確率変分解析の基礎研究

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はしがき

本研究はほとんど研究代表者が中心となり、平成７年度から平成９年度にかけて行った「確率変分解析の基礎研究」に関する研究成果をまとめたものである。研究計画調書の研究計画・方法のところでも述べたように、研究成果を単に学会や国際会議や関連する国内の研究集会で講演・発表等により公表するだけでなく、成果が得られ次第、随時論文等として発表することにも気を配った。研究実績の細かい説明を掲載する代わりに、本成果報告書の終わりに論文のコピーを掲載した。それをみれば一目瞭然であるが、比較的短い期間に量的にはかなりのことを行えたと自負している。なお本研究の研究組織および研究経費は下記の通りである。

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目次

研 究 成 果

1. 研究概要と研究成果

無限次元の解析学のひとつであるホワイトノイズ解析の枠組みの下で、「無限次元の調和解析」を展開しようという試みは、２１世紀に向けての数学の無限次元化の流れの中にあっては、現代解析学の発展の自然な方向性の一翼を担う重要な課題である。そのための第一ステップとして、無限次元の各種ラプラシアンのホワイトノイズ解析におけるそれぞれの異なる役割を数学的に見極めることは重大な意義をもつ。そのひとつの方法として、主部をラプラシアンにもつ無限次元方程式の解の性質を調べることが考えられるが、特にホワイトノイズ解析では確率変分をとるほうがより自然な場合も多いので、確率変分方程式に関する研究が重要となってくる背景がある。以下に本研究における成果の概要を述べるが、大体大まかに云って、次の４点に分けられる。１）新しいタイプの無限次元ラプラス作用素の研究、２）擬フーリエ・メーラー変換（Pseudo-Fourier-Mehler 変換、以下擬ＦＭ変換と略記）に関する研究、３）確率変分と確率変分方程式の研究、４）ホワイトノイズ解析の確率境界値問題への応用に関する研究である。

ホワイトノイズ解析における飛田微分に付随した新しいタイプのラプラシアンΔ_pを無限次元多様体上に構成した。それは従来の飛田のラプラシアンΔ_Hとは異なり、C^∞不変性を有する良い性質の作用素として実現出来たため、ド・ラーム＝ホッジ＝小平の分解定理の無限次元版を示すことに成功した。

ホワイトノイズ解析における無限次元フーリエ変換の変種として、擬ＦＭ変換Ψを構成し、基本的な諸性質を導出した。また特に Intertwining Properties と呼ばれる性質やフォック展開表現などを導いた。さらにその性質を詳しく調べていくうちに、その族$\{\Psi_\theta; \theta \in \mathbf{R}\}$が超汎関数空間$(S)^*$上の線形同相写像群の正則１変数部分群を成し、しかもその対応する生成作用素が$iN + i\Delta_G^*$で与えられることが判明した。さらにはその擬ＦＭ変換の一般化の方法が分かり、一般化ＦＭ変換X_θを構成した。その族$\{X_\theta; \theta \in \mathbf{R}\}$はやはり$(S)^*$上の線形同相写像群の正則な１変数部分群であり、その生成作用素が$\alpha N + \beta\Delta_G^*$ $(\alpha, \beta \in \mathbf{C})$で与えられ、またこのことによって特徴付けられることが判明した。実際、このX_θは無限次元のＦ変換$\mathcal{F}$、ＦＭ変換$\mathcal{F}_\theta$、擬ＦＭ変換Ψ_θをすべて含む、非常に大きなクラスである。

確率変分方程式の解として、確率場$X(s)$がホワイトノイズ空間に実現され、そのＳ変換に対して古典的変分法が適用可という条件を課すと、特別な場合に限り変分操作が可能で、変分δXの具体的表現が得られる。超汎関数のクラスの確率変分を考える際のパラメータ集合は、一般化球面と微分位相同形なd次元ユークリッド空間内の閉多様体に限られる。この枠から外れると、Deformation による極限操作で変分を計算する方法が適用できない。

確率境界値問題を考察し、ホワイトノイズ解析における手法を適用することにより、一般化された確率漸近解の定式化に基づき、確率解析的視点から解の構成を行った。また確率系の振動論の観点から、新しいタイプの極限定理を導いた。

2. 研究実績と関連資料

2.1 擬ＦＭ変換の諸性質と抽象方程式への応用

ホワイトノイズ解析におけるフーリエ・メーラー変換論の一般化として擬ＦＭ変換を定義し、その基本的な諸性質を導出した。特に

1）ガウス白色雑音汎関数との関係の究明、
2）変換論と類似の重要な諸関係式の導出、
3）テスト汎関数に対する擬ＦＭ変換のImage領域の決定、
4）同変換の全単射かつ強連続性の証明、
5）同変換族の半群性、
6）対応する生成作用素の導出，等

を行った。また擬ＦＭ変換の応用として，ホワイトノイズ空間上の近似コーシー問題を考察し，擬ＦＭ変換を施すことによって，近似解の収束性，弱解の存在性，強収束性を議論し，解の存在・一意性定理を証明した。さらに，擬ＦＭ変換とホワイトノイズ解析における典型的な作用素，例えば，飛田微分作用素，Kubo作用素，ガトー微分作用素，その共役作用素，掛け算作用素などとの間のIntertwining Propertiesを調べた。それに基づき擬ＦＭ変換の特徴付けを行った。また積分核作用素理論を用いて，擬ＦＭ変換とその双対変換のフォック展開表現を求めた。

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Pseudo-Fourier-Mehler Transform in White Noise Analysis and Application of Lifted Convergence to A Certain Approximate Cauchy Problem

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Dedicated to Professor Yoshio Nakamura on the occasion of his retirement from Saitama University

Abstract

We consider a certain approximate Cauchy problem which is deeply connected with the abstract Cauchy problem for heat equation on the white noise space. We also propose the pseudo-Fourier-Mehler transform and lifted convergence with a concept of *lifting*. By using the former we can obtain an explicit form of the solution, and an application of the latter leads to a precise discussion of the convergence problem.

§ 1. Introduction

Let us consider the following Cauchy problem for the heat equation on the white noise space, i.e.,

$$\text{(P1)}\qquad \begin{aligned}&\frac{\partial u}{\partial t}(t,y)=\frac{1}{2}\Delta_G u(t,y), \qquad t>0,\ y\in S'(\mathbf{R})\\ &u(0,y)=f(y),\end{aligned}$$

where Δ_G is the Gross Laplacian and $S'(\mathbf{R})$ is the space of tempered distributions on $\mathbf{R}$. The Cauchy problem of this type has been studied by many researchers (e.g. [9,13] ; also [5,6]). The operator part is also written as

$$\text{(1.1)}\qquad \Delta_G=\Xi_{0,2}(\mathrm{Tr})=\int_{\mathbf{R}}\partial_t^2 dt,$$

according to the theory of kernel operator [12] . ∂_t is the Hida differentiation [10] (see also [4]). We set $P=\frac{1}{2}\Delta_G$ for convention. As well known, the Gross Laplacian is a singular operator in the sense that it has a singular kernel (see Eq.(1.1)), so one of the basic ideas of treating the above equation is usually a certain approximate approach in a proper manner. In fact, we may rewrite the above problem by using the formula (e.g. [12]):

$$\text{(1.2)}\qquad \Delta_G u=\langle x,\nabla u(x)\rangle-Nu(x),\quad u\in(S),$$

where N is the number operator and (S) is the space of white noise test functionals [17] .

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Hence we shall adopt the ε-approximation for instance. our approximation is as follows. For $y \in S'(\mathbf{R})$, there exists a sequence of $\eta_\varepsilon \equiv \eta_\varepsilon(y)$ in $S(\mathbf{R})$ such that

(1.3) $$\eta_\varepsilon \to y \quad \text{strongly in} \quad S'(\mathbf{R}),$$

where η_ε is given by $\eta_\varepsilon \equiv \eta_{1/n} = \pi_n y$, and π_n is the projection from $S'(\mathbf{R})$ into the subspace of Hermite polynomials of order n. Then we have

(1.4) $$\langle y, \nabla\varphi(y)\rangle = \lim_{\varepsilon\to 0}\langle \eta_\varepsilon, \nabla\varphi(y)\rangle = \lim_{\varepsilon\to 0} \mathrm{D}_{\eta_\varepsilon}\varphi(y), \text{ in } (S)_p \ (p>0),$$

where $(S)_p$ is the completion of L^2-space (L^2) on $S'(\mathbf{R})$ relative to the norm $\|\cdot\|_p = \|\Gamma(A)^p\cdot\|$ (with the second quantization $\Gamma(A)$, $A = -d^2/dt^2 + t^2 + 1$, and (L^2)-norm $\|\cdot\|$ relative to the white noise measure $d\mu$) [10,17] . Consequenty we have the following first approximate problem:

(P2) $$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t}(t,y) = \frac{1}{2}(\mathrm{D}\eta_\varepsilon u_\varepsilon(t,y) - Nu_\varepsilon(t,y)) \equiv P_\varepsilon u_\varepsilon(t,y), \ t>0, \ x \in S'(\mathbf{R}), \\ &u_\varepsilon(0,y) = f(y). \end{aligned}$$

Note that $P_\varepsilon \to P$ in operator sense (actually more strongly, in uniform convergence sense) when $\eta_\varepsilon \to y$ strongly in $S'(\mathbf{R})$.

Let us consider its Fourier transformed problem. Generally speaking, we cannot apply the Fourier transform F(= F-transform for short) [14,15] in White Noise Analysis for (P1) because $F(\Delta_G u(t,\cdot))(x)$ dose faol to exist, while the F-tansformation is valid for the operator P_ε. Hence we have

(P3) $$\begin{aligned} &\frac{\partial v_\varepsilon}{\partial t}(t,x) = -\frac{1}{2}(N + \Delta_G{}^* - i\langle\cdot,\eta_\varepsilon\rangle(\cdot))v_\varepsilon(t,x) \equiv \hat{P}_\varepsilon v_\varepsilon(t,x), \\ &v_\varepsilon(0) = \hat{f} \in (L^2), \end{aligned}$$

where $v_\varepsilon(t,x)$ is the F-transform of $u_\varepsilon(t,y)$, i.e., $v_\varepsilon(t,x), = F(v_\varepsilon(t,\cdot))(x)$, and $\Delta_G{}^*$ is the adjoint of Δ_G. (N.B. The foundation of F is based upon the theory of U-functionals [19] .) Since the F-transform is a one-to-one mapping in the space $(S)^*$ of generalized white noise functionals (or Hida distributions) [19] , consideration of (P2) is almost equivalent to thinking of (P3). We are very interested in this approximated problem (P3) and some convergence problems associated with (P3). Notice that there still exists a singular term $\langle x,\eta_\varepsilon\rangle v_\varepsilon(t,x)$ in (P3). In the following, unless otherwise stated, we interpret the solution as like this: for example, as to the case of (P3), $v_\varepsilon \equiv v_\varepsilon(t,x)$ is said to be a solution of the peoblem (P3) if v_ε satisfies

(1.5) $$\begin{aligned} &\langle v_\varepsilon(t),\varphi\rangle - \langle \hat{f},\varphi\rangle + \frac{1}{2}\int_0^t \langle Nv_\varepsilon(s),\varphi\rangle ds + \frac{1}{2}\int_0^t \langle \Delta_G{}^* v_\varepsilon(s),\varphi\rangle ds \\ &= \frac{i}{2}\int_0^t \langle v_\varepsilon(s), \langle\cdot,\eta_\varepsilon\rangle\varphi\rangle ds, \end{aligned}$$

for any $\varphi \in (S) \cap \tilde{F}$, and any $t>0$, where $\tilde{F}$ is a proper functional space, (which will be described precisely later).

The purpose of this paper consists in studying the F-transformed problem (P3) and discussing some convergence problems in white noise space. In § 2, we shall introduce some notations used in succeeding sections, and state some results on existence and uniqueness of solutions for problems (P2) and (P3). The proofs for those assertions will given in § 3—§ 7. In § 3 we will discuss amoothing and finite-dimensional approximation of the problem (P3). § 4 gives convergence result of approximate solutions $z_\varepsilon^{m,\delta}(t,x)$. The existence and uniqueness theorems for weak solutions of the smoothened problem (P5) are proved in § 5. Its strong convergence is discussed in §6. Then we will show in § 7 the existence and uniqueness of solutions for the modified Cauchy problem (P4). We define the Pseudo-Fourier-Mehler transform in § 8 and study the properties of the transformation precisely, and also discuss the S_3-transformed problem in connection with the newly introduced ΨFM transform. In § 9 we introduce the concept of a lifting and discuss lifted convergence of the F-transformed problem. As for other applications of the F-transform related to (P1), see [5,6] (cf. also [3]).

§ 2. Notation and Some Results

Define $|||\varphi|||_{2,r} := \|(I+N)^{r/2}\varphi\|$ for $\varphi\in P$ where P is algebra of complex polynomials over $S'(\mathbf{R})$, and we denote by $D^{2,r}$ the completion of P with respect to the norm $|||\cdot|||_{2,r}$, and $(D^{2,r},|||\cdot|||_{2,r})$ becomes a Hibert space of (L^2). For simplicity, we also write $G\equiv\Delta_G{}^*$. Let $\|\varphi\|_{N,2}$ (resp. $\|\varphi\|_{G,2}$) denote the graph norm of the operator N(resp. $\Delta_G{}^*$) in (L^2) for the element φ of Dom (N) (resp. Dom$(\Delta_G{}^*)$), and also H_N^2 (rest. H_G^2) denotes the completion of Dom (N) (rest. Dom$(\Delta_G{}^*)$) with respect to the graph norm $\|\cdot\|_{N,2}$ (rest. $\|\cdot\|_{G,2}$). Notice that $(H_N^2, \|\cdot\|_{N,2})$ becomes a Hibert space of (L^2) and that the space $D^{2,2}$ is canonically isomorphic to H_N^2. Then the dual space $(H_N^2)^*$ is written as H_N^{-2}, and this type of notation would be used extensively for other functional spaces unless otherwise stated. Occasionally the equivalent norm $\|\varphi\|_X:=(\|\varphi\|^2+\|X\varphi\|^2)^{1/2}$ will be used in what follows for $X = N$ or $X=\Delta_G{}^*$ instead of $\|\cdot\|_{N,2}$, $\|\cdot\|_{G,2}$ and there will be no confusion in terms of the context. We will use $(\cdot,\cdot)$ for the inner product, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing whereas we mean the canonical bilinear form.

Let T be an open interval $(0,t_e)$ for finite t_e, and $\bar{T}$ denotes its closure, i.e., the closed interval $[0,t_e]$. We fix $p>\frac{1}{4}$ throughout the paper. When X is an operator on a Hilbert space, then $\tilde{X}$ indicates the corresponding bilinear form associated with the operator X such that

$$\tilde{X}(\varphi, \psi)=\langle X\varphi, \psi\rangle;$$

for instance, $X=N$ or $X = G$. Next we introduce the regularized spaces, suggested by the theory of abstract evolution equations [18] . We define $R_{PN}^{\lambda}(H_N^2)$ (resp. $R_{PG}^{\lambda}(H_G^2)$) as the whole space of $\varphi\in H_N^2\cap(S)_p$ (resp. $\varphi\in H_G^2\cap(S)_p$) satisfying that there exists some constant λ_N (resp. λ_G) such that the inequality $\mathrm{Re}\tilde{N}(\varphi,\varphi)+\lambda_N\|\varphi\|^2\geq\alpha\|\varphi\|_{N,2}^2$, $\alpha>0$ (resp. $\mathrm{Re}\tilde{G}(\varphi,\varphi)+\lambda_G\|\varphi\|^2\geq\beta\|\varphi\|_{G,2}^2$, $\beta>0$) holds. When V is a topological vector space, then $L^2(T ; V)$ is the whole space of V-valued square integrable functions on T. We assume that a mapping : $t\rightarrow f(t)$ be strongly measurable for any element $f\in L^2(T;V)$ so that the Bochner integral [7] of f can make sense. Moreover, notive that N (resp. $\Delta_G{}^*$) proves to be a continuous linear operator from $L^2(T;R_{PN}^{\lambda}(H_N^2))$ (rest. $L^2(T;R_{PG}^{\lambda}(H_G^2))$) into $L^2(T;H_N^{-2}\cap(S)_{-p})$ (resp. $L^2(T;H_G^{-2}\cap(S)_{-p})$). In what follows we set

$$V:=R_{PN}^{\lambda}(H_N^2)\cap R_{PG}^{\lambda}(H_G^2)$$

and denote by V^* the dual space of V. Define (cf. [18])

$$W_2^1(T;V,V^*):=\{f|f\in L^2(T;V),\ \frac{d\tilde{f}}{dt}\in L^2(T;V^*)\}.$$

Further when $\tilde{F}$ is a functional space, then $F^{-1}\tilde{F}$ is the space of $f\in(S)^*$ such that $\hat{f}=Ff\in\tilde{F}$.

Now we introduce some results on the existence and uniqueness of solutions for the problems

(P3) and (P4). The proofs of those theorems below will be given in § 3 through § 7.

Theorem 2.1. *There exists a unique solution v_ε of (P3) in $W_2^1(T;V, V^*)$ such that v_ε satisfies*

$$\frac{\partial v_\varepsilon}{\partial t}=\hat{P}_\varepsilon v_\varepsilon(t,x) \tag{2.1}$$

with the initial condition $v_\varepsilon(0)=\hat{f}\in(L^2)$.

Theorem 2.2. *There exists a function $\tilde{v}_\varepsilon\in C^0(\bar{T};H)$ such that*

$$v_\varepsilon(t)=\tilde{v}_\varepsilon(t) \text{ holds in } (L^2),\ dt-a.e.t,$$

where we put $H=(L^2)$.

By virtue of the bijectivity of the Fourier transform F in $(S)^*$, the following theorem is derived almost obviously from the above assertion.

Theorem 2.3. *There exists a unique solution u_ε (except dt-measure null set) of (P2) in $\mathrm{F}^{-1}W_2^1(T;V,V^*)\cap \mathrm{F}^{-1}C^0(\bar{T};H)$ such that*

$$\frac{\partial u_\varepsilon}{\partial t}(t,y)=P_\varepsilon u_\varepsilon(t,y),\ t>0,\ y\in S'(\mathbf{R})$$

with $u_\varepsilon(0)=f(y)\in \mathrm{F}^{-1}(L^2)$.

§ 3. Smoothing and Finite-dimensional Approximation

The purpose of this section is to exclude the singularlity of coefficient of the operator in question by smoothing procedure and to make a finite-dimensional approximation of the smoothened equation. First of all we will begin with introducing the definition of S_1-transformation, which gives an equivalent problem.

We define $z_\varepsilon(t,x)$ (for $\varepsilon>0$, $t>0$, $x\in S'(\mathbf{R})$) as follows:

$$\begin{aligned} z_\varepsilon(t,x)&:=S_1^{-1}v_\varepsilon(t,x) \\ &=\exp\left\{-\frac{1}{2}(\lambda_N+\lambda_G)t\right\}\cdot v_\varepsilon(t,x), \end{aligned} \tag{3.1}$$

Then we see that $z_\varepsilon(t,x)$ satisfies the following modified equation:

$$\begin{aligned} &\frac{\partial z_\varepsilon}{\partial t}(t,x)=-\frac{1}{2}(N_1+G_1-i\langle x,\eta_\varepsilon\rangle)z_\varepsilon(t,x)\equiv \mathrm{P}_\varepsilon z_\varepsilon(t,x), \\ &z_\varepsilon(0)=\hat{\mathrm{f}}\in(L^2),\ t>0,\ x\in S'(\mathbf{R}), \end{aligned} \tag{P4}$$

where $N_1=N+\lambda_N I$ and $G_1=G+\lambda_G I$. We consider the smoothing of the coefficient $\langle x,\eta_\varepsilon\rangle$. Let π_m be the projection from $S'(\mathbf{R})$ into the subspace of Hermite polynomials of order m, and set $\zeta_\delta\equiv\zeta_{1/m}=\pi_m x$ (e.g. [11]). Then it follows that ζ_δ converges to x in strong topology as δ tends to zero, hence, of course, it holds that $\Phi(\varepsilon,\delta):=(\zeta_\delta,\eta_\varepsilon)\to\langle x,\eta_\varepsilon\rangle$as $\delta\downarrow 0$. If we write the smoothing operation as $SM(\delta)$, then the new operator is given by $\mathrm{P}_{\varepsilon,\delta}:=SM(\delta)\mathrm{P}_\varepsilon=-\frac{1}{2}(N_1+G_1-i\Phi(\varepsilon,\delta))$. The smoothened modified Cauchy problem is as follows:

$$\begin{aligned} &\frac{\partial z_\varepsilon^\delta}{\partial t}(t,x)=\mathrm{P}_{\varepsilon,\delta}z_\varepsilon^\delta(t,x),\ t>0,\ x\in S'(\mathbf{R}) \\ &z_\varepsilon^\delta(0)=\hat{f}\in(L^2). \end{aligned} \tag{P5}$$

Next we shall consider a finite-dimensional approximation of the problem by the Faedo-Galerkin method (e.g. [2]). P_m denotes m-dimensional approximation map from (L^2) into an m-dimensional vector space V_m. Then the smoothened modified Cauchy problem for P_m is given by

$$\text{(P6)}\qquad \begin{aligned}&\frac{\partial z_\varepsilon^{m,\delta}}{\partial t}(t,x)=-\frac{1}{2}(N_1+G_1-i\Phi(\varepsilon,\delta))z_\varepsilon^{m,\delta}(t,x)\equiv P_mP_{\varepsilon,\delta}z_\varepsilon^\delta(t,x),\\ &z_\varepsilon^{m,\delta}(0)=\hat{f}\in(L^2),\ t>0,\ x\in S'(\mathbf{R}),\end{aligned}$$

and note that $P_m P_{\varepsilon,\delta}=P_{\varepsilon,\delta}P_m$, while $P_m P_\varepsilon \neq P_\varepsilon P_m$, implying that the smoothing $SM(\delta)$ and the finite-dimensional approximation P_m is not commutative. As a matter of fact, this finite-dimensional approximation is realized as follows:

$$P_m z_\varepsilon^\delta(t,x)=z_\varepsilon^{m,\delta}(t,x)=\sum_{i=1}^m g_{i,m}^{\varepsilon,\delta}(t)w_i(x),\ \text{and} \tag{3.2}$$

$$\hat{f}^m(x)=\sum_{i=1}^m \xi_{i,m}w_i(x). \tag{3.3}$$

The sequence $\{w_i\}$, $\{g_{i,m}^{\varepsilon,\delta}(t)\}$, and $\{\xi_{i,m}\}$ are described below and are settled in a well-defined manner. In fact, since the space V is separable, there exists a dense linear span $\{w_1,w_2,\cdots,w_m,\cdots\}$ in V. For a fixed integer m, $\xi_{i,m}$ is defined by $(\hat{f},\ w_i)$ for $1\le i\le m$, and $\hat{f}^m(x)$ in (3.3) converges strongly to $\hat{f}(x)$ in $H=(L^2)$. Moreover, $\vec{g}_m^{\varepsilon,\delta}(t)=\{g_{i,m}^{\varepsilon,\delta}(t)\}_{1<i<m}\in \mathbf{C}^m(\forall t\in T)$ satisfies the following m-linear ordinary differential equation system, i.e.,

$$\mathrm{W}_m\frac{d}{dt}\vec{g}_m^{\varepsilon,\delta}(t)+\frac{1}{2}\mathrm{N}_m\vec{g}_m^{\varepsilon,\delta}(t)+\frac{1}{2}\mathrm{g}_m\vec{g}_m^{\varepsilon,\delta}(t)=\frac{i}{2}\mathrm{H}_m^{\varepsilon,\delta}\vec{g}_m^{\varepsilon,\delta}(t), \tag{3.4}$$
$$\text{with } \vec{g}_m^{\varepsilon,\delta}(0)=\vec{\xi}_m,$$

where W_m is an $m\times m$ matrix given by $\|(w_i,w_j)_H\|$, and N_m (resp. g_m) is an $m\times m$ matrix respectively given by $\|\tilde{N}_1(w_i,w_j)\|$, $\|G_1(w_i,w_j)\|$. $\mathrm{H}_m^{\varepsilon,\delta}$ is also an m-square matrix defined by $\|(\Phi(\varepsilon,\delta)w_i,w_j)_H\|$ and $\vec{\xi}_m$ is a vector given by $\{\xi_{i,m}\}_{1<i<m}$.

The well-definedness of $\vec{g}_m^{\varepsilon,\delta}$ (t) and the validity of m-dimensional spproximation P_m for each $m\in\mathbf{N}$ is due to the following assertion.

Theorem 3.1. (i) *For each $m\in\mathbf{N}$, $\varepsilon>0$, $\delta>0$ (fixed), there exists a unique solution vector $\vec{g}_m^{\varepsilon,\delta}$ (t) such that $\vec{g}_m^{\varepsilon,\delta}(\cdot)\in C^0(\bar{T};\mathbf{C}^m)\cap C^1(T;\mathbf{C}^m)$ satisfying Eq.(3.4).* (ii) *Moreover, the solution can be expressed by*

$$\vec{g}_m^{\varepsilon,\delta}(t)=\exp(tC_m^{\varepsilon,\delta})\cdot\vec{\xi}_m, \tag{3.5}$$

where the matrix $C_m^{\varepsilon,\delta}$ is given by

$$i\Phi(\varepsilon,\delta)E_m-\frac{1}{2}\mathrm{W}_m^{-1}\mathrm{N}_m-\frac{1}{2}\mathrm{W}_m^{-1}\mathrm{g}_m, \tag{3.6}$$

and E_m is an m-square unit matrix.

Remark 3.2. When $C_m^{\varepsilon,\delta}=\|c_{ij}(m;\varepsilon,\delta)\|$, then the convergence criterion of a function $exp(\cdot)$ is given in the usual manner, that is, $\exp(tC_m^{\varepsilon,\delta})$ is said to be convergent if and only if the series $\sum_{k=0}^\infty\frac{1}{k!}t^kc_{ij}^{(k)}(m;\varepsilon,\delta)$ converges absolutely, where $c_{ij}^{(k)}$ indicates the (i,j)-component of the k-th power matrix $(C_m^{\varepsilon,\delta})^k$. Actually it is guaranteed by the following estimate:

$$|c_{nl}^{(k)}(m;\varepsilon,\delta)|<m^{k-1}\left(\max_{1<i,j<m}|c_{ij}(m;\varepsilon,\delta)|\right)^k$$

for $1\le n,l\le m$.

Proof of Theorem 3.1. We first note that for each m, an aggregate of $\{w_1,w_2,\cdots,w_m\}$ is linearly independent, so the Gram determinant $\det \mathrm{W}_m$ is nonzero, hence the non-singular matrix W_m has its inverse W_m^{-1}. Consequently when we write $\mathrm{W}_m^{-1}=\|v_{ij}{}^m\|$, then the (i,j)-component c_{ij} $(m;\varepsilon,\delta)$ of the matrix $C_m^{\varepsilon,\delta}$ defined by (3.6) is expressed by

$$c_{ij}(m;\varepsilon,\delta)=\frac{i}{2}\Phi(\varepsilon,\delta)\delta_{ij}-\frac{1}{2}\sum_{k=1}^{m} v_{ik}^{m}\tilde{N}_1(w_k,w_j)-\frac{1}{2}\sum_{k=1}^{m} v_{ik}^{m}\tilde{G}_1(w_k,w_j), \tag{3.7}$$

where δ_{ij} is the Kronecker's delta. The assertion is direct result of the following lemma with $\vec{g}(t)=\vec{g}_m^{\varepsilon,\delta}(t)$, $M=C_m^{\varepsilon,\delta}$, and $\vec{\xi}=\vec{\xi}_m$:

Lemma 3.3. *For $m\in\mathbf{N}$, let $\vec{g}(t)$ be an $\mathbf{R}^m$-valued vector function on $\bar{T}$ and M an $m\times m$-matrix. Then the liear differential equation*

$$\frac{d}{dt}\vec{g}(t)=M\vec{g}(t) \quad \text{with} \quad \vec{g}(0)=\vec{\xi}\in\mathbf{R}^m \tag{3.8}$$

has a unique solution satisfying $\vec{g}(\cdot)\in C^0(T;\mathbf{C}^m)\cap C^1(T;\mathbf{C}^m)$, and moreover, it has an expression $\vec{g}(t)=\exp(tM)\cdot\vec{\xi}$.

§ 4. Convergence of Approximate Solutions $z_\varepsilon^{m,\delta}(t)$

According to the results in the previous section, for each m, the vector function $\vec{g}_m^{\varepsilon,\delta}(t)$ is well-defined and uniquely determined, and

$$g_{i,m}^{\varepsilon,\delta}(\cdot)\in C^0(\bar{T};\mathbf{C})\cap C^1(T;\mathbf{C}).$$

So that, as far as $m\in\mathbf{N}$, $\varepsilon>0$, ani $\delta>0$ are fixed, we know from (3.2) and (3.3) that there exists a unique approximate solution $z_\varepsilon^{m,\delta}(t,x)$ such that

$$z_\varepsilon^{m,\delta}(\cdot)\in C^0(\bar{T};V)\cap C^1(T;V) \quad (\subset L^2(T;V)), \tag{4.1}$$

satisfying Eq.(P6). Then we have the following estimate.

Lemma 4.1. *For each $m\in\mathbf{N}$ ($m\gg1$, sufficiently large), $\varepsilon>0$, $\delta>0$ (fixed), for any $\hat{f}\in H=(L^2)$ we can find some positive constant C_0 and*

$$|||z_\varepsilon^{m,\delta}|||_{2,T,V}^2\le\frac{C_0}{\gamma(\alpha,\beta;\varepsilon,\delta)}\|\hat{f}\|^2<\infty \tag{4.2}$$

holds, where the constant $\gamma(\alpha,\beta;\varepsilon,\delta)=\alpha+\beta-2|\Phi(\varepsilon,\delta)|$, and $|||g|||_{2,T,V}$ is the usual norm of the space $L^2(T;V)$.

Remark 4.2. In the above (4.2), the constant $\gamma(\alpha,\beta;\varepsilon,\delta)$ is able to remain positive for sufficiently large α and β. By the passage to limit $\delta\downarrow0$, the term $\Phi(\varepsilon,\delta)$ approaches to a certain finite number for fixed $\varepsilon>0$, so the inequality (4.2) is still valid. However, we cannot expect the existence of the limit when ε tends to zero, as γ proves to be$-\infty$ even for any α, $\beta>0$.

Proof of Lemma 4.1. From (P6) we get immediately

$$\begin{aligned}&\langle\frac{\partial}{\partial t}z_\varepsilon^{m,\delta}(t),w_j\rangle+\frac{1}{2}\langle N_1z_\varepsilon^{m,\delta}(t),w_j\rangle+\frac{1}{2}\langle G_1z_\varepsilon^{m,\delta}(t),w_j\rangle\\&=\frac{i}{2}\langle\Phi(\varepsilon,\delta)z_\varepsilon^{m,\delta}(t),w_j\rangle\end{aligned} \tag{4.3}$$

for $w_j\in V$, $1\le j\le m$, $m\in\mathbf{N}$. The linearlity of the approximate smoothened modified equation allows us to derive

(4.4) $$(\frac{\partial}{\partial t}z_\varepsilon^{m,\delta}(t),z_\varepsilon^{m,\delta}(t))_H+\frac{1}{2}{}_{V^*}\langle N_1 z_\varepsilon^{m,\delta}(t),z_\varepsilon^{m,\delta}(t)\rangle_V$$
$$+\frac{1}{2}{}_{V^*}\langle G_1 z_\varepsilon^{m,\delta}(t),z_\varepsilon^{m,\delta}(t)\rangle_V=\frac{i}{2}(\Phi(\varepsilon,\delta)z_\varepsilon^{m,\delta}(t),z_\varepsilon^{m,\delta}(t))_H,$$

by multiplying (4.4) by the previously obtained solution $g_{i,m}^{\varepsilon,\delta}(t)$ and taking a summation from 1 up to m. Recall the following formula:

Lemma 4.3 *Let* H *be a real Hilbert space. Then*

(4.5) $$2\Big(\frac{\partial}{\partial t}f(t),f(t)\Big)_H=\frac{d}{dt}\|f(t)\|_H^2$$

holds for any $f(t)\in C^1(T;H)$.

