

If we substitute the above result and

$$\sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k = (-\Delta_s)^{\frac{3}{2}} \boldsymbol{f}$$

into (5.10), noting that

$$|\mathrm{si}\,(t)| = \left| \left[-\frac{\cos \lambda}{\lambda} \right]_t^\infty - \int_t^\infty \frac{\cos \lambda}{\lambda^2} \, d\lambda \right| = \mathcal{O}(t^{-1}) \quad (t \to \infty),$$

then we conclude that

$$\sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \mathrm{si}(|k\pi|) \langle \cdot, \varphi_k \rangle_{L^2} \, \varphi_k$$

is a second-order pseudo-differential operator, and it can be estimated as

$$\left\| \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \mathrm{si}(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \, \varphi_k \right\|_{L^2} \le C \left(\sum_{k \in \mathbb{Z}} \frac{|k\pi|^4}{\mathcal{L}^6} |\langle \boldsymbol{f}, \varphi_k \rangle_{L^2}|^2 \right)^{\frac{1}{2}}$$

$$\le C \mathcal{L}^{-1} \|(-\Delta_s) \boldsymbol{f}\|_{L^2}.$$

Consequently, under the assumption that $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$,

$$\|L_i \boldsymbol{f}\|_{L^2} \le C \left(\|(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f}\|_{L^2} + \mathcal{L}^{-1} \|(-\Delta_s) \boldsymbol{f}\|_{L^2} + \mathcal{L}^{-3} \|\boldsymbol{f}\|_{L^2} \right) < \infty.$$

$\square$

Thus, we have obtained the linear part of the L^2-*gradient* of $\mathcal{E}_i$.

5.4.3 The nonlinear part

In this section, we consider the part of $G_{i1k}(\boldsymbol{f}, \boldsymbol{\phi})$ ($k = 2$, 3) or $G_{i2}(\boldsymbol{f}, \boldsymbol{\phi})$ that is nonlinear with respect to $\boldsymbol{f}$.

We let $G(\boldsymbol{f}, \boldsymbol{\phi})$ be either $G_{i1k}(\boldsymbol{f}, \boldsymbol{\phi})$ ($k = 2$, 3) or $G_{i2}(\boldsymbol{f}, \boldsymbol{\phi})$, and we observe that it has the following form:

$$G(\boldsymbol{f}, \boldsymbol{\phi}) = \boldsymbol{G}_A(\boldsymbol{f}) \cdot \Delta \boldsymbol{\phi}' + \boldsymbol{G}_B(\boldsymbol{f}) \cdot \Delta \boldsymbol{\phi} + \boldsymbol{G}_C(\boldsymbol{f}) \cdot \boldsymbol{\phi}'(s_1) + \boldsymbol{G}_D(\boldsymbol{f}) \cdot \boldsymbol{\phi}'(s_2).$$

Lemma 5.5 *The following identities hold for any function $\boldsymbol{\zeta}$ on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$.*

1.

$$\iint_{|s_1 - s_2| \geq \varepsilon} \boldsymbol{\zeta}(s_1, s_2) \cdot \boldsymbol{\phi}'(s_1) \, ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{\zeta}(s, s + \varepsilon) - \boldsymbol{\zeta}(s, s - \varepsilon)) \cdot \boldsymbol{\phi}(s) \, ds$$

$$- \iint_{|s_1 - s_2| \geq \varepsilon} \frac{\partial}{\partial s_1} \boldsymbol{\zeta}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1) \, ds_1 ds_2$$

2.

$$\iint_{|s_1 - s_2| \geq \varepsilon} \boldsymbol{\zeta}(s_1, s_2) \cdot \boldsymbol{\phi}'(s_2) \, ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{\zeta}(s + \varepsilon, s) - \boldsymbol{\zeta}(s - \varepsilon, s)) \cdot \boldsymbol{\phi}(s) \, ds$$

$$- \iint_{|s_1 - s_2| \geq \varepsilon} \frac{\partial}{\partial s_2} \boldsymbol{\zeta}(s_1, s_2) \cdot \boldsymbol{\phi}(s_2) \, ds_1 ds_2$$

3.

$$\iint_{|s_1 - s_2| \geq \varepsilon} \boldsymbol{\zeta}(s_1, s_2) \cdot \Delta \boldsymbol{\phi} \, ds_1 ds_2 = \iint_{|s_1 - s_2| \geq \varepsilon} (\boldsymbol{\zeta}(s_1, s_2) - \boldsymbol{\zeta}(s_2, s_1)) \cdot \boldsymbol{\phi}(s_1) \, ds_1 ds_2$$

Proof. First note that

$$\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\phi'(s_1)\,ds_1ds_2$$

$$=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left(\int_{s_2-\frac{\mathcal{L}}{2}}^{s_2-\varepsilon}+\int_{s_2+\varepsilon}^{s_2+\frac{\mathcal{L}}{2}}\right)\boldsymbol{\zeta}(s_1,s_2)\cdot\phi'(s_1)\,ds_1ds_2$$

$$=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{[\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_1)]_{s_2-\frac{\mathcal{L}}{2}}^{s_2-\varepsilon}+[\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_1)]_{s_2+\varepsilon}^{s_2+\frac{\mathcal{L}}{2}}\right.$$

$$\left.-\left(\int_{s_2-\frac{\mathcal{L}}{2}}^{s_2-\varepsilon}+\int_{s_2+\varepsilon}^{s_2+\frac{\mathcal{L}}{2}}\right)\frac{\partial}{\partial s_1}\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_1)\,ds_1\right\}ds_2$$

$$=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\boldsymbol{\zeta}(s_2-\varepsilon,s_2)\cdot\phi(s_2-\varepsilon)-\boldsymbol{\zeta}(s_2+\varepsilon,s_2)\cdot\phi(s_2+\varepsilon)\right.$$

$$\left.-\left(\int_{s_2-\frac{\mathcal{L}}{2}}^{s_2-\varepsilon}+\int_{s_2+\varepsilon}^{s_2+\frac{\mathcal{L}}{2}}\right)\frac{\partial}{\partial s_1}\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_1)\,ds_1\right\}ds_2$$

$$=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\boldsymbol{\zeta}(s_2-\varepsilon,s_2)\cdot\phi(s_2-\varepsilon)-\boldsymbol{\zeta}(s_2+\varepsilon,s_2)\cdot\phi(s_2+\varepsilon)\right\}ds_2$$

$$-\iint_{|s_1-s_2|\geq\varepsilon}\frac{\partial}{\partial s_1}\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_1)\,ds_1ds_2=(*).$$

By changing variables such that $s_2\pm\varepsilon=s$, we obtain

$$(*)=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{\zeta}(s,s+\varepsilon)-\boldsymbol{\zeta}(s,s-\varepsilon))\cdot\phi(s)ds$$

$$-\iint_{|s_1-s_2|\geq\varepsilon}\frac{\partial}{\partial s_1}\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_1)\,ds_1ds_2.$$

The third identity of this lemma follows from

$$\iint_{|s_1-s_2|\geq\varepsilon}\boldsymbol{\zeta}(s_1,s_2)\cdot\phi(s_2)\,ds_1ds_2=\iint_{|s_1-s_2|\geq\varepsilon}\boldsymbol{\zeta}(s_2,s_1)\cdot\phi(s_1)\,ds_1ds_2.$$

$\square$

Corollary 5.1 *For any function* $\boldsymbol{\zeta}$ *on* $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$ *it holds that*

$$\iint_{|s_1-s_2|\geq\varepsilon}\boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\phi'\,ds_1ds_2$$

$$=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{\zeta}(s,s+\varepsilon)-\boldsymbol{\zeta}(s+\varepsilon,s)-\boldsymbol{\zeta}(s,s-\varepsilon)+\boldsymbol{\zeta}(s-\varepsilon,s))\cdot\phi(s)\,ds$$

$$-\iint_{|s_1-s_2|\geq\varepsilon}\frac{\partial}{\partial s_1}(\boldsymbol{\zeta}(s_1,s_2)-\boldsymbol{\zeta}(s_2,s_1))\cdot\phi(s_1)\,ds_1ds_2.$$

Proof. Using Lemma 5.5, we have

$$
\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\phi}'\,ds_1ds_2
$$

$$
= \iint_{|s_1-s_2|\geq\varepsilon} (\boldsymbol{\zeta}(s_1,s_2)-\boldsymbol{\zeta}(s_2,s_1))\cdot\boldsymbol{\phi}'(s_1)\,ds_1ds_2
$$

$$
= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{\zeta}(s,s+\varepsilon)-\boldsymbol{\zeta}(s+\varepsilon,s)-\boldsymbol{\zeta}(s,s-\varepsilon)+\boldsymbol{\zeta}(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds
$$

$$
-\iint_{|s_1-s_2|\geq\varepsilon} \frac{\partial}{\partial s_1}(\boldsymbol{\zeta}(s_1,s_2)-\boldsymbol{\zeta}(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2.
$$

$\square$

From Lemma 5.5 and Corollary 5.1, we obtain

$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2
$$

$$
= \lim_{\varepsilon\to+0} \iint_{|s_1-s_2|\geq\varepsilon} G(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2
$$

$$
= \lim_{\varepsilon\to+0} \left(\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_A(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi}'ds_1ds_2 + \iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_B(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi}\,ds_1ds_2 \right.
$$

$$
\left. + \iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_C(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_1)\,ds_1ds_2 + \iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_D(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_2)\,ds_1ds_2 \right)
$$

$$
= \lim_{\varepsilon\to+0} \left\{ \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_A(\boldsymbol{f})(s,s+\varepsilon)-\boldsymbol{G}_A(\boldsymbol{f})(s+\varepsilon,s) \right.
$$

$$
-\boldsymbol{G}_A(\boldsymbol{f})(s,s-\varepsilon)-\boldsymbol{G}_A(\boldsymbol{f})(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds
$$

$$
- \iint_{|s_1-s_2|\geq\varepsilon} \frac{\partial}{\partial s_1}(\boldsymbol{G}_A(\boldsymbol{f})(s_1,s_2)-\boldsymbol{G}_A(\boldsymbol{f})(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2
$$

$$
+ \iint_{|s_1-s_2|\geq\varepsilon} (\boldsymbol{G}_B(\boldsymbol{f})(s_1,s_2)-\boldsymbol{G}_B(\boldsymbol{f})(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2
$$

$$
+ \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_C(\boldsymbol{f})(s,s+\varepsilon)-\boldsymbol{G}_C(\boldsymbol{f})(s,s-\varepsilon))\cdot\boldsymbol{\phi}(s)\,ds
$$

$$
- \iint_{|s_1-s_2|\geq\varepsilon} \frac{\partial}{\partial s_1}\boldsymbol{G}_C(\boldsymbol{f})(s_1,s_2)\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2
$$

$$
+ \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_D(\boldsymbol{f})(s+\varepsilon,s)-\boldsymbol{G}_D(\boldsymbol{f})(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds
$$

$$
\left. - \iint_{|s_1-s_2|\geq\varepsilon} \frac{\partial}{\partial s_2}\boldsymbol{G}_D(\boldsymbol{f})(s_1,s_2)\cdot\boldsymbol{\phi}(s_2)\,ds_1ds_2 \right\}
$$