By (4.5) we have

(4.6) $$\mathrm{Re}\Big(\frac{\partial}{\partial t}z_\varepsilon^{m,\delta}(t),\ z_\varepsilon^{m,\delta}(t)\Big)_H=\frac{1}{2}\frac{d}{dt}\|z_\varepsilon^{m,\delta}(t)\|^2.$$

We may apply (4.6) for (4.4) to obtain

(4.7) $$\frac{1}{2}\frac{d}{dt}\|z_\varepsilon^{m,\delta}(t)\|^2+\frac{1}{2}\mathrm{Re}_{V^*}\langle N_1 z_\varepsilon^{m,\delta}(t),\ z_\varepsilon^{m,\delta}(t)\rangle_V$$
$$+\frac{1}{2}\mathrm{Re}_{V^*}\langle G_1 z_\varepsilon^{m,\delta}(t),\ z_\varepsilon^{m,\delta}(t)\rangle_V=-\frac{1}{2}\mathrm{Im}(\Phi(\varepsilon,\delta)z_\varepsilon^{m,\delta}(t),z_\varepsilon^{m,\delta}(t))_H.$$

Likewise as to the initial condition we have $\langle z_\varepsilon^{m,\delta}(0),\ w_j\rangle=\langle \hat{f}^m,\ w_j\rangle$, $i<j<m$, from which we reads

(4.8) $$\|z_\varepsilon^{m,\delta}(0)\|_2=\|\hat{f}^m\|_2.$$

Taking (4.8) into consideration we may integrate Eq.(4.7) relative to the Lebesgue measure dt over the interval $(0,t_e)$, and get

(4.9) $$\|z_\varepsilon^{m,\delta}(t_e)\|^2-\|\hat{f}^m\|^2+\mathrm{Re}\int_0^{te}\tilde{N}_1(z_\varepsilon^{m,\delta}(t),\ z_\varepsilon^{m,\delta}(t))dt$$
$$+\mathrm{Re}\int_0^{te}\tilde{G}_1(z_\varepsilon^{m,\delta}(t),z_\varepsilon^{m,\delta}(t))dt=-\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)\|^2dt.$$

The definitions of the regularized spaces $\mathrm{R}_{PN}^{\lambda}(H_N^2)$, $\mathrm{R}_{PG}^{\lambda}(H_G^2)$ mentioned in § 2 imply that the inepualities of the form

(4.10) $$\mathrm{Re}\int_0^{te}\tilde{X}(\varphi(t),\ \varphi(t))dt>K_X\int_0^{te}\|\varphi(t)\|^2{}_V dt$$

holds respectively for $\tilde{X}=\tilde{N}_1$ (or $\tilde{X}=\tilde{G}_1$) and the constants $K_X=\alpha$(or$=\beta$), because we can find constants λ_N,λ_G for properly chosen $\alpha,\beta>0$ in accordance with the norm values of V. While, since $P_m\hat{f}$ converges to $\hat{f}$ in (L^2) as m approaches to infinity, the continuity of norm can provide woth some positive constant C_0 such that

(4.11) $$\|\hat{f}^m\|\le C_0^{1/2}\|\hat{f}\|$$

holds, while C_0 is chosen uniformly with respect to m as far as m is sufficiently large. An estimation of Eq.(4.9) together with (4.10) and (4.11) leads to

(4.12) $$\gamma(\alpha,\beta;\varepsilon,\delta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)\|^2{}_V dt\le C_0\|\hat{f}\|^2,$$

where $\gamma(\alpha,\beta\,;\,\varepsilon,\delta)$ (see Lemma 4.1) has a meaning for sufficiently large α,β. By Remak 4.2 we conclude the assertion. □

The estimate Eq.(4.2) in Lemma 4.1 means that $z_\varepsilon^{m,\delta}(\cdot)$ ranges in a bounded set in $L^2(T\,;\,V)$, so we can extract a subsequence $\{z_\varepsilon^{m(k),\delta}\}_k$ such that

(4.13) $$z_\varepsilon^{m(k),\delta}(\cdot)\to z_\varepsilon^{\delta}(\cdot)\quad\text{weakly in } L^2(T\,;\,V),$$

because a bounded set in the Hibert space is not compact, but is weakly compact.

§ 5. Existence of Weak Solutions to the Smoothened Problem

In this section we shall show that the weak solutions obtained as a weak limit lie in $W_2^1(T; V, V^*)$ and satisfy the smoothened modified Cauchy problem (P5) in distribution sense. The uniqueness is also proved (see Proposition 5.5).

Lemma 5.1. *The following equality*

$$\frac{d}{dt}(\tilde{z}_\varepsilon^\delta(t),\ w_j)+\frac{1}{2}\{\tilde{N}_1(z_\varepsilon^\delta(t),\ w_j)+\tilde{G}_1(z_\varepsilon^\delta(t),\ w_j)\}$$
$$=\frac{i}{2}\Phi(\varepsilon,\delta)(z_\varepsilon^\delta(t),\ w_j),\qquad \forall j \tag{5.1}$$

holds in $\mathrm{D}'(T)$ *sense with* $(z_\varepsilon^\delta(0),\ w_j)=(\hat{f},\ w_j)$, *where* $(\tilde{z}_\varepsilon^\delta(t),\ w_j)$ *is a modification of* $(z_\varepsilon^\delta(t),\ w_j)$ *except dt-measure null set.*

Proof of Lemma 5.1. For simplicity we set $\lambda=m(k)$. Take $\varphi(t)\in C^1(\bar{T})$ such that $\varphi(t_e)=0$, and define $\varphi_j(t):=\varphi(t)\cdot w_j$ for $\lambda>j$. From (4.3) in § 4, immediately

$$\langle\frac{\partial}{\partial t}z_\varepsilon^{\lambda,\delta}(t),\varphi_j(t)\rangle+\frac{1}{2}\tilde{N}_1(z_\varepsilon^{\lambda,\delta}(t),\varphi_j(t))$$
$$+\frac{1}{2}\tilde{G}_1(z_\varepsilon^{\lambda,\delta}(t),\ \varphi_j(t))=\frac{i}{2}(\Phi(\varepsilon,\delta)z_\varepsilon^{\lambda,\delta}(t),\ \varphi_j(t))_H. \tag{5.2}$$

On the other hand, an application of integration by parts formula allows

$$\int_T\int_{S'(R)}\frac{\partial}{\partial t}z_\varepsilon^{\lambda,\delta}(t,x)\bar{\varphi}_j(t,x)dt\times\mu(dx)$$
$$=-\int_0^{te}(z_\varepsilon^{\lambda,\delta}(t),\ \varphi'_j(t))dt-(\hat{f}^\lambda,\varphi_j(0))_H.$$

Hence we can rewrite (5.2) into

$$(\hat{f}^\lambda,\varphi_j(0))_H+\int_0^{te}i(\Phi(\varepsilon,\delta)z_\varepsilon^{\lambda,\delta}(t),\varphi_j(t))_H dt$$
$$=\int_0^{te}\Big\{\frac{1}{2}\tilde{N}_1(z_\varepsilon^{\lambda,\delta}(t),\varphi_j(t))+\frac{1}{2}\tilde{G}_1(z_\varepsilon^{\lambda,\delta}(t),\varphi_j(t))-\frac{1}{2}(z_\varepsilon^{\lambda,\delta}(t),\varphi'_j(t))\Big\}dt, \tag{5.3}$$

because we integrated (5.2) with respect to dt and substituted the above-mentioned result for it. If we take the limit $\lambda\to\infty$ in (5.3) by taking the weak convergence (4.13) and the (L^2) convergence of $\hat{f}^\lambda$ in § 4 into account, then the following equality

$$(\hat{f},\varphi_j(0))_H+\int_0^{te}i(\Phi(\varepsilon,\delta)z_\varepsilon^\delta(t),\varphi_j(t))_H dt$$
$$=\int_0^{te}\Big\{\frac{1}{2}\tilde{N}_1(z_\varepsilon^\delta(t),\varphi_j(t))+\frac{1}{2}\tilde{G}_1(z_\varepsilon^\delta(t),\varphi_j(t))-\frac{1}{2}(z_\varepsilon^\delta(t),\ \varphi'_j(t))\Big\}dt \tag{5.4}$$

holds. Actually the above is also true for any φ consisting in $\mathrm{D}(T)$ satisfying the condition that φ vanishes at the one end point $t=t_e$. We denote it by the symbol $\mathrm{D}_e(T)$, and put $\varphi_j(t)=\varphi(t)w_j(x)$ for $\varphi\in\mathrm{D}_e(T)$. Exchanging $\varphi_j(t)$ for this one we may rewrite (5.4) and readily get

$$\langle\frac{d}{dt}(\tilde{z}_\varepsilon^\delta(t),w_j),\varphi\rangle+\frac{1}{2}\langle\{\tilde{N}_1(z_\varepsilon^\delta(t),w_j)+\tilde{G}_1(z_\varepsilon^\delta(t)\rangle w_j)\},\varphi\rangle$$
$$=\langle\frac{i}{2}\Phi(\varepsilon,\delta)(z_\varepsilon^\delta(t),w_j),\varphi\rangle, \tag{5.5}$$

which implies the assertion. Note that $(z_\varepsilon^\delta(t),w_j)$ is regarded as an element of $\mathrm{D}'(T)$ in the left hand side of the above expression (5.5). □

Lemma 5.2. *The equality*

$$\frac{d}{dt}z_\varepsilon^\delta(t)+\frac{1}{2}(N_1+G_1)z_\varepsilon^\delta(t)=\frac{i}{2}\Phi(\varepsilon,\delta)z_\varepsilon^\delta(t),\quad with\quad z_\varepsilon^\delta(0)=\hat{f}, \tag{5.6}$$

holds in V^*.

Proof of Lemma 5.2. Recall that $\sum_j\xi_j w_j$ is dense in V for a basis $\{w_j\}$ of V. From (5.1) of Lemma 5.1 it is easy to see by virtue of linearity that

$$\begin{aligned}&\frac{d}{dt}(\tilde{z}_\varepsilon^\delta(t),\psi)+\frac{1}{2}\{\tilde{N}_1(z_\varepsilon^\delta(t),\psi)+\tilde{G}_1(z_\varepsilon^\delta(t),\psi)\}\\&=\frac{i}{2}\Phi(\varepsilon,\delta)(z_\varepsilon^\delta(t),\psi)\end{aligned} \tag{5.7}$$

holds for arbitrary element ψ of V. Immediately Eq.(5.7) implies that

$${}_{V^*}\langle\frac{d}{dt}z_\varepsilon^\delta(t)+\frac{1}{2}(N_1+G_1)z_\varepsilon^\delta(t)-\frac{i}{2}\Phi(\varepsilon,\delta)z_\varepsilon^\delta(t),\psi\rangle_V=0,\forall\psi\in V,$$

which completes the proof. □

Remark 5.3. Recall that the number operator N is a continuous linear operator from $L^2(T;\mathrm{R}^\lambda_{pN}(H_N^2))$ into $L^2(T;H_N^{-2}\cap(\mathrm{S})_{-p})$, and that $G=\Delta_G^*$ is a continuous linear operator from $L^2(T;\mathrm{R}^\lambda_{pG}(H_G^2))$ into $L^2(T;H_G^{-2}\cap(\mathrm{S})_{-p})$, which is stated in § 2. We used this fact in the proof of Lemma 5.2.

If we combine Remark 5.3 with the fact that z_ε^δ lies in $L^2(T\,;\,V)$, then it follows immediately that

$$\frac{d}{dt}z_\varepsilon^\delta(t)=\frac{d\tilde{z}_\varepsilon^\delta}{dt}(t)=\frac{i}{2}\Phi(\varepsilon,\delta)z_\varepsilon^\delta(t)-\frac{1}{2}(N_1+G_1)z_\varepsilon^\delta(t)\in L^2(T;V^*),$$

whre $\frac{d\tilde{z}_\varepsilon^\delta}{dt}(t)$ is a L^2-modification of $\frac{d}{dt}z_\varepsilon^\delta(t)$. Summing up, we therefore obtain

Proposition 5.4. *The smoothened modified Cauchy problem* (P5) *has a solution* $z_\varepsilon^\delta(t,x)$ in $W_2^1(T;V,V^*)$, regarded as an evolution system.

Proposition 5.5. *The solution* $z_\varepsilon^\delta(t,x)\in W_2^1(T;V,V^*)$ *obtained in Proposition 5.4 is unique.*

Proof of Proposition 5.5. Assume that there are two solutions $z_\varepsilon^{\delta,1},z_\varepsilon^{\delta,2}\in W_2^1(T;V,V^*)$ for the smoothened modified Cauchy problem (P5) (see § 3). Set $w_\varepsilon^\delta:=z_\varepsilon^{\delta,1}-z_\varepsilon^{\delta,2}\in W_2^1(T;V,V^*)$. From Lemma 5.2 w_ε^δ satisfies

$$\frac{d}{dt}w_\varepsilon^\delta(t)+\frac{1}{2}(N_1+G_2)w_\varepsilon^\delta(t)=\frac{i}{2}\Phi(\varepsilon,\delta)w_\varepsilon^\delta(t) \tag{5.8}$$

In V^* with the initial condition $w_\varepsilon^\delta(0)=0$. Hence we get

$$\begin{aligned}&{}_{V^*}\langle\frac{d}{dt}w_\varepsilon^\delta(t),w_\varepsilon^\delta(t)\rangle_V+\frac{1}{2}\tilde{N}_1(w_\varepsilon^\delta(t),w_\varepsilon^\delta(t))\\&\qquad+\frac{1}{2}\tilde{G}_1(w_\varepsilon^\delta(t),w_\varepsilon^\delta(t))=\frac{1}{2}{}_{V^*}\langle i\Phi(\varepsilon,\delta)\tilde{w}_\varepsilon^\delta(t),w_\varepsilon^\delta(t)\rangle_V.\end{aligned}$$

Integrating it with respect to dt, and repeating the similar argument in the proof of Lemma 4. 1, we can obtain

$$\begin{aligned}&\|w_\varepsilon^\delta(t_e)\|^2+\int_T\mathrm{Re}\tilde{N}_1(w_\varepsilon^\delta(t),w_\varepsilon^\delta(t))dt\\&\qquad+\int_T\mathrm{Re}\tilde{G}_1(w_\varepsilon^\delta(t),w_\varepsilon^\delta(t))dt=-\mathrm{Im}\Phi(\varepsilon,\delta)\int_T\|w_\varepsilon^\delta(t)\|^2dt.\end{aligned}$$

The properties of operators N, Δ_G^* yield at once

$$|||w_\varepsilon^\delta|||_{2,T,V}^2=\int_T\|w_\varepsilon^\delta(t)\|_V^2dt\leq 0,$$

which implies that $w_\varepsilon^\delta(\cdot)=0$ in $L^2(T\,;\,V)$, i.e., $w_\varepsilon^\delta(t,\cdot)=0\in V$, dt-a.e. □

By virtue of uniqueness (stated in Proposition 5.5) the limit $z_\varepsilon^\delta(j)$ for each subsequence $\{z_\varepsilon^{m_j(k),\delta}\}_k$ must be all conincident for distinct j's, so that, it deduces that

$$z_\varepsilon^{m,\delta}\to z_\varepsilon^\delta \text{ weakly in } L^2(T,V), \tag{5.9}$$

as m tends to infinity, instead of (4.13) in § 4. Moreover we can claim a stronger assertion as to the convergence of $\{z_\varepsilon^{m,\delta}\}_m$ (see § 6).

§ 6. Strong Convergence

We shall show in this section the strong convergence of $\{z_\varepsilon^{m,\delta}\}_m$ in $L^2(T\,;\,V)$. In fact we have

Proposition 6.1. *The sequence $\{z_\varepsilon^{m,\delta}\}_m$ of approximate solutions converges strongly to z_ε^δ in $L^2(T\,;\,V)$.*

Remark 6.2. It is interesting to note that by the strong convergence in Proposition 6.1 we can deduce directly from Remark 5.3 and $P_m\mathrm{P}_{\varepsilon,\delta}=\mathrm{P}_{\varepsilon,\delta}P_m$ that

$$\frac{\partial z_\varepsilon^{m,\delta}}{\partial t}\to\frac{\partial z_\varepsilon^\delta}{\partial t} \text{ strongly in } L^2(T\,;\,V^*), \tag{6.1}$$

as m tends to infinity, if we take advantage of the equation in (P6).

Proof of Proposition 6.1. It is sufficient to verify that $\int_T\|z_\varepsilon^{m,\delta}(t)-z_\varepsilon^\delta(t)\|_V^2dt\to 0$ as $m\to\infty$, and equivalently we have only to show that

$$(\alpha+\beta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)-z_\varepsilon^\delta(t)\|_V^2dt+\|z_\varepsilon^{m,\delta}(t_e)-z_\varepsilon^\delta(t_e)\|_H^2$$
$$+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)-z_\varepsilon^\delta(t)\|^2dt$$

vanishes as m approaches to infinity. We have the lemma:

Lemma 6.3. *The following equality holds:*

$$\begin{aligned}&\|z_\varepsilon^\delta(t_e)\|^2-\|\hat f\|^2+\mathrm{Re}\int_0^{te}\tilde N_1(z_\varepsilon^\delta(t),z_\varepsilon^\delta(t))dt\\&+\mathrm{Re}\int_0^{te}\tilde G_1(z_\varepsilon^\delta(t),z_\varepsilon^\delta(t))dt+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}\|z_\varepsilon^\delta(t)\|^2dt=0.\end{aligned}\tag{6.2}$$

Note that we have a similar type equality (4.9) for $\{z_\varepsilon^{m,\delta}\}$ as we have seen in § 4. As a matter of fact, from Lemma 5.2

$${}_{V^*}\langle\frac{d}{dt}z_\varepsilon^\delta(t)+\frac{1}{2}(N_1+G_1)z_\varepsilon^\delta(t)-\frac{i}{2}\Phi(\varepsilon,\delta)z_\varepsilon^\delta(t),z_\varepsilon^{m,\delta}(t)\rangle_V=0$$

holds for each $t\in T$ because $z_\varepsilon^{m,\delta}(t)\in V$. From (5.9) a weak convergence of $\{z_\varepsilon^{m,\delta}\}$ in $L^2(T\,;\,V)$ allows

$$\begin{aligned}&\int_T\langle\frac{d}{dt}z_\varepsilon^\delta(t),z_\varepsilon^\delta(t)\rangle dt+\frac{1}{2}\int_T\langle(N_1+G_1)z_\varepsilon^\delta(t),z_\varepsilon^\delta(t)\rangle dt\\&=\int_T\frac{1}{2}\langle i\Phi(\varepsilon,\delta)z_\varepsilon^\delta(t),z_\varepsilon^\delta(t)\rangle dt.\end{aligned}$$

Therefore the assertion (6.2) follows directly from the same discussion in the proof of Lemma 4.1. To go back to the proof of Proposition 6.1, we may then apply the inequalities (4.10) and the equality (4.9) to get

$$
\begin{aligned}
&(\alpha+\beta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)-z_\varepsilon^\delta(t)\|_V^2dt+\|z_\varepsilon^{m,\delta}(t_e)-z_\varepsilon^\delta(t_e)\|_H^2+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)-z_\varepsilon^\delta(t)\|^2dt\\
&\le\int_0^{te}\mathrm{Re}\tilde{N}_1(z_\varepsilon^{m,\delta}-z_\varepsilon^\delta,z_\varepsilon^{m,\delta}-z_\varepsilon^\delta)dt+\|z_\varepsilon^{m,\delta}(t_e)-z_\varepsilon^\delta(t_e)\|_H^2\\
&\quad+\int_0^{te}\mathrm{Re}\tilde{G}_1(z_\varepsilon^{m,\delta}-z_\varepsilon^\delta,z_\varepsilon^{m,\delta}-z_\varepsilon^\delta)dt+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}\|z_\varepsilon^{m,\delta}(t)\|^2dt\\
&\quad-\mathrm{Im}\,\Phi(\varepsilon,\delta)\left\{\int_0^{te}(z_\varepsilon^\delta(t),z_\varepsilon^{m,\delta}(t)-z_\varepsilon^\delta(t))_Hdt+\int_0^{te}(z_\varepsilon^{m,\delta}(t),z_\varepsilon^\delta(t))_Hdt\right\}\\
&\quad=\left\{\int_0^{te}\mathrm{Re}\tilde{N}_1(z_\varepsilon^{m,\delta},z_\varepsilon^{m,\delta})dt+\int_0^{te}\mathrm{Re}\tilde{G}_1(z_\varepsilon^{m,\delta},z_\varepsilon^{m,\delta})dt\right.\\
&\qquad\left.+\|z_\varepsilon^{m,\delta}(t_e)\|_H^2+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}|z_\varepsilon^{m,\delta}(t)\|^2dt\right\}\\
&\quad-\left\{\int_0^{te}\mathrm{Re}\tilde{N}_1(z_\varepsilon^\delta,z_\varepsilon^{m,\delta}-z_\varepsilon^\delta)dt+\int_0^{te}\mathrm{Re}\tilde{G}_1(z_\varepsilon^\delta,z_\varepsilon^{m,\delta}-z_\varepsilon^\delta)dt\right.\\
&\qquad\left.+(z_\varepsilon^\delta(t_e),z_\varepsilon^{m,\delta}(t_e)-z_\varepsilon^\delta(t_e))_H+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}(z_\varepsilon^\delta,z_\varepsilon^{m,\delta}-z_\varepsilon^\delta)_Hdt\right\}\\
&\quad-\left\{\int_0^{te}\mathrm{Re}\tilde{N}_1(z_\varepsilon^{m,\delta},z_\varepsilon^\delta)dt+\int_0^{te}\mathrm{Re}\tilde{G}_1(z_\varepsilon^{m,\delta},z_\varepsilon^\delta)dt\right.\\
&\qquad\left.+(z_\varepsilon^{m,\delta}(t_e),z_\varepsilon^\delta(t_e))_H+\mathrm{Im}\,\Phi(\varepsilon,\delta)\int_0^{te}(z_\varepsilon^{m,\delta},z_\kappa^\delta)_Hdt\right\},
\end{aligned}
$$

and the last terms $\to 0$ as m tends to infinity, because the group of expressions in the first brace is equal to $\|\hat{f}^m\|^2$ and the term in the third brace together with $\|\hat{f}^m\|^2$ converges to Eq. (6.2), implying the convergence toward null, and also because each term in the second brace converges to zero where we just employed the weak convergence result (5.9). □

On this account a direct limit operation for (P6) gives the strong solution, i.e.,

Theorem 6.4. *The problem (P5) has a unique strong solution in* $W_2^1(T;V,V^*)$.

Furthermore we assert

Theorem 6.5. *The map* : $\hat{f}\to z_\varepsilon^\delta$ is continuous from $(L^2)\to W_2^1\ (T\ ;\ V,\ V^*)$.

Proof of Theorem 6.5. The continuity of norm and strong convergence (cf. Proposition 6.1) give

$$\big|\|z_\varepsilon^{m,\delta}\|_{2,T,V}-\|z_\varepsilon^\delta\|_{2,T,V}\big|<\|z_\varepsilon^{m,\delta}-z_\varepsilon^\delta\|_{2,T,V}\to 0\quad(\text{as } m\to\infty)$$

Consequently it follows from (4.2) of Lemma 4.1 that

$$\|z_\varepsilon^\delta\|_{2,T,V}=\lim_{m\to\infty}\|z_\varepsilon^{m,\delta}\|_{2,T,V}\le\sqrt{\frac{C_0}{\gamma(\alpha,\beta;\varepsilon,\delta)}}\|\hat{f}\|<\infty \tag{6.3}$$

for each $\varepsilon>0,\delta>0$ and for sufficiently large $\alpha,\beta>0$. Eq.(6.3) implies the assertion. □

§ 7 Existence and Uniqueness of Solutions for the Cauchy Problem

The purpose of this section is chiefly to give the proof of Theorem 2.1 (stated in § 2), one of the principal results of this paper. As is clear in the viewpoint of the properties of S_1-transform (3. 1) defined in § 3, the problem (P3) with $\hat{P}_\varepsilon$ is equivalent to the problem (P4) with P_ε, so that, as to the existence part of Theorem 2.1 it is sufficient to show the following, i.e.,

Theorem 7.1. *There exists a solution z_ε of (P4) (see in § 3) in* $W_2^1\ (T:\ V,V^*)$.

N.B. As a consequence, by virtue of Theorem 7.1 it proves to be true that (P3) has a solution $v_\varepsilon=S_1z_\varepsilon\in W_2^1\ (T:\ V,V^*)$.

Proof of Theorem 7.1 Recall Remark 4.2 in § 4 and the smoothing argument stated in the beginning of § 3. We have $\lim_{\delta\to 0}\Phi(\varepsilon,\delta) = s'\langle x,\eta_\varepsilon\rangle s$, and, for $\varepsilon>0$, for sufficiently large $\alpha,\beta>0$, $\lim_{\delta\to 0}\gamma(\alpha,\beta;\varepsilon,\delta)=\gamma'(\alpha,\beta;\varepsilon)$ where $\gamma'(\alpha,\beta;\varepsilon)$ is given by the number $\alpha+\beta-2|\langle x,\eta_\varepsilon\rangle|>0$. It is thus follows from (6.3) of Theorem 6.5 that

$$\lim_{\delta\to 0}\|\|z_\varepsilon^\delta\|\|_{2,T,V}\leq\sqrt{\frac{C_0}{\gamma'(\alpha,\beta;\varepsilon)}}\|\hat{f}\|<\infty. \tag{7.1}$$

It is interesting to note that

$$\lim_{\delta\to 0}\mathrm{P}_{\varepsilon,\delta}=\lim_{\delta\to 0}SM(\delta)\,\mathrm{P}_\varepsilon=\mathrm{P}_\varepsilon \tag{7.2}$$

holds in strong sense of operator, and more strictly, it goes in uniform sense as well.
On the one hand, (6.3) suggests that a bounded subset of $\{Z_\varepsilon^\delta\}_\delta$ is weakly compact relative to the topology of $L^2(T\,;\,V)$. Clearly we have

$$\lim_{k\to\infty}\mathrm{Re}\tilde{N}_1(z_\varepsilon^{\delta(k)}(t),\varphi)=\mathrm{Re}_{V^*}\langle N_1z_\varepsilon(t),\varphi\rangle,\text{ and}$$
$$\lim_{k\to\infty}\mathrm{Re}\tilde{G}_1(z_\varepsilon^{\delta(k)}(t),\varphi)=\mathrm{Re}_{V^*}\langle G_1z_\varepsilon(t),\varphi\rangle_V,$$

where z_ε(t) denotes a weak limit of $\{z_\varepsilon^\delta\}_\delta$ in terms of $L^2(T\,;\,V)$-topology and the element $N_1 z_\varepsilon$ (t) (resp. G_1z_ε (t)) is at least contained in $H_N^{-2}\cap(\mathrm{S})_{-p}$ (resp. $H_G^{-2}\cap(\mathrm{S})_{-p}$) for each $t\in T$. As a consequence we deduce together with (7.2) that z_ε(t) satisfies

$$\int_0^{te}\langle(N_1+G_1)z_\varepsilon(t),\varphi(t)\rangle dt$$
$$=\int_0^{te}\langle i(x,\eta_\varepsilon\rangle z_\varepsilon(t),\varphi(t)\rangle dt+2\int_0^{te}\langle z_\varepsilon(t),\varphi'(t)\rangle dt$$

for any $\varphi\in\mathrm{C}_0(\bar{T};V)$, which is a weak version of the problem (P4). One the other hand, from (5.6) in § 5, it is obvious that

$$\lim_{\delta\to 0}\frac{d}{dt}z_\varepsilon^\delta(t)=\frac{i}{2}\langle x,\eta_\varepsilon\rangle z_\varepsilon(t)-\frac{1}{2}(N_1+G_1)z_\varepsilon(t)\in V^*$$

for each $t\in T$, and further its $L^2(T)$-limit also exists and we know that the solution z_ε(t) of (P4) lies in $W_2^1(T;V,V^*)$. □

As a consequence, immediately we obtain

Proposition 7.2. *The solution z_ε has a continuous modification $\tilde{z}_\varepsilon$ such that $\tilde{z}_\varepsilon\in\mathrm{C}^0(T;H)$*, and it satisfies the following equation:

$$\begin{aligned}&\langle z_\varepsilon(t),\varphi\rangle-\langle\hat{f},\varphi\rangle+\frac{1}{2}\int_0^t\langle(N_1+G_1)z_\varepsilon(t),\varphi\rangle dt\\&=\frac{i}{2}\int_0^t\langle\langle x,\eta_\varepsilon\rangle z_\varepsilon(t),\varphi\rangle dt\end{aligned}\tag{7.3}$$

for each $t\in T$, any $\varphi\in(\mathrm{S})$.

N.B. The proof of the former part in Proposition 7.2 goes similarly as in that of Theorem 2. 2, see the argument given below.

As to the uniqueness of solutions of (P3), we consider S_2-transform and we can prove the uniqueness of solution for the transformed problem, which is equivalent to (P3). We need some truncation technique here. Take $M>0$ and let it be foxed. We set $B_M:=\{x\in\mathrm{S}'(\mathbf{R}):\|x\|_q\leq M,\ \forall q\}$, $K_0^M\equiv K_0^M(\varepsilon):=\sup_{x\in BM}|s'\langle x,\eta_\varepsilon\rangle s|(<\infty)$, and $\lambda(\varepsilon):=\lambda_N+2K_0^M(\varepsilon)$. Note that this number K_0^M is well-defined and remains finite as far as $\varepsilon>0$. $\lambda(\varepsilon)$ eventually diverges as ε(or M) tends to zero (or infinity). Then there can be found some functional $\phi^M(x)\in C^\infty_{Fréchet}(\mathrm{S}'(\mathbf{R}))$ such that $\phi_M(x)=1$, for $x\in B_M$, and $=0$, for $x\in B_{M+1}^C$, and we set for each $t\in T, w^M(t,x)=w(t,x)\phi^M(x)$. Then clearly w^M coincides with w itself for x within the ball B_M and it vanishes for x outside the ball B_{M+1}. The S_2-transform is defined by

$$g_\varepsilon(t,x):=S_2^{-1}v_\varepsilon(t,x) \tag{7.4}$$
$$=\exp\left\{-\frac{1}{2}(\lambda(\varepsilon)+\lambda_G)t\right\}\cdot v_\varepsilon(t,x),$$

for each t, x. Then the problen (P3) is changed into

$$\frac{\partial g_\varepsilon}{\partial t}(t,x)=-\frac{1}{2}(N_2^\varepsilon+G_1)g_\varepsilon(t,x)+\frac{i}{2}\langle x,\eta_\varepsilon\rangle g_\varepsilon(t,x), \tag{7.5}$$
$$g_\varepsilon(0)=\hat{f}\in(L^2),\quad t>0, x\in S'(\mathbf{R}),$$

where $N_2^\varepsilon=N+\lambda(\varepsilon)\cdot I$. Hence we have its truncated problem

$$\frac{\partial g_\varepsilon^M}{\partial t}(t,x)=-\frac{1}{2}(N_2^\varepsilon+G_1)g_\varepsilon^M(t,x)+\frac{i}{2}\langle x,\eta_\varepsilon\rangle g_\varepsilon^M(t,x), \tag{7.6}$$
$$g_\varepsilon^M(0)=\hat{f}(x)\in(L^2).$$

According to Theorem 7.1, suppose that there are two solutions $g_\varepsilon^{M,1}, g_\varepsilon^{M,2}$, and set $w_\varepsilon^M;=g_\varepsilon^{M,1}-g_\varepsilon^{M,2}$ $\in W_2^1(T;V,V^*)$, with $w_\varepsilon^M(0)=0$. This is obvious, because the problem (P3) is equivalent to the above (7.5) and the S_2-transform is $W_2^1(T;V,V^*)$-invariant, and $g_\varepsilon\in W_2^1(T;V,V^*)$ if so is vv_ε. Then it is easy to see that

$$_{V^*}\langle\frac{\partial}{\partial t}\tilde{w}_\varepsilon^M(t),w_\varepsilon^M(t))\rangle_V+\frac{1}{2}\tilde{N}_2^\varepsilon(w_\varepsilon^M(t),w_\varepsilon^M(t))+\frac{1}{2}\tilde{G}_1(w_\varepsilon^M(t),w_\varepsilon^M(t))$$
$$\frac{1}{2}(i\langle x,\eta_\varepsilon\rangle w_\varepsilon^M(t),w_\varepsilon^M(t))_H.$$

By integrating it with respect to dt we readily obtain

$$\frac{1}{2}\|\tilde{w}_\varepsilon^M(t_e)\|^2+\frac{1}{2}\int_0^{te}\mathrm{Re}\tilde{N}_2^\varepsilon(w_\varepsilon^M,w_\varepsilon^M)dt+\frac{1}{2}\int_0^{te}\mathrm{Re}\tilde{G}_1(w_\varepsilon^M,w_\varepsilon^M)dt \tag{7.7}$$
$$=\int_0^{te}\frac{1}{2}\mathrm{Re}(i\langle x,\eta_\varepsilon\rangle w_\varepsilon^M(t),w_\varepsilon^M(t))_H dt.$$

Because $|(i\langle x,\eta_\varepsilon\rangle w_\varepsilon^M(t),w_\varepsilon^M(t))_H|\leq \sup_{x\in BM}|\langle x,\eta_\varepsilon\rangle|\cdot\int w_\varepsilon^M(t,x)\bar{w}_\varepsilon^M(t,x)\mu(dx)$, we may combine definition of the regularized spaces in § 2 with Eq.(4.10) type estimates to get

$$0\geq\frac{1}{2}\|\tilde{w}_\varepsilon^M(t_e)\|^2-K_0^M(\varepsilon)\cdot\int_0^{te}\|w_\varepsilon^M(t)\|^2dt+\frac{1}{2}\int_0^{te}\mathrm{Re}\tilde{N}_1(w_\varepsilon^M,w_\varepsilon^M)dt$$
$$+\frac{1}{2}\int_0^{te}\mathrm{Re}\tilde{G}_1(w_\varepsilon^M,w_\varepsilon^M)dt+\int_0^{te}(K_0^M(\varepsilon)w_\varepsilon^M,w_\varepsilon^M)_H dt$$
$$\geq\frac{1}{2}\|\tilde{w}_\varepsilon^M(t_e)\|^2+\frac{\alpha}{2}\int_0^{te}\|w_\varepsilon^M(t)\|_{N,2}^2dt+\frac{\beta}{2}\int_0^{te}\|w_\varepsilon^M(t)\|_{G,2}^2dt.$$

By the passage to limit $M\to\infty$, we can easily get

$$\int_T\{a\|w_\varepsilon(t)\|_{N,2}^2+\beta\|w_\varepsilon(t)\|_{G,2}^2\}dt\leq0$$

for $\alpha,\beta>0$, implying that $w_\varepsilon(t)=0$ in $H_N^2\cap H_G^2\cap(S)_p$, dt-a.e., because

$$\lim_{M\to\infty}\mu(\{\|x\|_q\geq M,\forall q\}\cup\{\|w_\varepsilon^M(t)-w_\varepsilon(t)\|_V>\varepsilon'\})=0$$

holds for $\varepsilon'>0$ except the Lebesgue measure dt-null set. Therefore the problem (7.5) has a unique solution g_ε in $W_2^1(T;V,V^*)$. We thus attain the assertion of Theorem 2.1 stated in § 2 and know that the problem (P3) has a unique solution v_ε in $W_2^1(T;V,V^*)$ satisfying (2.1).