$$
= (\dagger).
$$

Here, we shall prove that if $\boldsymbol{f} \in H^3$, then

$$(\dagger) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left\{ -\frac{\partial}{\partial s_1} \left(\boldsymbol{G}_A(\boldsymbol{f})(s_1, s_2) - \boldsymbol{G}_A(\boldsymbol{f})(s_2, s_1) \right) + \boldsymbol{G}_B(\boldsymbol{f})(s_1, s_2) - \boldsymbol{G}_B(\boldsymbol{f})(s_2, s_1) \right.$$
$$\left. -\frac{\partial}{\partial s_1} \boldsymbol{G}_C(\boldsymbol{f})(s_1, s_2) - \frac{\partial}{\partial s_1} \boldsymbol{G}_D(\boldsymbol{f})(s_2, s_1) \right\} \cdot \boldsymbol{\phi}(s_1) \, ds_1 ds_2$$
$$= (\ddagger)$$

holds. Furthermore, we shall check that

$$\boldsymbol{N}(\boldsymbol{f})(s) = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ -\frac{\partial}{\partial s} \left(\boldsymbol{G}_A(\boldsymbol{f})(s, s_2) - \boldsymbol{G}_A(\boldsymbol{f})(s_2, s) \right) + \boldsymbol{G}_B(\boldsymbol{f})(s, s_2) - \boldsymbol{G}_B(\boldsymbol{f})(s_2, s) \right.$$
$$\left. -\frac{\partial}{\partial s} \boldsymbol{G}_C(\boldsymbol{f})(s, s_2) - \frac{\partial}{\partial s} \boldsymbol{G}_D(\boldsymbol{f})(s_2, s) \right\} \, ds_2$$

is well defined at $\mathscr{L}^1$-a.e. $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$; we do this so that we can use Fubini's theorem, which gives rise to

$$(\ddagger) = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \boldsymbol{N}(\boldsymbol{f})(s) \cdot \boldsymbol{\phi}(s) \, ds = \langle \boldsymbol{N}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2}.$$

We shall also show that $\boldsymbol{N}(\boldsymbol{f})$ is a lower-order term of order less than three, and
$$\|\boldsymbol{N}(\boldsymbol{f})\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}, \lambda).$$

We use $\boldsymbol{N}_{i1k}$ or $\boldsymbol{N}_{i2}$ as counterparts of $\boldsymbol{N}$ when $G = G_{i1k}$ or G_{i2}, respectively.

As preparation, we need some lemmas and corollaries.

Lemma 5.6 *For $\boldsymbol{\kappa} \in L^\infty$, it holds that*

$$(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2 = \mathcal{O}(\Delta s)^4.$$

Proof. From a direct calculation, we arrive at

$$(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2 = \int_{s_2}^{s_1} \int_{s_2}^{s_1} (1 - \boldsymbol{\tau}(s_3) \cdot \boldsymbol{\tau}(s_4)) \, ds_3 ds_4$$
$$= \frac{1}{2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_4)\|_{\mathbb{R}^n}^2 \, ds_3 ds_4$$
$$= \frac{1}{2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_4}^{s_3} \int_{s_4}^{s_3} \boldsymbol{\kappa}(s_5) \cdot \boldsymbol{\kappa}(s_6) \, ds_5 ds_6 ds_3 ds_4$$
$$= \mathcal{O}(\Delta s)^4.$$

$\square$

Corollary 5.2 *Assume that $\boldsymbol{f}$ is bi-Lipschitz and $\boldsymbol{\kappa} \in L^\infty$. Then for $k \geq 1$,*

$$\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k - 1 = \mathcal{O}(\Delta s)^2$$

holds.

Proof. To begin, we note that with appropriate modifications, the claim can be shown not only for natural numbers but also for all real numbers greater than one. However, we need the result only for natural numbers, and so we will assume that k is a natural number, in which case, we have

$$
\begin{aligned}
\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k - 1 &= \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} - 1\right) \sum_{\ell=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^\ell \\
&= \frac{\left(\frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - 1\right)}{\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} + 1} \sum_{\ell=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^\ell \\
&= \frac{(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}{(\Delta s)^2} \frac{1}{\frac{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}{|\Delta s|} + \frac{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}{|\Delta s|^2}} \sum_{\ell=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^\ell \\
&= \mathcal{O}(\Delta s)^2.
\end{aligned}
$$

$\square$

Lemma 5.7 *Let $\boldsymbol{\kappa} \in L^\infty$. It follows that*

$$
T_i^0 \boldsymbol{f} = \mathcal{O}(\Delta s), \quad T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) = \mathcal{O}(\Delta s)^2, \quad T_i^0 \boldsymbol{f} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} = \mathcal{O}(\Delta s)^2.
$$

Proof. Recall that T_i^k is given by (5.1). By the following computations, we obtain

$$
\begin{aligned}
T_i^0 \boldsymbol{f} &= \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_i) = \frac{1}{\Delta s} \int_{s_2}^{s_1} (\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_i)) \, ds_3 \\
&= \frac{1}{\Delta s} \int_{s_2}^{s_1} \int_{s_i}^{s_3} \boldsymbol{\kappa}(s_4) \, ds_4 ds_3 = \mathcal{O}(\Delta s),
\end{aligned}
$$

$$
\begin{aligned}
T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) &= \frac{1}{\Delta s} \int_{s_2}^{s_1} \int_{s_i}^{s_3} \boldsymbol{\kappa}(s_4) \cdot \boldsymbol{\tau}(s_j) \, ds_4 ds_3 \\
&= \frac{1}{\Delta s} \int_{s_2}^{s_1} \int_{s_i}^{s_3} \boldsymbol{\kappa}(s_4) \cdot (\boldsymbol{\tau}(s_j) - \boldsymbol{\tau}(s_4)) \, ds_4 ds_3 \\
&= \frac{1}{\Delta s} \int_{s_2}^{s_1} \int_{s_i}^{s_3} \int_{s_4}^{s_j} \boldsymbol{\kappa}(s_4) \cdot \boldsymbol{\kappa}(s_5) \, ds_5 ds_4 ds_3 \\
&= \mathcal{O}(\Delta s)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
T_i^0 \boldsymbol{f} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} &= T_i^0 \boldsymbol{f} \cdot \left(\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_j)\right) + T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) \\
&= T_i^0 \boldsymbol{f} \cdot T_j^0 \boldsymbol{f} + T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) \\
&= \mathcal{O}(\Delta s)^{1+1} + \mathcal{O}(\Delta s)^2 = \mathcal{O}(\Delta s)^2.
\end{aligned}
$$

$\square$

Corollary 5.3 *Suppose that $\boldsymbol{f}$ is bi-Lipschitz and $\boldsymbol{\kappa} \in L^\infty$. Then for $k \geq 1$,*

$$T_i^k \boldsymbol{f} = \mathcal{O}(\Delta s), \quad T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) = \mathcal{O}(\Delta s)^2, \quad T_i^k \boldsymbol{f} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} = \mathcal{O}(\Delta s)^2$$

hold.

Proof. From the following calculations, we obtain

$$T_i^k \boldsymbol{f} = \left\{ \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k - 1 \right\} \frac{\Delta \boldsymbol{f}}{\Delta s} + T_i^0 \boldsymbol{f} = \mathcal{O}(\Delta s)^2 + \mathcal{O}(\Delta s) = \mathcal{O}(\Delta s),$$

$$\begin{aligned}
T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) &= \left\{ \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k - 1 \right\} \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_j) + T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) \\
&= \mathcal{O}(\Delta s)^2 + \mathcal{O}(\Delta s)^2 = \mathcal{O}(\Delta s)^2,
\end{aligned}$$

and

$$\begin{aligned}
T_i^k \boldsymbol{f} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} &= T_i^k \boldsymbol{f} \cdot \left(\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_j) \right) + T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) \\
&= T_i^k \boldsymbol{f} \cdot T_j^0 \boldsymbol{f} + T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) \\
&= \mathcal{O}(\Delta s)^{1+1} + \mathcal{O}(\Delta s)^2 = \mathcal{O}(\Delta s)^2.
\end{aligned}$$

$\square$

Lemma 5.8 *We also have the following results with regard to the derivatives of $\mathcal{M}(\boldsymbol{f})$ and $T_i^k(\boldsymbol{f})$:*

$$\frac{\partial}{\partial s_j} \mathcal{M}(\boldsymbol{f}) = \frac{2(-1)^j}{(\Delta s)^3} T_j^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j), \quad \frac{\partial}{\partial s_j} \frac{\Delta \boldsymbol{f}}{\Delta s} = \frac{(-1)^j}{\Delta s} T_j^0 \boldsymbol{f},$$

$$\begin{aligned}
\frac{\partial}{\partial s_j} T_i^k \boldsymbol{f} = &- \frac{(-1)^j k}{\Delta s} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^{k+2} \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot T_j^0 \boldsymbol{f} \right) \frac{\Delta \boldsymbol{f}}{\Delta s} \\
&+ \frac{(-1)^j}{\Delta s} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k T_j^0 \boldsymbol{f} - \delta_{ij} \boldsymbol{\kappa}(s_j).
\end{aligned}$$

Proof. These can be shown by direct calculation, as follows :

$$\begin{aligned}
\frac{\partial}{\partial s_j} \mathcal{M}(\boldsymbol{f}) &= - \frac{2\Delta \boldsymbol{f} \cdot (-1)^{j-1} \boldsymbol{\tau}(s_j)}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} + \frac{(-1)^{j-1} 2}{(\Delta s)^3} \\
&= \frac{2(-1)^j}{(\Delta s)^3} \left\{ \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_j) - 1 \right\} \\
&= \frac{2(-1)^j}{(\Delta s)^3} \left\{ \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_j) \right\} \cdot \boldsymbol{\tau}(s_j) \\
&= \frac{2(-1)^j}{(\Delta s)^3} T_j^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j),
\end{aligned}$$