Consequently, Theorem 2.1 is a direct result of Theorem 2.1. Actually it is derived by a kind of routine work which is well known in the theory of evolution equations. We shall give a sketch of the proof below. First of all note that the Density theorem for $W_2^1(T;V,V^*)$ is true:

$D(T;V)$ is dense in $W_2^1(T;V,V^*)$. Then it follows easily from the above fact and a method of extension by reflection [18] that there exists an continuous linear operator $E: W_2^1(T;V,V^*) \to W_2^1(\mathbf{R};V,V^*)$ such that $E(u)=u$, dt-a.e. on T. Hence the localized situation can be attributed to the global case so that we may take much advantage of the standard theory of vector-valued distributions [20] . An application of the diagonalization argument [1] (see also [8]) together with the aforementioned Schwartz theory [20] leads to the following assertion:

Proposition 7.3. *If* $u \in W_2^1(T;V,V^*)$, them

$$u \in C^0(\bar{T};[V,V^*]_{1/2}),$$

where $[V,V^*]_{1/2}$ *is the intermediate space.*

In particular, since $[V,V^*]_{1/2}=H$ in our case, we can conclude that there exists $\tilde{v}_\varepsilon \in C^0(\bar{T};H)$ such that $\tilde{v}_\varepsilon(t)=v_\varepsilon(t)$ in H, dt-a.e. t if v_ε lies in $W_2^1(T;V,V^*)$. Moreover, as we have stated in § 2, we immediately get Theorem 2.3 by virtue of the bijectivity of the F-transform on $(S)^*$.

Remark 7.4. The assertion in Proposition 7.3 means that if $u \in W_2^1(T;V,V^*)$, then the function $u(t)$ is an intermediate space $[V,V^*]_{1/2}$ valued continuous mapping to $\bar{T}$ after a possible modification on a set of Lebesgue dt-measure zero.

Remark7.5. In fact, by the above-mentioned Extension theorem, u in Proposition 7.3 (as a continuous mapping on **R**) is the function with modification appearing in $C_b^0(\mathbf{R};[V,V^*]_{1/2})$. However, there may be an ambiguity in the set on which u is modified, i.e., the set on which the function is modified depends on $u \in W_2^1(T;V,V^*)$. As a matter of fact, we can restate the above assertion more precisely: A map: $D(\bar{T};V) \to D(\bar{T};V)$ extends by continuity to a map: $W_2^1(T;V,V^*) \to C^0(\bar{T};[V,V^*]_{1/2})$. This follows immediately from

$$\|u\|_{C_b^0(\mathbf{R};[V,V^*]_{1/2})} \leq C\|u\|_{W_2^1(\mathbf{R};V,V^*)},$$

because we have only to rewrite it into $u \in D(\mathbf{R};V)$ by making use of the extension map E [1, 20] .

Remark7.6. More generally, our assertion (Proposition 7.3) is a Corollary of the so-called Intermediate Derivative theorem. If u belongs to the space

$$\check{W}^m(T):=\left\{u \in L^2(T;V), \frac{d^m u}{dt^m}=u^{(m)} \in L^2(T;V^*)\right\},$$

then we first know that $u^{(j)} \in L^2(T;[V,V^*]_{j/m})(1 \leq j \leq m-1)$; furthermore, the map: $W^m(T) \to L^2(T;[V,V^*]_{j/m}])$ is a continuous linear operator, where $[V,V^*]_{j/m}$ is the intermediate space [18] . But if you look at it in a larger space $[V,V^*]_{(j+1/2)/m} (\supset [V,V^*]_{j/m})$, then it can be regarded as a continuous function with values in the larger space, and bedides, the map: $u \to u^{(j)}$ is continuous from $W^m(T) \to C^0(\bar{T};[V,V^*]_{(j+1/2)/m})$.

§ 8. Pseudo-Fourier-Mehler Transform

The study of the Fourier transform F in white noise calculus was initiated and has been developed to a mature level by H.-H. Kuo [14,15] (also [17]). While, the Fouriet-Mehler transform F_θ is a kind of generalization of F [16] (also [12]), which furnishes the theory of infinite dimensional Fourier transforms in white noise space with adequately fruitful and profitable ingradients. In this section we introduce the Pseudo-Fourier-Mehler (ΨFM for short) transform having quite similar nice properties as the Fourier-Mehler transform in white noise analysis possesses. In connection with the ΨFM transform, we shall define later the $S_3(\varepsilon)$-transformation and discuss the $S_3(\varepsilon)$-transformed problem. Consideration of the $S_3(\varepsilon)$-transformed problem gives us a fecund suggestion sbout a concept "lifting", which is the main theme in the last section.

We begin with introducing the ΨFM transform.

Definition 8.1. $\{\mathrm{F}^{\theta}_{\Psi}, \theta\in\mathbf{R}\}$ *is said to be the Pseudo-Fourier-Mehler* (ΨFM) *transform if* $\mathrm{F}^{\theta}_{\Psi}$ *is a mapping from* $(\mathrm{S})^*$ *into itself for* $\theta\in\mathbf{R}$,*whose U-functional is given by*

$$S(\mathrm{F}^{\theta}_{\Psi}\Phi)(\xi)=F(e^{i\theta}\xi)\cdot\exp[ie^{i\theta}\sin\theta|\xi|^2],\quad \xi\in\mathrm{S}(\mathbf{R}), \tag{8.1}$$

or equivalently

$$S(\mathrm{F}^{\theta}_{\Psi}\Phi)(\xi)=\langle\Phi,\exp[e^{i\theta}\langle\cdot,\xi\rangle-\frac{1}{2}|\xi|^2]\rangle,\quad \xi\in\mathrm{S}(\mathbf{R}), \tag{8.2}$$

for $\Phi\in(\mathrm{S})^*$, *where S is the S-transform in white noise analysis and F denotes the U-fiunctional of* Φ [19] .

Proposition 8.2. *The following properties hold :*

(i) $\mathrm{F}^{0}_{\Psi}=Id$; *(Id denotes the identity operator.)*

(ii) $\mathrm{F}^{\theta}_{\Psi}\neq\mathrm{F}$ *for any* $\theta\in\mathbf{R}\backslash\{0\}$;

(iii) $\mathrm{F}^{\theta}_{\Psi}\neq\mathrm{F}_{\theta}$ *for any* $\theta\in\mathbf{R}\backslash\{0\}$;

where F *is the Fourier transform and* F_{θ} *is the Fourier-Mehler transform* (cf. [16]).

Proof. As to (i), it is easy to see that $\mathrm{S}(\mathrm{F}^{0}_{\Psi}\Phi)(\xi)=S\Phi(\xi)=F(\xi)$. By virtue of the characterization theorem (e.g. [12, Theorem 4.38, p.123]), we obtain $\mathrm{F}^{0}_{\Psi}=Id$. (iii) is obvious from definitions (see Definition 8.1 and [16]). Since $\mathrm{F}_0=Id$ and $\mathrm{F}_{-\pi/2}=\mathrm{F}$ (e.g. [16]), it follows clearly from (iii) that $\mathrm{F}\neq\mathrm{F}^{\theta}_{\Psi}$ for any $\theta\in\mathbf{R}$ except $\theta=0$. □

Proposition 8.3. *The inverse operator of the* Ψ*FM transform is given by* $(\mathrm{F}^{\theta}_{\Psi})^{-1}=\mathrm{F}^{-\theta}_{\Psi}$ *for* $\theta\in\mathbf{R}$.

Proof. It is sufficient to show that $\mathrm{F}^{-\theta}_{\Psi}\mathrm{F}^{\theta}_{\Psi}=\mathrm{F}^{\theta}_{\Psi}\mathrm{F}^{-\theta}_{\Psi}=Id$. As a matter of fact, for $\Phi\in(\mathrm{S})^*$ we get from the definition (8.1)

$$\begin{aligned} S(\mathrm{F}^{-\theta}_{\Psi}(\mathrm{F}^{\theta}_{\Psi}\Phi))(\xi)&=S(\mathrm{F}^{\theta}_{\Psi}\Phi)(e^{-i\theta}\xi)\cdot\exp(-ie^{-i\theta}\sin\theta|\xi|^2)\\ &=(S\Phi)(e^{i\theta}(e^{-i\theta}\xi))\cdot\exp(ie^{i\theta}\sin\theta|e^{-i\theta}\xi|^2)\cdot\exp(-ie^{-i\theta}\sin\theta|\xi|^2)\\ &=(S\Phi)(\xi)\cdot\exp(0)=S(Id\cdot\Phi)(\xi),\quad (\xi\in\mathrm{S}(\mathbf{R}))\end{aligned} \tag{8.3}$$

because we used the relation $S(\mathrm{F}^{-\theta}_{\Psi}\Phi)(\xi)=S\Phi(e^{-i\theta}\xi)\cdot\exp(-ie^{-i\theta}\sin\theta|\xi|^2)$ so as to obtain the second line of Eq.(8.3). An application of the chatacterization theorem to Eq.(8.3) gives $\mathrm{F}^{-\theta}_{\Psi}\mathrm{F}^{\theta}_{\Psi}=Id$. As for the other part of the desired equalities, it goes almost similarly. □

Next let us consider what the image of the space (S) under $\mathrm{F}^{\theta}_{\Psi}$ is like (see Corollary 8.6 below). The ΨFM transform $\mathrm{F}^{\theta}_{\Psi}$ also enjoys some interesting properties on the product of Gaussian white noise functionals (see Theorem 8.4 and Theorem 8.5).

Theorem 8.4. *Let* g_c *be a Gaussian white noise functional, i.e.,* $g_c(\cdot):=\mathrm{N}\exp\left(-\frac{1}{2c}|\cdot|^2\right)$ *with renormalization* N. *For* $\theta\in\mathbf{R}$ *the following equalities hold :*

(i) $\mathrm{F}^{\theta}_{\Psi}\Phi:g_{c(\theta)}=\Gamma(e^{i\theta}Id)\Phi,\quad \forall\Phi\in(\mathrm{S})^*$;

(ii) *for any* $p\in\mathbf{R}$, $\quad\|\mathrm{F}^{\theta}_{\Psi}\Phi:g_{c(\theta)}\|_p=\|\Phi\|_p,\quad \forall\Phi\in(\mathrm{S})_p$;

where : *denotes the Wick product (e.g.* [12,p.101]) *and the parameter* $c(\theta)$ *is given by* $c(\theta)=-(2^{-1}ie^{-i\theta}\mathrm{cosec}\,\theta+1)$.

Proof. Noting that the U-functional of g_c is given by $\exp(-2^{-1}(1+c)^{-1}|\xi|^2)$, we readily obtain

(8.4)
$$S(\mathrm{F}^{\theta}_{\Psi}\Phi{:}g_{c(\theta)})(\xi)=S(\mathrm{F}^{\theta}_{\Psi}\Phi)(\xi)\cdot(Sg_{c(\theta)})(\xi)$$
$$=S\Phi(e^{i\theta}\xi)\cdot\Xi(\theta,\xi),\quad \xi\in S(\mathbf{R}),$$

bevause we employed Eq. (8.1) and put

$$\Xi(\theta,\xi):=\exp\Big(ie^{i\theta}\sin\theta|\xi|^2-\frac{1}{2(1+c(\theta))}|\xi|^2\Big).$$

Then we cannot find any $\theta\in\mathbf{R}$ such that $(8.4)=S\Phi(\xi)=e^{-\frac{1}{2}|\xi|^2}\langle\Phi,e^{\langle\cdot,\xi\rangle}\rangle$ may hold, which implies that $\mathrm{F}^{\theta}_{\Psi}\Phi{:}g_{c(\theta)}\neq\Phi$ for any $\Phi\in(S)^*$. However, when $\Phi(x)=\sum_{n=0}^{\infty}\langle{:}x^{\otimes n}{:},f_n\rangle$, $f_n\in\hat{S}_{-p}(\mathbf{R}^n)$ (the symmetric $S_{-p}(\mathbf{R}^n)$ space), then its U-functional $S\Phi(\xi)$ is given by $\sum_{n=0}^{\infty}\langle\xi^{\otimes n},f_n\rangle$, so that, we easily get from definition of the second quantization operator Γ(e.g. [17])

$$\text{r.h.s. of }(8.4)=\sum_{n=0}^{\infty}\langle(e^{i\theta})^n\xi^{\otimes n},f_n\rangle\cdot\Xi(\theta,\xi)=S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\Xi(\theta,\xi).$$

Hence, if $2i(1+c(\theta))e^{i\theta}\sin\theta=1$ holds, then clearly $\Xi(\theta,\xi)$ proves to be 1, suggesting with the characterization theorem that $\mathrm{F}^{\theta}_{\Psi}\Phi{:}g_{c(\theta)}=\Gamma(e^{i\theta}Id)\Phi$. Moreover, it is easy to see that $\|\mathrm{F}^{\theta}_{\Psi}\Phi{:}g_{c(\theta)}\|_p=\|\Gamma(e^{i\theta}Id)\Phi\|_p=\|\Phi\|_p$ holds for any $p\in\mathbf{R}$. □

If we take the assertion obtained in Theorem 8.4 into account, then the following questions will arise naturally: whether the ΨFM-transformed Φ(i.e. $\mathrm{F}^{\theta}_{\Psi}\Phi$) can be represented by the Wick product of something like a transformed Φ and a Gaussian white noise functional g_c ; furthermore, if so, what is the parameter $c=c(\theta)$ then? First of all, on the supposition that $\Phi(x)=\sum_{n=0}^{\infty}\langle{:}x^{\otimes n}{:},f_n\rangle\in(S)^*$, a simple computation gives, for $\xi\in S(\mathbf{R})$

(8.5)
$$S(\mathrm{F}^{\theta}_{\Psi}\Phi)(\xi)=S\Phi(e^{i\theta}\xi)\cdot\exp(ie^{i\theta}\sin\theta|\xi|^2)$$
$$=S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\exp(ie^{i\theta}\sin\theta|\xi|^2).$$

We know from Eq. (8.5) that there is no possibility that $\mathrm{F}^{\theta}_{\Psi}\Phi$ may coincide with $\Phi : g_{K(\theta)}$ *even for any* $K(\theta),\theta\in\mathbf{R}$, because

(8.6)
$$S(\Phi{:}g_{K(\theta)})(\xi)=S\Phi(\xi)\cdot(Sg_{K(\theta)})(\xi)=S\Phi(\xi)\cdot\Lambda(K(\theta),\xi)$$

with $\Lambda(r,\xi):=\exp\{-2^{-1}(1+r)^{-1}|\xi|^2\}$. On the other hand, since the S-transform of $\Gamma(e^{i\theta})\Phi : g_{K(\theta)}$ is given by $S(\Gamma(e^{i\theta}Id)\ \Phi\ )(\xi)\cdot\Lambda(K(\theta),\xi)$, it is true from (8.5) that $\mathrm{F}^{\theta}_{\Psi}\Phi=\Gamma(e^{i\theta}Id)\Phi{:}g_{K(\theta)}$ may possibly hold for $\Phi\in(S)^*$, $\theta\in\mathbf{R}$ as far as $2i(1+K(\theta))e^{i\theta}\sin\theta+1=0$ is satisfied. Let us next consider the evaluation of the term $\mathrm{F}^{\theta}_{\Psi}\Phi(\Phi\in(S)_p)$ relative to the $(S)_p$-norm ($p\in\mathbf{R}$). We need to determine the parameter $A(\theta)$, which comes from the relation between $\Gamma(e^{i\theta})\Phi : g_{K(\theta)}$ and $\Gamma(e^{i\theta})(\Phi : g_{A(\theta)})$. By a similar calculation in (8.5) we readily obtain

(8.7)
$$S(\Gamma(e^{i\theta}Id)(\Phi{:}g_{A(\theta)}))(\xi)=S(\Phi{:}g_{A(\theta)})(e^{i\theta}\xi)$$
$$=(S\Phi)(e^{i\theta}\xi)\cdot\Lambda(A(\theta),e^{i\theta}\xi)=(S\theta)(e^{i\theta}\xi)\cdot\exp\Big\{-\frac{e^{2i\theta}}{2(1+A(\theta))}|\xi|^2\Big\},$$

by making use of Eq.(8.6). A comparison of (8.7) with $S(\Gamma(e^{i\theta})\Phi)(\xi)\cdot\Lambda(K(\theta),\xi)$ provides with

$$\Gamma(e^{i\theta}Id)\Phi{:}g_{K(\theta)}=\Gamma(e^{i\theta}Id)(\Phi{:}g_{A(\theta)})$$

if the equality $e^{2i\theta}(1+K(\theta))=A(\theta)+1$ holds, i.e., as far as $A(\theta)=2^{-1}ie^{-3i\theta}\cdot\mathrm{cosec}\,\theta-1$. It therefore follows that

$$\| F^{\theta}_{\Psi}\Phi \|_p = \| \Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)} \|_p$$
$$= \| \Gamma(e^{i\theta}Id)(\Phi : g_{A(\theta)}) \|_p = \| \Phi : g_{A(\theta)} \|_p$$

for all $\Phi \in (S)_p, p \in \mathbf{R}$, and any $\theta \in \mathbf{R}$. Summing up, we thus obtain

Theorem 8.5. *The following equalities hold for any* $\theta \in \mathbf{R}$;

(i) *if* $K(\theta) = 2^{-1} i e^{-i\theta} \mathrm{cosec}\,\theta - 1$, *then*

$$F^{\theta}_{\Psi}\Phi = \Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)}, \quad \Phi \in (S)^*;$$

(ii) *if* $A(\theta) = 2^{-1} i e^{-3i\theta} \mathrm{cosec}\,\theta - 1$, *then*

$$\| F^{\theta}_{\Psi}\Phi \|_p = \| \Phi : g_{A(\theta)} \|_p, \quad \Phi \in (S)_p$$

for all $p \in \mathbf{R}$.

Let us think of the image of $\varphi \in (S)$ under the Pseudo-Fourier-Mehler transform. It is easily checked that $g_c : g_d = 1$ holds with $c + d = -2$. So we have

$$g_{c(\theta)} : g_{K(\theta)} = 1. \tag{8.8}$$

From (ii) of Theorem 8.4, immediately, $\varphi \in (S)$ if and only if $F^{\theta}_{\Psi}\varphi : g_{c(\theta)} \in (S)$, so that, it is equivalent to

$$F^{\theta}_{\Psi}\varphi : g_{c(\theta)} : g_{K(\theta)} \in (S) : g_{K(\theta)},$$

Where $(S) : g_{K(\theta)}$ denotes the whole space of elements $\varphi : g_{K(\theta)}$ for $\varphi \in (S)$. Consequently, it is obvious that $F^{\theta}_{\Psi}\varphi \in (S) : g_{K(\theta)}$, by virtue of Eq.(8.8.). Therefore we obtain

Corollary 8.6. *For* $\theta \in \mathbf{R}$,

$$Im\, F^{\theta}_{\Psi}(S) = (S) : g_{K(\theta)} := \{\varphi : g_{K(\theta)} ; \varphi \in (S)\},$$

where $K(\theta) = 2^{-1} i e^{-i\theta} \mathrm{cosec}\,\theta - 1$.

Remark 8.7. The results in Theorem 8.4 and Theorem 8.5 are quite similar to those of the Fourier-Mehler transform. In fact, for $p \in \mathbf{R}$, $\Phi \in (S)_p$, $\|(F_\theta \Phi) : g_{c_1(\theta)}\|_p = \|\Phi\|_p$ and $\|F_\theta \Phi\|_p = \|\Phi : g_{c_2(\theta)}\|_p$ hold with $c_1(\theta) = -i\cot\theta - 2$, and $c_2(\theta) = i\cot\theta - 2$ (e.g. [12, § 9.H]).

Remark 8.8. The image of (S) under the Fourier-Mehler transform F_θ is given by $(S) : g_{i\cot\theta}$, while that of (S) under the Fourier transform F coincides with the space $(S) : \tilde{\delta}_0 := \{\varphi : \tilde{\delta}_0 ; \varphi \in (S)\}$, where $\tilde{\delta}_0$ is the delta function at 0 and $\lim_{c \to 0} g_c = \tilde{\delta}_0$ (e.g. [12, Chapter 9]).

From Proposition 8.3 our Pseudo-Fourier-Mehler transform F^{θ}_{Ψ} is injective and surjective. Moreover, it is easy to check that F^{θ}_{Ψ} is strongly continuous from $(S)^*$ into itself. Thus we have the following theorem.

Theorem 8.9. *The* ΨFM *transform* $F^{\theta}_{\Psi} : (S)^* \to (S)^*$ *is a bijective and strongly continuous linear operator.*

Proposition 8.10. $\{F^{\theta}_{\Psi} ; \theta \in \mathbf{R}\}$ *is a semigroup.*

N.B. In other words, with Theorem 8.9, the set $\{F^{\theta}_{\Psi} ; \theta \in \mathbf{R}\}$ forms a one parameter group of strongly continuous linear operators acting in $(S)^*$.

Proof of Proposition 8.10. For $\Phi \in (S)^*$, $\xi \in S(\mathbf{R})$, any $\theta, \eta \in \mathbf{R}$, from (8.2) we have

$$S(F^{\theta+\eta}_{\Psi}\Phi)(\xi) = \langle \Phi, \exp\left\{e^{i(\theta+\eta)}\langle \cdot, \xi \rangle - \frac{1}{2}|\xi|^2\right\}\rangle. \tag{8.9}$$

While, from (8.1)

(8.10)

$$S(F^{\theta}_{\Psi}(F^{\eta}_{\Psi}))(\xi) = S(F^{\eta}_{\Psi}\Phi)(e^{i\theta}\xi) \cdot \exp[ie^{i\theta}\sin\theta|\xi|^2]$$
$$= F(e^{i\eta}(e^{i\theta}\xi)) \cdot \exp[ie^{i\eta}\sin\eta|e^{i\theta}\xi|^2] \cdot \exp[ie^{i\theta}\sin\theta|\xi|^2]$$

$$=e^{-\frac{1}{2}|e^{i(\theta+\eta)}\xi|^2}\langle \Phi, e^{\langle \cdot, e^{i(\theta+\eta)}\xi\rangle}\rangle \cdot \exp\{ie^{i\theta}(e^{i(\theta+\eta)}\sin\eta+\sin\theta)|\xi|^2\}$$
$$=\langle \Phi, \exp\left\{e^{i(\theta+\eta)}\langle\cdot,\xi\rangle-\frac{1}{2}|\xi|^2\right\}\rangle,$$

with the U-functional F of Φ. By Comparing (8.9) with (8.10), we get

$$S(\mathrm{F}_{\Psi}^{\theta+\eta}\Phi)(\xi)=S(\mathrm{F}_{\Psi}^{\theta}\mathrm{F}_{\Psi}^{\eta}\Phi)(\xi).$$

Consequently, the characterization theorem leads to $\mathrm{F}_{\Psi}^{\theta+\eta}\Phi=\mathrm{F}_{\Psi}^{\theta}\cdot\mathrm{F}_{\Psi}^{\eta}\Phi$, $\Phi\in(\mathrm{S})^*$, which completes the proof. □

We are now in a position to state the principal result in this section. This is a very important property of the ΨFM transform, especially on an applicational basis.

Theorem 8.11 *The infinitesimal generator of* $\{\mathrm{F}_{\Psi}^{\theta};\theta\in\mathbf{R}\}$ *is given by* $i(N+\Delta_G{}^*)$, *where* N *is the number opreator and* $\Delta_G{}^*$ *is the adjoint of the Gross Laplacian* Δ_G.

Remark 8.12 It is well known that the infinitesimal generator of the Fourier-Mehler transforms $\{\mathrm{F}_{\theta};\theta\in\mathbf{R}\}$ is $iN+\frac{i}{2}\Delta_G{}^*$ while the adjoint operator of $\{\mathrm{F}_{\theta};\theta\in\mathbf{R}\}$ has $iN+\frac{i}{2}\Delta_G$ as its infinitesimal generator (e.g. see [12]). The proof of Theorem 8.11 is almost similar to the above ones.

Proof of Theorem 8.11. First of all we set $F_\theta(\xi):=S(\mathrm{F}_{\Psi}^{\theta}\Phi)(\xi)$ and $\mathrm{F}_0(\xi):=S(\Phi)(\xi)$ for $\Phi\in(\mathrm{S})^*$, $\xi\in\mathrm{S}(\mathbf{R})$, paying attention to (i) of Proposition 8.2. From (8.1) we have $F_\theta(\xi)=F_0(e^{i\theta}\xi)\cdot\exp[ie^{i\theta}\sin\theta|\xi|^2]$. Since $F_0$is Frēchet differentiable, the functional $F_\theta(\xi)$ is differentiable in θ as well, and it is easy to check that

(8.11)

$$\lim_{\theta\to0}\frac{1}{\theta}\{F_\theta(\xi)-F_0(\xi)\}$$
$$=\langle F_0'(e^{i\theta}\xi), ie^{i\theta}\xi\rangle\cdot\exp[ie^{i\theta}\sin\theta|\xi|^2]|_{\theta=0}+F_0(e^{i\theta}\xi)\cdot\frac{d}{dt}\exp[ie^{i\theta}\sin\theta|\xi|^2]|_{\theta=0}$$
$$=i\langle F'(\xi),\xi\rangle+i|\xi|^2\cdot F(\xi).$$

While, we can easily check that the U-functional $\theta^{-1}\{F_\theta(\xi)-F_0(\xi)\},\theta\in\mathbf{R}$ satisfies the uniform bounded criterion : $\exists\, C_0>0$ so that

$$\sup_{\substack{z\in C\\ |z|=R}}|\theta^{-1}\{\tilde{F}_\theta(z\xi)-\tilde{F}_0(z\xi)\}|\le C_0\exp\{c_1R^{c_2}|\xi|_p^2\},$$

holds for all $R>0$, all $\xi\in\mathrm{S}(\mathbf{R})$ with $c_1>0$, $c_2>0$, where $\tilde{F}_*$ denotes an entire analytic extension of F. Hence, the strong convergence criterion theorem [19] (see also [12, Chapter 4]) allows convergence of $S^{-1}(\theta^{-1}\{F_\theta(\cdot)-F_0(\cdot)\})(x)=\theta^{-1}\{\mathrm{F}_{\Psi}^{\theta}\Phi(x)-\Phi(x)\}$ in $(\mathrm{S})^*$as θ tends to zero. We need the following two lemmas.

Lemma 8.13 (cf. [12, *Theorem* 6.11, *p*.196]) *Let* $F(\xi)=S\Phi(\xi)$, $\xi\in\mathrm{S}(\mathbf{R})$ *for* $\Phi\in(\mathrm{S})^*$. Then

(i) *F is Fréchet differentiable* ;

(ii) *the S-transform of* $N\,\Phi(x)$ *is given by* $\langle F'(\xi),\xi\rangle$, $\xi\in\mathrm{S}(\mathbf{R})$;

where N *is the number operator.*

Lemma 8.14. (cf. [12, *Theorem*6.20, p.206]) *For any* Φ *in* $(\mathrm{S})^*$, *the S-transform of* $\Delta_G{}^*\Phi(x)$ *is given by* $|\xi|^2S\Phi(\xi)$, $\xi\in\mathrm{S}(\mathbf{R})$.

We may deduce at once that

(8.12) $$S(N\Phi+\Delta_G{}^*\Phi)(\xi)=\langle F'(\xi),\xi\rangle+|\xi|^2F(\xi),\quad \xi\in\mathrm{S}(\mathbf{R}),$$

with simple applications of Lemma 8.13 and Lemma 8.14 ; moreover, it is easily verified from (8.11) anf (8.12) together with the above-mentioned convergence result that

$$\lim_{\theta\to 0}\frac{1}{\theta}(\mathrm{F}^{\theta}_{\Psi}-Id)\Phi(x)=\lim_{\theta\to 0}S^{-1}\Big(\frac{1}{\theta}\{F_{\theta}(\cdot)-F_{0}(\cdot)\}\Big)(x)$$
$$=S^{-1}(i\langle F'(\xi),\ \xi\rangle+i|\xi|^{2}\cdot F(\xi))=i(N+\Delta_{G}^{*})\Phi(x),\quad \text{in}\quad (\mathrm{S})^{*},$$
which concludes the assertion. □

Finally we define the $S_3(\varepsilon)$-transformation. When we define $g(t,x):=S_3(\varepsilon)^{-1}\cdot v_\varepsilon(t,x)=\exp(-i\langle x,\eta_\varepsilon\rangle t)\cdot v_\varepsilon(t,x)$, then the problem (P.3) is rewritten into

(8.13)
$$\frac{\partial g}{\partial t}(t,x)=-\frac{1}{2}(N+\Delta_G^{*})g(t,x),$$
$$g(0)=\hat{f}\in(L^2).$$

We have already proved Theorem 2.1-Theorem 2.3 in § 7. According to Theorem 2.1 (or Theorem 7.1) and Theorem 2.2 (see also § 7), the solution $v_\varepsilon(t,x)$ of (P.3) lies in $W_2^1(T;V,V^*)$, hence it follows immediately from definition of g that there exists a unique solution $g(\cdot)\in W_2^1(T;V,V^*)$ satisfying Eq.(8.13) and $\exists\tilde{g}\in C^0(\bar{T};H)$ such that $\tilde{g}(t,x)=g(t,x)$ in H, dt-a.e., since $\exp(-i\langle\cdot,\eta_\varepsilon\rangle t)\in(\mathrm{S})_x$ and the space V is a (S)-module. Then it is easy to see that such a solution g satisfies formally $\frac{\partial g}{\partial t}(t,x)=i(N+\Delta_G^{*})g+h(t,x)$ with $g(0)=\hat{f}\in(L^2)$, where $h(t,x)=K_0(N+G)\tilde{g}\in C^0(\bar{T};H)$, $K_0:=-(i+\frac{1}{2})\in\mathbf{C}$. So that, we may apply Theorem 8.11 to obtain

$$g(t,x)=\mathrm{F}^{t}_{\Psi}\hat{f}(x)+\int_0^t\mathrm{F}^{t-s}_{\Psi}h(s,x)ds,$$

and if we set $\Phi_\varepsilon(t,x):=\exp(i\langle x,\eta_\varepsilon\rangle t)$, then clearly $v_\varepsilon(t,x)$ can be expressed by

(8.14)
$$\Phi_\varepsilon(t,x)\mathrm{F}^{t}_{\Psi}\hat{f}(x)+\Phi_\varepsilon(t,x)\int_0^t\mathrm{F}^{t-s}_{\Psi}h(s,x)ds.$$

§ 9. The Limit of ε-Approximation and Lifted Convergence

In this section we shall explain the concept of *lifting* as well as how and from what it comes out and give the definition of its lifted convergence. Based upon it, we shall discuss in a sense the limit of ε-approximation which was introduced in § 1, and state our main result in this paper on the convergence problem for the approximate problem (P.3). We start with introducing some notations to be used in this section. We denote the renormalization of Φ relative to x by the symbol $N_x\Phi(x)$ instead of $:\Phi(x):_{(x)}$. Recall that we fix $p>\frac{1}{4}$ (see § 2), then we have $Tr:=\int\delta_t^{\otimes 2}dt\in\hat{\mathcal{S}}_{-p}(\mathbf{R}^2)$ and $|Tr|_{-p,\otimes 2}=\|A^{-2p}\|_{HS}$, moreover $:x^{\otimes n}:\in\hat{\mathcal{S}}_{-p}(\mathbf{R}^n)$ holds for $n\in\mathbf{N}$ with

(9.1)
$$|:x^{\otimes n}:|_{-p}\leq\sqrt{n!}(|x|_{-p}+\|A^{-2p}\|_{HS}^{\frac{1}{2}})^n$$

(e.g. [12]), where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. So we put $A_n(p)=|Tr|^n_{-p,\otimes 2}\cdot|\xi|_p^{2n}$, $\xi\in\mathrm{S}(\mathbf{R})$. For simplicity, we set

$r_\varepsilon(n)=:\langle\eta_\varepsilon,\xi\rangle^n:$ for $\varepsilon>0$, $r_0(n)=:\langle y,\xi\rangle^n:$,

$R_\varepsilon(n)=|r_\varepsilon(n)-r_0(n)|$, and $Q_n^\varepsilon(p)\sum_{k=0}^{G(n)}A_k(p)R_\varepsilon(n-2k)$,

where $G(n)$ denotes the Gauss symbol for the number $n/2$. We write

$E_n^\varepsilon(p)=\{Q_{n+1}^\varepsilon(p)-Q_n^\varepsilon(p)\}/Q_n^\varepsilon(p)$ and $E_n(p)=\sup_{0<\varepsilon<1}E_n^\varepsilon(p)$.

Let $I(n,k)=n!\{2^k k!(n-2k)!\}^{-1}$ and $I_0(n)=\max\{I(n,k);0\leq k\leq G(n)\}$, and besides we put

$M(n)=I_0(n)^{-1}$. Assume that $\exists\, t_p>0$ and $\exists\{K_n(p)\}_n$, a strictly decreasing sequence ($p\in\mathbf{N}$) such that $K_n(p)>0$ and

$$t(1+E_n(p))\cdot K_{n+1}(p)<(n+1)K_n(p) \tag{9.2}$$

holds for any $t\in(0,t_p]$. For each $p\in\mathbf{N}$, we set $T_p(t):=\tan(2\pi t/t_p)$, $t>0$. H_p acts for $A=\sum_n^\infty a_n$ as $H_pA=\sum_n^\infty M(n)\cdot K_n(p)a_n$ for $p\in\mathbf{N}$ (cf. Remark 9.8 below).

Roughly speaking, a lifting is something like a new functional obtained by a sort of cancellation method of singularlity. Suggested by Eq.(8.14), we write $N_\varepsilon(t,x)=(\mathrm{N}_x\Phi_\varepsilon/\Phi_\varepsilon)(t,x)$, $\varepsilon>0$ and $N_0(t,x)=(\mathrm{N}_x\Phi_0/\Phi_0)(t,x)$, where formally $\Phi_0(t,x)=\exp(i\langle x,y\rangle t)$ for $x,y\in S'(\mathbf{R})$. Note that only the renormalized Φ_0 has a meaning. Actually we have $\mathrm{N}_x\Phi_\varepsilon(t,\cdot)\in(\mathrm{S})$ and $\mathrm{N}_x\Phi_0(t,\cdot)\in(\mathrm{S})^*$ for fixed y,for each $t>0$. Let v_ε be the solution of the problem (P.3) (cf. Therem 2.1 and the latter part of § 8). When we write an multiplicative operation of the singular weighted factors $\{N_\varepsilon(t,x)\}_{\varepsilon>0}$ by L^ε, then $\mathrm{L}^\varepsilon v_\varepsilon$, $\varepsilon>0$, is in fact given by $\mathrm{N}_x\Phi_\varepsilon\cdot g$, while L^0v_0 is expressed as $\mathrm{N}_x\Phi_0\cdot g$, where we may regard that v_0 is given a priori by the intuitively formal expression:

$$\Phi_0(t,x)\,\mathrm{F}_\Psi^t f(x)+\Phi_0(t,x)\int_0^t \mathrm{F}_\Psi^{t-s}h(s,x)ds.$$

Let us consider now the above each term. $\mathrm{L}^\varepsilon v_\varepsilon(t,\cdot)$ lies in V at least for each t since $\Phi_\varepsilon(t,\cdot)$ itself belongs to (S) for any t. So canonically, $\mathrm{L}^\varepsilon v_\varepsilon(t,\cdot)$ is always well-defined as an element of $(\mathrm{S})^*$ for any $\varepsilon>0$. On the other hand, there is a problem indeed in defining L^0v^0. Although $(\mathrm{S})^*$ is a (S)-module, we do not know generally whether $V\cdot(\mathrm{S})^*\subset(\mathrm{S})^*$ or not. However, if we think of $\mathrm{N}_x\Phi_0(t,x)$: $g(t,x)$ instead of the aboce L^0v_0, then the validity easily follows. As a matter of fact, we have

(9.3)

$$\begin{aligned}&S(\mathrm{N}_x\Phi_0(t,\cdot) : g(t,\cdot))(\xi)\\&=S(\mathrm{N}_x\Phi_0(t,\cdot))(\xi)\Big\{S(\mathrm{F}_\Psi^t f)(\xi)+S\Big(\int_0^t \mathrm{F}_\Psi^{t-s}h(s,\cdot)ds\Big)(\xi)\Big\},\ \xi\in\mathrm{S}(\mathbf{R}),\end{aligned}$$

because we employed the expression of g given in the last part of § 8.

Lemma 9.1. *We can find some positive constant C_1 such that for $\xi\in\mathrm{S}(\mathbf{R})$*

$$\sup_{\substack{z\in\mathbf{C}\\|z|R}}\Big|\sum_{n=0}^\infty\frac{1}{n!}\langle x^{\otimes n},(z\xi)^{\otimes n}\rangle\Big|\le C_1\cdot\exp(aR|\xi|_p),\ (\text{for fixed } x\in\mathrm{S}'(\mathbf{R}))$$

holds for any $R>0$, and for some $a>0$, $p\in\mathbf{N}_0$.

Proof. $x^{\otimes n}$ has the decomposition [12, p334] :

$$x^{\otimes n}=\sum_{k=0}^{G(n)}\binom{n}{2k}(2k-1)!!{:}x^{\otimes(n-2k)}{:}\hat{\otimes}\,Tr^{\otimes k}. \tag{9.4}$$

The assertion follows immediately from the following estimate.

$$\begin{aligned}&\sum_{n=0}^\infty\frac{1}{n!}|\langle x^{\otimes n},(z\xi)^{\otimes n}\rangle|\\&\le\sum_{n=0}^\infty\frac{|z|^n}{n!}\sum_{k=0}^{G(n)}\binom{n}{2k}(2k-1)!!|\langle Tr,\xi\otimes\xi\rangle|^k\cdot|\langle{:}x^{\otimes(n-2k)}{:},\xi^{\otimes(n-2k)}\rangle|\\&\le\sum_{n=0}^\infty\frac{|z|^n}{n!}\Big\{\sum_{k=0}^{G(n)}\binom{n}{2k}(2k-1)!!|Tr|_{-p,\otimes2}^k|{:}x^{\otimes(n-2k)}{:}|_{\hat{S}_{-p}(\mathbf{R}^{n-2k})}\Big\}|\xi|_p^n,\end{aligned}$$

where we used (9.1) and (9.4). □

Lemma 9.2. *Let φ be in (L^2). Then there can be found some positive constant C_2 such that*

$$\sup_{\substack{z\in R\\ |z|=R}} |\tilde{S}\varphi(z\xi)| \le C_2 \cdot \exp(aR^2|\xi|_p^2)$$

holds for any $R>0$, and for some $a>0$, $p\in\mathbf{N}_0$, where $\sim$ indicates an entire analytic extension of $S\varphi(\xi)$, $\xi\in S(\mathbf{R})$.

Proof. Because the Wiener-Itô decomposition in $L^2(S'(\mathbf{R}), d\mu)$ allows the expression $\varphi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:, f_n\rangle$, $f_n\in\hat{L}^2(\mathbf{R}^n)$ (the symmetric L^2-space), we have the following estimate of $S\varphi(\lambda\xi)$ as a function of $\lambda\in\mathbf{R}$,

$$|(S\varphi)(\lambda\xi)| \le \sum_{n=0}^{\infty}|\langle f_n, (\lambda\xi)^{\otimes n}\rangle|$$

$$\le \left(\sum_{n=0}^{\infty}\frac{1}{n!}|\lambda|^{2n}|\xi|_p^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}n!|f_n|_{-p}^2\right)^{\frac{1}{2}} = \|\varphi\|_{-p}\exp\left(\frac{1}{2}|\lambda|^2|\xi|_p^2\right),$$

by employing the Schwarz inequality. Hence it follows that $S\varphi(\cdot\xi)$ has an entire analytic exension to $z\in\mathbf{C}$ for every $\xi\in S(\mathbf{R})$, and the inequality $|(\tilde{S}\varphi)(z\xi)|\le C_2\exp\left(\frac{1}{2}|z|^2(\xi|_p^2\right)$, $z\in\mathbf{C}$ holds with $C_2=\|\varphi\|$, whereby can easily follow the assertion. $\square$

By applying Lemma 9.1, we can easily see that for fixed $y\in S'(\mathbf{R})$, $\exists C_1>0$ such that

$$\sup_{\substack{z\in C\\ |z|=R}}|S(\mathrm{N}_x\Phi_0(t,\cdot))(z\xi)| \le C_1\exp(aR|\xi|_p), \quad R>0, \tag{9.5}$$

holds for some $a>0$, $p\in\mathbf{N}_0$. While, the estimate

$$\sup_{\substack{z\in C\\ |z|=R}}|S(\mathrm{F}_{\Psi}^{t}\hat{f})(z\xi)| \le C_2\exp(aR^2|\xi|_p^2) \tag{9.6}$$

is a direct result of Lemma 9.2 with an easy inequality $|S(\mathrm{F}_{\Psi}^{t}\hat{f})(\xi)|\le\|\mathrm{F}_{\Psi}^{t}\hat{f}\|_{-p}\, e^{\frac{1}{2}|\xi|_p^2}e^{|\xi|^2}$, because $\mathrm{F}_{\Psi}^{t}\hat{f}(\cdot)$ is contained at least in V for any t except dt-null set. As to the team $S(\int_0^t \mathrm{F}_{\Psi}^{t-s}h(s,\cdot)ds)(\xi)$, it is sufficient to find the estimate of $S(\mathrm{F}_{\Psi}^{t-s}H(s,\cdot))(\xi)$, since

$$S(\lim_{N\to\infty}\sum_{k}^{N}\mathrm{F}_{\Psi}^{t-s_k}h(s_k,\cdot)\Delta_{s_k})(\xi) = \lim_{N\to\infty}\sum_{k}^{N}S(\mathrm{F}_{\Psi}^{t-s_k}h(s_k,\cdot))(\xi)\cdot\Delta_{s_k}$$

from the continuity of the S-transform. Derivation of the same type estimate for this term again turns out to be the attribution to Lemma 9.2. Hence, if we regard the multiplication $*$ of $\mathrm{N}_x\Phi * g$ as the Wick product, then it is well-defined and makes sense as an element of $(S)^*$ for each $t>0$. On this account, we attain the following definitions.

Definition 9.3. (*Lifting*) *We define the lifting of v_ε as*

$$\mathrm{L}^{\varepsilon}v_\varepsilon(t,\cdot) := \mathrm{N}_x\Phi_\varepsilon(t,\cdot){:}g(t,\cdot)\in(S)^*, \ \varepsilon>0, \ t>0,$$

and the lifting of v_0 as

$$\mathrm{L}^{0}v_0(t,\cdot) := \mathrm{N}_x\Phi_0(t,\cdot){:}g(t,\cdot)\in(S)^*, \ t>0.$$

Definition 9.4. (*Lifted Convergence*) Let $\Psi_n(t)$, $\Psi_0(t)\in(S)^*$, $t>0$, *be liftings in the sence of Definition 9.3. Then we say that $\Psi_n(t)$ converges towards $\Psi_0(t)$ in lifting sense with H_p as n approaches to ∞ if for each t,*

$$H_p(S\Psi_n(t))(\xi)\to H_p(S\Psi_0(t))(\xi), \quad \xi\in S(\mathbf{R}),$$

as $n\to\infty$ for some $p\in\mathbf{N}$.

Remark 9.5. It seems to be quite natural and reasonable that we should define the convergence of liftings as $H_p(S\Psi_n)(\xi)\to H_p(S\Psi_0)(\xi)$, $\xi\in S(\mathbf{R})$ as $n\to\infty$ for some $p\in\mathbf{N}$. However, on an applicational basis, it would be better and more useful to formulate it in the framework including the parameter, because our liftings in question have the parameter t.

For convention, taking (9.2) into consideration, we may give the definition of lifted convergence as a concept attached to the parameter given. When $\Psi(t,\cdot)\in(S)^*$ for each t and we write the U-functional of $\Psi(t,\cdot)$ as $F(t,\xi)$, $\xi\in S(\mathbf{R})$, then we define an operator H_p by $\mathrm{H}_pF(t,\xi)=H_pF(T_p^{-1}(t),\xi)$.

Definition 9.6. *(Lifted Convergence associated with the parameter) Let $\Psi_n(t)$, $\Psi_0(t)\in(S)^*$ be liftings for each $t>0$. We say that $\Psi_n(t)$ converges toward $\Psi_0(t)$ in lifting sense with (H_p, T_p) as n approaches to ∞ if*

$$\mathrm{H}_p(S\Psi_n(t))(\xi)\to\mathrm{H}_p(S\Psi_0(t))(\xi),\quad \xi\in S(\mathbf{R}),$$

as $n\to\infty$ for each $t>0$. and for some $p\in\mathbf{N}$.

Now we shall state our principal result in this section.

Theorem 9.7. *Let $\{v_\varepsilon\}_\varepsilon$ be the solution of the problem (P.3). Then, under the assertion (9.2), $v_\varepsilon(t,\cdot)$ converges to $v_0(t,\cdot)$ in lifting sence with (H_p, T_p) as $\varepsilon\downarrow 0$, for any $t>0$, and some $p>\frac{1}{4}$.*

Remark 9.8. It is interesting to note that our assumption (9.2) is always true as far as $K_{n+1}(p)/K_n(p)\ll 1$. In fact, wa have

$$Q^\varepsilon_{n+1}(p)-Q^\varepsilon_n(p)=\sum_{n=0}^{m}A_k(p)\{R_\varepsilon(n-2k+1)-R_\varepsilon(n-2k)\}$$

with $n=2m$, $m\in\mathbf{N}$ (essentially the same expression is given for $n=m+1$), where we put $A_0(p)=1$ and $R_\varepsilon(0)=0$ as a matter of convenience. As a consequence, $Q^\varepsilon_{n+1}(p)-Q^\varepsilon_n(p)=O(R_\varepsilon(n+1))$ as $R_\varepsilon(n)\to 0$, hence $E_n(p)$ is finite for any p, any n.

Lemma 9.9. *For each $n\in\mathbf{N}$,*

$$:\langle\eta_\varepsilon,\xi\rangle^n:\to:\langle y,\xi\rangle^n:,\quad \xi\in S(\mathbf{R}),$$

as ε tends to zero.

Proof. Since η_ε converges strongly to y in $S'(\mathbf{R})$ from our major premise (cf. Eq.(1.3) in § 1), it is true that $\langle\eta_\varepsilon,\xi\rangle\to\langle y,\xi\rangle$ as $\varepsilon\downarrow 0$ for any $\xi\in S(\mathbf{R})$, and so is the assertion. □

Lemma 9.10. *The inequality*

$$|\langle\eta_\varepsilon^{\otimes n}-y^{\otimes n},\xi^{\otimes n}\rangle|\le I_0(n)\cdot Q^\varepsilon_n(p),\quad \xi\in S(\mathbf{R}),$$

holds for $n\in\mathbf{N}$, and $\varepsilon>0$, and $p>\frac{1}{4}$.

Proof. A direct computation gives

$$\begin{aligned}
&|\langle\eta_\varepsilon^{\otimes n}-y^{\otimes n},\xi^{\otimes n}\rangle|\\
&\le\sum_{k=0}^{G(n)}\binom{n}{2k}(2k-1)!!|\langle(:\eta_\varepsilon^{\otimes(n-2k)}:-:y^{\otimes(n-2k)}:)\hat{\otimes}Tr^{\otimes k},\xi^{\otimes n}\rangle|\\
&\le\sum_{k=0}^{G(n)}I(n,k)|\langle Tr,\xi\otimes\xi\rangle|^k\cdot|\langle:\eta_\varepsilon^{\otimes(n-2k)}:-:y^{\otimes(n-2k)}:,\xi^{\otimes(n-2k)}\rangle|\\
&\le I_0(n)\sum_{a=0}^{G(n)}|Tr|^k_{-p,2}|\xi|_p^{2k}|:\langle\eta_\varepsilon,\xi\rangle^{n-2k}:-:\langle y,\xi\rangle^{n-2k}:|\\
&=I_0(n)\sum_{k=0}^{G(n)}A_k(p)|r_\varepsilon(n-2k)-r_0(n-2k)|=I_0\sum_{k=0}^{G(n)}A_p(p)\cdot R_\varepsilon(n-2k),
\end{aligned}$$

because we employed the formula(9.4) and (9.1). □

Proposition 9.11. *Under the assumption* (9.2), the series

$$\sum_{k=0}^{\infty}\frac{1}{n!}t^n K_n(p)\cdot Q_n^{\varepsilon}(p)$$

is sbsolutely convergent uniformly in $\varepsilon\in(0,1)$ *for any* $t\in(t,t_p]$, *and for some* $p>\frac{1}{4}$.

Proof. Set $a_n\equiv a_n(p,t,\varepsilon):=\frac{1}{n!}t^n K_n(p)\cdot Q_n^{\varepsilon}(p)$. Then we have

$$\frac{a_{n+1}}{a_n}=\frac{t}{n+1}\frac{K_{n+1}(p)}{K_n(p)}(1+E_n^{\varepsilon}(p)).$$

In fact, under(9.2), as far as $K_{n+1}(p)/K_n(p)\ll 1$(sufficiently small), the inequality $(a_{n+1}/a_n)(\mathrm{p,t},\varepsilon)<(n+1)^{-1}t_p\cdot(1+E_n(\mathrm{p}))<1$ holds for $\forall\varepsilon\in(0,1)$, $\forall t\in(0,t_p]$, $p>\frac{1}{4}$, together with Remark 9.8. □

Proof of Theorem 9.7. For $\xi\in S(\mathbf{R})$, we have

$$\begin{aligned}
(9.7)\quad & |H_p[S(\mathrm{L}^{\varepsilon}v_{\varepsilon}(t))(\xi)]-H_p[S(\mathrm{L}^{0}v_0(t))(\xi)]| \\
&=|H_p[S(\mathrm{N}_x\Phi_{\varepsilon}(t):g(t))(\xi)]-H_p[S(\mathrm{N}_x\Phi_0(t):g(t))(\xi)]| \\
&=|H_p[\{S(\mathrm{N}_x\Phi_{\varepsilon}(t))(\xi)-S(\mathrm{N}_x\Phi_0(t))(\xi)\}\cdot S(g(t))(\xi)]| \\
&=\leq|H_p[S(\mathrm{N}_x\Phi_{\varepsilon}(t))(\xi)-S(\mathrm{N}_x\Phi_0(t))(\xi)]|\cdot|S(g(t))(\xi)|.
\end{aligned}$$

An application of Lemma 9.1 and Lemma 9.2 allows the estimate

$$\sup_{\substack{z\in C\\ |z|=R}}|(\tilde{S}g(t))(z\xi)|\leq C\exp(aR^2|\xi|_p^2),\quad \xi\in S(\mathbf{R}),\ R>0,$$

together with(9.3), (9.5), and(9.6), so that it is sufficient to estimate the first term in the right hand side of Eq.(9.7). As a matter of fact, it is easy to see from the formula [12, p.333] that

$$\begin{aligned}
(9.8)\quad & \text{1st term in r.h.s of Eq.(9.7)} \\
&=|H_p[S(\sum_{n=0}^{\infty}\frac{1}{n!}\langle :x^{\otimes n}:,(it\eta_{\varepsilon})^{\otimes n}\rangle)(\xi)-S(\sum_{n=0}^{\infty}\frac{1}{n!}\langle :x^{\otimes n}:,(ity)^{\otimes n}\rangle)(\xi)]| \\
&\leq\sum_{n=0}^{\infty}\frac{t^m}{n!}M(n)\cdot K_n(P)|\langle\eta_{\varepsilon}^{\otimes n}-y^{\otimes n},\ \xi^{\otimes n}\rangle|.
\end{aligned}$$

Since the inequality in Lemma 9.10 is valid for each $n\in\mathbf{N}$, Eq.(9.8) can be estimated majorantly by the series $\sum_{n=0}^{\infty}\frac{1}{n!}t^n K_n(p)\cdot Q_n^{\varepsilon}(p)$, the convergence of which is guaranteed by Proposition 9.11.

Therefore it follows from(9.2) that

$$\begin{aligned}
(9.9)\quad & |\mathrm{H}_p[S(\mathrm{L}^{\varepsilon}v_{\varepsilon}(t))(\xi)]-\mathrm{H}_p[S(\mathrm{L}^{0}v_0(t))(\xi)]| \\
&=|H_p[S(\mathrm{L}^{\varepsilon}v_{\varepsilon}(T_p^{-1}(t)))(\xi)]-H_p[S(\mathrm{L}^{0}v_0(T_p^{-1}(t)))(\xi)]| \\
&\leq C|H_p[S(\mathrm{N}_x\Phi_{\varepsilon}(T_p^{-1}(t)))(\xi)-S(\mathrm{N}_x\Phi_0(T_p^{-1}(t)))(\xi)]| \\
&\leq\sum_{n=0}^{\infty}\frac{1}{n!}\{T_p^{-1}(t)\}^n K_n(p)\cdot Q_n^{\varepsilon}(p)<\infty,
\end{aligned}$$

uniformly in $\varepsilon\in(0,1)$, for any $t>0$, $p>\frac{1}{4}$. Consequently,

$$(9.10)\qquad \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\frac{1}{n!}\{T_p^{-1}(t)\}^n K_n(p)\cdot Q_n^{\varepsilon}(p)=\sum_{n=0}^{\infty}\lim_{\varepsilon\to 0}\frac{1}{n!}\{T_p^{-1}(t)\}^n K_n(p)\cdot Q_n^{\varepsilon}(p),$$

where we paid attention to Remark 9.8. Moreover,

$$(9.11)\qquad \lim_{\varepsilon\to 0}Q_n^{\varepsilon}(p)=\sum_{k=0}^{G(n)}A_k(p)\cdot\lim_{\varepsilon\to 0}R_{\varepsilon}(n-2k),\quad\text{and}$$

$$(9.12)\qquad \lim_{\varepsilon\to 0}R_{\varepsilon}(n-2k)=\lim_{\varepsilon\to 0}|r_{\varepsilon}(n-2k)-r_0(n-2k)|=0,$$

because we applied Lemma 9.9. Combining all the results from Eq.(9.9) through Eq.(9.12), we finally obtain that $H_p[S(L^{\varepsilon}v_{\varepsilon}(t))(\xi)]$ converges to $H_p[S(L^0v_0(t))(\xi)]$, $\xi\in S(\mathbf{R})$, as ε tends to zero for any $t>0$, some $p>\frac{1}{4}$, which completes the proof. □

Remark 9.12. In checking the criterion of series convergence, the seight *M(n)* associated with the operator H_p provides with the same effect as to put the condition $\{I_0(n+1)/I_0(n)\}\wedge 1$, which made a contribution to simplifying the computation.

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ON PSEUDO-FOURIER-MEHLER TRANSFORMS AND INFINITESIMAL GENERATORS IN WHITE NOISE CALCULUS*)

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§1. Introduction

The study of the Fourier transform $\mathcal{F}$ in white noise calculus was initiated and has been developed to a mature level by H.-H. Kuo [16,17] (also [19]). While, the Fourier-Mehler transform $\mathcal{F}_\theta$ is a kind of generalization of $\mathcal{F}$ [18] (also [11]), which furnishes the theory of infinite dimensional Fourier transforms in white noise space with adequately fruitful and profitable ingradients.

In this article we introduce Pseudo-Fourier-Mehler (PFM for short) transform having quite similar nice properties as the Fourier-Mehler transform possesses. It was originally defined in [5] and used for application to abstract equations in infinite dimensional spaces. In connection with other Fourier type transforms in white noise analysis, we can compute the infinitesimal generator of the PFM transform directly and show that our Pseudo-Fourier-Mehler transform enjoys intertwining properties. We shall state the characterization theorem for PFM transforms, which is one of our main results in this article. The Fock expansion of PFM transform can be derived as well. Lastly we shall introduce a generalization idea of PFM transform and investigate some properties that the generalized transform should satisfy.

The Pseudo-Fourier-Mehler transform is a very important and interesting operator in the standpoint of how to express the solutions for the Fourier-transformed abstract Cauchy problems ([5,6]; see also [4,8]).

In [1] they have studied the two dimensional complex Lie group $\mathcal{G}$ explicitly and succeeded in describing every one parameter subgroup with infinitesimal generator $(\frac{2a+b}{2})\Delta_G + bN$, where N is the number operator and Δ_G is the Gross Laplacian. Furthermore, one can find in [24] another related work, especially on a systematic study of Lie algebras containing infinite dimensional Laplacians.

We are able to state our results in the general setting (e.g., [23]; see also [7]) of white noise analysis. As a matter of fact, almost all statements in our theory

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remains valid under non-minor change of the basic setting. However, just for simplicity we adopt in this article the so-called original standard setting [11] in white noise analysis or Hida calculus to state our results related to the PFM transform.

§2. Notation and Preliminaries

Let $\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ be the Schwartz class space on $\mathbb{R}$ and $\mathcal{S}^* \equiv \mathcal{S}'(\mathbb{R})$ its dual space. Then $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ is a Gelfand triple. We define the family of norms given by $|\xi|_p = |A^p\xi|$, $p > 0$, $\xi \in \mathcal{S}(\mathbb{R})$, where the operator $A = -d^2/dt^2 + t^2 + 1$ and $|\cdot|$ is the $L^2(\mathbb{R})$-norm. Let $\mathcal{S}_p \equiv \mathcal{S}_p(\mathbb{R})$ be the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm $|\cdot|_p, p > 0$. We denote its dual space by $\mathcal{S}_p^* \equiv \mathcal{S}_p'(\mathbb{R})$, and we have $\mathcal{S}_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}_p'(\mathbb{R})$. Let μ be the standard Gaussian measure on $\mathcal{S}'(\mathbb{R})$ such that

$$\int_{\mathcal{S}^*} \exp(\sqrt{-1}\langle x, \xi\rangle)\mu(dx) = \exp\Big(-\frac{1}{2}|\xi|^2\Big),$$

for any $\xi \in \mathcal{S}(\mathbb{R})$. (L^2) denotes the Hilbert space of complex-valued μ-square integrable functionals with norm $\|\cdot\|$. The Wiener-Itô decomposition theorem gives the unique representation of φ in (L^2), i.e.,

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n \in \hat{L}^2_{\mathbb{C}}(\mathbb{R}^n), \tag{1}$$

where I_n denotes the multiple Wiener integral of order n and $\hat{L}^2_{\mathbb{C}}(\mathbb{R}^n)$ the space of symmetric complex valued L^2-functions on $\mathbb{R}^n$. The second quantization operator $\Gamma(A)$ is densely defined on (L^2) as follows: for $\varphi = \sum_{n=0}^{\infty} I_n(f_n) \in \mathrm{Dom}(\Gamma(A))$,

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n). \tag{2}$$

For $p \in \mathbb{N}$, define $\|\varphi\|_p = \|\Gamma(A)^p\varphi\|$ and let $(\mathcal{S})_p \equiv \{\varphi \in (L^2);\ \|\varphi\|_p < \infty\}$ and the dual space of $(\mathcal{S})_p$ is denoted by $(\mathcal{S})_p^*$. Let $(\mathcal{S})$ be the projective limit of $\{(\mathcal{S})_p; p \in \mathbb{N}\}$. It is called a space of test white noise functionals. The elements in the dual space $(\mathcal{S})^*$ of $(\mathcal{S})$ are called generalized white noise functionals or Hida distributions. In fact, $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$ is a Gelfand triple [11]. For convention all dual pairings $\langle\cdot,\cdot\rangle$, resp. $\langle\langle\cdot,\cdot\rangle\rangle$ mean the canonical bilinear forms on $\mathcal{S}^* \times \mathcal{S}$ (resp. $(\mathcal{S})^* \times (\mathcal{S})$) unless otherwise stated.

The S-transform of $\Phi \in (\mathcal{S})^*$ is a function on $\mathcal{S}$ defined by

$$(S\Phi)(\xi) := \langle\langle \Phi, :\exp\langle\cdot,\xi\rangle:\rangle\rangle, \qquad \xi \in \mathcal{S}(\mathbb{R}), \tag{3}$$

where $:\exp\langle\cdot,\xi\rangle: \equiv \exp\langle\cdot,\xi\rangle \cdot \exp\left(-\frac{1}{2}|\xi|^2\right)$. Then note that a mapping $: \mathbb{C} \ni z \mapsto (S\Phi)(z\xi + \eta)$ is entire holomorphic for any $\xi, \eta \in \mathcal{S}$. A complex valued function F on $\mathcal{S}$ is called a *U-functional* if and only if it is ray entire on $\mathcal{S}$ and if there exist constants $C_1, C_2 > 0$, and $p \in \mathbb{N} \cup \{0\}$ so that the estimate

$$|F(z\xi)| \leq C_1 \exp\Big(C_2 |z|^2 {|\xi|_p}^2\Big)$$

may hold for all $z \in \mathbb{C}, \xi \in \mathcal{S}$. We have the following Characterization Theorem [25]:

Theorem 1. *If $\Phi \in (\mathcal{S})^*$, then $S\Phi$ is a U-functional. Conversely, if F is a U-functional, then there exists a unique element Φ in $(\mathcal{S})^*$ such that $S\Phi = F$ holds.*

Based upon the above characterization we are able to give rigorous definitions to Fourier type transforms of infinite dimensions. The Kuo type Fourier transform $\mathcal{F}$ [16,17] of a generalized white noise functional Φ in $(\mathcal{S})^*$ is the generalized white noise functional, S-transformation of which is given by

$$(4) \qquad S(\mathcal{F}\Phi)(\xi) = \langle\langle \Phi, \exp(-i\langle\cdot,\xi\rangle)\rangle\rangle, \qquad \xi \in \mathcal{S}.$$

Likewise, the Fourier-Mehler transform $\mathcal{F}_\theta$ ($\theta \in \mathbb{R}$) [18] of a generalized white noise functional Φ in $(\mathcal{S})^*$ is the generalized white noise functional, S-transformation of which is given by

$$(5) \qquad S(\mathcal{F}_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\Big\{\mathrm{e}^{i\theta}\langle\cdot,\xi\rangle - \frac{1}{2}\mathrm{e}^{i\theta}\cos\theta|\xi|^2\Big\}\rangle\rangle, \qquad \xi \in \mathcal{S}.$$

The Fourier-Mehler transform $\mathcal{F}_\theta, \theta \in \mathbb{R}$ is a generalization of the Kuo type Fourier transform $\mathcal{F}$. Actually, $\mathcal{F}_0 = Id$, and $\mathcal{F}_{-\pi/2}$ is coincident with the Fourier transform $\mathcal{F}$. It is easy to see that $\mathcal{F}_{\pi/2}$ is the inverse Fourier transform $\mathcal{F}^{-1}$. Hence we have

$$S(\mathcal{F}^{-1}\Phi)(\xi) = (S\Phi)(i\xi)\exp\Big(-\frac{1}{2}|\xi|^2\Big), \qquad \xi \in \mathcal{S}.$$

§3. Pseudo-Fourier-Mehler Transform

We begin with introducing the Pseudo-Fourier-Mehler transform in white noise analysis.