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$$\frac{\partial}{\partial s_j}\frac{\Delta \boldsymbol{f}}{\Delta s} = \frac{\partial}{\partial s_j}\frac{\boldsymbol{f}_1 - \boldsymbol{f}_2}{s_1 - s_2} = \frac{(-1)^{j-1}\boldsymbol{\tau}(s_j)}{s_1 - s_2} - \frac{(-1)^{j-1}(\boldsymbol{f}_1 - \boldsymbol{f}_2)}{(s_1 - s_2)^2}$$

$$= \frac{(-1)^{j-1}}{\Delta s}\left(\boldsymbol{\tau}(s_j) - \frac{\Delta \boldsymbol{f}}{\Delta s}\right) = \frac{(-1)^j}{\Delta s}T_j^0\boldsymbol{f},$$

$$\frac{\partial}{\partial s_j}T_i^k\boldsymbol{f} = \frac{\partial}{\partial s_j}\left\{\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}_i\right\}$$

$$= \frac{\partial}{\partial s_j}\left\{\left(\left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2\right)^{-\frac{k}{2}}\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}_i\right\}$$

$$= \left\{\frac{\partial}{\partial s_j}\left(\left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2\right)^{-\frac{k}{2}}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s} + \left(\left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2\right)^{-\frac{k}{2}}\frac{\partial}{\partial s_j}\frac{\Delta \boldsymbol{f}}{\Delta s} - \frac{\partial}{\partial s_j}\boldsymbol{\tau}_i$$

$$= -\frac{k}{2}\left(\left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2\right)^{-\frac{k}{2}-1}\left(2\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\partial}{\partial s_j}\frac{\Delta \boldsymbol{f}}{\Delta s}\right)\frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$\quad + \left(\left\|\frac{\Delta \boldsymbol{f}}{\Delta s}\right\|_{\mathbb{R}^n}^2\right)^{-\frac{k}{2}}\frac{\partial}{\partial s_j}\frac{\Delta \boldsymbol{f}}{\Delta s} - \delta_{ij}\boldsymbol{\kappa}(s_j)$$

$$= -\frac{(-1)^j k}{\Delta s}\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^{k+2}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot T_j^0\boldsymbol{f}\right)\frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$\quad + \frac{(-1)^j}{\Delta s}\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k T_j^0\boldsymbol{f} - \delta_{ij}\boldsymbol{\kappa}(s_j).$$

$$\square$$

Then let us consider how to derive and estimate $\boldsymbol{N}_{i12}$. If we set

$$G_{112}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{1}{2}\mathscr{M}(\boldsymbol{f})(\tilde{Q}_{11}\boldsymbol{f}\cdot\tilde{Q}_{12}\boldsymbol{\phi} + \tilde{Q}_{12}\boldsymbol{f}\cdot\tilde{Q}_{11}\boldsymbol{\phi})$$

$$= \mathscr{M}(\boldsymbol{f})(\Delta \boldsymbol{f}'\cdot\Delta \boldsymbol{\phi}')$$

$$= \boldsymbol{g}_{112}(\boldsymbol{f})\cdot\Delta \boldsymbol{\phi}',$$

then $\boldsymbol{g}_{112}$ can be expressed as

$$\boldsymbol{g}_{112}(\boldsymbol{f})(s_1, s_2) = \mathscr{M}(\boldsymbol{f})\Delta \boldsymbol{f}'.$$

Lemma 5.9 *Let* $\alpha \in (0, \frac{1}{2})$. *If* $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, *then*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{112}(\boldsymbol{f},\boldsymbol{\phi})(s_1, s_2)ds_1 ds_2 = \langle\boldsymbol{N}_{112}(\boldsymbol{f}),\boldsymbol{\phi}\rangle_{L^2},$$

where

$$\boldsymbol{N}_{112}(\boldsymbol{f})(s_1) = 2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} - \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\}ds_2$$

satisfies

$$\|\boldsymbol{N}_{112}(\boldsymbol{f})\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Proof. In what follows, we denote $\boldsymbol{g}_{112}(\boldsymbol{f})(s_1, s_2)$ by $\boldsymbol{g}_{112}(s_1, s_2)$ for simplicity. Using Corollary 5.1, we obtain

$$\iint_{|s_1-s_2|\geq\varepsilon} G_{112}(\boldsymbol{f}, \boldsymbol{\phi}) ds_1 ds_2$$

$$= \iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{g}_{112}(\boldsymbol{f}) \cdot \Delta\boldsymbol{\phi}' ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{g}_{112}(s, s+\varepsilon) - \boldsymbol{g}_{112}(s+\varepsilon, s) - \boldsymbol{g}_{112}(s, s-\varepsilon) + \boldsymbol{g}_{112}(s-\varepsilon, s)) \cdot \boldsymbol{\phi}(s) ds$$

$$\qquad - \iint_{|s_1-s_2|\geq\varepsilon} \frac{\partial}{\partial s_1}(\boldsymbol{g}_{112}(s_1, s_2) - \boldsymbol{g}_{112}(s_2, s_1)) \cdot \boldsymbol{\phi}(s_1) ds_1 ds_2$$

$$= (*).$$

Noting that

$$\boldsymbol{g}_{112}(s_1, s_2) = \mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{f}' = \mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{\tau} = \mathcal{O}(\Delta s),$$

we can show that

$$\boldsymbol{g}_{112}(s, s+\varepsilon) - \boldsymbol{g}_{112}(s+\varepsilon, s) - \boldsymbol{g}_{112}(s, s-\varepsilon) + \boldsymbol{g}_{112}(s-\varepsilon, s) = \mathcal{O}(\varepsilon) \quad (\varepsilon \to 0).$$

$\mathcal{O}(\varepsilon)$ is uniform with respect to $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, and in what follows, we will use this notation. Since

$$\boldsymbol{g}_{112}(s_1, s_2) - \boldsymbol{g}_{112}(s_2, s_1) = 2\mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{\tau}$$

holds, if we use Lemma 5.8 and Corollary 5.3, we obtain

$$\frac{\partial}{\partial s_1}\{\boldsymbol{g}_{112}(s_1, s_2) - \boldsymbol{g}_{112}(s_2, s_1)\} = 2\left(\frac{\partial\mathscr{M}(\boldsymbol{f})}{\partial s_1}\Delta\boldsymbol{\tau} + \mathscr{M}(\boldsymbol{f})\frac{\partial\Delta\boldsymbol{\tau}}{\partial s_1}\right)$$

$$= 2\left\{-\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} + \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\}$$

$$= \mathcal{O}(\Delta s)^{-3+2+1} + \mathcal{O}(1)$$

$$= \mathcal{O}(1) \quad \text{as} \quad \Delta s \to 0,$$

which implies $\dfrac{\partial}{\partial s_1}\{\boldsymbol{g}_{112}(s_1, s_2) - \boldsymbol{g}_{112}(s_2, s_1)\}$ is bounded and hence absolutely integrable on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$. We then use Fubini's theorem to obtain

$$(*) \to -\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} 2\left\{-\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} + \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\} \cdot \boldsymbol{\phi}(s_1) ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left[2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} - \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\} ds_2\right] \cdot \boldsymbol{\phi}(s_1) ds_1$$

$$= \langle \boldsymbol{N}_{112}(\boldsymbol{f}), \boldsymbol{\phi}\rangle_{L^2}$$

by letting $\varepsilon \to +0$ in $(*)$. Since

$$\|\boldsymbol{\kappa}\|_{L^\infty} = \|\boldsymbol{f}''\|_{L^\infty} \leq C_\alpha\|\boldsymbol{f}\|_{H^{3-\alpha}}$$

for $\alpha \in \left(0, \frac{1}{2}\right)$, the bound on the integrand of $\boldsymbol{N}_{112}$ follows from Corollary 5.3. Thus, we have

$$\|\boldsymbol{N}_{112}(\boldsymbol{f})\|_{L^2} \leq C\|\boldsymbol{f}\|_{H^{3-\alpha}}.$$

$\square$

Next, we consider G_{212}, which may be decomposed as

$$
\begin{aligned}
G_{212}(\boldsymbol{f},\phi) &= \frac{1}{2}\mathscr{M}(\boldsymbol{f})(\tilde{Q}_{21}\boldsymbol{f}\cdot\tilde{Q}_{22}\phi + \tilde{Q}_{22}\boldsymbol{f}\cdot\tilde{Q}_{21}\phi) \\
&= \frac{1}{2}\mathscr{M}(\boldsymbol{f})\left\{(-2T_1^0\boldsymbol{f})\cdot 2\left(\phi'(s_2)-\frac{\Delta\phi}{\Delta s}\right) + (-2T_2^0\boldsymbol{f})\cdot 2\left(\phi'(s_1)-\frac{\Delta\phi}{\Delta s}\right)\right\} \\
&= 2\mathscr{M}(\boldsymbol{f})(T_1^0\boldsymbol{f}+T_2^0\boldsymbol{f})\cdot\frac{\Delta\phi}{\Delta s} - 2\mathscr{M}(\boldsymbol{f})T_2^0\boldsymbol{f}\cdot\phi'(s_1) - 2\mathscr{M}(\boldsymbol{f})T_1^0\boldsymbol{f}\cdot\phi'(s_2) \\
&= \boldsymbol{G}_{212B}(\boldsymbol{f})\cdot\Delta\phi + \boldsymbol{G}_{212C}(\boldsymbol{f})\cdot\phi'(s_1) + \boldsymbol{G}_{212D}(\boldsymbol{f})\cdot\phi'(s_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{G}_{212B}(\boldsymbol{f}) &= \frac{2}{\Delta s}\mathscr{M}(\boldsymbol{f})(T_1^0\boldsymbol{f}+T_2^0\boldsymbol{f}), \\
\boldsymbol{G}_{212C}(\boldsymbol{f}) &= -2\mathscr{M}(\boldsymbol{f})T_2^0\boldsymbol{f}, \\
\boldsymbol{G}_{212D}(\boldsymbol{f}) &= -2\mathscr{M}(\boldsymbol{f})T_1^0\boldsymbol{f}.
\end{aligned}
$$

For simplicity, we will let $\boldsymbol{G}_{212B}(s_1,s_2)$ denote $\boldsymbol{G}_{212B}(\boldsymbol{f})(s_1,s_2)$, and similarly for $\boldsymbol{G}_{212C}$ and $\boldsymbol{G}_{212D}$. Then we have

$$
\boldsymbol{G}_{212B}(s_1,s_2) = -\boldsymbol{G}_{212B}(s_2,s_1), \quad \boldsymbol{G}_{212C}(s_1,s_2) = \boldsymbol{G}_{212D}(s_2,s_1).
$$

Lemma 5.10 *Let $\alpha\in(0,\frac{1}{2})$. If $\boldsymbol{f}\in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then*

$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z}_\varsigma)^2}\boldsymbol{G}_{212B}(\boldsymbol{f})(s_1,s_2)\cdot\Delta\phi\, ds_1 ds_2 = \langle\boldsymbol{N}_{212B}(\boldsymbol{f}),\phi\rangle_{L^2},
$$

where

$$
\boldsymbol{N}_{212B}(\boldsymbol{f})(s_1) = 4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\frac{1}{\Delta s}\mathscr{M}(\boldsymbol{f})(T_1^0\boldsymbol{f}+T_2^0\boldsymbol{f})ds_2
$$

satisfies

$$
\|\boldsymbol{N}_{212B}(\boldsymbol{f})\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).
$$

Proof. From Lemma 5.5.3, we obtain

$$
\begin{aligned}
&\iint_{|s_1-s_2|\geq\varepsilon}\boldsymbol{G}_{212B}(\boldsymbol{f})\cdot\Delta\phi\, ds_1 ds_2 \\
&= \iint_{|s_1-s_2|\geq\varepsilon}(\boldsymbol{G}_{212B}(\boldsymbol{f})(s_1,s_2)-\boldsymbol{G}_{212B}(\boldsymbol{f})(s_2,s_1))\cdot\phi(s_1)ds_1 ds_2 \\
&= 2\iint_{|s_1-s_2|\geq\varepsilon}\boldsymbol{G}_{212B}(s_1,s_2)\cdot\phi(s_1)ds_1 ds_2,
\end{aligned}
$$

and from Lemma 5.7, we obtain

$$
\boldsymbol{G}_{212B}(s_1,s_2) = \frac{2}{\Delta s}\mathscr{M}(\boldsymbol{f})(T_1^0\boldsymbol{f}+T_2^0\boldsymbol{f}) = \mathcal{O}(\Delta s)^{-1+0+1} = \mathcal{O}(1).
$$

In particular, $\boldsymbol{G}_{212B}$ belongs to $L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. We apply Fubini's theorem, and then we arrive at

$$
\begin{aligned}
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\boldsymbol{G}_{212B}(\boldsymbol{f})\cdot\Delta\phi\, ds_1 ds_2 &= \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})}\left(2\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}G_{212B}(\boldsymbol{f},\phi)(s_1,s_2)ds_2\right)\cdot\phi(s_1)ds_1 \\
&= \langle\boldsymbol{N}_{212B}(\boldsymbol{f}),\phi\rangle_{L^2}
\end{aligned}
$$

by letting $\varepsilon \to +0$ in the above identity. The claimed estimate on $\|\boldsymbol{N}_{212B}(\boldsymbol{f})\|_{L^2}$ is obtained in a similar way to that of $\|\boldsymbol{N}_{112}(\boldsymbol{f})\|_{L^2}$, and so we omit the details.