Definition 1. *$\{\Psi_\theta, \theta \in \mathbb{R}\}$ is said to be the Pseudo-Fourier-Mehler (PFM) transform [5,6] if Ψ_θ is a mapping from $(\mathcal{S})^*$ into itself for $\theta \in \mathbb{R}$, whose U-functional is given by*

$$(6) \qquad S(\Psi_\theta\Phi)(\xi) = F(e^{i\theta}\xi)\cdot\exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big), \qquad \xi \in \mathcal{S},$$

or equivalently

$$(7) \qquad S(\Psi_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\Big(e^{i\theta}\langle\cdot,\xi\rangle - \frac{1}{2}|\xi|^2\Big)\rangle\rangle, \qquad \xi \in \mathcal{S},$$

for $\Phi \in (\mathcal{S})^*$, where S is the S-transform in white noise analysis and F denotes the U-functional of Φ.

By virtue of Theorem 1, the right hand sides in Eq.(6) and Eq.(7) are U-functionals, and $\Psi_\theta\Phi$ exists for each Φ in $(\mathcal{S})^*$. Therefore the above-mentioned Pseudo-Fourier-Mehler transform is well-defined. Hence we have

Proposition 2. *The following properties hold:*
(i) $\Psi_0 = Id$; *(Id denotes the identity operator.)*
(ii) $\Psi_\theta \neq \mathcal{F}$ *for any* $\theta \in \mathbb{R}\setminus\{0\}$;
(iii) $\Psi_\theta \neq \mathcal{F}_\theta$ *for any* $\theta \in \mathbb{R}\setminus\{0\}$.

Proof. As to (i), it is easy to see that $S(\Psi_0\Phi)(\xi) = S\Phi(\xi) = F(\xi)$. The characterization theorem allows the equality $\Psi_0 = Id$. (iii) is obvious from definitions. Since $\mathcal{F}_0 = Id$ and $\mathcal{F}_{-\pi/2} = \mathcal{F}$, it follows clearly from (iii) that $\mathcal{F}$ never coincides with Ψ_θ for any $\theta \in \mathbb{R}$ except $\theta = 0$. □

Proposition 3. *The invese operator of the Pseudo-Fourier-Mehler transform* Ψ_θ *is given by* $(\Psi_\theta)^{-1} = \Psi_{-\theta}$ *for* $\theta \in \mathbb{R}$.

Proof. It is sufficient to show that $\Psi_{-\theta}\Psi_\theta = \Psi_\theta\Psi_{-\theta} = Id$. As a matter of fact, for $\Phi \in (\mathcal{S})^*$ we get from the definition (6)

$$
\begin{aligned}
(8)\qquad S(\Psi_{-\theta}(\Psi_\theta\Phi))(\xi) &= S(\Psi_\theta\Phi)(\mathrm{e}^{-i\theta}\xi)\cdot\exp\Big(-i\mathrm{e}^{-i\theta}\sin\theta|\xi|^2\Big)\\
&= (S\Phi)(\mathrm{e}^{i\theta}(\mathrm{e}^{-i\theta}\xi))\cdot\exp\Big(i\mathrm{e}^{i\theta}\sin\theta|\mathrm{e}^{-i\theta}\xi|^2\Big)\cdot\exp\Big(-i\mathrm{e}^{-i\theta}\sin\theta|\xi|^2\Big)\\
&= (S\Phi)(\xi)\cdot\exp(0) = S(Id\cdot\Phi)(\xi), \qquad \xi \in \mathcal{S},
\end{aligned}
$$

because we used the relation

$$S(\Psi_{-\theta}\Phi)(\xi) = S\Phi(\mathrm{e}^{-i\theta}\xi)\cdot\exp\Big(-i\mathrm{e}^{-i\theta}\sin\theta|\xi|^2\Big)$$

so as to obtain the second line of Eq.(8). An application of the characterization theorem to Eq.(8) gives $\Psi_{-\theta}\Psi_\theta = Id$. As for the other part of the desired equalities, it goes almost similarly. □

Next let us consider what the image of the space $(\mathcal{S})$ under Ψ_θ is like (see Corollary 6 below). The Pseudo-Fourier-Mehler transform Ψ_θ also enjoys some interesting properties on the product of Gaussian white noise functionals (see Theorem 4 and Theorem 5).

Theorem 4. *Let* g_c *be a Gaussian white noise functional, i.e.,* $g_c(\cdot) := \mathcal{N}\exp(-|\cdot|^2/2c)$ *with renormalization* $\mathcal{N}$ *and* $c \in \mathbb{C}$, $c \neq 0, -1$. *For* $\theta \in \mathbb{R}$ *the following equalities hold:*

$$(i)\ \Psi_\theta\Phi : g_{c(\theta)} = \Gamma(e^{i\theta}Id)\Phi, \qquad \forall\Phi \in (\mathcal{S})^*;$$

$$(ii)\ \text{for any } p \in \mathbb{R}, \quad \|\Psi_\theta\Phi : g_{c(\theta)}\|_p = \|\Phi\|_p, \quad \forall\Phi \in (\mathcal{S})_p;$$

where : *denotes the Wick product (e.g. [11,p.101]) and the parameter* $c(\theta)$ *is given by* $c(\theta) = -(2^{-1}i\ e^{-i\theta}\csc\theta + 1)$.

Proof. Noting that the U-functional of g_c is given by $\exp(-2^{-1}\ (1+c)^{-1}\ |\xi|^2)$, we readily obtain

$$
\begin{aligned}
(9)\qquad S(\Psi_\theta\Phi : g_{c(\theta)})(\xi) &= S(\Psi_\theta\Phi)(\xi)\cdot(Sg_{c(\theta)})(\xi)\\
&= S\Phi(\mathrm{e}^{i\theta}\xi)\cdot\Xi(\theta,\xi), \qquad \xi \in \mathcal{S},
\end{aligned}
$$

because we employed Eq.(6) and put

$$\Xi(\theta,\xi) := \exp\Big(ie^{i\theta}\sin\theta|\xi|^2 - \frac{1}{2(1+c(\theta))}|\xi|^2\Big).$$

Then we cannot find any $\theta \in \mathbb{R}$ such that

$$(9) = S\Phi(\xi) = \exp\Big(-\frac{1}{2}|\xi|^2\Big)\cdot\langle\langle\Phi, e^{\langle\cdot,\xi\rangle}\rangle\rangle$$

may hold, which implies that $\Psi_\theta\Phi : g_{c(\theta)} \neq \Phi$ for any $\Phi \in (\mathcal{S})^*$. However, when $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle, f_n \in \hat{\mathcal{S}}_{-p}(\mathbb{R}^n)$ (the symmetric space $\mathcal{S}_{-p}(\mathbb{R}^n)$), then its U-functional $S\Phi(\xi)$ is given by $\sum_{n=0}^{\infty}\langle\xi^{\otimes n}, f_n\rangle$, so that, we easily get from definition of the second quantization operator Γ

$$\text{r.h.s. of } (9) = \sum_{n=0}^{\infty}\langle(e^{i\theta})^n\xi^{\otimes n}, f_n\rangle\cdot\Xi(\theta,\xi) = S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\Xi(\theta,\xi).$$

Hence, if $2i(1+c(\theta))e^{i\theta}\sin\theta = 1$ holds, then clearly $\Xi(\theta,\xi)$ proves to be 1, suggesting with the characterization theorem that

$$\Psi_\theta\Phi : g_{c(\theta)} = \Gamma(e^{i\theta}Id)\Phi.$$

Moreover, it is easy to see that

$$\|\Psi_\theta\Phi : g_{c(\theta)}\|_p = \|\Gamma(e^{i\theta}Id)\Phi\|_p = \|\Phi\|_p$$

holds for any $p \in \mathbb{R}$. □

If we take the assertion obtained in Theorem 4 into account, then the following questions will arise naturally: whether the PFM transformed Φ (i.e. $\Psi_\theta\Phi$) can be represented by the Wick product of something like a transformed Φ and a Gaussian white noise functional g_c; furthermore, if so, what is the parameter $c = c(\theta)$ then? First of all, on the assumption that $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle \in (\mathcal{S})^*$, a simple computation gives, for $\xi \in \mathcal{S}$

$$(10) \qquad \begin{aligned} S(\Psi_\theta\Phi)(\xi) &= S\Phi(e^{i\theta}\xi)\cdot\exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big) \\ &= S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big). \end{aligned}$$

We know from Eq.(10) that there is no possibility that $\Psi_\theta\Phi$ may coincide with $\Phi : g_{K(\theta)}$ even for any $K(\theta), \theta \in \mathbb{R}$, because

$$(11) \qquad S(\Phi : g_{K(\theta)})(\xi) = S\Phi(\xi)\cdot(Sg_{K(\theta)})(\xi) = S\Phi(\xi)\cdot\Lambda(K(\theta),\xi)$$

with

$$\Lambda(r,\xi) := \exp\Big\{-\frac{1}{2(1+r)}|\xi|^2\Big\}.$$

On the other hand, since the S-transform of $\Gamma(e^{i\theta})\Phi : g_{K(\theta)}$ is given by

$$S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\Lambda(K(\theta),\xi),$$

it is true from (10) that

$$\Psi_\theta\Phi = \Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)}$$

may possibly hold for $\Phi\in(\mathcal{S})^*, \theta\in\mathbb{R}$ as far as

$$2i(1+K(\theta))e^{i\theta}\sin\theta + 1 = 0$$

is satisfied. Let us next consider the evaluation of the term $\Psi_\theta\Phi$ ($\Phi\in(\mathcal{S})_p$) relative to the $(\mathcal{S})_p$-norm ($p\in\mathbb{R}$). We need to determine the parameter $A(\theta)$, which comes from the relation between $\Gamma(e^{i\theta})\Phi : g_{K(\theta)}$ and $\Gamma(e^{i\theta})(\Phi : g_{A(\theta)})$. By a similar calculation in (10) we readily obtain

$$\begin{aligned}(12)\qquad S\big(\Gamma(e^{i\theta}Id)(\Phi : g_{A(\theta)})\big)(\xi) &= S(\Phi : g_{A(\theta)})(e^{i\theta}\xi)\\ &= (S\Phi)(e^{i\theta}\xi)\cdot\Lambda(A(\theta),e^{i\theta})\\ &= (S\Phi)(e^{i\theta}\xi)\cdot\exp\Big\{-\frac{e^{2i\theta}}{2(1+A(\theta))}|\xi|^2\Big\},\end{aligned}$$

by making use of Eq.(11). A comparison of (12) with $S(\Gamma(e^{i\theta})\Phi)(\xi)\cdot\Lambda(K(\theta),\xi)$ provides with

$$\Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)} = \Gamma(e^{i\theta}Id)(\Phi : g_{A(\theta)})$$

as far as $A(\theta) = 2^{-1}ie^{-i\theta}\csc\theta - 1$. It therefore follows that

$$\begin{aligned}\|\Psi_\theta\Phi\|_p &= \|\Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)}\|_p\\ &= \|\Gamma(e^{i\theta}Id)(\Phi : g_{A(\theta)})\|_p = \|\Phi : g_{A(\theta)}\|_p\end{aligned}$$

for all $\Phi\in(\mathcal{S})_p, p\in\mathbb{R}$, and any $\theta\in\mathbb{R}$. Summing up, we thus obtain

Theorem 5. *The following equalities hold for any $\theta\in\mathbb{R}$:*
(i) if $K(\theta) = 2^{-1}ie^{-i\theta}\csc\theta - 1$, then

$$\Psi_\theta\Phi = \Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)},\qquad \Phi\in(\mathcal{S})^*;$$

(ii) if $A(\theta) = 2^{-1}ie^{-i\theta}\csc\theta - 1$, then

$$\|\Psi_\theta\Phi\|_p = \|\Phi : g_{A(\theta)}\|_p,\qquad \Phi\in(\mathcal{S})_p$$

for all $p\in\mathbb{R}$.

Let us think of the image of $\varphi\in(\mathcal{S})$ under the Pseudo-Fourier-Mehler transform. It is easily checked that $g_c : g_d = 1$ holds with $c+d=-2$. So we have

$$(13)\qquad g_{c(\theta)} : g_{K(\theta)} = 1.$$

From (ii) of Theorem 4, immediately, $\varphi\in(\mathcal{S})$ if and only if

$$\Psi_\theta\varphi : g_{c(\theta)}\in(\mathcal{S}),$$

so that, it is equivalent to

$$\Psi_\theta\varphi : g_{c(\theta)} : g_{K(\theta)}\in(\mathcal{S}) : g_{K(\theta)},$$

where $(\mathcal{S}) : g_{K(\theta)}$ denotes the whole space of elements $\varphi : g_{K(\theta)}$ for $\varphi\in(\mathcal{S})$. Consequently, it is obvious that $\Psi_\theta\varphi\in(\mathcal{S}) : g_{K(\theta)}$, by virtue of Eq.(13). Therefore we obtain

Corollary 6. *For $\theta \in \mathbb{R}$,*

$$\operatorname{Im} \Psi_\theta(\mathcal{S}) = (\mathcal{S}) : g_{K(\theta)} \equiv \{\varphi : g_{K(\theta)};\ \varphi \in (\mathcal{S})\},$$

where $K(\theta) = 2^{-1} i e^{-i\theta} \csc\theta - 1$.

Remark 2. The results in Theorem 4 and Theorem 5 are quite similar to those of the Fourier-Mehler transform. In fact, for $p \in \mathbb{R}$, $\Phi \in (\mathcal{S})_p$,

$$\|(\mathcal{F}_\theta \Phi) : g_{c_1(\theta)}\|_p = \|\Phi\|_p \text{ and } \|\mathcal{F}_\theta \Phi\|_p = \|\Phi : g_{c_2(\theta)}\|_p$$

hold with $c_1(\theta) = -i\cot\theta - 2$, and $c_2(\theta) = i\cot\theta - 2$ (e.g. [11,§9.H]).

Remark 3. The image of $(\mathcal{S})$ under the Fourier-Mehler transform $\mathcal{F}_\theta$ is given by $(\mathcal{S}) : g_{i\cot\theta}$, while that of $(\mathcal{S})$ under the Fourier transform $\mathcal{F}$ coincides with the space

$$(\mathcal{S}) : \tilde{\delta}_0 \equiv \{\varphi : \tilde{\delta}_0;\ \varphi \in (\mathcal{S})\},$$

where $\tilde{\delta}_0$ is the delta function at 0 and

$$\lim_{c \to 0} g_c = \tilde{\delta}_0$$

(e.g. [11, Chapter 9]).

§4. Infinitesimal Generators

First of all, for all $\theta \in \mathcal{S}$ we define

$$\varphi_\xi(x) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle : x^{\otimes n} :, \xi^{\otimes n} \rangle$$

with $x \in \mathcal{S}^*, \xi \in \mathcal{S}$. We call it an exponential vector. Then $\{G_\theta, \theta \in \mathbb{R}\}$ is an operator on $(\mathcal{S})$ defined by

$$(G_\theta \varphi_\xi)(x) := \varphi_{e^{i\theta}\xi}(x) \cdot \exp\left(ie^{i\theta} \sin\theta |\xi|^2\right). \tag{14}$$

Let τ denote the distribution in $(\mathcal{S} \otimes \mathcal{S})^*$ given by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \qquad \xi, \eta \in \mathcal{S}.$$

Note that it can be expressed as

$$\tau = \int_{\mathbb{R}} \delta_t \otimes \delta_t dt = \sum_{j=0}^{\infty} e_j \otimes e_j \in (\mathcal{S} \otimes \mathcal{S})^*,$$

where $\{e_n\}$ denotes a complete orthonormal basis for $L^2(\mathbb{R})$. Moreover we have

$$\tau^{\otimes n} = \int_{\mathbb{R}^n} \delta_{t_1} \otimes \delta_{t_1} \otimes \cdots \otimes \delta_{t_n} \otimes \delta_{t_n} dt_1 \cdots dt_n.$$

The following is an easy exercise. The next lemma provides with a general expression for elements of general form in $(\mathcal{S})$.

Lemma 7. *When* $\varphi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle \in (\mathcal{S})$ *with* $f_n \in \hat{\mathcal{S}}(\mathbb{R}^n)$,*(the symmetric* $\mathcal{S}(\mathbb{R}^n)$*), then* $G_\theta\varphi$ *is given by*

$$(G_\theta\varphi)(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, g_n\rangle,$$

and

$$g_n \equiv g_n(\varphi) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!}(i\sin\theta)^m e^{i(n+m)\theta}\tau^{\otimes m} * f_{2m+n},$$

where for the element f_{2m+n} *in* $\hat{\mathcal{S}}(\mathbb{R}^{2m+n})$ *the term* $\tau^{\otimes m} * f_{2m+n}$ *actually has the following integral expression*

$$\begin{aligned}&(\tau^{\otimes m} * f_{2m+n})(t_1, \cdots, t_n)\\ &= \int_{\mathbb{R}^m} f_{2m+n}(s_1, s_1, \cdots, s_m, s_m, t_1, \cdots, t_n)ds_1 \cdots ds_m.\end{aligned}$$

On this account, we obtain immediately

Proposition 8. *The Pseudo-Fourier-Mehler transform* $\{\Psi_\theta; \theta \in \mathbb{R}\}$ *is given by the adjoint operator of* $\{G_\theta;\ \theta \in \mathbb{R}\}$, *i.e.,*

$$\Psi_\theta = G_\theta^*$$

holds in operator equality sense for all $\theta \in \mathbb{R}$.

The next proposition gives an explicit action of the PFM transform Ψ_θ for the generalized white noise functionals of general form. It is due to a direct computation.

Proposition 9. *For* $\Phi \in (\mathcal{S})^*$ *given as* $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, F_n\rangle$, *it holds that*

$$\Psi_\theta\Phi(x) = \sum_{n=0}^{\infty}\Big\langle : x^{\otimes n} :, \sum_{l+2m=n} a(l,m,\theta)\cdot F_l\hat{\otimes}\tau^{\otimes m}\Big\rangle,$$

where the constant $a(l,m,\theta)$ *is given by*

$$a(l,m,\theta) = \frac{1}{m!}e^{i(l+m)\theta}(i\sin\theta)^m.$$

Remark 4. Similar results for Fourier-Mehler transform as the above can be found in [23]. For the proof of Proposition 9, it is almost the same as those given in [23].

It follows from Proposition 3 that the Pseudo-Fourier-Mehler transform Ψ_θ is injective and surjective. Moreover, it is easy to check that Ψ_θ is a strongly continuous operator from $(\mathcal{S})^*$ into itself, when we take Lemma 7 and Proposition 8 into consideration. Thus we have the following theorem.

Theorem 10 [5]. *The Pseudo-Fourier-Mehler transform $\Psi_\theta : (\mathcal{S})^* \to (\mathcal{S})^*$ is a bijective and strongly continuous linear operator.*

Theorem 11 [5]. *The set $\{\Psi_\theta; \theta \in \mathbb{R}\}$ forms a one parameter group of strongly continuous linear operator acting on the space $(\mathcal{S})^*$ of Hida distributions.*

Proof. For $\Phi \in (\mathcal{S})^*$, $\xi \in \mathcal{S}$, and any $\theta, \eta \in \mathbb{R}$, from (7) of Definition 1 we have

$$S(\Psi_{\theta+\eta}\Phi)(\xi) = \langle\langle \Phi, \exp\Big\{ e^{i(\theta+\eta)}\langle \cdot, \xi\rangle - \frac{1}{2}|\xi|^2 \Big\}\rangle\rangle. \tag{15}$$

While, from (6)

$$\begin{aligned} S(\Psi_\theta(\Psi_\eta))(\xi) &= S(\Psi_\eta\Phi)(e^{i\theta}\xi) \cdot \exp\Big(ie^{i\theta}\sin\theta|\xi|^2 \Big) \\ &= F(e^{i\eta}(e^{i\theta}\xi)) \cdot \exp\Big(ie^{i\eta}\sin\eta|e^{i\theta}\xi|^2 \Big) \cdot \exp\Big(ie^{i\theta}\sin\theta|\xi|^2 \Big) \\ &= e^{-\frac{1}{2}\left|e^{i(\theta+\eta)}\xi\right|^2} \langle\langle \Phi, e^{\langle \cdot, e^{i(\theta+\eta)}\xi\rangle}\rangle\rangle \cdot \exp\Big\{ ie^{i\theta}\big(e^{i(\theta+\eta)}\sin\eta + \sin\theta\big)|\xi|^2 \Big\} \\ &= \langle\langle \Phi, \exp\Big\{ e^{i(\theta+\eta)}\langle \cdot, \xi\rangle - \frac{1}{2}|\xi|^2 \Big\}\rangle\rangle, \end{aligned} \tag{16}$$

with the U-functional F of Φ. By comparing (15) with (16), we get

$$S(\Psi_{\theta+\eta}\Phi)(\xi) = S(\Psi_\theta\Psi_\eta\Phi)(\xi).$$

Consequently, the characterization theorem leads to

$$\Psi_{\theta+\eta}\Phi = \Psi_\theta \cdot \Psi_\eta\Phi, \qquad \Phi \in (\mathcal{S})^*,$$

which completes the proof. □

We are now in a position to state one of the principal results in this paper. This is a very important property of the Pseudo-Fourier-Mehler transform, especially on an applicational basis.

Theorem 12 [5]. *The infinitesimal generator of $\{\Psi_\theta; \theta \in \mathbb{R}\}$ is given by $i(N + \Delta_G^*)$, where N is the number operator and Δ_G^* is the adjoint of the Gross Laplacian Δ_G.*

Remark 5. It is well known that the infinitesimal generator of the Fourier-Mehler transforms $\{\mathcal{F}_\theta; \theta \in \mathbb{R}\}$ is $iN + \frac{i}{2}\Delta_G^*$, while the adjoint operator of $\{\mathcal{F}_\theta;\ \theta \in \mathbb{R}\}$ has $iN + \frac{i}{2}\Delta_G$ as its infinitesimal generator (e.g. see [11]). The proof of Theorem 12 is almost similsr to the above ones.

Proof of Theorem 12. First of all we set

$$F_\theta(\xi) := S(\Psi_\theta\Phi)(\xi) \qquad \text{and} \qquad F_0(\xi) := S(\Phi)(\xi)$$

for $\Phi \in (\mathcal{S})^*$, $\xi \in \mathcal{S}$, paying attention to (i) of Proposition 2. From (6) we have $F_\theta(\xi) = F_0(e^{i\theta}) \cdot \exp\left[ie^{i\theta}\sin\theta|\xi|^2\right]$. Since F_0 is Fréchet differentiable, the functional

$F_\theta(\xi)$ is differentiable in θ as well, and it is easy to check that

$$\begin{aligned}
(17)\qquad & \lim_{\theta\to 0}\frac{1}{\theta}\{F_\theta(\xi)-F_0(\xi)\} \\
&= \langle F_0'(e^{i\theta}\xi), ie^{i\theta}\rangle \cdot \exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big)\Big|_{\theta=0} \\
&\qquad\qquad + F_0(e^{i\theta}\xi)\cdot \frac{d}{dt}\exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big)\Big|_{\theta=0} \\
&= i\langle F'(\xi),\xi\rangle + i|\xi|^2\cdot F(\xi).
\end{aligned}$$

While, we can easily check that the U-functional $\theta^{-1}\cdot\{F_\theta(\xi)\ -F_0(\xi)\}$, $\theta\in\mathbb{R}$ satisfies the uniform bounded criterion: $\exists C_0>0$ so that

$$\sup_{\substack{z\in\mathbb{C}\\|z|=R}}\left|\frac{1}{\theta}\{\tilde F_\theta(z\xi)-\tilde F_0(z\xi)\}\right| \le C_0\exp\big(c_1R^{c_2}|\xi|_p^2\big)$$

holds for all $R>0$, all $\xi\in\mathcal{S}$ with $c_1>0, c_2>0$, where $\tilde F_*$ denotes an entire analytic extension of F. Hence, the strong convergence criterion theorem [25] (see also [11, Chapter 4]) allows convergence of

$$S^{-1}\Big(\frac{1}{\theta}\{F_\theta(\cdot)-F_0(\cdot)\}\Big)(x) = \frac{1}{\theta}\{\Psi_\theta\Phi(x)-\Phi(x)\}$$

in $(\mathcal{S})^*$ as θ tends to zero. We need the following two lemmas.

Lemma 13. *(cf. [11,Theorem 6.11,p.196]) Let $F(\xi)=S\Phi(\xi),\xi\in\mathcal{S}$ for $\Phi\in(\mathcal{S})^*$. Then*
(i) F is Fréchet differentiable;
(ii) the S-transform of $N\Phi(x)$ is given by $\langle F'(\xi),\xi\rangle$, $\xi\in\mathcal{S}$;
where N is the number operator.

Lemma 14. *(cf. [11,Theorem 6.20,p.206]) For any Φ in $(\mathcal{S})^*$, the S-transform of $\Delta_G^*\Phi(x)$ is given by $|\xi|^2\ S\Phi(\xi)$, $\xi\in\mathcal{S}$.*

We may deduce at once that

$$(18)\qquad S(N\Phi+\Delta_G^*\Phi)(\xi) = \langle F'(\xi),\xi\rangle + |\xi|^2F(\xi),\quad \xi\in\mathcal{S},$$

with simple applications of Lemma 13 and Lemma 14. Moreover, it is easily verified from (17) and (18) together with the above-mentioned convergence result that

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1}{\theta}(\Psi_\theta - Id)\Phi(x) &= \lim_{\theta\to 0} S^{-1}\Big(\frac{1}{\theta}\{F_\theta(\cdot)-F_0(\cdot)\}\Big)(x) \\
&= S^{-1}\big(i\langle F'(\xi),\xi\rangle + i|\xi|^2\cdot F(\xi)\big) \\
&= i(N+\Delta_G^*)\Phi(x), \qquad \text{in}\quad (\mathcal{S})^*,
\end{aligned}$$

which completes the proof. □

§5. Application of PFM Transform

The purpose of this section is to show a typical example of application of the Pseudo-Fourier-Mehler transform Ψ_θ to the Cauchy problem.

Example 6. (A simple application of the PFM transform) Let us consider the following abstract Cauchy problem on the white noise space:

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = iNu(t,x) + \varphi(x), \\ &u(0,\cdot) = f(\cdot) \in (\mathcal{S}), \end{aligned} \tag{19}$$

with $t > 0$, where N denotes the number operator. One of the most remarkable benefits of white noise analysis consists in its application to differential equation theory and how to solve the problem (cf. [1], [2,3], [4,8]). Especially in [4,8], by resorting to the analogy in the finite dimensional cases we have applied the infinite dimensional Kuo type Fourier transform to the Cauchy problem for heat equation type with Gross Laplacian, and have succeeded in derivation of the general solution and also in direct verification for existence and uniqueness of the solution. On this account, we think of using the Fourier transform to the aforementioned problem. Recall the formula:

$$\mathcal{F}(N\Phi) = N(\mathcal{F}\Phi) + \Delta_G^*(\mathcal{F}\Phi), \qquad \text{for all } \Phi \in (\mathcal{S})^*. \tag{20}$$

We set $v(t,y) \equiv (\mathcal{F}u(t,\cdot))(y)$ for each $t \in \mathbb{R}_+$. We may employ the Fourier transform $\mathcal{F}$ for (19) so as to obtain

$$\begin{aligned} &\frac{\partial v(t,y)}{\partial t} = iNv(t,y) + i\Delta_G^* v(t,y) + \hat{\varphi}(y), \\ &\text{with} \qquad v(0,y) = \hat{f}(y), \end{aligned} \tag{21}$$

because we made use of the formula (20) and set $\hat{F} = \mathcal{F}F$. The operator part of the Fourier transformed problem (21) is exactly equivalent to the infinitesimal generator of PFM transform with parameter t (see Theorem 12). Hence, the semigroup theory in functional equation theory allows immediately the following explicit exression of the solution in question:

$$v(t,y) = \Psi_t \hat{f}(y) + \int_0^t \Psi_{t-s}\hat{\varphi}(y)ds. \tag{22}$$

We can show the existence and uniqueness of the solution by applying Theorem 4 and Theorem 5 to (22) under a certain condition on the initial data φ, f. In that case the integral term appearing in (22) should be interpreted as Bochner type one. So much for the Cauchy problem, because this is not our main topic in this article. We shall go back to the PFM transform and proceed further in the next section.

§6. Intertwining Properties

In this section we shall investigate some intertwining properties between the Pseudo-Fourier-Mehler transform Ψ_θ and other typical operators in white noise analysis, such as Gâteaux differential, the adjoint of Gâteaux differential, Hida differential operator, and Kubo operator (the adjoint of Hida differential), etc. Furthermore, we shall introduce the characterization theorem for PFM transforms, which is one of our main results in this paper.

We begin with definition of the Gâteaux differential D_y in the direction $y \in \mathcal{S}^*$. For $y \in \mathcal{S}^*$ fixed, for the element φ in $(\mathcal{S})$ given by $\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$, we put

$$D_y\varphi(x) = \lim_{\theta \to 0} \frac{\varphi(x+\theta y) - \varphi(x)}{\theta}, \qquad x \in \mathcal{S}^*. \tag{23}$$

The limit existence in the right hand side of (23) is always guaranteed, and $D_y\varphi(x)$ is actually given by

$$D_y\varphi(x) = \sum_{n=0}^{\infty} n \langle : x^{\otimes(n-1)} :, y \hat{\otimes}_1 f_n \rangle, \qquad x \in \mathcal{S}^*. \tag{24}$$

In fact, D_y becomes a continuous linear operator from $(\mathcal{S})$ into itself. Since the Dirac delta function δ_t lies in $\mathcal{S}^*$, adoption of δ_t instead of y does make sense in the above (23) and (24). On the other hand, the Hida differential operator ∂_t $(= \partial/\partial x(t))$ is originally proposed by T. Hida [9] and defined by

$$\partial_t := S^{-1} \frac{\delta}{\delta \xi(t)} S, \qquad \xi \in \mathcal{S}$$

(cf. [15]; see also [7]). It is well known that the action of ∂_t is equivalent to that of D_{δ_t} on the dense domain [11] (or [7],[14]). So we can define

$$\partial_t = D_{\delta_t}, \qquad t \in \mathbb{R}.$$

The Kubo operator ∂_t^* [15] is the adjoint of Hida differential ∂_t, defined by

$$\langle\langle \partial_t^* \Phi, \varphi \rangle\rangle = \langle\langle \Phi, \partial_t \varphi \rangle\rangle,$$

for $\Phi \in (\mathcal{S})^*$, $\varphi \in (\mathcal{S})$. As a matter of fact, ∂_t (resp. ∂_t^*) can be considered as a continuous linear operator from $(\mathcal{S})$ (resp. $(\mathcal{S})^*$) into itself with respect to the weak or strong topology. More precisely, the Hida differential proves to be a continuous mapping from $(\mathcal{S})_{p+q}$ into $(\mathcal{S})_q$ for $q > \frac{1}{4}$, $p \geq 0$, while the Kubo operator turns out to be the one from $(\mathcal{S})_{-p}$ into $(\mathcal{S})_{-(p+q)}$ for the same pair p, q as given above. For $\xi \in \mathcal{S}$, $\varphi \in (\mathcal{S})$, the derivative $(D_\xi\varphi)(x)$ is defined in the usual manner, and there exists its extension $\tilde{D}_\xi : (\mathcal{S})^* \to (\mathcal{S})^*$. Even for that, we shall henceforth use the same notation D_ξ for brevity, as far as there is no confusion in the context. We set $q_\xi := i(D_\xi + D_\xi^*)$, where D_ξ^* is the adjoint of D_ξ.