$\square$

Lemma 5.11 *Let $\alpha \in (0, \frac{1}{2})$. If $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \boldsymbol{G}_{212C}(\boldsymbol{f})(s_1, s_2) \cdot \boldsymbol{\phi}'(s_1) ds_1 ds_2 = \langle \boldsymbol{N}_{212C}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2}$$

holds, where

$$\boldsymbol{N}_{212C}(\boldsymbol{f})(s_1) = -2 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{2}{(\Delta s)^3} (T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) T_2^0 \boldsymbol{f} + \frac{\mathscr{M}(\boldsymbol{f})}{\Delta s} T_1^0 \boldsymbol{f} \right\} ds_2$$

satisfies

$$\|\boldsymbol{N}_{212C}(\boldsymbol{f})\|_{L^2} \le C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Proof. From Lemma 5.5.1,

$$\iint_{|s_1-s_2| \ge \varepsilon} \boldsymbol{G}_{212C}(\boldsymbol{f}) \cdot \boldsymbol{\phi}'(s_1) ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_{212C}(\boldsymbol{f})(s, s+\varepsilon) - \boldsymbol{G}_{212C}(\boldsymbol{f})(s, s-\varepsilon)) \cdot \boldsymbol{\phi}(s) ds$$

$$\qquad - \iint_{|s_1-s_2| \ge \varepsilon} \frac{\partial}{\partial s} \boldsymbol{G}_{212C}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1) ds_1 ds_2$$

$$= (*),$$

and from Lemma 5.7, we have

$$\boldsymbol{G}_{212C}(\boldsymbol{f})(s_1, s_2) = -2\mathscr{M}(\boldsymbol{f}) T_2^0 \boldsymbol{f} = \mathcal{O}(\Delta s).$$

Therefore, we have

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_{212C}(\boldsymbol{f})(s, s+\varepsilon) - \boldsymbol{G}_{212C}(\boldsymbol{f})(s, s-\varepsilon)) \cdot \boldsymbol{\phi}(s_1) ds_1 = \mathcal{O}(\varepsilon) \to 0 \quad (\varepsilon \to 0),$$

and combining this with Lemma 5.8, Lemma 5.7, and Corollary 5.3, we obtain

$$\frac{\partial}{\partial s_1} \boldsymbol{G}_{212C}(\boldsymbol{f}) = 2 \left\{ \frac{2}{(\Delta s)^3} (T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) T_2^0 \boldsymbol{f} + \frac{\mathscr{M}(\boldsymbol{f})}{\Delta s} T_1^0 \boldsymbol{f} \right\}$$

$$= \mathcal{O}(\Delta s)^{-3+2+1} + \mathcal{O}(\Delta s)^{-1+1} = \mathcal{O}(1).$$

We apply Fubini's theorem and obtain

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \boldsymbol{G}_{212C}(\boldsymbol{f}) \cdot \boldsymbol{\phi}'(s_1) ds_1 ds_2$$

$$= - \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\partial}{\partial s_1} \boldsymbol{G}_{212C}(\boldsymbol{f}) \cdot \boldsymbol{\phi}(s_1) ds_1 ds_2$$

$$= - \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{\partial}{\partial s_1} \boldsymbol{G}_{212C}(\boldsymbol{f}) \cdot \boldsymbol{\phi}(s_1) ds_2 \right) ds_1$$

$$= \langle \boldsymbol{N}_{212C}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2}$$

in the limit as $\varepsilon \to +0$ in $(*)$. The claimed estimate on $\|\boldsymbol{N}_{212C}(\boldsymbol{f})\|_{L^2}$ again can be proved in a similar way to the proof of the estimate for $\boldsymbol{N}_{112}$; we skip the details. $\qquad\square$

Lemma 5.12 *Let* $\alpha \in (0, \frac{1}{2})$. *If* $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, *then*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \boldsymbol{G}_{212D}(\boldsymbol{f})(s_1, s_2) \cdot \phi'(s_2) ds_1 ds_2 = \langle \boldsymbol{N}_{212C}(\boldsymbol{f}), \phi \rangle_{L^2}$$

holds.

Proof. From $G_{212D}(\boldsymbol{f})(s_1, s_2) = G_{212C}(\boldsymbol{f})(s_2, s_1)$, it is straightforward to show that

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \boldsymbol{G}_{212D}(\boldsymbol{f}) \cdot \phi'(s_2) ds_1 ds_2 = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \boldsymbol{G}_{212C}(\boldsymbol{f}) \cdot \phi'(s_1) ds_1 ds_2$$
$$= \langle \boldsymbol{N}_{212C}(\boldsymbol{f}), \phi \rangle_{L^2} .$$

$\qquad\square$

Next we consider how to derive and estimate $\boldsymbol{N}_{i13}$. As we showed above, G_{i13} can be written using the operations $\tilde{Q}$ and $\bar{Q}$; that is,

$$G_{i13}(\boldsymbol{f}, \phi) = \frac{\tilde{Q}_{i1}\boldsymbol{f} \cdot \bar{Q}_{i2}\phi + \bar{Q}_{i1}\boldsymbol{f} \cdot \tilde{Q}_{i2}\phi + \bar{Q}_{i1}\boldsymbol{f} \cdot \bar{Q}_{i2}\phi}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$+ \frac{\tilde{Q}_{i2}\boldsymbol{f} \cdot \bar{Q}_{i1}\phi + \bar{Q}_{i2}\boldsymbol{f} \cdot \tilde{Q}_{i1}\phi + \bar{Q}_{i2}\boldsymbol{f} \cdot \bar{Q}_{i1}\phi}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

However, we can also rewrite $\boldsymbol{G}_{i13}$ using the new operations T_i^k:

$$G_{113}(\boldsymbol{f}, \phi) = 0,$$

$$G_{213}(\boldsymbol{f}, \phi) = \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\{ T_1^0 \boldsymbol{f} \cdot (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \phi}{\Delta s} + (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \left(\frac{\Delta \phi}{\Delta s} - \phi'(s_2) \right) \right.$$
$$+ (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \phi}{\Delta s} + T_2^0 \boldsymbol{f} \cdot (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \frac{\Delta \phi}{\Delta s}$$
$$+ (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \left(\frac{\Delta \phi}{\Delta s} - \phi'(s_1) \right) + (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \frac{\Delta \phi}{\Delta s} \right\}$$
$$= \boldsymbol{G}_{213B}(\boldsymbol{f}) \cdot \Delta \phi + \boldsymbol{G}_{213C}(\boldsymbol{f}) \cdot \phi'(s_1) + \boldsymbol{G}_{213D}(\boldsymbol{f}) \cdot \phi'(s_2),$$

$$\boldsymbol{G}_{213B}(\boldsymbol{f}) = \frac{2}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\{ (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \left(T_1^0 \boldsymbol{f} + \frac{\Delta \boldsymbol{f}}{\Delta s} \right) + (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \left(T_2^0 \boldsymbol{f} + \frac{\Delta \boldsymbol{f}}{\Delta s} \right) \right.$$
$$\left. + 2(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s} \right\},$$

$$\boldsymbol{G}_{213C}(\boldsymbol{f}) = -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s},$$

$$\boldsymbol{G}_{213D}(\boldsymbol{f}) = -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \frac{\Delta \boldsymbol{f}}{\Delta s},$$

and

$$\boldsymbol{G}_{213B}(s_1, s_2) = -\boldsymbol{G}_{213B}(s_2, s_1), \quad \boldsymbol{G}_{213C}(s_1, s_2) = \boldsymbol{G}_{213D}(s_2, s_1).$$

In order to consider these functions systematically, we decompose $\boldsymbol{G}_{213B}$ as follows:

$$\boldsymbol{G}_{213B}(\boldsymbol{f}) = \boldsymbol{G}_{213B1}(\boldsymbol{f}) + \boldsymbol{G}_{213B2},$$

$$\boldsymbol{G}_{213B1}(\boldsymbol{f}) = \frac{2}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) + T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\frac{\Delta \boldsymbol{f}}{\Delta s},$$

$$\boldsymbol{G}_{213B2}(\boldsymbol{f}) = \frac{2}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}\left\{(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))T_1^0 \boldsymbol{f} + (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_2^0 \boldsymbol{f}\right.$$

$$\left. +2(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\frac{\Delta \boldsymbol{f}}{\Delta s}\right\}.$$

From Lemma 5.5.3, we have

$$\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{213B}(\boldsymbol{f}) \cdot \Delta\boldsymbol{\phi}\, ds_1 ds_2$$

$$= \iint_{|s_1-s_2|\geq\varepsilon} (\boldsymbol{G}_{213B}(s_1, s_2) - \boldsymbol{G}_{213B}(s_2, s_1)) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2$$

$$= 2\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{213B}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2$$

$$= 2\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{213B1}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2 + 2\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{213B2}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2,$$

and using this along with Lemma 5.7 and Corollary 5.3, we obtain

$$\boldsymbol{G}_{213B2}(\boldsymbol{f}) = \mathcal{O}(\Delta s)^{-3}\left\{\mathcal{O}(\Delta s)^{2+1} + \mathcal{O}(\Delta s)^{2+1} + \mathcal{O}(\Delta s)^{2+2}\right\} = \mathcal{O}(1),$$

which gives rise to

$$\lim_{\varepsilon\to+0} 2\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{213B2}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2$$

$$= 2\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \boldsymbol{G}_{213B2}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2$$

$$= 2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \boldsymbol{G}_{213B2}(s_1, s_2)\, ds_2\right) .\boldsymbol{\phi}(s_1)\, ds_1.$$

On the other hand, we cannot likewise consider $\boldsymbol{G}_{213B1}(\boldsymbol{f})$, since

$$\boldsymbol{G}_{213B1}(\boldsymbol{f}) = \mathcal{O}(\Delta s)^{-1},$$

that is, it needs to be combined with the other terms. By Lemma 5.5.1,

$$\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2) \cdot \boldsymbol{\phi}'(s_1)\, ds_1 ds_2$$