Lemma 15. *For each $\theta \in \mathbb{R}$, $t \in \mathbb{R}$,*

$$\Psi_\theta(\partial_t^* \Phi) = e^{i\theta} \partial_t^* (\Psi_\theta \Phi)$$

holds for all $\Phi \in (\mathcal{S})^*$.

Proof. First of all, note that $S(\partial_t^*\Phi)(\xi) = \xi(t) \cdot S(\Phi)(\xi)$. So, for the generalized white noise functional $\Phi \in (\mathcal{S})^*$ given in the form $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$, $x \in \mathcal{S}^*$ we readily get

$$S(\Psi_\theta(\partial_t^* \Phi))(\xi) = \mathrm{e}^{i\theta} \xi(t) \cdot \sum_{n=0}^{\infty} \langle f_n, \mathrm{e}^{in\theta} \xi^{\otimes n} \rangle \cdot \exp\left(i \mathrm{e}^{i\theta} \sin\theta |\xi|^2\right). \tag{25}$$

While we establish

$$S(\Psi_\theta(\partial_t^* \Phi))(\xi) = \mathrm{e}^{i\theta} S(\partial_t^* (\Psi_\theta \Phi))(\xi) \tag{26}$$

by applying (25), because we made use of the relation

$$S(\partial_t^*(\Psi_\theta \Phi))(\xi) = \xi(t) \cdot (S\Phi)(\mathrm{e}^{i\theta}\xi) \cdot \exp\left(i \mathrm{e}^{i\theta} \sin\theta |\xi|^2\right).$$

An application of the Potthoff-Streit characterization theorem (Theorem 1) to (26) leads to the required equality in Hida distribution sense. □

Proposition 16. *For each $\theta \in \mathbb{R}$, $t \in \mathbb{R}$*

(i) $\Psi_\theta(\partial_t \Phi) = e^{-i\theta} \partial_t (\Psi_\theta \Phi) - 2i \sin\theta \partial_t^* (\Psi_\theta \Phi)$;

(ii) $\Psi_\theta(x(t)\Phi) = e^{-i\theta} x(t)(\Psi_\theta \Phi)$;

hold for all $\Phi \in (\mathcal{S})^*$.

Remark 7. The assertion (i) of Proposition 16 follows from a direct computation. We have only to employ the following two rules:

$$S\partial_t(\cdot) = \frac{\delta}{\delta\xi(t)} S(\cdot), \qquad \partial_t^*(\cdot) = S^{-1}\xi(t) S(\cdot).$$

The second assertion (ii) is also due to a simple computation together with the first assertion (i) and Lemma 15. Moreover, we need to apply the multiplication operator: $x(t)(\cdot) = (\partial_t + \partial_t^*)\,(\cdot)$ (e.g. [19]). Those proofs go almost similarly as in the proof of Lemma 15 and are very easy, hence omitted.

The next proposition indicates some intertwining property between the PFM transform and Gâteaux differential operator.

Proposition 17. *For each parameter $\theta \in \mathbb{R}$,$t \in \mathbb{R}$*

(i) $e^{-i\theta} \tilde{D}_\xi (\Psi_\theta \Phi) = \Psi_\theta(\tilde{D}_\xi \Phi) + 2i \sin\theta \cdot D_\xi^* (\Psi_\theta \Phi)$;

(ii) $\tilde{D}_\xi(\Psi_\theta \Phi) + D_\xi^*(\Psi_\theta \Phi) = e^{i\theta} \Psi_\theta(\widetilde{\langle \cdot, \xi \rangle} \Phi)$;

hold for all generalized white noise functionals in $(\mathcal{S})^*$.

Proof. It is interesting to note that Gâteaux differential D_ξ and its adjoint D_ξ^* enjoy the integral kernel operator theoretical expressions in white noise analysis (see the next section; or [11,12], [23]). Namely,

$$\tilde{D}_\xi := \left(\int_{\mathbb{R}} \xi(t) \partial_t dt \right)^{\sim}, \quad \text{and} \quad D_\xi^* := \int_{\mathbb{R}} \xi(t) \partial_t^* dt, \quad \forall \xi \in \mathcal{S}. \tag{27}$$

Let $\Delta = \{t_k\}$ be a proper finite partition of the t parameter space, and $|\Delta|$ denotes the maximum of increment Δt_k over $1 \leq k \leq m$. The assertion (i) yields from (i) of Proposition 16. In fact, by linearity of the PFM transform we get

$$(28) \qquad \sum_{k=1}^{m} \Delta t_k\, \xi(t_k) \cdot \Psi_\theta(\partial_{t_k}\Phi) = \Psi_\theta\Big(\sum_{k=1}^{m} \xi(t_k)\partial_{t_k}\Delta t_k \cdot \Phi\Big),$$

for $\forall \xi \in \mathcal{S}$. Consider the same type finite summation for the other terms in (i) of Proposition 16. By taking the limit $m \to \infty$ and by continuity of Ψ_θ (Theorem 10), we can obtain the desired result with consideration of Eq.(27).
As to (ii), note first that we can have the expression

$$(29) \qquad \tilde{q}_\xi = i\widetilde{\langle x, \xi\rangle} = \Big(i \int_{\mathbb{R}} x(t)\xi(t)dt\Big)^{\sim},$$

by virtue of the multiplication operator $x(t)(\cdot)$ (cf. Remark 7). With (ii) of Proposition 16, we may take advantage of continuity of Ψ_θ and (29) to deduce that

$$\begin{aligned} e^{-i\theta}(D_\xi + D_\xi^*)(\Psi_\theta\Phi) &= e^{-i\theta}\Big(\int_{\mathbb{R}} x(t)\xi(t)dt\Big)(\Psi_\theta\Phi) \\ &= \lim_{m\to\infty} \Psi_\theta\Big(\sum_{k=0}^{m} \Delta t_k \xi(t_k)x(t_k) \cdot \Phi\Big) \\ &= \Psi_\theta(\langle x, \xi\rangle \cdot \Phi) \end{aligned}$$

by passage to the limit $|\Delta| \to 0$. □

The following theorem gives the characterization for Pseudo-Fourier-Mehler transforms $\{\Psi_\theta;\ \theta \in \mathbb{R}\}$, which is one of our main results in this paper.

Theorem 18 [6]. *The Pseudo-Fourier-Mehler transform $\{\Psi_\theta; \theta \in \mathbb{R}\}$ satisfies the following conditions:*
(P1) $\Psi_\theta : (\mathcal{S})^* \to (\mathcal{S})^*$ is a continuous linear operator for $forall \theta \in \mathbb{R}$;
(P2) $\Psi_\theta(\tilde{D}_\xi\Phi) = e^{i\theta}\tilde{D}_\xi(\Psi_\theta\Phi) - 2\sin\theta \cdot \tilde{q}_\xi(\Psi_\theta\Phi)$;
(P3) $\Psi_\theta(\tilde{q}_\xi\Phi) = e^{-i\theta}\tilde{q}_\xi(\Psi_\theta\Phi)$;
where $\Phi \in (\mathcal{S})^*$, $\xi \in \mathcal{S}$. Conversely, if a continuous linear operator $A_\theta : (\mathcal{S})^* \to (\mathcal{S})^*$ satisfies the above conditions: (P1) $\sim$ (P3), then A_θ is a constant multiple of Ψ_θ.

Proof. (P1) is obvious (Theorem 10). (P2)(resp. (P3)) yields from (i)(resp. (ii)) of Proposition 17. It is due to a simple computation. Conversely, suppose that the operator A_θ be compatible with (P1),(P2) and (P3). We need the following results.

Lemma 19. *We assume that A_θ be a continuous linear operator from $(\mathcal{S})^*$ into itself, satisfying the three conditions (P1) $\sim$ (P3). Then the following relations*
(i) $(\Psi_\theta^{-1}\Xi_\theta)D_\xi = D_\xi(\Psi_\theta^{-1}\Xi_\theta)$;
(ii) $(\Psi_\theta^{-1}\Xi_\theta)q_\xi = q_\xi(\Psi_\theta^{-1}\Xi_\theta)$;
(iii) $(\Psi_\theta^{-1}\Xi_\theta)D_\xi^* = D_\xi^*(\Psi_\theta^{-1}\Xi_\theta)$;
hold for all $\xi \in \mathcal{S}$, $\theta \in \mathbb{R}$.

The proof will be given below. The next theorem is well known (e.g. [12, Theorem 3.6, p.267] or [23, Prop.5.7.6, p.148]).

Theorem 20. *Let Λ be a continuous linear operator on $(\mathcal{S})^*$, satisfying*
(i) $\Lambda\tilde{q}_\xi = \tilde{q}_\xi\Lambda$, for any $\xi \in \mathcal{S}$;
(ii) $\Lambda D_\xi^* = D_\xi^*\Lambda$, for any $\xi \in \mathcal{S}$.
Then the operator Λ is a scalar operator.

Thus, by taking (ii),(iii) of Lemma 19 into account, we may apply Theorem 20 for A_θ to obtain the assertion: $\Psi_\theta^{-1}A_\theta$ is a scalar operator. □

Proof of Lemma 19. Basically it is due to a direct computation. Each proof goes similarly, so we shall show only (iii) below. For the other two we will give just rough instructions. First of all, note that we have only to consider $\Psi_{-\theta}$ instead of Ψ_θ^{-1} by virtue of Proposition 3. As to (i), it is sufficient to calculate it with (P2) for both and (P3) for the PFM transform. As for (ii), simply (P3) for both A_θ and Ψ_θ. As to (iii), for $\forall\Phi \in (\mathcal{S})^*$, $\forall\xi \in \mathcal{S}$

$$
\begin{aligned}
(\Psi_\theta^{-1}A_\theta)D_\xi^*\Phi &= -i\Psi_\theta^{-1}(A_\theta q_\xi)\Phi - \Psi_\theta^{-1}(A_\theta D_\xi)\Phi, \\
&= -e^{i\theta}(\Psi_\theta^{-1}q_\xi)A_\theta\Phi - e^{i\theta}(\Psi_\theta^{-1}D_\xi)A_\theta\Phi
\end{aligned} \tag{30}
$$

because we used a relation

$$
D_\xi^* = -i\tilde{q}_\xi - \tilde{D}_\xi \tag{31}
$$

in the first equality and also employed (P2),(P3) in the second one. An application of (P2),(P3) to the last expression in (30), together with (31) again, gives

$$
\begin{aligned}
(30) &= -iq_\xi(\Psi_\theta^{-1}A_\theta)\Phi - D_\xi(\Psi_\theta^{-1}A_\theta)\Phi \\
&= (-iq_\xi - D_\xi)(\Psi_\theta^{-1}A_\theta)\Phi = D_\xi^*(\Psi_\theta^{-1}A_\theta)\Phi,
\end{aligned}
$$

which completes the proof. □

§7. Fock Expansion

Let $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ denote the space of continuous linear operators from $(\mathcal{S})$ into $(\mathcal{S})^*$. The space $\hat{\mathcal{S}}'_{l,m}(\mathbb{R}^{l+m})$ is a symmtrized space of $\mathcal{S}'(\mathbb{R}^{l+m})$ with respect to the first l, and the second m variables independently. By virtue of the symbol characterization theorem for operators on white noise functionals [21](see also [23]), for the operator Π lying in $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ there exists uniquely a kernel distribution $\kappa_{l,m}$ in $\hat{\mathcal{S}}'_{l,m}(\mathbb{R}^{l+m})$ such that the operator Π may have the Fock expansion:

$$\Pi = \sum_{l,m=0}^{\infty} \Pi_{l,m}(\kappa_{l,m}).$$

Moreover, the series $\Pi\varphi$, $\varphi \in (\mathcal{S})$ converges in $(\mathcal{S})^*$ [21]. Generally, each component $\Pi_{l,m}$ of the Fock expansion has a formal integral expression:

$$\Pi_{l,m}(\kappa) = \int_{\mathbb{R}^{l+m}} \kappa(s_1,\cdots,s_l,t_1,\cdots,t_m)\cdot \\ \cdot\, \partial^*_{s_1}\cdots\partial^*_{s_l}\partial_{t_1}\cdots\partial_{t_m}ds_1\cdots ds_l dt_1\cdots dt_m.$$

Remark 8. We call it an integral kernel operator with kernel distribution κ. The theory of integral kernel operators and the general expansion theory in white noise analysis were proposed and have been developed enthusiastically by N. Obata [21-23] (see also [11]). Those topics are closely related to quantum stochastic calculus, which has been greatly investigated in chief by Hudson, Meyer, and Parthasarathy. More details on this topic will be found in, for instance, (i) K.R.Parthasarathy: *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel, 1992; (ii) P.A.Meyer: *Quantum Probability for Probabilists*, Lecture Notes in Mathematics Vol.1538, Springer-Verlag, Heidelberg, 1993.

We shall give below two typical examples of the integral kernel operators in white noise analysis.

Example 9. (The number operator N) Let $\tau \in (\mathcal{S}\otimes\mathcal{S})^*$ be the trace operator defined by

$$\langle\tau,\xi\otimes\eta\rangle = \langle\xi,\eta\rangle, \qquad \xi,\eta\in\mathcal{S}.$$

The number operator N is usually expressed as

$$\int_{\mathbb{R}} \partial^*_t\partial_t dt$$

by Kuo's notation in white noise analysis. By the Obata theory, N has the following representation as a continuous linear operator from $(\mathcal{S})$ into itself, namely,

$$N = \Pi_{1,1}(\tau) = \int_{\mathbb{R}^2} \tau(s,t)\partial^*_s\partial_t ds dt.$$

Example 10. (The Gross Laplacian Δ_G) By the usual notation in white noise analysis we have the expression

$$\Delta_G = \int_{\mathbb{R}} \partial^2_t dt.$$

Then the Gross Laplacian Δ_G can be also expressed by

$$\Delta_G = \Pi_{0,2}(\tau) = \int_{\mathbb{R}^2} \tau(s_1, s_2)\partial_{s_1}\partial_{s_2} ds_1 ds_2$$

as a continuous linear operator from $(\mathcal{S})$ into $(\mathcal{S})$.

Let us consider the general expansion of our Pseudo-Fourier-Mehler transform. We may take advantage of Obata's integral kernel operator theory in order to obtain Fock expansion representations of Ψ_θ and its adjoint G_θ. That is to say,

Theorem 21. *For $\theta \in \mathbb{R}$, the PFM transform Ψ_θ and the adjoint operator G_θ have the following Fock expansions:*

$$(i) \quad \Psi_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}(ie^{i\theta}\sin\theta)^l(e^{i\theta}-1)^m \cdot \Pi_{2l+m,m}(\tau^{\otimes l}\otimes\lambda_m);$$

$$(ii) \quad G_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}(ie^{i\theta}\sin\theta)^m(e^{i\theta}-1)^l \cdot \Pi_{l,l+2m}(\lambda_l\otimes\tau^{\otimes m});$$

where the kernel $\lambda_m \in (\mathcal{S}^{\otimes 2m})^*$ is given by

$$\lambda_m := \sum_{i_1,i_2,\cdots,i_m=0}^{\infty} e_{i_1}\otimes\cdots\otimes e_{i_m}\otimes e_{i_1}\otimes\cdots\otimes e_{i_m}.$$

§8. Generalization

Let $GL((\mathcal{S}))$ be the group of all linear homeomorphisms from $(\mathcal{S})$ into $(\mathcal{S})$. Then we have

Proposition 22. *$\{G_\theta; \theta \in \mathbb{R}\}$ is a regular one parameter subgroup of $GL((\mathcal{S}))$ with infinitesimal generator $i(N + \Delta_G)$.*

Let us consider some generalization. Suggested by [1], for example we propose to define the generalized PFM transform X_θ, $\theta \in \mathbb{R}$ as operator on $(\mathcal{S})^*$ whose U-functional is given by

$$S(X_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\Big(e^{\alpha\theta}\langle\cdot,\xi\rangle - \frac{1}{2}J(\alpha,\beta;\theta)|\xi|^2\Big)\rangle\rangle, \tag{32}$$

(cf. Eq. (7) in Definition 1 of PFM transform), for $\xi \in \mathcal{S}$, $\Phi \in (\mathcal{S})^*$. We set

$$J(\alpha,\beta;\theta) = e^{2\alpha\theta} - 2H(\alpha,\beta;\theta),$$
$$\text{with} \quad H(\alpha,\beta;\theta) = h(\alpha,\beta)\cdot(e^{2\alpha\theta}-1),$$

where $h(\alpha,\beta) = \beta/2\alpha$, for α, $\beta \in \mathbb{C}$, $\alpha \neq 0$. Then we denote the adjoint operator of X_θ by Z_θ.

Claim 23. *The set $\{Z_\theta; \theta \in \mathbb{R}\}$ is a regular one parameter subgroup of $GL((\mathcal{S}))$.*

Claim 24. *The infinitesimal generator of* $\{Z_\theta; \theta \in \mathbb{R}\}$ *is given by the operator* $\alpha N + \beta \Delta_G$.

Claim 25. *The generalized PFM transform* $\{X_\theta;\ \theta \in \mathbb{R}\}$ *is a one parameter subgroup of* $GL((\mathcal{S})^*)$.

Claim 26. *The infinitesimal generator of* $\{X_\theta; \theta \in \mathbb{R}\}$ *is given by the operator* $\alpha N + \beta \Delta_G^*$.

Remark 11. The above definition (32) of generalized PFM transform X_θ can be alternatively replaced by the following expression:

$$S(X_\theta \Phi)(\xi) = F(\mathrm{e}^{\alpha\theta}\xi) \cdot \exp\Big(H(\alpha, \beta; \theta) \cdot |\xi|^2\Big),$$

where F denotes the U-functional of Φ in $(\mathcal{S})^*$, i.e., $S\Phi = F$.

Remark 12. Especially when $\alpha = \beta = i(\in \mathbb{C})$, then the above-defined generalized PFM transforms X_θ are, of course, attributed to the simple PFM transforms Ψ_θ given by (6), (7) in the section 3.

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Some Intertwining Properties of the Pseudo-Fourier-Mehler Transform*

ISAMU DÔKU**

§ 1. Introduction

The study of the Fourier transform in white noise calculus was initiated and has been developed to a mature level by H. -H. Kuo [12]. While, the Fourier-Mehler transform is a kind of generalization of Fourier transform [13], which furnishes the theory of infinite dimensional Fourier transforms in white noise space.

In this article we investigate Pseudo-Fourier-Mehler (PFM for short) transform having quite similar nice properties as the Fourier-Mehler transform possesses. It was originally defined in [3] and used for application to abstract equations in infinite dimensional spaces [3] (see also [1], [2], [4], [6-9]). In connection with other Fourier type transforms in white noise analysis, we can compute the infinitesimal generator of the PFM transform directly and show that our Pseudo-Fourier-Mehler transform enjoys intertwining properties.

§ 2. Notation and Preliminaries

Let $S\equiv S(\mathbf{R})$ be the Schwartz class space on $\mathbf{R}$ and $S^*\equiv S'(\mathbf{R})$ its dual space. Then $S(\mathbf{R})\subset L^2(\mathbf{R})\subset S'(\mathbf{R})$ is a Gelfand triple. We define the family of norms given by $|\xi|_p=|A^p\xi|$, $p>0$, $\xi\in S(\mathbf{R})$, where the operator $A=-d^2/dt^2+t^2+1$ and $|\cdot|$ is the $L^2(\mathbf{R})$-norm. Let $S_p\equiv S_p(\mathbf{R})$ be the completion of $S(\mathbf{R})$ with respect to the norm $|\cdot|_p$, $p>0$. We denote its dual space by $S_p^*\equiv S_p'(\mathbf{R})$, and we have $S_p(\mathbf{R})\subset L^2(\mathbf{R})\subset S_p'(\mathbf{R})$. Let μ be the standard Gaussian measure on $S'(\mathbf{R})$ such that

$$\int_{S^*}\exp\left(\sqrt{-1}\langle x,\xi\rangle\right)\mu(dx)=\exp\left(-\frac{1}{2}|\xi|^2\right),$$

for any $\xi\in S(\mathbf{R})$. (L^2) denotes the Hilbert space of complex-valued μ-square integrable functionals with norm $\|\cdot\|$. The Wiener-Itô decomposition theorem gives the unique representation of φ in (L^2), i.e.,

$$\varphi=\sum_{n=0}^{\infty}I_n(f_n),\qquad f_n\in\hat{L}^2_{\mathrm{C}}(\mathbf{R}^n), \tag{1}$$

where I_n denotes the multiple Wiener integral of order n and $\hat{L}^2_{\mathrm{C}}(\mathbf{R}^n)$ the space of symmetric complex valued L^2-functions on $\mathbf{R}^n$. The second quantization operator $\Gamma(A)$ is densely defined on

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(L^2) as follows: for $\varphi=\sum_{n=0}^{\infty} I_n(f_n)\in \mathrm{Dom}(\Gamma(A))$,

(2) $$\Gamma(A)\varphi=\sum_{n=0}^{\infty} I_n(A^{\otimes n}f_n).$$

For $p\in\mathbf{N}$, define $\|\varphi\|_p=\|\Gamma(A)^p\varphi\|$ and let $(S)_p\equiv\{\varphi\in(L^2)\ ;\ \|\varphi\|_p<\infty\}$ and the dual space of $(S)_p$ is denoted by $(S)^*_p$. Let (S) be the projective limit of$\{(S)_p\ ;\ \mathrm{P}\in\mathbf{N}\}$. It is called a space of test white noise functionals. The elements in the dual space $(S)^*$ of (S) are called generalized white noise functionals or Hida distributions. In fact, $(S)\subset(L^2)\subset(S)^*$is a Gelfand triple [10]. For convention all dual pairings$\langle\cdot,\cdot\rangle$, resp. $\langle\langle\cdot,\cdot\rangle\rangle$mean the canonical bilinear forms on $S^*\times S$(resp. $(S)^*\times(S)$) unless otherwise stated.

The S-transform of $\Phi\in(S)^*$ is a function on S defined by

(3) $$(S\Phi)(\xi):=\langle\langle\Phi,:\exp\langle\cdot,\xi\rangle:\rangle\rangle,\qquad \xi\in S(\mathbf{R}),$$

where $:\exp\langle\cdot,\xi\rangle:\ \equiv\exp\langle\cdot,\xi\rangle\cdot\exp(-\frac{1}{2}|\xi|^2)$. Then note that a mapping: $\mathbf{C}\ni z\mapsto(S\Phi)(z\xi+\eta)$ is entire holomorphic for any ξ, $\eta\in S$. A complex valued function F on S is called a *U-functional* if and only if it is ray entire on S and if there exist constants C_1, $C_2>0$, and $p\in\mathbf{N}\cup\{0\}$ so that the estimate

$$|F(z\xi)|\le C_1\exp(C_2|z|^2|\xi|_p^2)$$

may hold for all $z\in\mathbf{C}$, $\xi\in S$. We have the following Potthoff-Streit Characterization [10, 14]:

THEOREM 1. *If $\Phi\in(S)^*$, then $S\Phi$ is a U-functional. Conversely, if F is a U-functional, then there exists a unique element Φ in $(S)^*$ such that $S\Phi$-F holds.*

Based upon the above characterization we are able to give rigorous definitions to Fourier type transforms of infinite dimensions. The Kuo type Fourier transform $\mathcal{F}$ [12] of a generalized white noise functional Φ in $(S)^*$ is the generalized white noise functional, S-transformation of which is given by

(4) $$S(\mathcal{F}\Phi)(\xi)=\langle\langle\Phi,\exp(-i\langle\cdot,\xi\rangle)\rangle\rangle,\qquad \xi\in S.$$

Likewise, the Fourier-Mehler transform $\mathcal{F}_\theta$ $(\theta\in\mathbf{R})$ [13] of a generalized white noise functional Φ in $(S)^*$ is the generalized white noise functional, S-transformation of which is given by

(5) $$S(\mathcal{F}_\theta\Phi)(\xi)=\langle\langle\Phi,\exp\{e^{i\theta}\langle\cdot,\xi\rangle-\frac{1}{2}e^{i\theta}\cos\theta|\xi|^2\}\rangle\rangle,\quad \xi\in S.$$

The Fourier-Mehler transform $\mathcal{F}_\theta$, $\theta\in\mathbf{R}$ is a generalization of the Kuo type Fourier transform $\mathcal{F}$. Actually, $\mathcal{F}_0=Id$, and $\mathcal{F}_{-\pi/2}$ is coincident with the Fourier transform $\mathcal{F}$. It is easy to see that $\mathcal{F}_{\pi/2}$ is the inverse Fourier transform $\mathcal{F}^{-1}$. Hence we have

$$S(\mathcal{F}^{-1}\Phi)(\xi)=(S\Phi)(i\xi)\exp(-\frac{1}{2}|\xi|^2),\qquad \xi\in S.$$

§ 3. Pseudo-Fourier-Mehler Transform

We begin with introducing the Pseudo-Fourier-Mehler transform in white noise analysis.

Definition 1. *$\{\Psi_\theta,\ \theta\in\mathbf{R}\}$ is said to be the Pseudo-Fourier-Mehler (PFM) transform* [3] *if Ψ_θ is a mapping from $(S)^*$ into itself for $\theta\in\mathbf{R}$, whose U-functional is given by*

(6) $$S(\Psi_\theta\Phi)(\xi)=F(e^{i\theta}\xi)\cdot\exp(ie^{i\theta}\sin\theta|\xi|^2),\quad \xi\in S,$$

or equivalently

(7) $$S(\Psi_\theta\Phi)(\xi)=\langle\langle\Phi, \exp(e^{i\theta}\langle\cdot, \xi\rangle-\frac{1}{2}|\xi|^2)\rangle\rangle, \quad \xi\in S,$$

for $\Phi\in(S)^*$, *where* S *is the S-transform in white noise analysis and F denotes the U-functional of* Φ.

By virtue of Theorem 1, the right hand sides in Eq.(6) and Eq.(7) are U-functionals, and $\Psi_\theta\Phi$ exists for each Φ in $(S)^*$. Therefore the above-mentioned Pseudo-Fourier-Mehler transform is well-defined. Hence we have

Proposition 2 [3]. *The following properties hold:*

(i) $\Psi_0=Id$; (*Id denotes the identity operator.*)

(ii) $\Psi_\theta\neq\mathcal{F}$ *for any* $\theta\in\mathbf{R}\setminus\{0\}$;

(iii) $\Psi_\theta\neq\mathcal{F}_\theta$ *for any* $\theta\in\mathbf{R}\setminus\{0\}$.

Proposition 3 [3]. *The invese operator of the Pseudo-Fourier-Mehler transform* Ψ_θ *is given by* $(\Psi_\theta)^{-1}=\Psi_{-\theta}$ *for* $\theta\in\mathbf{R}$.

Next let us consider what the image of the space (S) under Ψ_θ is like (see Corollary 5 below). The Pseudo-Fourier-Mehler transform Ψ_θ also enjoys some interesting properties on the product of Gaussian white noise functionals (see Theorem 4).

Theorem 4. *The following equalities hold for any* $\theta\in\mathbf{R}$:

(i) *if* $K(\theta)=2^{-1}ie^{-i\theta}\csc\theta-1$, *then*

$$\Psi_\theta\Phi=\Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)}, \qquad \Phi\in(S)^*;$$

(ii) *if* $A(\theta)=2^{-1}ie^{-i\theta}\csc\theta-1$, *then*

$$\|\Psi_\theta\Phi\|_p=\|\Phi : g_{A(\theta)}\|_p, \qquad \Phi\in(S)_p$$

for all $p\in\mathbf{R}$.

Let us think of the image of $\varphi\in(S)$ under the Pseudo-Fourier-Mehler transform. It is easily checked that $g_c : g_d=1$ holds with $c+d=-2$. So we have

(8) $\quad g_{c(\theta)} : g_{K(\theta)}=1.$

Immediately, $\varphi\in(S)$ if and only if

$$\Psi_\theta\varphi : g_{c(\theta)}\in(S),$$

so that, it is equivalent to

$$\Psi_\theta\varphi : g_{c(\theta)} : g_{K(\theta)}\in(S) : g_{K(\theta)},$$

where $(S) : g_{K(\theta)}$ denotes the whole space of elements $\varphi : g_{K(\theta)}$ for $\varphi\in(S)$. Consequently, it is obvious that $\Psi_\theta\varphi\in(S) : g_{K(\theta)}$, by virtue of Eq.(8). Therefore we obtain

Corollary 5. *For* $\theta\in\mathbf{R}$,

$$\mathrm{Im}\Psi_\theta(S)=(S) : g_{K(\theta)}\equiv\{\varphi : g_{K(\theta)};\ \varphi\in(S)\}$$

where $K(\theta)=2^{-1}ie^{i\theta}\csc\theta-1$.

Remark 2. The results in Theorem 4 is quite similar to that of the Fourier-Mehler transform. In fact, for $p\in\mathbf{R}$, $\Phi\in(S)_p$,

$$\|(\mathcal{F}_\theta\Phi) : g_{c_1(\theta)}\|_p=\|\Phi\|_p \text{ and } \|F_\theta\Phi\|_p=\|\Phi : g_{c_2(\theta)}\|_p$$

hold with $c_1(\theta)=-i\cot\theta-2$, and $c_2(\theta)=i\cot\theta-2$ (e.g. [10, §9. H]).

Remark 3. The image of (S) under the Fourier-Mehler transform $\mathcal{F}_\theta$ is given by $(S) : g_{i\cot\theta}$, while that of (S) under the Fourier transform $\mathcal{F}$ coincides with the space

$$(S) : \tilde{\delta}_0\equiv\{\varphi : \tilde{\delta}_0;\ \varphi\in(S)\},$$

where $\tilde{\delta}_0$ is the delta function at 0 and

$$\lim_{c\to 0} g_c = \tilde{\delta}_0$$

(e.g. [10, Chapter 9]).

§ 4. Infinitesimal Generators

First of all, for all $\theta \in S$ we define

$$\varphi_\xi(x) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle : x^{\otimes n} :, \xi^{\otimes n} \rangle$$

with $x \in S^*$, $\xi \in S$. We call it an exponential vector. Then $\{G_\theta, \theta \in \mathbf{R}\}$ is an operator on (S) defined by

(9) $\quad (G_\theta \varphi_\xi)(x) := \varphi e^{i\theta} \xi(x) \cdot \exp(ie^{i\theta} \sin\theta |\xi|^2).$

Let τ denote the distribution in $(S \otimes S)^*$ given by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in S.$$

Note that it can be expressed as

$$\tau = \int_{\mathbf{R}} \delta_t \otimes \delta_t dt = \sum_{j=0}^{\infty} e_j \otimes e_j \in (S \otimes S)^*,$$

where $\{e_n\}$ denotes a complete orthonormal basis for $L^2(\mathbf{R})$. Moreover we have

$$\tau^{\otimes n} = \int_{\mathbf{R}} \delta_{t1} \otimes \delta_{t1} \otimes \cdots \otimes \delta_{tn} \otimes \delta_{tn} d_{t1} \cdots dt_n.$$

The following is an easy exercise. The next lemma provides with a general expression for elements of general form in (S).

Lemma 6. *When $\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle \in (S)$ with $f_n \in \hat{S}(\mathbf{R}^n)$, (the symmetric $S(\mathbf{R}^n)$), then $G_\theta \varphi$ is given by*

$$(G_\theta \varphi)(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, g_n \rangle,$$

and

$$g_n \equiv g_n(\varphi)$$
$$= \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} (i\sin\theta)^m e^{i(n+m)\theta} \tau^{\otimes m} * f_{2m+n},$$

where for the element f_{2m+n} in $\hat{S}(\mathbf{R}^{2m+n})$ the term $\tau^{\otimes m} * f_{2m+n}$ actually has the following integral expession

$$(\tau^{\otimes m} * f_{2m+n})(t_1, \cdots, t_n)$$
$$= \int_{\mathbf{R}^m} f_{2m+n}(s_1, s_1 \cdots, s_m, s_m, t_1, \cdots, t_n) ds_1 \cdots ds_m.$$

On this account, we obtain immediately

Proposition 7. *The Pseudo-Fourier-Mehler transform $\{\Psi_\theta ; \theta \in \mathbf{R}\}$ is given by the adjoint operator of $\{G_\theta ; \theta \in \mathbf{R}\}$, i.e.,*

$$\Psi_\theta = G_\theta^*$$

holds in operator equality sense for all $\theta \in \mathbf{R}$.

The next proposition gives an explicit action of the PFM transform Ψ_θ for the generalized white noise functionals of general form. It is due to a direct computation.

Proposition 8. *For $\Phi \in (S)^*$ given as $\Phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:, F_n\rangle$, it holds that*

$$\Psi_\theta \Phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:, \sum_{l+2m=n} a(l, m, \theta)\cdot F_l \hat{\otimes} \tau^{\otimes m}\rangle,$$

where the constant $a(l, m, \theta)$ is given by

$$a(l, m, \theta)=\frac{1}{m!}e^{i(l+m)\theta}(i\sin\theta)^m.$$

Remark 4. Similar results for Fourier-Mehler transform as the above can be found in [14]. For the proof of Proposition 8, it is almost the same as those given in [14].

It follows from Proposition 3 that the Pseudo-Fourier-Mehler transform Ψ_θ is injective and surjective. Moreover, it is easy to check that Ψ_θ is strongly continuous operator from $(S)^*$ into itself, when we take Lemma 6 and Proposition 7 into consideration. Thus we have the following theorem.

Theorem 9 [3] [4] [6]. *The Pseudo-Fourier-Mehler transmform $\Psi_\theta : (S)^* \to (S)^*$ is a bijective and strongly continuous linear operator.*

Theorem 10 [3] [4] [6]. *The set $\{\Psi_\theta ; \theta \in \mathbf{R}\}$ forms a one parameter group of strongly continuous linear operator acting on the space $(S)^*$ of Hida distributions.*

We are now in a position to state the following remarkable result on PFM transform. This is a very important property of the Pseudo-Fourier-Mehler transform, especially on an applicational basis [3] [4] [6-8] (see also [1], [2], [9]).

Theorem 11 [3] [4] [7]. *The infinitesimal generator of $\{\Psi_\theta ; \theta \in \mathbf{R}\}$ is given by $i(N+\Delta_G^*)$, where N is the number operator and Δ_G^* is the adjoint of the Gross Laplacian Δ_G.*

Remark 5. It is well known that the infinitesimal generator of the Fourier-Mehler transforms $\{\mathcal{F}_\theta ; \theta \in \mathbf{R}\}$ is $iN+\frac{i}{2}\Delta_G^*$, while the adjoint operator of $\{\mathcal{F}_\theta ; \theta \in \mathbf{R}\}$has $iN+\frac{i}{2}\Delta_G$ as its infinitesimal generator (e.g. see [10]). The proof of Theorem 11 is almost similsr to the above ones.

§ 5. Intertwining Properties

In this section we shall investigate some intertwining properties between the Pseudo-Fourier-Mehler transform Ψ_θ and other typical operators in white noise analysis, such as Gâteaux differential, the adjoint of Gâteaux differential, Hida differential operator, and Kubo operator (the adjoint of Hida differential), etc. Furthermore, we shall introduce the characterization theorem for PFM transforms, which is one of our main results in this paper.

We begin with definition of the Gâteaux differential D_y in the direction $y \in S^*$. For $y \in S^*$ fixed, for the element φ in (S) given by $\varphi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:, f_n\rangle$, we put

$$D_y\varphi(x)=\lim_{\theta\to 0}\frac{\varphi(x+\theta y)-\varphi(x)}{\theta}, \qquad x \in S^*. \tag{10}$$

The limit existence in the right hand side of (10) is always guaranteed, and $D_y\varphi(x)$ is actually given by

(11) $\qquad D_y\varphi(x)=\sum_{n=0}^{\infty} n\langle :x^{\otimes(n-1)}:, y\widehat{\otimes}_1 f_n\rangle, \qquad x\in S^*.$

In fact, D_y becomes a continuous linear operator from (S) into itself. Since the Dirac delta function δ_t lies in S^*, adoption of δ_t instead of y does make sense in the above (10) and (11).On the other hand, the Hida differential operator $\partial_t(=\partial/\partial x(t))$ is originally proposed by T. Hida [10] and defined by

$$\partial_t:=\mathrm{S}^{-1}\frac{\delta}{\delta\xi(t)}\mathrm{S}, \quad \xi\in S$$

(cf. [5]). It is well known that the action of ∂_t is equivalent to that of $\mathrm{D}_{\delta t}$ on the dense domain [10] (or [5]). So we can define

$$\partial_t=D_{\delta t}, \qquad t\in\mathbf{R}.$$

The Kubo operator ∂_t^* is the adjoint of Hida differential ∂_t, defined by

$$\langle\langle\partial_t^*\Phi, \varphi\rangle\rangle=\langle\langle\Phi, \partial_t\varphi\rangle\rangle ,$$

for $\Phi\in(S)^*_;$ $\varphi\in(S)$. As a matter of fact, ∂_t (resp. ∂_t^*) can be considered as a continuous linear operator from (S) (resp. $(S)^*$) into itself with respect to the weak or strong topology. More precisely, the Hida differential proves to be a continuous mapping from $(S)_{p+q}$ into $(S)_q$ for $q>\frac{1}{4}$, $p>0$, while the Kubo operator turns out to be the one from $(S)_{-p}$ into $(S)_{-(p+q)}$ for the same pair p, q as given above. For $\xi\in S$, $\varphi\in(S)$, the derivative $(D_\xi\varphi)(x)$ is defined in the usual manner, and there exists its extension $\tilde{D}_\xi:(S)^*\to(S)^*$. Even for that, we shall henceforth use the same notation D_ξ for brevity, as far as there is no confusion in the context. We set $q_\xi:=i(D_\xi+D_\xi^*)$, where D_ξ^* is the adjoint of D_ξ.

Lemma 12. *For each* $\theta\in\mathbf{R}$, $t\in\mathbf{R}$,

$$\Psi_\theta(\partial_t^*\Phi)=e^{i\theta}\partial_t^*(\Psi_\theta\Phi)$$

holds for all $\Phi\in(S)^*$.

Proof. First of all, note that $S(\partial_t^*\Phi)(\xi)=\xi(t)\cdot S(\Phi)(\xi)$. So, for the generalized white noise functional $\Phi\in(S)^*$ given in the form $\Phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:, f_n\rangle$, $x\in S^*$ we readily get

(12) $\qquad S(\Psi_\theta(\partial_t^*\Phi))(\xi)$

$$=e^{i\theta}\xi(t)\cdot\sum_{n=0}^{\infty}\langle f_n, e^{in\theta}\xi^{\otimes n}\rangle\cdot\exp(ie^{i\theta}\sin\theta|\xi|^2).$$

While we establish

(13) $\qquad S(\Psi_\theta(\partial_t^*\Phi))(\xi)=e^{i\theta}S(\partial_t^*(\Psi_\theta\Phi))(\xi)$

by applying (12), because we made use of the relation

$$S(\partial_t^*(\Psi_\theta\Phi))(\xi)=\xi(t)\cdot(S\Phi)(e^{i\theta}\xi)\cdot\exp(ie^{i\theta}\sin\theta|\xi|^2).$$

An application of the Potthoff-Streit characterization theorem (Theorem 1) to (13) leads to the required equality in Hida distribution sense.

Proposition 13. *For each* $\theta\in\mathbf{R}$, $t\in\mathbf{R}$

(i) $\Psi_\theta(\partial_t\Phi)=e^{-i\theta}\partial_t(\Psi_\theta\Phi)-2i\sin\theta\partial_t^*(\Psi_\theta\Phi)$;

(ii) $\Psi_\theta(x(t)\Phi)=e^{-i\theta}x(t)(\Psi_\theta\Phi)$;

hold for all $\Phi\in(S)^*$.

Remark 6. The assertion (i) of Proposition 13 follows from a direct computation. We have only to employ the following two rules:

$$S\partial_t(\cdot)=\frac{\delta}{\delta\xi(t)}S(\cdot), \qquad \partial_t^*(\cdot)=S^{-1}\xi(t)S(\cdot).$$

The second assertion (ii) is also due to a simple computation together with the first assertion (i) and Lemma 12. Moreover, we need to apply the multiplication operator: $x(t)(\cdot)=(\partial_t+\partial_t^*)(\cdot)$ (e.g. [10]). Those proofs go almost similarly as in the proof of Lemma 12 and are very easy, hence omitted.

The next proposition indicates some intertwining property between the PFM transform and Gâteaux differential operator.

Proposition 14. *For each parameter* $\theta\in\mathbf{R}$, $t\in\mathbf{R}$

(i) $e^{-i\theta}\tilde{D}_\xi(\Psi_\theta\Phi)=\Psi_\theta(\tilde{D}_\xi\Phi)+2i\sin\theta\cdot D_\xi^*(\Psi_\theta\Phi)$;

(ii) $\tilde{D}_\xi(\Psi_\theta\Phi)+D_\xi^*(\Psi_\theta\Phi)=e^{i\theta}\Psi_\theta(\langle\tilde{\cdot};\xi\rangle\Phi)$;

hold for all generalized white noise functionals in $(S)^*$.

Proof. It is interesting to note that Gâteaux differential D_ξ and its adjoint D_ξ^* enjoy the integral kernel operator theoretical expressions in white noise analysis (see [14]; or [10, 11]). Namely,

$$\tilde{D}_\xi:=\left(\int_{\mathbf{R}}\xi(t)\,\partial_t dt\right)^{\sim}, \text{ and } D_\xi^*:=\int_{\mathbf{R}}\xi(t)\,\partial_t^* dt,\ \forall\xi\in S. \tag{14}$$

Let $\Delta=\{t_k\}$be a proper finite partition of the t parameter space, and $|\Delta|$ denotes the maximum of increment Δt_k over $1\le k\le m$. The assertion (i) yields from (i) of Proposition 13. In fact, by linearity of the PFM transform we get

$$\sum_{k=1}^{m}\Delta t_k\xi(t_k)\cdot\Psi_\theta(\partial_{t_k}\Phi)=\Psi_\theta\left(\sum_{k=1}^{m}\xi(t_k)\,\partial_{t_k}\Delta t_k\cdot\Phi\right), \tag{15}$$

for $\forall\xi\in S$. Consider the same type finite summation for the other terms in (i) of Proposition 13. By taking the limit $m\to\infty$ and by continuity of Ψ_θ (Theorem 9), we can obtain the desired result with consideration of Eq. (14).

As to (ii), note first that we can have the expression

$$\tilde{q}_\xi=i\langle\tilde{x},\ \xi\rangle=\left(i\int_{\mathbf{R}}x(t)\,\xi(t)\,dt\right)^{\sim}, \tag{16}$$

by virtue of the multiplication operator $x(t)(\cdot)$ (cf. Remark 6). With (ii) of Proposition 13, we may take advantage of continuity of Ψ_θ and (16) to deduce that

$$\begin{aligned} e^{-i\theta}(D_\xi+D_\xi^*)(\Psi_\theta\Phi)&=e^{-i\theta}\left(\int_{\mathbf{R}}x(t)\,\xi(t)\,dt\right)(\Psi_\theta\Phi)\\ &=\lim_{m\to\infty}\Psi_\theta\left(\sum_{k=0}^{m}\Delta t_k\xi(t_k)x(t_k)\cdot\Phi\right)\\ &=\Psi_\theta(\langle x,\ \xi\rangle\cdot\Phi)\end{aligned}$$

by passage to the limit $|\Delta|\to 0$.

The following theorem gives the characterization for Pseudo-Fourier- Mehler transforms $\{\Psi_\theta\ ;\ \theta\in\}\mathbf{R}$, which is one of our main results in this paper.

Theorem15 [4] [6] [8]. *The Pseudo-Fourier-Mehler transform* $\{\Psi_\theta\ ;\ \theta\in\mathbf{R}\}$ *satisfies the following conditions:*

(P1) $\Psi_\theta:(S)^*\to(S)^*$ *is a continuous linear operator for for all* $\theta\in\mathbf{R}$;

(P2) $\Psi_\theta(\tilde{D}_\xi\Phi)=e^{i\theta}\tilde{D}_\xi(\Psi_\theta\Phi)-2\sin\theta\cdot\tilde{q}_\xi(\Psi_\theta\Phi)$;

(P3) $\Psi_\theta(\tilde{q}_\xi\Phi)=e^{-i\theta}\tilde{q}_\xi(\Psi_\theta\Phi)$;

where $\Phi\in(S)^*$, $\xi\in S$.

Conversely, if a continuous linear operator $A_\theta:(S)^*\to(S)^*$ *satisfies the above conditions*: (P1) $\sim$(P3), *then* A_θ *is a constant multiple of* Ψ_θ.

Proof. (P1) is obvious (Theorem 9). (P2) (resp. (P3)) yields from (i) (resp. (ii)) of Proposition 14. It is due to a simple computation. Conversely, suppose that the operator A_θ be compatible with (P1), (P2) and (P3). We need the following results.

Lemma 16. *We assume that* A_θ *be a continuous linear operator from* $(S)^*$ *into itself, satisfying the three conditions.* (P1)$\sim$(P3). *Then the following relations*

(i) $(\Psi_\theta^{-1}A_\theta)D_\xi=D_\xi(\Psi_\theta^{-1}A_\theta)$;

(ii) $(\Psi_\theta{}^{-1}A_\theta)q_\xi=q_\xi(\Psi_\theta{}^{-1}A_\theta)$;

(iii) $(\Psi_\theta{}^{-1}A_\theta)D_\xi^*=D_\xi(\Psi_\theta{}^{-1}A_\theta)$;

hold for for all $\xi\in S$, $\theta\in\mathbf{R}$.

The proof will be given below. The next theorem is well known (e.g. [11, Theorem 3.6, p. 267] or [14, Prop. 5.7.6, p. 148]).

Theorem17. *Let* Λ *be a continuous linear operator on* $(S)^*$, *satisfying*

(i) $\Lambda\tilde{q}_\xi=\tilde{q}_\xi\Lambda$, *for any* $\xi\in S$;

(ii) $\Lambda D_\xi^*=D_\xi^*\Lambda$, *for any* $\xi\in S$.

Then the operator Λ *is a scalar operator.*

Thus, by taking (ii), (iii) of Lemma 16 into account, we may apply Theorem 17 for A_θ to obtain the assertion: $\Psi_\theta^{-1}A_\theta$ is a scalar operator.

Proof of Lemma 16. Basically it is due to a direct computation. Each proof goes similarly, so we shall show only (iii) below. For the other two we will give just rough instructions. First of all, note that we have only to consider $\Psi_{-\theta}$ instead of $\Psi_\theta{}^{-1}$ by virtue of Proposition 3. As to (i), it is sufficient to calculate it with (P2) for both and (P3) for the PFM transform. As for (ii), simply (P3) for both A_θ and Ψ_θ. As to (iii), for $\forall\Phi\in(S)^*$, $\forall\xi\in S$

$$(\Psi_\theta^{-1}A_\theta)D_\xi^*\Phi=-i\Psi_\theta^{-1}(A_\theta q_\xi)\Phi-\Psi_\theta^{-1}(A_\theta D_\xi)\Phi, \tag{17}$$
$$=-e^{i\theta}(\Psi_\theta^{-1}q_\xi)A_\theta)\Phi-e^{i\theta}(\Psi_\theta^{-1}D_\xi)A_\theta\Phi$$

because we used a relation

$$D_\xi^*=-i\tilde{q}_\xi-\tilde{D}_\xi \tag{18}$$

in the first equality and also employed (P2), (P3) in the second one. An application of (P2), (P3) to the last expression in (17), together with (18) again, gives

$$(17)=-iq_\xi(\Psi_\theta^{-1}A_\theta)\Phi-D_\xi(\Psi_\theta^{-1}A_\theta)\Phi$$
$$=(-iq_\xi-D_\xi)(\Psi_\theta^{-1}A_\theta)\Phi=D_\xi^*(\Psi_\theta^{-1}A_\theta)\Phi,$$

which completes the proof.

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A Fock Expansion of the Pseudo-Fourier-Mehler Transform*

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§ 1. Introduction

In this article we study Pseudo-Fourier-Mehler (PFM for short) transform having quite similar nice properties as the Fourier-Mehler transform [12] possesses. It was originally defined in [5] and used for application to abstract equations in infinite dimensional spaces. In connection with other Fourier type transforms in white noise analysis, we can compute the infinitesimal generator of the PFM transform directly and show that our Pseudo-Fourier-Mehler transform enjoys intertwining properties [9]. We have also obtained in [9] the characterization theorem for PFM transforms. In this article, we shall introduce an example of PFM transform applied to the abstract Cauchy problem and also state the Fock expansion of PFM transform. Lastly we shall introduce a generalization idea of PFM transform and investigate some properties that the generalized transform should satisfy.

The Pseudo-Fourier-Mehler transform is a very important and interesting operator in the standpoint of how to express the solutions for the Fourier-transformed abstract Cauchy problems. Related works on application of infinite dimensional Fourier type transforms can be found ([4,11] ; see also [2, 3], [7, 8]).

In [1] they have studied the two dimensional complex Lie group $\mathcal{G}$ explicitly and succeeded in describing every one parameter subgroup with infinitesimal generator $(\frac{2a+b}{2})\Delta_G + bN$, where N is the number operator and Δ_G is the Gross Laplacian. Furthermore, one can find in [15] another related work, especially on a systematic study of Lie algebras containing infinite dimensional Laplacians.

We are able to state our results in the general setting (e.g. [14] ; see also [6]) of white noise analysis. As a matter of fact. almost all statements in our theory remains valid even without minor change of the basic setting. However, just for simplicity we adopt in this article the so-called original standard setting [12] in white noise analysis or Hida calculus to state our results related to the PFM transform.

§ 2. Notation

Let $S \equiv S(\mathbf{R})$ be the Schwartz class space on $\mathbf{R}$ and $S^* \equiv S'(\mathbf{R})$ its dual space. Then $S(\mathbf{R}) \subset L^2$

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$(\mathbf{R}) \subset S'(\mathbf{R})$ is a Gelfand triple. We define the family of norms given by $|\xi|_p = |A^p\xi|$, $p>0$, $\xi \in S(\mathbf{R})$, where the operator $A = -d^2/dt^2 + t^2 + 1$ and $|\cdot|$ is the $L^2(\mathbf{R})$-norm. Let $S_p \equiv S_p(\mathbf{R})$ be the completion of $S(\mathbf{R})$ with respect to the norm $|\cdot|_p$, $p>0$. We denote its dual space by $S_p^* \equiv S'_p(\mathbf{R})$, and we have $S_p(\mathbf{R}) \subset L^2(\mathbf{R}) \subset S'_p(\mathbf{R})$. Let μ be the white noise measure in White Noise Analysis. Actually it is a standard Gaussian probability measure on $S'(\mathbf{R})$. (L^2) denotes the Hilbert space of complex-valued μ-square integrable functionals with norm $\|\cdot\|$. The second quantization operator in white noise analysis is denoted by $\Gamma(A)$, which is densely defined on (L^2). For $p \in \mathbf{N}$, define $\|\varphi\|_p = \|\Gamma(A)^p\varphi\|$ and let $(S)_p \equiv \{\varphi \in (L^2) ; \|\varphi\|_p < \infty\}$ and the dual space of $(S)_p$ is denoted by $(S)_p^*$. Let (S) be the projective limit of $\{(S)_p;\ p \in \mathbf{N}\}$. It is called a space of test white noise functionals. The elements in the dual space $(S)^*$ of (S) are called generalized white noise functionals or Hida distributions. In fact, $(S) \subset (L^2) \subset (S)^*$ is a Gelfand triple [12]. For convention all dual pairings $\langle\cdot,\cdot\rangle$, resp. $\langle\langle\cdot,\cdot\rangle\rangle$ mean the canonical bilinear forms on $S^* \times S^2$(resp. $(S)^* \times (S)$) unless otherwise stated.

The S-transform of $\Phi \in (S)^*$ is a function on S defined by

(1) $$(S\Phi)(\xi) := \langle\langle \Phi, :\exp\langle :, \xi\rangle : \rangle\rangle, \qquad \xi \in S(\mathbf{R}),$$

where: $\exp\langle\cdot,\xi\rangle : \equiv \exp\langle\cdot, \xi\rangle \cdot \exp(-\frac{1}{2}|\xi|^2)$.

We now introduce the Pseudo-Fourier-Mehler transform in white noise analysis.

Definition 1 [5] [9]. $\{\Psi_\theta,\ \theta \in \mathbf{R}\}$ *is said to be the Pseudo-Fourier-Mehler (PFM) transform if* Ψ_θ *is a mapping from* $(S)^*$ into itself for $\theta \in \mathbf{R}$, *whose U-functional is given by*

(2) $$S(\Psi_\theta \Phi)(\xi) = F(e^{i\theta}\xi) \cdot (\exp(ie^{i\theta}\sin\theta |\xi|^2), \qquad \xi \in S,$$

or equivalently

(3) $$S(\Psi_\theta \Phi)(\xi) = \langle\langle \Phi, \exp(e^{i\theta}\langle\cdot, \xi\rangle - \frac{1}{2}|\xi|^2)\rangle\rangle, \quad \xi \in \mathrm{S},$$

for $\Phi \in (S)^*$, *where S is the S-transform in white noise analysis and F* denotes the U-functional of Φ(see also [7, 8]).

By virtue of Potthoff-Streit characterization [12], the right hand sides in Eq. (2) and Eq. (3) are U-functionals, and $\Psi_\theta \Phi$ exists for each Φ in $(S)^*$. Therefore the above-mentioned Pseudo-Fourier-Mehler transform is well-defined.

§ 3. Application of PFM Transform

The purpose of this section is to show a typical example of application of the Pseudo-Fourier-Mehler transform Ψ_θ to the Cauchy problem.

Example 2. (A simple application of the PFM transform)

Let us consider the following abstract Cauchy problem on the white noise space:

(4) $$\frac{\partial u(t, x)}{\partial t} = iNu(t, x) + \varphi(x),$$
$$u(0, \cdot) = f(\cdot) \in (\mathrm{S}),$$

with $t>0$, where N denotes the number operator. One of the most remarkable benefits of white noise analysis consists in its application to differential equation theory and how to solve the problem (cf. [1], [2, 3], [4, 11]). Especially in [4, 11], by resorting to the analogy in the finite dimensional cases we have applied the infinite dimensional Kuo type Fourier transform to the Cauchy problem for heat equation type with Gross Laplacian, and have succeeded in derivation of the general solution and also in direct verification for existence and uniqueness of the solution. On this account, we think of using the Fourier transform to the aforementioned problem. Recall the formula:

(5) $$\mathcal{F}(N\Phi) = N(\mathcal{F}\Phi) + \Delta_G^*(\mathcal{F}\Phi), \qquad \text{for all } \Phi \in (S)^*.$$

We set $v(t, y) \equiv (\mathcal{F}u(t, \cdot))(y)$ for each $t \in \mathbf{R}_+$. We may employ the Fourier transform $\mathcal{F}$ for (4) so as to obtain

(6) $$\frac{\partial V(t, y)}{\partial t} = iNv(t, y) + i\Delta_G^* v(t, y) + \hat{\varphi}(y), \qquad \text{with} \qquad v(0, y) = \hat{f}(y),$$

because we made use of the formula (5) and set $\hat{F} = \mathcal{F}F$, The operator part of the Fourier transformed problem (6) is exactly equivalent to the infinitesimal generator of PFM transform with parameter t (see [5], [9]). Hence, the semigroup theory in functional equation theory allows immediately the following explicit exression of the solution in question:

(7) $$v(t, y) = \Psi_t \hat{f}(y) + \int_0^t \Psi_{t-s} \hat{\varphi}(y)\, ds.$$

We can show the existence and uniqueness of the solution by applying Theorem 4 in [9] to (7) under a certain condition on the initial data φ, f. In that case the integral term appearing in (7) should be interpreted as Bochner type one. So much for the Cauchy problem, because this is not our main topic in this article. We shall go back to the PFM transform and proceed further in the next section.

§ 4. Fock Expansion

Let $\mathcal{L}((S), (S)^*)$ denote the space of continuous linear operators from (S) into $(S)^*$. The space $\hat{S}'_{l,m}(\mathbf{R}^{l+m})$ is a symmtrized space of $S'(\mathbf{R}^{l+m})$ with respect to the first l, and the second m variables independently. By virtue of the symbol characterization theorem for operators on white noise functionals [13](see also [14]), for the operator Π lying in $\mathcal{L}((S), (S)^*)$ there exists uniquely a kernel distribution $\varkappa_{l,m}$ in $\hat{S}'_{l,m}(\mathbf{R}^{l+m})$ such that the operator Π may have the Fock expansion:

$$\Pi = \sum_{l,m=0}^{\infty} \Pi_{l,m}(\varkappa_{l,m}).$$

Moreover, the series $\Pi\varphi$, $\varphi \in (S)$ converges in $(S)^*$ [13]. Generally, each component $\Pi_{l,m}$ of the Fock expansion has a formal integral expression:

$$\Pi_{l,m}(\varkappa) = \int_{\mathbf{R}^{l+m}} \varkappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \cdot \partial_{s1}^* \cdots \partial_{sl}^* \partial_{t1} \cdots \partial_{tm} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

Remark3. We call it an integral kernel operator with kernel distribution $\varkappa$. The theory of integral kernel operators and the general expansion theory in white noise analysis were proposed and have been developed enthusiastically by N. Obata [13, 14](see also [12]). Those topics are closely related to quantum stochastic calculus, which has been greatly investigated in chief by Hudson, Meyer, and Parthasarathy. More details on this topic will be found in, for instance, (i) K. R.Parthasarathy: *An Introduction to Quantum Stochastic Calculus,* Birkhäuser, Basel, 1992; (ii) P. A.Meyer: *Quantum Probability for Probabilists,* Lecture Notes in Mathematics Vol.1538, Springer -Verlag, Heidelberg, 1993.

We shall give below two typical examples of the integral kernel operators in white noise analysis.

Example 4. (The number operator N)

Let $\tau \in (S \otimes S)^*$ be the trace operator defined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in S.$$

The number operator N is usually expressed as

$$\int_{\mathbf{R}} \partial_t^* \partial_t dt$$

by Kuo's notation in white noise analysis. By the Obata theory, N has the following representation as a continuous linear operator from (S) into itself, namely,

$$N = \Pi_{1,1}(\tau) = \int_{\mathbf{R}^2} \tau(\mathrm{s}, \mathrm{t}) \partial_s^* \partial_t ds dt.$$

Example 5. (The Gross Laplacian Δ_G)

By the usual notation in white noise analysis we have the expression

$$\Delta_G = \int_{\mathbf{R}} \partial_t^2 dt.$$

Then the Gross Laplacian Δ_G can be also expressed by

$$\Delta_G = \Pi_{0,2}(\tau) = \int_{\mathbf{R}^2} \tau(\mathrm{s}_1, \mathrm{s}_2) \partial \mathrm{s}_1 \partial \mathrm{s}_2 d\mathrm{s}_1 ds_2$$

as a continuous linear operator from (S) into (S).

Let us consider the general expansion of our Pseudo-Fourier-Mehler transform. We may take advantage of Obata's integral kernel operator theory in order to obtain Fock expansion representations of Ψ_θ and its adjoint G_θ. That is to say,

Theorem 1. *For $\theta \in \mathbf{R}$, the PFM transform Ψ_θ and the adjoint operator G_θ have the following Fock expansions :*

(i) $$\Psi_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!} (ie^{i\theta} \sin\theta)^l (e^{i\theta} - 1)^m \cdot \Pi_{2l+m,m}(\tau^{\otimes l} - \otimes \lambda_m) ;$$

(ii) $$\mathrm{G}_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!} (ie^{i\theta} \sin\theta)^m (e^{i\theta} - 1)^l \cdot \Pi_{l,l+2m}(\lambda_l \otimes \tau^{\otimes m} ;$$

where the kernel $\lambda_m \in (S^{\otimes 2m})^*$ is given by

$$\lambda_m := \sum_{i_1, i_2, \cdots, i_m = 0}^{\infty} e_{i1} \otimes \cdots \otimes e_{im} \otimes e_{i1} \otimes \cdots \otimes e_{im}.$$

§ 5. Generalization

Let $GL((S))$ be the group of all linear homeomorphisms from (S) into (S). Then we have

Proposition 2. *$\{\mathrm{G}_\theta \ ; \ \theta \in \mathbf{R}\}$ is a regular one parameter subgroup of $GL((S))$ with infinitesimal generator $i(N + \Delta_G)$.*

Let us consider some generalization. Suggested by [1], for example we propose to define the generalized PFM transform X_θ, $\theta \in \mathbf{R}$ as operator on $(S)^*$ whose U-functional is given by

(8) $$S(X_\theta \Phi)(\xi) = \langle\langle \Phi, \exp(e^{\alpha\theta} \langle \cdot, \xi \rangle - \frac{1}{2} J(\alpha, \beta,; \theta) |\xi|^2) \rangle\rangle ,$$

(cf. Eq.(3) in Definition 1 of PEM transform), for $\xi \in S$, $\Phi \in (S)^*$. We set

$$J(\alpha, \beta ; \theta) = e^{2\alpha\theta} - 2\mathrm{H}(\alpha, \beta : \theta), \qquad \text{with } H(\alpha, \beta ; \theta) = h(\alpha, \beta) \cdot (e^{2\alpha\theta} - 1),$$

where $h(\alpha, \beta) = \beta / 2\alpha$, for $\alpha, \beta \in \mathbf{C}$, $\alpha \neq 0$. Then we denote the adjoint operator of X_θ by Z_θ.

Claim 3. *The set $\{\mathrm{Z}_\theta \ ; \ \theta \in \mathbf{R}\}$ is a regular one parameter subgroup of $GL((S))$.*

Claim 4. *The infinitesimal generator of $\{\mathrm{Z}_\theta \ ; \ \theta \in \mathbf{R}\}$ is given by the operator $\alpha N + \beta \Delta_G$.*

Claim 5. *The generalized PFM transform $\{X_\theta \ ; \ G \in \mathbf{R}\}$ is a one parameter subgroup of $GL((S)^*)$.*

Claim 6. *The infinitesimal generator of $\{X_\theta \ ; \ \theta \in \mathbf{R}\}$ is given by the operator $\alpha N + \beta \Delta_G^*$.*

Remark 6. The above definition (8) of generalized PFM transform X_θ can be alternatively replaced by the following expression:

$$S(X_\theta \Phi)(\xi) = F(e^{\alpha\theta}\xi) \cdot \exp(H(\alpha, \beta; \theta) \cdot |\xi|^2),$$

where F denotes the U-functional of Φ in $(S)^*$, i.e., $S\Phi = F$.

Remark 7. Especially when $\alpha = \beta = i (\in \mathbf{C})$, then the above-defined generalized PFM transforms X_θ are, of course, attributed to the simple PFM transforms Ψ_θ given by (2), (3) in the section 2.

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2.2 擬ＦＭ変換の一般化

無限次元のフーリエ型変換をホワイトノイズ解析における一般的枠組みの下で統合的に論じた。特に，擬ＦＭ変換については詳しくその性質を調べ，その特徴付け定理を導いた。さらに代数的観点から作用素解析を行い，その生成作用素によって特徴付けられる無限次元の変換群について論じた。またラプラシアンとNumber作用素から決まる生成作用素をもつ超汎関数空間上の位相同形写像群の可微分部分群として無限次元のＦ型変換を定義することを提唱し，その新しい定式化の下では，ホワイトノイズ解析のクオ型フーリエ変換，フーリエ・メーラー変換，および擬ＦＭ変換等がすべて典型例として含まれてしまい，統一的に扱うことができることを指摘した。

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PREFACE

The seventh Japan-Russia Symposium on Probability Theory and Mathematical Statistics was held at The Meiji Mutual Life Insurance Co. Corporate training Center in Tokyo, July 26-30, 1995, under the joint auspicies of the Mathematical Society of Japan and Institute of Statistical Mathematics. There were 35 participants from Russia, Ukraina, Lithuania and Georgia, 142 from Japan, and 4 from other European and Asian countries.

This volume contains papers presented at the Symposium. Records of the meetings and a list of the Organizing Committee are attached at the end of the volume. The Symposium proved to be very fruitful in promoting scientific exchanges among probabilists and statisticians from various countries including Japan and Russia.

Previous six (Japan-USSR) Symposia were held in Khabarovsk(1969), Kyoto(1972), Tashkent(1975), Tbilisi(1982), Kyoto(1986) and Kiev(1991). The Proceedings of the Symposia were published from Springer-Verlag as Vol.330, 550, 1021 and 1299 of Lecture Notes in Mathematics. The Proceedings of the last one at Kiev was published from World Scientific in 1992.

We are very grateful to all those who have contributed to the success of the Symposium. Thanks are due to Professors H.Nagai and A.A.Novikov for their great efforts in preparing this Proceedings. We would like to express our sincere gratitude to The Meiji Mutual Life Insurance Co. and Japan World Exposition ('70) Commemorative Fund for their truly generous support.