$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_{213C}(\boldsymbol{f})(s, s+\varepsilon) - \boldsymbol{G}_{213C}(\boldsymbol{f})(s, s-\varepsilon)) \cdot \boldsymbol{\phi}(s)\, ds$$

$$\quad - \iint_{|s_1-s_2|\geq\varepsilon} \frac{\partial}{\partial s_1}\boldsymbol{G}_{213C}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2$$

$$= (*)$$

holds, and we will show that

$$\boldsymbol{G}_{213C}(\boldsymbol{f})(s, s+\varepsilon) - \boldsymbol{G}_{213C}(\boldsymbol{f})(s, s-\varepsilon) \to 0$$

uniformly with respect to s as $\varepsilon \to +0$. For this, we calculate $\boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2)$ as follows:

$$\boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2) = -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_1^0 \boldsymbol{f} - \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1)$$

$$= -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_1^0 \boldsymbol{f} - 2\mathscr{M}(\boldsymbol{f})(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1) - \frac{2}{(\Delta s)^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1)$$

$$= -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_1^0 \boldsymbol{f} - 2\mathscr{M}(\boldsymbol{f})(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1)$$

$$\quad - \frac{2}{(\Delta s)^2}\left\{\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2 - 1\right\}\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_2)\right)\boldsymbol{\tau}(s_1) - \frac{2}{(\Delta s)^2}(T_2^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1)$$

$$= \mathcal{O}(\Delta s)^{-2+2+1} + \mathcal{O}(\Delta s)^{0+2+0} + \mathcal{O}(\Delta s)^{-2+2+2}$$

$$\quad - \frac{2}{(\Delta s)^3}\left(\int_{s_2}^{s_1}\int_{s_2}^{s_3}\int_{s_4}^{s_2}\boldsymbol{\kappa}(s_4) \cdot \boldsymbol{\kappa}(s_5)\, ds_5 ds_4 ds_3\right)\boldsymbol{\tau}(s_1)$$

$$= \mathcal{O}(\Delta s) - \frac{2}{(\Delta s)^3}\left\{\int_{s_2}^{s_1}\int_{s_2}^{s_3}\int_{s_4}^{s_2}\left(\boldsymbol{\kappa}(s_4) \cdot \boldsymbol{\kappa}(s_5) - \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right)\, ds_5 ds_4 ds_3\right\}\boldsymbol{\tau}(s_1)$$

$$\quad - \frac{2}{(\Delta s)^3}\left(\int_{s_2}^{s_1}\int_{s_2}^{s_3}\int_{s_4}^{s_2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\, ds_5 ds_4 ds_3\right)\boldsymbol{\tau}(s_1)$$

$$= \mathcal{O}(\Delta s) + \mathcal{O}(\|\boldsymbol{\kappa}\|_{L^\infty}\omega_\kappa(\Delta s)) + \frac{1}{3}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\boldsymbol{\tau}(s_1),$$

where we set

$$\omega_\kappa(\delta) = \sup_{|s_a - s_b| \le \delta}\|\boldsymbol{\kappa}(s_a) - \boldsymbol{\kappa}(s_b)\|_{\mathbb{R}^n}$$

and use the estimate

$$\left|\boldsymbol{\kappa}(s_4) \cdot \boldsymbol{\kappa}(s_5) - \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right| = |(\boldsymbol{\kappa}(s_4) - \boldsymbol{\kappa}(s_1)) \cdot \boldsymbol{\kappa}(s_5) + \boldsymbol{\kappa}(s_1) \cdot (\boldsymbol{\kappa}(s_5) - \boldsymbol{\kappa}(s_1))|$$
$$\le 2\|\boldsymbol{\kappa}\|_{L^\infty}\omega_\kappa(\Delta s).$$

We also used

$$\int_{s_2}^{s_1}\int_{s_2}^{s_3}\int_{s_4}^{s_2}ds_5 ds_4 ds_3 = -\frac{1}{6}(\Delta s)^3$$

in the above computation. Recall that $\alpha \in (0, \frac{1}{2})$ and $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then, from $\boldsymbol{\kappa} \in C^{\frac{1}{2}-\alpha}$, we obtain

$$\boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2) = \frac{1}{3}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\boldsymbol{\tau}(s_1) + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},$$

and therefore it follows that

$$\boldsymbol{G}_{213C}(\boldsymbol{f})(s, s+\varepsilon) - \boldsymbol{G}_{213C}(\boldsymbol{f})(s, s-\varepsilon) = \mathcal{O}(\varepsilon)^{\frac{1}{2}-\alpha}.$$

Thus, we arrive at

$$\lim_{\varepsilon \to +0} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{G}_{213C}(\boldsymbol{f})(s, s+\varepsilon) - \boldsymbol{G}_{213C}(\boldsymbol{f})(s, s-\varepsilon)) \cdot \boldsymbol{\phi}(s) ds = 0.$$

Next, we consider $\dfrac{\partial}{\partial s_1} \boldsymbol{G}_{213C}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)$. Using Lemma 5.8, we obtain

$$\frac{\partial}{\partial s_1} \boldsymbol{G}_{213C}(s_1, s_2) = -\frac{\partial}{\partial s_1} \left\{ \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s} \right\}$$

$$= -\left(\frac{\partial}{\partial s_1} \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \right) (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s} - \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left(\frac{\partial}{\partial s_1} T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) \right) \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$\quad - \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\partial}{\partial s_1} \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= \boldsymbol{G}'_{213C1}(\boldsymbol{f}) + \boldsymbol{G}'_{213C2}(\boldsymbol{f}) + \boldsymbol{G}'_{213C3}(\boldsymbol{f}),$$

where

$$\boldsymbol{G}'_{213C1}(\boldsymbol{f}) = \frac{4\Delta \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= \frac{4}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^2}^2} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}} \right)^2 \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_1) \right) (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= \frac{4}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^2}^2} \left\{ (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) + 1 \right\} (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \frac{\Delta \boldsymbol{f}}{\Delta s},$$

$$\boldsymbol{G}'_{213C2}(\boldsymbol{f}) = -\frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left[\left\{ \frac{2}{\Delta s} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot T_1^0 \boldsymbol{f} \right) \frac{\Delta \boldsymbol{f}}{\Delta s} \right. \right.$$

$$\left. \left. - \frac{1}{\Delta s} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^2 T_1^0 \boldsymbol{f} \right\} \cdot \boldsymbol{\tau}(s_2) \right] \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= -\frac{2}{(\Delta s)^3} \left[\left\{ 2 \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_2) \right) \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^2 \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2) \right\} \cdot T_1^0 \boldsymbol{f} \right] \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \frac{\Delta \boldsymbol{f}}{\Delta s}$$

$$= -\frac{2}{(\Delta s)^3} \left[\left\{ 2 \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_2) \right) \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^2 \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2) \right\} \cdot T_1^0 \boldsymbol{f} \right] (T_1^4 \boldsymbol{f} + \boldsymbol{\tau}(s_1)),$$

$$\boldsymbol{G}'_{213C3}(\boldsymbol{f}) = \frac{2}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) T_1^0 \boldsymbol{f}.$$

We will show the following (in the order listed):

(5.11) $\qquad \boldsymbol{G}'_{213C1}(\boldsymbol{f}) = \boldsymbol{G}_{213B1}(\boldsymbol{f}) + \mathcal{O}(|\Delta s|^{-\frac{1}{2}-\alpha}),$

(5.12) $\qquad \boldsymbol{G}'_{213C2}(\boldsymbol{f}) = \dfrac{1}{3\Delta s} \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \boldsymbol{\tau}(s_1) + \mathcal{O}(|\Delta s|^{-\frac{1}{2}-\alpha}),$

(5.13) $\qquad \boldsymbol{G}'_{213C3}(\boldsymbol{f}) = \mathcal{O}(1).$

Proof of (5.11). Using Lemma 5.7 and Corollary 5.3, we have

$$
\boldsymbol{G}'_{213C1}(\boldsymbol{f}) = \frac{4}{(\Delta s)\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^2}} \left\{(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))+1\right\}(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\frac{\Delta\boldsymbol{f}}{\Delta s}
$$

$$
= \frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)+T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\frac{\Delta\boldsymbol{f}}{\Delta s}
$$

$$
+ \frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}\left\{(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)-T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\right.
$$

$$
\left.+2(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\right\}\frac{\Delta\boldsymbol{f}}{\Delta s}
$$

$$
= \boldsymbol{G}_{213B1}(\boldsymbol{f}) - \frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|}\right)^2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\tau}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} + \mathcal{O}(\Delta s)^{-3+2+2}
$$

$$
= \boldsymbol{G}_{213B1}(\boldsymbol{f}) - \frac{2(\Delta s)}{\|\Delta\boldsymbol{f}\|^4_{\mathbb{R}^n}}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\tau}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} + \mathcal{O}(\Delta s).
$$

Noting that

$$
\int_{s_1}^{s_2}\int_{s_1}^{s_2}\int_{s_4}^{s_3} ds_5 ds_4 ds_3 = 0,
$$

we obtain

$$
\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\tau} = \frac{1}{\Delta s}\int_{s_1}^{s_2}\int_{s_1}^{s_2}\boldsymbol{\tau}(s_3)\cdot\boldsymbol{\kappa}(s_4)\,ds_4 ds_3
$$

$$
= \frac{1}{\Delta s}\int_{s_1}^{s_2}\int_{s_1}^{s_2}\int_{s_4}^{s_3}\boldsymbol{\kappa}(s_5)\cdot\boldsymbol{\kappa}(s_4)\,ds_5 ds_4 ds_3
$$

$$
= \frac{1}{\Delta s}\int_{s_1}^{s_2}\int_{s_1}^{s_2}\int_{s_4}^{s_3}\left(\boldsymbol{\kappa}(s_5)\cdot\boldsymbol{\kappa}(s_4)-\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\right)ds_5 ds_4 ds_3
$$

$$
= \mathcal{O}(\Delta s)^{-1+3+\frac{1}{2}-\alpha}
$$

$$
= \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha},
$$

and so

$$
-\frac{2|\Delta s|}{\|\Delta\boldsymbol{f}\|^4_{\mathbb{R}^n}}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\tau}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} = \mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}.
$$

$\square$

Proof of (5.12). By Lemma 5.7 and Corollary 5.3,

$$
\left[\left\{2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s}-\boldsymbol{\tau}(s_2)\right\}\cdot T_1^0\boldsymbol{f}\right]T_1^4\boldsymbol{f}
$$

$$
= \left(\mathcal{O}(\Delta s)^2 + \mathcal{O}(\Delta s)^2\right)\mathcal{O}(\Delta s)
$$

$$
= \mathcal{O}(\Delta s)^3
$$

holds, and then we have

$$
\boldsymbol{G}'_{213C2}(\boldsymbol{f}) = -\frac{2}{(\Delta s)^3}\left[\left\{2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s}-\boldsymbol{\tau}(s_2)\right\}\cdot T_1^0\boldsymbol{f}\right](T_1^4\boldsymbol{f}+\boldsymbol{\tau}(s_1))
$$

$$
= -\frac{2}{(\Delta s)^3}\left[\left\{2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s}-\boldsymbol{\tau}(s_2)\right\}\cdot T_1^0\boldsymbol{f}\right]\boldsymbol{\tau}(s_1)+\mathcal{O}(1).
$$

In a similar way, from

$$\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_2) = \left(\boldsymbol{\tau}(s_2) + T_2^0 \boldsymbol{f}\right) \cdot \boldsymbol{\tau}(s_2) = 1 + \mathcal{O}(\Delta s)^2$$

and Corollary 5.3,

$$
\begin{aligned}
\boldsymbol{G}'_{213C2}(\boldsymbol{f}) = & -\frac{2}{(\Delta s)^3}\left[\left\{2\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot \boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2 \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2)\right\}\cdot T_1^0 \boldsymbol{f}\right]\boldsymbol{\tau}(s_1) + \mathcal{O}(1) \\
= & -\frac{2}{(\Delta s)^3}\left[\left\{2\left(1+\mathcal{O}(\Delta s)^2\right)^2\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2)\right\}\cdot T_1^0 \boldsymbol{f}\right]\boldsymbol{\tau}(s_1) + \mathcal{O}(1) \\
= & -\frac{2}{(\Delta s)^3}\left\{\left(2\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2)\right)\cdot T_1^0 \boldsymbol{f}\right\}\boldsymbol{\tau}(s_1) + \mathcal{O}(1)
\end{aligned}
$$

holds. Then, we obtain

$$
\begin{aligned}
\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot T_1^0 \boldsymbol{f} = & \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \left(\frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_1)\right) \\
= & \frac{1}{(\Delta s)^2}\int_{s_2}^{s_1}\int_{s_2}^{s_1}\boldsymbol{\tau}(s_3)\cdot(\boldsymbol{\tau}(s_4) - \boldsymbol{\tau}(s_1))\,ds_4 ds_3 \\
= & \frac{1}{(\Delta s)^2}\int_{s_2}^{s_1}\int_{s_2}^{s_1}\int_{s_1}^{s_4}\boldsymbol{\tau}(s_3)\cdot \boldsymbol{\kappa}(s_5)\,ds_5 ds_4 ds_3 \\
= & \frac{1}{(\Delta s)^2}\int_{s_2}^{s_1}\int_{s_2}^{s_1}\int_{s_1}^{s_4}\int_{s_5}^{s_3}\boldsymbol{\kappa}(s_6)\cdot \boldsymbol{\kappa}(s_5)\,ds_6 ds_5 ds_4 ds_3 \\
= & \frac{1}{(\Delta s)^2}\int_{s_2}^{s_1}\int_{s_2}^{s_1}\int_{s_1}^{s_4}\int_{s_5}^{s_3}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\,ds_6 ds_5 ds_4 ds_3 \\
& -\frac{1}{(\Delta s)^2}\int_{s_2}^{s_1}\int_{s_2}^{s_1}\int_{s_1}^{s_4}\int_{s_5}^{s_3}\left(\boldsymbol{\kappa}(s_6)\cdot \boldsymbol{\kappa}(s_5) - \|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\right)\,ds_6 ds_5 ds_4 ds_3 \\
= & \frac{(\Delta s)^2}{12}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{-2+4+\frac{1}{2}-\alpha} \\
= & \frac{(\Delta s)^2}{12}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha},
\end{aligned}
$$

$$
\begin{aligned}
T_1^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) = & \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_1}^{s_3}\int_{s_4}^{s_2}\boldsymbol{\kappa}(s_4)\cdot \boldsymbol{\kappa}(s_5)\,ds_5 ds_4 ds_3 \\
= & \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_1}^{s_3}\int_{s_4}^{s_2}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\,ds_5 ds_4 ds_3 \\
& +\frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_1}^{s_3}\int_{s_4}^{s_2}\left(\boldsymbol{\kappa}(s_4)\cdot \boldsymbol{\kappa}(s_5) - \|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\right)\,ds_5 ds_4 ds_3 \\
= & \frac{(\Delta s)^2}{3}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{-1+3+\frac{1}{2}-\alpha} \\
= & \frac{(\Delta s)^2}{3}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\boldsymbol{G}'_{213C2}(\boldsymbol{f}) =\ & -\frac{2}{(\Delta s)^3}\left\{\frac{2(\Delta s)^2}{12}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha}\right.\\
& \left. -\frac{(\Delta s)^2}{3}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha}\right\}\boldsymbol{\tau}(s_1) + \mathcal{O}(1)\\
=\ & \frac{1}{3(\Delta s)}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\boldsymbol{\tau}(s_1) + \mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}.
\end{aligned}
$$

$\square$

Proof of (5.13). Using Lemma 5.7 and Corollary 5.3,

$$
\boldsymbol{G}'_{213C3}(\boldsymbol{f}) = \frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f} = \mathcal{O}(\Delta s)^{-3+2+1} = \mathcal{O}(1)
$$

can be proved. $\square$

From (5.11)–(5.13) and noting that

$$
\int_{\{s_2\,|\,|s_1-s_2|\geq\varepsilon\}}\frac{1}{\Delta s}ds_2 = 0,
$$

we arrive at

$$
\begin{aligned}
&\lim_{\varepsilon\to+0}\iint_{|s_1-s_2|\geq\varepsilon}\left(\boldsymbol{G}_{213C}(\boldsymbol{f})(s_1,s_2)\cdot\boldsymbol{\phi}'(s_1) + \boldsymbol{G}_{213B1}(\boldsymbol{f})(s_1,s_2)\cdot\boldsymbol{\phi}(s_1)\right)ds_1ds_2\\
&= -\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\left[\left(\boldsymbol{G}'_{213C1}(\boldsymbol{f}) - \boldsymbol{G}_{213B1}(\boldsymbol{f})\right)\right.\\
&\qquad\qquad \left. + \left\{\boldsymbol{G}'_{213C2}(\boldsymbol{f}) - \frac{1}{3(\Delta s)}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\boldsymbol{\tau}(s_1)\right\} + \boldsymbol{G}'_{213C3}(\boldsymbol{f})\right]\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2\\
&= -\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left[\left(\boldsymbol{G}'_{213C1}(\boldsymbol{f}) - \boldsymbol{G}_{213B1}(\boldsymbol{f})\right)\right.\right.\\
&\qquad\qquad \left.\left. + \left\{\boldsymbol{G}'_{213C2}(\boldsymbol{f}) - \frac{1}{3(\Delta s)}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\boldsymbol{\tau}(s_1)\right\} + \boldsymbol{G}'_{213C3}(\boldsymbol{f})\right]ds_2\right)\cdot\boldsymbol{\phi}(s_1)\,ds_1.
\end{aligned}
$$

Lemma 5.13 *Let* $\alpha \in (0,\frac{1}{2})$. *If* $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$,

$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{213}(\boldsymbol{f},\boldsymbol{\phi})(s_1,s_2)ds_1ds_2 = \langle\boldsymbol{N}_{213}(\boldsymbol{f}),\boldsymbol{\phi}\rangle_{L^2}
$$

holds, where

$$
\begin{aligned}
&\boldsymbol{N}_{213}(\boldsymbol{f})(s_1)\\
&= -4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left(\frac{1}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) - T_2^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1) + \frac{1}{(\Delta s)\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}\right.\\
&\qquad\qquad + \frac{1}{(\Delta s)^3}\left[2\{(T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)) + 1\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)) + (T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\right](T_1^4\boldsymbol{f} + \boldsymbol{\tau}(s_1))\\
&\qquad\qquad \left. - \frac{1}{6(\Delta s)}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\boldsymbol{\tau}(s_1)\right)ds_2
\end{aligned}
$$

satisfies

$$\|\boldsymbol{N}_{213}(\boldsymbol{f})\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Proof.   From $\boldsymbol{G}_{213D}(\boldsymbol{f})(s_2, s_1) = \boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2)$, we obtain

$$
\lim_{\varepsilon \to +0} \iint_{|s_1-s_2|\geq\varepsilon} G_{213}(\boldsymbol{f}, \boldsymbol{\phi}) ds_1 ds_2
$$

$$
= \lim_{\varepsilon \to +0} \iint_{|s_1-s_2|\geq\varepsilon} \big(\boldsymbol{G}_{213B}(\boldsymbol{f})(s_1, s_2) \cdot \Delta\boldsymbol{\phi}
$$
$$
+ \boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2) \cdot \boldsymbol{\phi}'(s_1) + \boldsymbol{G}_{213D}(\boldsymbol{f})(s_1, s_2) \cdot \boldsymbol{\phi}'(s_2)\big) ds_1 ds_2
$$

$$
= \lim_{\varepsilon \to +0} 2 \iint_{|s_1-s_2|\geq\varepsilon} \big\{(\boldsymbol{G}_{213B1}(\boldsymbol{f})(s_1, s_2) + \boldsymbol{G}_{213B2}(\boldsymbol{f})(s_1, s_2)) \cdot \Delta\boldsymbol{\phi}
$$
$$
+ \boldsymbol{G}_{213C}(\boldsymbol{f})(s_1, s_2) \cdot \boldsymbol{\phi}'(s_1)\big\} ds_1 ds_2
$$

$$
= 2 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \boldsymbol{G}_{213B2}(s_1, s_2)\, ds_2 \right) \boldsymbol{\phi}(s_1)\, ds_1
$$

$$
- 2 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \big[(\boldsymbol{G}'_{213C1}(\boldsymbol{f}) - \boldsymbol{G}_{213B1}(\boldsymbol{f})) \right.
$$
$$
\left. + \left\{ \boldsymbol{G}'_{213C2}(\boldsymbol{f}) - \frac{1}{3(\Delta s)} \|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} \right\} + \boldsymbol{G}'_{213C3}(\boldsymbol{f}) \big]\, ds_2 \right) \cdot \boldsymbol{\phi}(s_1)\, ds_1
$$

$$
= \langle \boldsymbol{N}_{213}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2},
$$

and we compute

$$2\left\{\boldsymbol{G}_{213B2}(\boldsymbol{f}) - (\boldsymbol{G}'_{213C1}(\boldsymbol{f}) - \boldsymbol{G}_{213B1}(\boldsymbol{f}) + \boldsymbol{G}'_{213C3}(\boldsymbol{f}))\right\}$$

$$= 2\left[\frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left\{(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))T_1^0\boldsymbol{f} + (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_2^0\boldsymbol{f} + 2(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\frac{\Delta\boldsymbol{f}}{\Delta s}\right\}\right.$$

$$-\frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) + T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\frac{\Delta\boldsymbol{f}}{\Delta s}$$

$$-\frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left\{(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2) - T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)) + 2(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\right\}\frac{\Delta\boldsymbol{f}}{\Delta s}$$

$$+\frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) + T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\frac{\Delta\boldsymbol{f}}{\Delta s}$$

$$\left.-\frac{2}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}\right]$$

$$= \frac{4}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left[\left\{(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))T_1^0\boldsymbol{f} + (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_2^0\boldsymbol{f}\right\}\right.$$

$$\left.- (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2) - T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\frac{\Delta\boldsymbol{f}}{\Delta s} - (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}\right]$$

$$= \frac{4}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left\{(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) - T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\left(T_1^0\boldsymbol{f} - \frac{\Delta\boldsymbol{f}}{\Delta s}\right) - (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}\right\}$$

$$= -\frac{4}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left\{(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) - T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1) + (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}\right\}$$

$$= -\frac{4}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) - T_2^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1) - \frac{4}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}.$$

From the calculation

$$\left\{2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2)\right\}\cdot T_1^0\boldsymbol{f}$$

$$= \left\{2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_2)\right\}\cdot\left(\frac{\Delta\boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_1)\right)$$

$$= \frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2) - 2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_1)\right) + \boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2)$$

$$= -2\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}(s_2)\right)\left\{\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_1)\right\}\cdot\boldsymbol{\tau}(s_1) - \left(\frac{\Delta\boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_1)\right)\cdot\boldsymbol{\tau}(s_2)$$

$$= -2\{(T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)) + 1\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)) - (T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)),$$

we have

$$\boldsymbol{G}'_{213C2}(\boldsymbol{f}) = \frac{2}{(\Delta s)^3}\left[2\{(T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)) + 1\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)) + (T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\right](T_1^4\boldsymbol{f} + \boldsymbol{\tau}(s_1)),$$

and therefore

$$2\left\{\boldsymbol{G}_{213B2}(\boldsymbol{f}) - (\boldsymbol{G}'_{213C1}(\boldsymbol{f}) - \boldsymbol{G}_{213B1}(\boldsymbol{f}) + \boldsymbol{G}'_{213C3}(\boldsymbol{f}) + \boldsymbol{G}'_{213C2}(\boldsymbol{f}))\right\}$$

$$= -\frac{4}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) - T_2^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1) - \frac{4}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))T_1^0\boldsymbol{f}$$

$$- \frac{4}{(\Delta s)^3}\left[2\{(T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)) + 1\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)) + (T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\right](T_1^4\boldsymbol{f} + \boldsymbol{\tau}(s_1)).$$

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We omit the proof of the estimate on $\|\boldsymbol{N}_{213}(\boldsymbol{f})\|_{L^2}$ since it can be shown in a way similar to those of the previous cases.

$\square$

Finally, let us consider how to derive and estimate $\boldsymbol{N}_{i2}$ and derive the L^2-gradient expression of $\iint_{|s_1-s_2|\geq\varepsilon} G_{i2}(\boldsymbol{f},\boldsymbol{\phi})ds_1ds_2$. Let

$$\boldsymbol{G}_{i2B}(\boldsymbol{f}) = -\frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

From Lemma 5.5.3 and

$$\boldsymbol{G}_{i2B}(\boldsymbol{f})(s_1,s_2) = -\boldsymbol{G}_{i2B}(\boldsymbol{f})(s_2,s_1),$$

we have

$$\iint_{|s_1-s_2|\geq\varepsilon} G_{i2}(\boldsymbol{f},\boldsymbol{\phi})ds_1ds_2$$
$$= -2\iint_{|s_1-s_2|\geq\varepsilon} \frac{\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\cdot\Delta\boldsymbol{\phi}ds_1ds_2$$
$$= \iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{i2B}(s_1,s_2)\cdot\Delta\boldsymbol{\phi}ds_1ds_2$$
$$= \iint_{|s_1-s_2|\geq\varepsilon} (\boldsymbol{G}_{i2B}(s_1,s_2)-\boldsymbol{G}_{i2B}(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)ds_1ds_2$$
$$= 2\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{i2B}(s_1,s_2)\cdot\boldsymbol{\phi}(s_1)ds_1ds_2.$$

We also note that

$$2\boldsymbol{G}_{i2B}(s_1,s_2) = -4\frac{\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}$$
$$= -\frac{4\mathscr{M}_i(\boldsymbol{f})}{\Delta s}\left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^2\frac{\Delta\boldsymbol{f}}{\Delta s}$$
$$= -\frac{4\mathscr{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} - \frac{4\mathscr{M}_i(\boldsymbol{f})}{\Delta s}\boldsymbol{\tau}(s_1).$$

Lemma 5.14 *Let* $\alpha\in(0,\frac{1}{2})$*. If* $\boldsymbol{f}\in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$*, then*

$$\mathscr{M}_i(\boldsymbol{f}) + \frac{(-1)^i}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 = \mathcal{O}(|\Delta s|^{\frac{1}{2}-\alpha})$$

holds.

Proof. If $i = 1$, we have

$$\mathscr{M}_1(\boldsymbol{f}) - \frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2$$

$$= \frac{1}{2}\left\{\frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right\}$$

$$= \frac{1}{2}\left[\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2\left\{\frac{1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{1}{(\Delta s)^2}\right\} + \frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{(\Delta s)^2} - \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right]$$

$$= \frac{1}{2}\left[\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2\mathscr{M}(\boldsymbol{f}) + \left(\frac{\Delta\boldsymbol{\tau}}{\Delta s} + \boldsymbol{\kappa}(s_1)\right)\cdot\left(\frac{\Delta\boldsymbol{\tau}}{\Delta s} - \boldsymbol{\kappa}(s_1)\right)\right]$$

$$= \mathcal{O}(1) + \mathcal{O}\left(|\Delta s|^{\frac{1}{2}-\alpha}\right) = \mathcal{O}\left(|\Delta s|^{\frac{1}{2}-\alpha}\right).$$

Here, we used the boundedness of $\mathscr{M}(\boldsymbol{f})$, which follows from Lemma 5.6.

We now consider the case in which $i = 2$, firstly by observing that

$$\tilde{Q}_1(\boldsymbol{f}) \cdot \tilde{Q}_2(\boldsymbol{f})$$
$$= -4\left\{\boldsymbol{\tau}(s_1) - (R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))R\boldsymbol{f}\right\}\cdot\left\{\boldsymbol{\tau}(s_2) - (R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))R\boldsymbol{f}\right\}$$
$$= -4\{\boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2) - (R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))(R\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))\}$$
$$= -4\left\{1 - \frac{1}{2}\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2 - \left(1 - \frac{1}{2}\|R\boldsymbol{f} - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2\right)\left(1 - \frac{1}{2}\|R\boldsymbol{f} - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2\right)\right\}$$
$$= 2\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2 - 2\|R\boldsymbol{f} - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2 - 2\|R\boldsymbol{f} - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2$$
$$\quad + \|R\boldsymbol{f} - \boldsymbol{\tau}(s_1)\|_{\mathbb{R}^n}^2\|R\boldsymbol{f} - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2.$$

By direct calculation, we have

$$\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2 = \int_{s_2}^{s_1}\int_{s_2}^{s_1}\boldsymbol{\kappa}(s_3)\cdot\boldsymbol{\kappa}(s_4)\,ds_3d_4$$

$$= (\Delta s)^2\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 + \int_{s_2}^{s_1}\int_{s_2}^{s_1}\left(\boldsymbol{\kappa}(s_3)\cdot\boldsymbol{\kappa}(s_4) - \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right)ds_3d_4$$

$$= (\Delta s)^2\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 + \mathcal{O}(|\Delta s|^{\frac{2}{5}-\alpha})$$

and

$$R\boldsymbol{f} - \boldsymbol{\tau}(s_j) = T_j^1\boldsymbol{f} = T_j^0\boldsymbol{f} + \left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}} - 1\right)\frac{\Delta\boldsymbol{f}}{\Delta s}$$

$$= \frac{1}{\Delta s}\int_{s_2}^{s_1}(\boldsymbol{\tau}(s_3) - \boldsymbol{\tau}(s_j))\,ds_3 + \mathcal{O}(\Delta s)^2$$

$$= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_j}^{s_3}\boldsymbol{\kappa}(s_4)\,ds_4ds_3 + \mathcal{O}(\Delta s)^2$$

$$= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_j}^{s_3}\boldsymbol{\kappa}(s_1)\,ds_4ds_3 + \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_j}^{s_3}(\boldsymbol{\kappa}(s_4) - \boldsymbol{\kappa}(s_1))\,ds_4ds_3 + \mathcal{O}(\Delta s)^2$$

$$= \frac{(-1)^j}{2}(\Delta s)\boldsymbol{\kappa}(s_1) + \mathcal{O}(|\Delta s|)^{\frac{3}{2}-\alpha},$$

which leads to

$$\|R\boldsymbol{f} - \boldsymbol{\tau}(s_j)\|_{\mathbb{R}^n}^2 = \frac{1}{4}(\Delta s)^2\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 + \mathcal{O}(|\Delta s|)^{\frac{5}{2}-\alpha},$$

and therefore

$$\tilde{Q}_1(\boldsymbol{f}) \cdot \tilde{Q}_2(\boldsymbol{f}) = (\Delta s)^2 \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 + \mathcal{O}(|\Delta s|)^{\frac{5}{2}-\alpha}.$$

From this fact and since

$$\frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} = \frac{1}{(\Delta s)^2} + \mathscr{M}(\boldsymbol{f}) = \frac{1}{(\Delta s)^2} + \mathcal{O}(1),$$

we obtain

$$\begin{aligned}
\mathscr{M}_2(\boldsymbol{f}) &= -\frac{1}{2}\frac{\tilde{Q}_1(\boldsymbol{f}) \cdot \tilde{Q}_2(\boldsymbol{f})}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
&= -\frac{1}{2}\left\{\frac{1}{(\Delta s)^2} + \mathcal{O}(1)\right\}\left\{(\Delta s)^2\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 + \mathcal{O}(|\Delta s|)^{\frac{5}{2}-\alpha}\right\} \\
&= -\frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 + \mathcal{O}(|\Delta s|)^{\frac{1}{2}-\alpha}.
\end{aligned}$$

$\square$

Lemma 5.15 Let $\alpha \in (0, \frac{1}{2})$. If $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{i2}(\boldsymbol{f}, \boldsymbol{\phi})(s_1, s_2)ds_1 ds_2 = \langle \boldsymbol{N}_{i2}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2},$$

where

$$\boldsymbol{N}_{i2}(\boldsymbol{f})(s_1) = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[-\frac{4\mathscr{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} - \frac{4}{\Delta s}\left\{\mathscr{M}_i(\boldsymbol{f}) + \frac{(-1)^i}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right\}\boldsymbol{\tau}(s_1)\right]ds_2$$

satisfies

$$\|\boldsymbol{N}_{i2}(\boldsymbol{f})\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Proof. Using

$$\begin{aligned}
2\boldsymbol{G}_{i2B}(\boldsymbol{f}) &= -\frac{4\mathscr{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} - \frac{4\mathscr{M}_i(\boldsymbol{f})}{\Delta s}\boldsymbol{\tau}(s_1) \\
&= -\frac{4\mathscr{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} - \frac{4}{\Delta s}\left\{\mathscr{M}_i(\boldsymbol{f}) + \frac{(-1)^i}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right\}\boldsymbol{\tau}(s_1) + \frac{2(-1)^i}{\Delta s}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\boldsymbol{\tau}(s_1)
\end{aligned}$$

and

$$\int_{\{s_2 \,|\, |s_1-s_2|\geq\varepsilon\}} \frac{1}{\Delta s}ds_2 = 0,$$

we have

$$\begin{aligned}
&\iint_{|s_1-s_2|\geq\varepsilon} G_{i2}(\boldsymbol{f}, \boldsymbol{\phi})ds_1 ds_2 \\
&= 2\iint_{|s_1-s_2|\geq\varepsilon} \boldsymbol{G}_{i2B}(s_1, s_2) \cdot \boldsymbol{\phi}(s_1)ds_1 ds_2 \\
&= \iint_{|s_1-s_2|\geq\varepsilon} \left[-\frac{4\mathscr{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} - \frac{4}{\Delta s}\left\{\mathscr{M}_i(\boldsymbol{f}) + \frac{(-1)^i}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right\}\boldsymbol{\tau}(s_1)\right] \cdot \boldsymbol{\phi}(s_1)ds_1 ds_2.
\end{aligned}$$

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From
$$-\frac{4\mathcal{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} = \mathcal{O}(1),$$
which is obtained from Lemma 5.8, and from
$$-\frac{4}{\Delta s}\left\{\mathcal{M}_i(\boldsymbol{f}) + \frac{(-1)^i}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right\}\boldsymbol{\tau}(s_1) = \mathcal{O}(|\Delta s|^{-\frac{1}{2}-\alpha}),$$
which is obtained from Lemma 5.14, the term $[\cdots]$ from the above integrand is absolutely integrable on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ with respect to s_2. The integral is bounded on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ with respect to s_1. As $\varepsilon \to +0$, we apply Fubini's theorem in order to obtain

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{i2}(\boldsymbol{f}, \phi)ds_1 ds_2$$
$$= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left[-\frac{4\mathcal{M}_i(\boldsymbol{f})T_1^2\boldsymbol{f}}{\Delta s} - \frac{4}{\Delta s}\left\{\mathcal{M}_i(\boldsymbol{f}) + \frac{(-1)^i}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2\right\}\boldsymbol{\tau}(s_1)\right]ds_2\right)\phi(s_1)\,ds_1$$
$$= \langle\boldsymbol{N}_{i2}(\boldsymbol{f}), \phi\rangle_{L^2}.$$

The claimed estimate on $\|\boldsymbol{N}_{i2}(\boldsymbol{f})\|_{L^2}$ can be shown in a way similar to that used for the other cases. $\qquad\square$

Combining the argument above, we can reach conclusion of the nonlinear part. From the above argument, we can write that

$$\boldsymbol{N}_1(\boldsymbol{f}) = \boldsymbol{N}_{112}(\boldsymbol{f}) + \boldsymbol{N}_{12}(\boldsymbol{f})$$
$$= 2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} - \mathcal{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\}ds_2$$
$$- 4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left[\frac{\mathcal{M}_1(\boldsymbol{f})}{\Delta s}T_1^2\boldsymbol{f} + \frac{1}{\Delta s}\left\{\mathcal{M}_1(\boldsymbol{f}) - \frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}}^2\right\}\boldsymbol{\tau}(s_1)\right]ds_2,$$

and
$$\boldsymbol{N}_2(\boldsymbol{f})(s_1) = \boldsymbol{N}_{212B}(\boldsymbol{f}) + 2\boldsymbol{N}_{212C}(\boldsymbol{f}) + \boldsymbol{N}_{213}(\boldsymbol{f}) + \boldsymbol{N}_{22}(\boldsymbol{f}).$$

Furthermore, from

$$\frac{1}{\Delta s}\mathcal{M}(\boldsymbol{f})(T_1^0\boldsymbol{f} + T_2^0\boldsymbol{f}) - \left\{\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_s)T_2^0\boldsymbol{f} + \frac{\mathcal{M}(\boldsymbol{f})}{\Delta s}T_1^0\boldsymbol{f}\right\}$$
$$= \frac{1}{\Delta s}\left\{\mathcal{M}(\boldsymbol{f}) - \frac{2}{(\Delta s)^2}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\right\}T_2^0\boldsymbol{f}$$

and

$$\mathscr{M}(\boldsymbol{f}) - \frac{2}{(\Delta s)^2}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))$$

$$= \left\{ \frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{1}{(\Delta s)^2} \right\} - \frac{2}{(\Delta s)^2} \left\{ \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}(s_1) \right\} \cdot \boldsymbol{\tau}(s_1)$$

$$= \frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} + \frac{1}{(\Delta s)^2} - \frac{2}{(\Delta s)^2} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_1)$$

$$= \frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\{ 1 - \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^2 \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_1) \right\} + \frac{1}{(\Delta s)^2} \left\{ 1 - \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^4 \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}(s_1) \right\}$$

$$= -\frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) - \frac{1}{(\Delta s)^2}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)),$$

we obtain

$$\boldsymbol{N}_{212B}(\boldsymbol{f}) + 2\boldsymbol{N}_{212C}(\boldsymbol{f})$$

$$= 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{1}{\Delta s} \mathscr{M}(\boldsymbol{f})(T_1^0 \boldsymbol{f} + T_2^0 \boldsymbol{f}) - \left\{ \frac{2}{(\Delta s)^3}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}_s) T_2^0 \boldsymbol{f} + \frac{\mathscr{M}(\boldsymbol{f})}{\Delta s} T_1^0 \boldsymbol{f} \right\} \right] ds_2$$

$$= -4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{\Delta s} \left\{ \frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) + \frac{1}{(\Delta s)^2}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \right\} T_2^0 \boldsymbol{f} \, ds_2.$$

Using these identities and Lemmas 5.13 and 5.15, we conclude that

$$
\boldsymbol{N}_2(\boldsymbol{f})(s_1)
$$
$$
= \boldsymbol{N}_{212B}(\boldsymbol{f}) + 2\boldsymbol{N}_{212C}(\boldsymbol{f}) + \boldsymbol{N}_{213}(\boldsymbol{f}) + \boldsymbol{N}_{22}(\boldsymbol{f})
$$
$$
= -4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{\Delta s} \left\{ \frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) + \frac{1}{(\Delta s)^2}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) \right\} T_2^0 \boldsymbol{f}\, ds_2
$$
$$
- 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\frac{1}{(\Delta s)^3}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) - T_2^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))\boldsymbol{\tau}(s_1) + \frac{1}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}(T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_1^0 \boldsymbol{f} \right.
$$
$$
+ \frac{1}{(\Delta s)^3} \left[2\{(T_2^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) + 1\}(T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) + (T_1^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) \right] (T_1^4 \boldsymbol{f} + \boldsymbol{\tau}(s_1))
$$
$$
\left. - \frac{1}{6(\Delta s)}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \boldsymbol{\tau}(s_1) \right) ds_2
$$
$$
- 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\mathscr{M}_2(\boldsymbol{f})}{\Delta s}T_1^2 \boldsymbol{f} + \frac{1}{\Delta s}\left\{ \mathscr{M}_2(\boldsymbol{f}) + \frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \right\} \boldsymbol{\tau}(s_1) \right] ds_2
$$
$$
= -4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\{ (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))T_2^0 \boldsymbol{f} + (T_2^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_1^0 \boldsymbol{f} \right\} ds_2
$$
$$
- 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)^3} \left[(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))T_2^0 \boldsymbol{f} + (T_1^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2))T_1^4 \boldsymbol{f} \right.
$$
$$
\left. + 2 \left\{ (T_2^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) + 1 \right\} (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))T_1^4 \boldsymbol{f} \right] ds_2
$$
$$
- 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)^3} \left[T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) - T_2^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) \right.
$$
$$
\left. + 2 \left\{ (T_2^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2)) + 1 \right\} (T_1^2 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)) + T_1^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) - \frac{(\Delta s)^2}{6}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \right] \boldsymbol{\tau}(s_1) ds_2
$$
$$
- 4 \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\mathscr{M}_2(\boldsymbol{f})}{\Delta s}T_1^2 \boldsymbol{f} + \frac{1}{\Delta s}\left\{ \mathscr{M}_2(\boldsymbol{f}) + \frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 \right\} \boldsymbol{\tau}(s_1) \right] ds_2.
$$

References

[1] A. Abrams, J. Cantarella, J. H. G. Fu, M. Ghomi & R. Howard, *Circles minimize most knot energies*, Topology **42** (2) (2003), 381–394.

[2] S. Blatt, *Boundedness and regularizing effects of O'Hara's knot energies*, J. Knot Theory Ramifications **21** (2012), 1250010.

[3] S. Blatt, *The gradient flow of the Möbius energy near local minimizers*, Calc. Var. Partial Differential Equations **43** (3–4) (2012), 403–439.

[4] S. Blatt, *The gradient flow of the Möbius energy: ε-regularity and consequences* preprint, 2015.

[5] S. Blatt, private communication.

[6] S. Blatt and Ph. Reiter, *Stationary points of O'Hara's knot energy*, Manuscripta Math. **140** (2013), 29–50.

[7] S. Blatt, Ph. Reiter & A. Schikorra, *Harmonic analysis meets critical knots. Critical points of the Möbius energy are smooth*, Trans. Amer. Math. Soc. **368** (9) (2016), 6391–6438.

[8] M. H. Freedman, Z.-X. He and Z. Wang, *Möbius energy of knots and unknots*, Ann. of Math. (2) **139** (1994), 1–50.

[9] Z.-X. He *The Euler-Lagrange equation and heat flow for the Möbius energy*, Comm. Pure Appl. Math. **53** (2000), 399–431.

[10] A. Ishizeki, *Some properties of the Möbius energy and its variational formulae*, Master Thesis, Saitama University, 2013.

[11] A. Ishizeki & T. Nagasawa, *A decomposition theorem of the Möbius energy I: Decomposition and Möbius invariance*, Kodai Math. J. **37** (3) (2014), 737–754.

[12] A. Ishizeki & T. Nagasawa, *A decomposition theorem of the Möbius energy II: Variational formulae and estimates*, Math. Ann., **363** (1-2) (2015), 617–635.

[13] A. Ishizeki & T. Nagasawa, *The invariance of decomposed Möbius energies under the inversions with center on curves*, J. Knot Theory Ramifications **26** (2016), 1650009.

[14] A. Ishizeki & T. Nagasawa, *The L^2-gradient of decomposed Möbius energies*, to appear in Calc. Var. Partial Differential Equations.

[15] R. Kusner and J. M. Sullivan, *On distortion and thickness of knots*, in "Ideal Knots" (Ed.: A. Stasiak, V. Katrich, L. H. Kauffman), World Scientific, Singapore, 1998, pp. 315–352.

[16] S. Mizohata, "The Theory of Partial Differential Equations", Cambridge University Press, New York, 1973 (transl. from Japanese Ed.: Iwanami, Tokyo, 1965).

[17] T. Nagasawa, *Analytic approaches to the Möbius energy: History and recent topics*, to appear in Sūrikaisekikenkyūsho Kōkyūroku.

[18] J. O'Hara, *Energy of a knot*, Topology **30** (2) (1991), 241–247.

[19] J. O'Hara, *Family of energy functionals of knots*, Topology Appl. **48** (2) (1992), 147–161.

[20] J. O'Hara, *Energy functionals of knots II*, Topology Appl. **56** (1) (1994), 45–61.

[21] J. O'Hara and G. Solanes, *Möbius invariant energies and average linking with circles*, Tôhoku Math. J. (2) **67** (1) (2015), 51–82.

[22] A. P. Prudnikov, Yu. A. Brychkov & O. I. Marichev, "Integrals and Series", Nauka, Moscow, 1981 (in Russian; Japanese Ed.: Maruzen, Tokyo, 1991).

[23] Ph. Reiter, *Regularity theory for the Mobius energy*, Comm. Pure Appl. Anal. **9** (5) (2010), 1463–1471.