S. Watanabe
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CONTENTS

ON A CLASS OF INFINITE DIMENSIONAL FOURIER TYPE TRANSFORMS IN WHITE NOISE CALCULUS*

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ABSTRACT

We consider the Pseudo-Fourier-Mehler transform in white noise analysis and study various kinds of properties such as intertwining properties, with result that its characterization theorem is proved. Through this, we propose a new concept on a class of infinite dimensional Fourier type transforms, which is a one parameter subgroup and is also characterized in terms of its infinitesimal generator.

1. Introduction

The study of the Fourier transform $\mathcal{F}$ [9] in white noise calculus was initiated by Kuo. While, the Fourier-Mehler (FM) transform is a kind of generalization of $\mathcal{F}$.[10] In this paper we introduce Pseudo-Fourier-Mehler (PFM) transform [3] having quite similar properties as the FM transform possesses. It was used for application to abstract equations in infinite dimensional spaces.[3] In connection with other Fourier type transforms in white noise analysis, we can compute the infinitesimal generator of the PFM transform (§2.2) and show that our PFM transform enjoys intertwining properties (§3.1). We shall state the characterization theorem for PFM transforms (§3.2), which is one of our main results in this article. The Fock expansion of PFM transform can be derived as well (§3.3). Lastly we shall introduce a generalization idea of PFM transform and investigate some properties that the generalized transform should satisfy. Then we come to a recognition of two dimensional complex Lie algebras naturally containing the adjoints of infinite dimensional Laplacians. Above all we propose a new class of infinite dimensional Fourier type transforms (§4). Actually it turns out to be a differentiable one parameter subgroup of linear homeomorphisms on the space of Hida distributions, having $\alpha N + \beta \Delta_G^*$ as its infinitesimal generator. As to related works, Chung and Ji [1] have studied a certain subgroup with generator $(\frac{2a+b}{2})\Delta_G + bN$ from a slightly different point of view.

1.1 Notation

We adopt the so-called general setting [4] in white noise analysis throughout this article. Let T be a separable topological space equipped with a σ-finite Borel measure

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$d\nu(t)$ on the topological Borel field $\mathcal{B}(T)$. Further suppose that ν be equivalent to the Lebegue type measure. We consider the real Hilbert space $H := L^2(T, d\nu; \mathbb{R})$ with norm $|\cdot|_0$. Let A be a positive selfadjoint operator in H with Hilbert-Schmidt inverse. Its eigenvalues and normalized eigenvectors are denoted by $\{\lambda_n\}$ and $\{e_n\}$ respectively. Then $\{e_n\}$ becomes a complete orthonormal basis of H. We assume that $1 < \lambda_0 \le \lambda_1 \le \cdots \to \infty$, where $\lambda_0 = \inf \mathrm{Spec}(A)$. Let $E = \mathcal{S}_A(T)$ be the standard countably Hilbert space [12] constructed from (H, A). In fact, E becomes a nuclear space and we have a Gelfand triple $E = \mathcal{S}_A(T) \subset H = L^2(T, d\nu; \mathbb{R}) \subset E^* = \mathcal{S}_A^*(T)$, where E^* denotes the dual space of E. It follows by construction that

$$E = \bigcap_{p \ge 0} E_p \cong \operatorname*{projlim}_{p \to \infty} E_p, \quad , \quad E^* = \bigcup_{p \ge 0} E_{-p} \cong \operatorname*{indlim}_{p \to \infty} E_{-p},$$

where E_p is the Hilbert space equipped with the norm $|\xi|_p = |A^p \xi|_0$. Suggested by Kubo-Takenaka formulation [8], we assume: (i) for every $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T which coincides with ξ up to ν-null functions; (ii) for each $t \in T$ the evaluation map $\delta_t : t \mapsto \xi(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$; (iii) the map $t \mapsto \delta_t$ is continuous from T into E^*. Let μ be the standard Gaussian measure on E^* such that

$$\int_{E^*} \exp(\sqrt{-1}\langle x, \xi \rangle) \mu(dx) = \exp\Big(-\frac{1}{2}|\xi|_0^2\Big),$$

for any $\xi \in E$. (L^2) denotes the Hilbert space of complex-valued μ-square integrable functionals with norm $\|\cdot\|_0$. The Wiener-Itô decomposition theorem gives the unique representation of φ in (L^2), i.e.,

$$\varphi(x) = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} \langle :x^{\hat{\otimes} n}:, f_n \rangle, \quad f_n \in H_{\mathbb{C}}^{\hat{\otimes} n}, \quad x \in E^*, \tag{1}$$

where I_n denotes the multiple Wiener integral of order n and $H_{\mathbb{C}}^{\hat{\otimes} n}$ denotes the n-fold symmetric tensor product of the complexification of H and the symbol $:x^{\hat{\otimes} n}:$ is the Wick ordering of the distribution $x^{\otimes n}$. The second quantization operator $\Gamma(A)$ is densely defined on (L^2) as follows: for $\varphi = \sum_{n=0}^{\infty} I_n(f_n) \in \mathrm{Dom}(\Gamma(A))$,

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n). \tag{2}$$

For $p \in \mathbb{N}$, define $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$ and let $(E)_p \equiv \{\varphi \in (L^2); \|\varphi\|_p < \infty\}$ and the dual space of $(E)_p$ is denoted by $(E)_p^*$. Let (E) be the projective limit of $\{(E)_p; p \in \mathbb{N}\}$. It is called a space of test white noise functionals. The elements in the dual space $(E)^*$ of (E) are called generalized white noise functionals or Hida distributions. In fact, $(E) \subset (L^2) \subset (E)^*$ is a Gelfand triple. For convention all dual pairings $\langle \cdot, \cdot \rangle$, resp. $\langle\langle \cdot, \cdot \rangle\rangle$ mean the canonical bilinear forms on $E^* \times E$ (resp. $(E)^* \times (E)$) unless otherwise stated. The space with subscript $\mathbb{C}$ means its complexification.

Remark 1. E is topologized by the projective limit of Hilbert spaces $\{E_p\}$ with inner products $(\cdot,\cdot)_p$ and E^* is equipped with the inductive limit convex topology. In fact, the strong dual topology of E^* coincides with the inductive limit topology in our setting.[4]

1.2 Preliminaries

The S-transform of $\Phi \in (E)^*$ is a function on $E_{\mathbf{C}}$ defined by

$$(S\Phi)(\xi) := \langle\langle \Phi, :\exp\langle\cdot,\xi\rangle:\rangle\rangle, \qquad \xi \in E_{\mathbf{C}}, \tag{3}$$

where $:\exp\langle\cdot,\xi\rangle: \equiv \exp\langle\cdot,\xi\rangle \cdot \exp\left(-\frac{1}{2}|\xi|^2\right)$. Based on the Potthoff-Streit theorem [13] we are able to give rigorous definitions to Fourier type transforms of infinite dimensions. The Kuo type Fourier transform $\mathcal{F}$ of Φ in $(E)^*$ is the generalized white noise functional, S-transformation of which is given by [9]

$$S(\mathcal{F}\Phi)(\xi) = \langle\langle \Phi, \exp(-i\langle\cdot,\xi\rangle)\rangle\rangle, \qquad \xi \in E_{\mathbf{C}}. \tag{4}$$

Likewise, the Fourier-Mehler transform $\mathcal{F}_\theta$ ($\theta \in \mathbb{R}$) of Φ in $(E)^*$ is the generalized white noise functional, S-transformation of which is given by [10]

$$S(\mathcal{F}_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\left\{e^{i\theta}\langle\cdot,\xi\rangle - \frac{1}{2}e^{i\theta}\cos\theta|\xi|^2\right\}\rangle\rangle, \qquad \xi \in E_{\mathbf{C}}. \tag{5}$$

The Fourier-Mehler transform $\mathcal{F}_\theta, \theta \in \mathbb{R}$ is a generalization of $\mathcal{F}$. Actually, $\mathcal{F}_0 = Id$, and $\mathcal{F}_{-\pi/2}$ is coincident with $\mathcal{F}$. We denote by $D_y\varphi$ the Gâteaux differential of $\varphi \in (E)$ in the direction $y \in E^*$. In fact, D_y becomes a continuous linear operator from (E) into itself. The symbol ∂_t indicates the Hida differential operator in white noise analysis. It is well known that the action of ∂_t is equivalent to that of D_{δ_t} on the dense domain.[6,4] So we can define $\partial_t = D_{\delta_t}$, $t \in \mathbb{R}$. The Kubo operator ∂_t^* [8] is the adjoint of Hida differential. For $\xi \in E$, $\varphi \in (E)$, the derivative $(D_\xi\varphi)(x)$ is defined in the usual manner, and there exists its extension $\tilde{D}_\xi : (E)^* \to (E)^*$. Even for that, we shall henceforth use the same notation D_ξ for simplicity, as far as there is no confusion. We set $q_\xi := i(D_\xi + D_\xi^*)$, where D_ξ^* is the adjoint of D_ξ.

2. Pseudo-Fourier-Mehler Transform

We begin with introducing the PFM transform in white noise analysis.

Definition 1. *$\{\Psi_\theta, \theta \in \mathbb{R}\}$ is said to be the Pseudo-Fourier-Mehler(PFM) transform*[3] *if Ψ_θ is a mapping from $(E)^*$ into itself for $\theta \in \mathbb{R}$, whose U-functional is given by*

$$S(\Psi_\theta \Phi)(\xi) = \langle\langle \Phi, \exp\Big(e^{i\theta}\langle \cdot, \xi\rangle - \frac{1}{2}|\xi|^2\Big)\rangle\rangle, \qquad \xi \in E_{\mathbb{C}}, \quad \Phi \in (E)^*. \tag{6}$$

By virtue of the Potthoff-Streit theorem, the above-mentioned PFM transform is well-defined. Immediately, $\Psi_0 = Id$ (Id denotes the identity); $\Psi_\theta \neq \mathcal{F}, \mathcal{F}_\theta$ for any $\theta \in \mathbb{R} \setminus \{0\}$. It is easy to see that the invese operator of the PFM transform Ψ_θ is given by $(\Psi_\theta)^{-1} = \Psi_{-\theta}$ for $\theta \in \mathbb{R}$.

2.1 Image of the Space (E) under PFM Transform

Let us consider what the image of the space (E) under Ψ_θ is like. The PFM transform Ψ_θ enjoys some interesting properties on the product of Gaussian white noise (GWN) functionals. Let g_c be a GWN functional, i.e., $g_c(\cdot) := \mathcal{N} \exp(-|\cdot|^2/2c)$ with renormalization $\mathcal{N}$ and $c \in \mathbb{C}$, $c \neq 0, -1$. The symbol $:$ denotes the Wick product.[6]

Theorem 1. *The following equalities hold for any $\theta \in \mathbb{R}$:*
(i) if $a(\theta) = 2^{-1}ie^{-i\theta}\csc\theta - 1$, then $\Psi_\theta\Phi = \Gamma(e^{i\theta}Id)\Phi : g_{a(\theta)}$, $\quad \Phi \in (E)^*$;
(ii) moreover, for all $p \in \mathbb{R}$, $\quad \|\Psi_\theta\Phi\|_p = \|\Phi : g_{a(\theta)}\|_p$, $\quad \Phi \in (E)_p$.

It is due to the following lemma. The proof goes almost similarly as Theorem 8.5.[3]

Lemma 2. *For $\theta \in \mathbb{R}$ the following equalities hold:*
(i) $\quad \Psi_\theta\Phi : g_{c(\theta)} = \Gamma(e^{i\theta}Id)\Phi$, $\quad \forall\Phi \in (E)^*$;
(ii) for any $p \in \mathbb{R}$, $\quad \|\Psi_\theta\Phi : g_{c(\theta)}\|_p = \|\Phi\|_p$, $\quad \forall\Phi \in (E)_p$;
where $c(\theta) = -(2^{-1}i\, e^{-i\theta}\csc\theta + 1)$.

Let us think of the image of $\varphi \in (E)$ under the PFM transform. It is easily checked that $g_c : g_d = 1$ holds with $c+d = -2$. So we have $g_{c(\theta)} : g_{a(\theta)} = 1$. From (ii) of Lemma 2, immediately, $\varphi \in (E)$ if and only if $\Psi_\theta\varphi : g_{c(\theta)} \in (E)$, so that, it is equivalent to $\Psi_\theta\varphi : g_{c(\theta)} : g_{a(\theta)} \in (E) : g_{a(\theta)}$, where $(E) : g_{a(\theta)}$ denotes the whole space of elements $\varphi : g_{a(\theta)}$ for $\varphi \in (E)$. Consequently, it is obvious that $\Psi_\theta\varphi \in (E) : g_{a(\theta)}$. Therefore we obtain

Theorem 3. *For $\theta \in \mathbb{R}$, $Im\Psi_\theta(E) = (E) : g_{a(\theta)} \equiv \{\varphi : g_{a(\theta)};\ \varphi \in (E)\}$.*

Remark 2. The results in Theorem 1 and Lemma 2 are quite similar to those of the Fourier-Mehler transform. In fact, for $p \in \mathbb{R}$, $\Phi \in (\mathcal{S})_p$,

$$\|(\mathcal{F}_\theta\Phi):g_{c_1(\theta)}\|_p = \|\Phi\|_p \quad \text{and} \quad \|\mathcal{F}_\theta\Phi\|_p = \|\Phi:g_{c_2(\theta)}\|_p$$

hold with $c_1(\theta) = -i\cot\theta - 2$, and $c_2(\theta) = i\cot\theta - 2$.

Remark 3. The image of $(\mathcal{S})$ under the FM transform $\mathcal{F}_\theta$ is given by $(\mathcal{S}) : g_{i\cot\theta}$, while that of $(\mathcal{S})$ under the Fourier transform $\mathcal{F}$ coincides with the space $(\mathcal{S}) : \tilde{\delta}_0 \equiv \{\varphi : \tilde{\delta}_0;\ \varphi \in (\mathcal{S})\}$, where $\tilde{\delta}_0$ is the delta function at 0 and $\lim_{c\to 0} g_c = \tilde{\delta}_0$.

2.2 Infinitesimal Generators

First of all, for all $\theta \in \mathbb{R}$ we define $\varphi_\xi(x) := \sum_{n=0}^\infty \frac{1}{n!}\langle : x^{\otimes n} :, \xi^{\otimes n}\rangle$ with $x \in E^*, \xi \in E_{\mathbb{C}}$. Then $\{G_\theta, \theta \in \mathbb{R}\}$ is an operator on (E) defined by

$$(G_\theta\varphi_\xi)(x) := \varphi_{e^{i\theta}\xi}(x) \cdot \exp\left(ie^{i\theta}\sin\theta|\xi|^2\right). \tag{7}$$

Let τ denote the distribution in $(E \otimes E)^*$ given by $\langle \tau, \xi \otimes \eta\rangle = \langle \xi, \eta\rangle$, $\xi, \eta \in E$. Note that it can be expressed as $\tau = \int_T \delta_t \otimes \delta_t \nu(dt) = \sum_{j=0}^\infty e_j \otimes e_j \in (E \otimes E)^*$. Moreover we have

$$\tau^{\otimes n} = \int\int\cdots\int_{T^n} \delta_{t_1} \otimes \delta_{t_1} \otimes \cdots \otimes \delta_{t_n} \otimes \delta_{t_n} \nu(dt_1)\cdots\nu(dt_n).$$

The following is an easy exercise. The next lemma provides with a general expression for elements of general form in (E).

Lemma 4. *When $\varphi(x) = \sum_{n=0}^\infty \langle : x^{\otimes n} :, f_n\rangle \in (E)$ with $f_n \in E_{\mathbb{C}}^{\hat{\otimes} n}$, then $G_\theta\varphi$ is given by $(G_\theta\varphi)(x) = \sum_{n=0}^\infty \langle : x^{\otimes n} :, g_n\rangle$, and*

$$g_n \equiv g_n(\varphi) = \sum_{m=0}^\infty \frac{(n+2m)!}{n!m!}(i\sin\theta)^m e^{i(n+m)\theta}\tau^{\otimes m} * f_{2m+n},$$

where $(\tau^{\otimes m} * f_{2m+1})(t) = \int_{T^m} f_{2m+1}(s_1, s_1, \cdots, s_m, s_m, t)\nu(ds_1)\cdots\nu(ds_m)$.

On this account, we obtain immediately

Proposition 5. *The PFM transform $\{\Psi_\theta; \theta \in \mathbb{R}\}$ is given by the adjoint operator of $\{G_\theta;\ \theta \in \mathbb{R}\}$, i.e., $\Psi_\theta = G_\theta^*$ holds in operator equality sense for all $\theta \in \mathbb{R}$.*

The next proposition gives an explicit action of the PFM transform Ψ_θ for the generalized white noise functionals of general form.

Proposition 6. *For $\Phi \in (E)^*$ given as $\Phi(x) = \sum_{n=0}^\infty \langle : x^{\otimes n} :, F_n\rangle$, $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{sym}$, it holds that*

$$\Psi_\theta\Phi(x) = \sum_{n=0}^\infty \Big\langle : x^{\otimes n} :, \sum_{l+2m=n} a(l,m,\theta)\cdot F_l \hat{\otimes} \tau^{\otimes m}\Big\rangle,$$

where the constant $a(l,m,\theta)$ *is given by* $a(l,m,\theta) = \frac{1}{m!}e^{i(l+m)\theta}(i\sin\theta)^m$.

The proof is greatly due to some computational techniques in §5.6.[12] It follows that the PFM transform Ψ_θ is injective and surjective. Moreover, it is easy to check that Ψ_θ is a strongly continuous operator from $(E)^*$ into itself, when we take Lemma 4 and Proposition 5 into consideration. Thus we have the following theorem.

Theorem 7. *(i) The Pseudo-Fourier-Mehler transform* $\Psi_\theta : (E)^* \to (E)^*$ *is a bijective and strongly continuous linear operator.*
(ii) The set $\{\Psi_\theta; \theta \in \mathbb{R}\}$ *forms a one parameter group of strongly continuous linear operator acting on the space* $(E)^*$ *of Hida distributions.*

We are now in a position to state one of the principal results in this paper. This is a very important property of the Pseudo-Fourier-Mehler transform, especially on an applicational basis.[3,5,2]

Theorem 8. *The infinitesimal generator of* $\{\Psi_\theta; \theta \in \mathbb{R}\}$ *is given by* $i(N+\Delta_G^*)$, *where* N *is the number operator and* Δ_G^* *is the adjoint of the Gross Laplacian* Δ_G.

Remark 4. The assertions in Theorem 7 and Theorem 8 have been proved under the standard setting.[3]

Proof of Theorem 8. Set $F_\theta(\xi) := S(\Psi_\theta\Phi)(\xi)$ and $F_0(\xi) := S(\Phi)(\xi)$ for $\Phi \in (E)^*$, $\xi \in E$. From definition we have $F_\theta(\xi) = F_0(e^{i\theta}) \cdot \exp\ [ie^{i\theta}\ \sin\theta|\xi|^2]$. Since F_0 is Fréchet differentiable, $F_\theta(\xi)$ is differentiable in θ as well, and it is easy to check that

$$\lim_{\theta\to 0}\frac{1}{\theta}\{F_\theta(\xi) - F_0(\xi)\} = i\langle F'(\xi),\xi\rangle + i|\xi|^2 \cdot F(\xi). \tag{8}$$

While, the U-functional $\theta^{-1}\cdot\{F_\theta(\xi) - F_0(\xi)\}$, $\theta \in \mathbb{R}$ satisfies the uniform bounded criterion: $\exists C_0 > 0$ so that $\sup\{|\frac{1}{\theta}\{\tilde{F}_\theta(z\xi) - \tilde{F}_0(z\xi)\}|; z \in \mathbb{C}, |z| = R\} \le C_0 \exp(c_1 R^{c_2}|\xi|_p^2)$ holds for all $R > 0$, all $\xi \in E$ with $c_1 > 0, c_2 > 0$, where $\tilde{F}_*$ denotes an entire analytic extension of F. Hence, the strong convergence criterion theorem [13] allows convergence of $S^{-1}\left(\frac{1}{\theta}\{F_\theta(\cdot) - F_0(\cdot)\}\right)(x) = \frac{1}{\theta}\{\Psi_\theta\Phi(x) - \Phi(x)\}$ in $(E)^*$ as θ tends to zero. Therefore the assertion follows immediately from Theorem 6.11(p.196) and Theorem 6.20(p.206).[6] □

3. Operator Analysis of PFM Transforms

3.1 Intertwining Properties

In this section we shall investigate some intertwining properties between the PFM transform Ψ_θ and other typical operators in white noise analysis, such as Gâteaux differential, its adjoint, Hida differential operator, and Kubo operator, etc.

Lemma 9. *For each $\theta \in \mathbb{R}$, $t \in \mathbb{R}$, $\Psi_\theta(\partial_t^*\Phi) = e^{i\theta}\partial_t^*(\Psi_\theta\Phi)$ holds for all $\Phi \in (E)^*$.*

Proof. Note that $S(\partial_t^*\Phi)(\xi) = \xi(t) \cdot S(\Phi)(\xi)$. So, for $\Phi \in (E)^*$ given in the form $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, F_n\rangle$, $x \in E^*$ we readily get

$$S(\Psi_\theta(\partial_t^*\Phi))(\xi) = e^{i\theta}\xi(t) \cdot \sum_{n=0}^{\infty}\langle F_n, e^{in\theta}\xi^{\otimes n}\rangle \cdot \exp\big(ie^{i\theta}\sin\theta|\xi|^2\big). \tag{9}$$

While we establish $S(\Psi_\theta(\partial_t^*\Phi))(\xi) = e^{i\theta}S(\partial_t^*(\Psi_\theta\Phi))(\xi)$ by applying Eq.(9), because we made use of the relation $S(\partial_t^*(\Psi_\theta\Phi))(\xi) = \xi(t) \cdot (S\Phi)(e^{i\theta}\xi) \cdot \exp\big(ie^{i\theta}\sin\theta|\xi|^2\big)$. Hence, an application of the Potthoff-Streit theorem leads to the required equality in Hida distribution sense. □

Proposition 10. *For each $\theta \in \mathbb{R}$, $t \in \mathbb{R}$*
(i) $\Psi_\theta(\partial_t\Phi) = e^{-i\theta}\partial_t(\Psi_\theta\Phi) - 2i\sin\theta\partial_t^*(\Psi_\theta\Phi)$;
(ii) $\Psi_\theta(x(t)\Phi) = e^{-i\theta}x(t)(\Psi_\theta\Phi)$;
hold for all $\Phi \in (E)^*$.

Remark 5. The above assertion (i) follows from a direct computation. We have only to employ the following two rules: $S\partial_t(\cdot) = \frac{\delta}{\delta\xi(t)}S(\cdot)$, $\partial_t^*(\cdot) = S^{-1}\xi(t)S(\cdot)$. (ii) is also due to a simple computation together with (i) and Lemma 9. Moreover, we need to apply the multiplication operator: $x(t)(\cdot) = (\partial_t + \partial_t^*)\ (\cdot)$. Those proofs go almost similarly as in the proof of Lemma 9 and are very easy, hence omitted.

The next proposition indicates some intertwining property between the PFM transform and Gâteaux differential operator.

Proposition 11. *For each parameter $\theta \in \mathbb{R}$, $t \in \mathbb{R}$*
(i) $e^{-i\theta}\tilde{D}_\xi(\Psi_\theta\Phi) = \Psi_\theta(\tilde{D}_\xi\Phi) + 2i\sin\theta \cdot D_\xi^*(\Psi_\theta\Phi)$;
(ii) $\tilde{D}_\xi(\Psi_\theta\Phi) + D_\xi^*(\Psi_\theta\Phi) = e^{i\theta}\Psi_\theta(\widetilde{\langle\cdot,\xi\rangle}\Phi)$;
hold for all generalized white noise functionals in $(E)^*$.

Proof. First of all, we have

$$\tilde{D}_\xi := \left(\int_T \xi(t)\partial_t\nu(dt)\right)^{\sim}, \quad \text{and} \quad D_\xi^* := \int_T \xi(t)\partial_t^*\nu(dt), \quad \forall\xi \in E. \tag{10}$$

Let $\Delta = \{t_k\}$ be a proper finite partition of T, and $|\Delta|$ denotes the maximum of increment Δt_k over $1 \le k \le m$. The assertion (i) yields from (i) of Proposition

10. In fact, by linearity of the PFM transform we get $\sum_{k=1}^{m} \Delta\nu(t_k)\,\xi(t_k) \cdot \Psi_\theta(\partial_{t_k}\Phi) = \Psi_\theta(\sum_{k=1}^{m} \xi(t_k)\partial_{t_k}\Delta\nu(t_k) \cdot \Phi)$, for $\forall\xi \in E$. By taking the limit $|\Delta| \to \infty$ and by continuity of Ψ_θ (Theorem 7), we can obtain the desired result with consideration of Eq.(10). As to (ii), it goes similarly. We have only to note

$$\tilde{q}_\xi = i\widetilde{\langle x, \xi\rangle} = \left(i \int_T x(t)\xi(t)\nu(dt)\right)^{\sim}, \tag{11}$$

by virtue of Remark 5. With (ii) of Proposition 10 we deduce that

$$e^{-i\theta}(D_\xi + D_\xi^*)(\Psi_\theta\Phi) = e^{-i\theta}\left(\int_T x(t)\xi(t)\nu(dt)\right)(\Psi_\theta\Phi) = \Psi_\theta(\langle x, \xi\rangle \cdot \Phi). \ \square$$

3.2 Characterization for PFM Transforms

In this section we shall introduce the characterization theorem for PFM transforms, which is one of our main results in this paper.

Theorem 12. *The PFM transform $\{\Psi_\theta; \theta \in \mathbb{R}\}$ satisfies the following conditions:*
(P1) $\Psi_\theta : (E)^* \to (E)^*$ is a continuous linear operator for $\forall\theta \in \mathbb{R}$;
(P2) $\Psi_\theta(\tilde{D}_\xi\Phi) = e^{i\theta}\tilde{D}_\xi(\Psi_\theta\Phi) - 2\sin\theta \cdot \tilde{q}_\xi(\Psi_\theta\Phi)$;
(P3) $\Psi_\theta(\tilde{q}_\xi\Phi) = e^{-i\theta}\tilde{q}_\xi(\Psi_\theta\Phi)$; where $\Phi \in (E)^*$, $\xi \in E_{\mathbb{C}}$.
Conversely, if a continuous linear operator $A_\theta : (E)^* \to (E)^*$ satisfies the above conditions: (P1) $\sim$ (P3), then A_θ is a constant multiple of Ψ_θ.

Proof. (P1) is obvious(Theorem 7). (P2)(resp. (P3)) yields from (i)(resp. (ii)) of Proposition 11. Conversely, suppose that the operator A_θ be compatible with (P1),(P2) and (P3). We need the following results.

Lemma 13. *We assume that Ξ_θ be a continuous linear operator from $(E)^*$ into itself, satisfying the three conditions (P1) $\sim$ (P3). Then the following relations hold for $\forall\xi \in E_{\mathbb{C}}$, $\theta \in \mathbb{R}$.*
(i) $(\Psi_\theta^{-1}\Xi_\theta)D_\xi = D_\xi(\Psi_\theta^{-1}\Xi_\theta)$; (ii) $(\Psi_\theta^{-1}\Xi_\theta)q_\xi = q_\xi(\Psi_\theta^{-1}\Xi_\theta)$;
(iii) $(\Psi_\theta^{-1}\Xi_\theta)D_\xi^* = D_\xi^*(\Psi_\theta^{-1}\Xi_\theta)$.

The proof will be given below. The next result(Prop.5.7.6,p.148)[12] is well known.

Theorem 14. *Let Λ be a continuous linear operator on $(E)^*$, satisfying*
(i) $\Lambda\tilde{q}_\xi = \tilde{q}_\xi\Lambda$, for any $\xi \in E$;
(ii) $\Lambda D_\xi^* = D_\xi^*\Lambda$, for any $\xi \in E$.
Then the operator Λ is a scalar operator.

Thus, by taking (ii),(iii) of Lemma 13 into account with $A_\theta = \Xi_\theta$, we may apply Theorem 14 for A_θ to obtain the assertion: $\Psi_\theta^{-1}A_\theta$ is a scalar operator. $\square$

Proof of Lemma 13. Basically it is due to a direct computation. Each proof goes similarly, so we shall show only (iii) below. For the other two we will give just rough instructions. First of all, note that we have only to consider $\Psi_{-\theta}$ instead of Ψ_θ^{-1}. As to (i), it is sufficient to calculate it with (P2) for both A_θ and Ψ_θ, and with (P3) for the PFM transform. As for (ii), use simply (P3) for both. As to (iii), for $\forall\Phi \in (E)^*$,

$$(12) \qquad (\Psi_\theta^{-1}A_\theta)D_\xi^*\Phi = -i\Psi_\theta^{-1}(A_\theta q_\xi)\Phi - \Psi_\theta^{-1}(A_\theta D_\xi)\Phi,$$
$$= -e^{i\theta}(\Psi_\theta^{-1}q_\xi)A_\theta\Phi - e^{i\theta}(\Psi_\theta^{-1}D_\xi)A_\theta\Phi, \qquad \forall\xi \in E,$$

because we used a relation $D_\xi^* = -i\tilde{q}_\xi - \tilde{D}_\xi$ in the first equality and also employed (P2),(P3) in the second one. An application of (P2),(P3) to the last expression in Eq.(12) gives

$$(12) = -iq_\xi(\Psi_\theta^{-1}A_\theta)\Phi - D_\xi(\Psi_\theta^{-1}A_\theta)\Phi = (-iq_\xi - D_\xi)(\Psi_\theta^{-1}A_\theta)\Phi = D_\xi^*(\Psi_\theta^{-1}A_\theta)\Phi,$$

which completes the proof. □

3.3 Fock Expansion

Let $\mathcal{L}((E),(E)^*)$ denote the space of continuous linear operators from (E) into $(E)^*$. The space $(E_{\mathbf{C}}^{\otimes(l+m)})^*_{sym(l,m)}$ is a symmtrized space of $(E_{\mathbf{C}}^{\otimes(l+m)})^*$ with respect to the first l, and the second m variables independently. By virtue of the symbol characterization theorem for operators on white noise functionals,[11] for the operator Π lying in $\mathcal{L}((E), (E)^*)$ there exists uniquely a kernel distribution $\kappa_{l,m}$ in $(E_{\mathbf{C}}^{\otimes(l+m)})^*_{sym(l,m)}$ such that the operator Π may have the Fock expansion: $\Pi = \sum_{l,m=0}^\infty \Pi_{l,m}(\kappa_{l,m})$. Moreover, the series $\Pi\varphi$, $\varphi \in (E)$ converges in $(E)^*$. Generally, each component $\Pi_{l,m}$ of the Fock expansion has a formal integral expression:

$$\int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \cdot \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \nu(ds_1) \cdots \nu(ds_l)\nu(dt_1) \cdots \nu(dt_m).$$

We call it an integral kernel operator with kernel distribution κ. We shall give below two typical examples in white noise analysis.

Example. The number operator N has the following representation as a continuous linear operator from (E) into itself, namely,

$$N = \Pi_{1,1}(\tau) = \iint_{T^2} \tau(s,t)\partial_s^*\partial_t\nu(ds)\nu(dt).$$

While, the Gross Laplacian Δ_G can be also expressed by

$$\Delta_G = \Pi_{0,2}(\tau) = \iint_{T^2} \tau(s_1, s_2)\partial_{s_1}\partial_{s_2}\nu(ds_1)\nu(ds_2)$$

as a continuous linear operator from (E) into (E). □

Let us consider the general expansion of our PFM transform. We may take advantage of Obata's integral kernel operator theory in order to obtain Fock expansion representations of Ψ_θ and its adjoint G_θ. That is to say,

Theorem 15. *For $\theta \in \mathbb{R}$, we have the following Fock expansions:*

$$(i)\quad \Psi_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}(ie^{i\theta}\sin\theta)^l(e^{i\theta}-1)^m \cdot \Pi_{2l+m,m}(\tau^{\otimes l}\otimes\lambda_m);$$

$$(ii)\quad G_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}(ie^{i\theta}\sin\theta)^m(e^{i\theta}-1)^l \cdot \Pi_{l,l+2m}(\lambda_l\otimes\tau^{\otimes m});$$

where $\lambda_m \in (E_{\mathbb{C}}^{\otimes 2m})^*$ *is given by* $\sum_{i_1,i_2,\cdots,i_m=0}^{\infty} e_{i_1}\otimes\cdots\otimes e_{i_m}\otimes e_{i_1}\otimes\cdots\otimes e_{i_m}$.

4. A Class of Generalized F-Type Transforms

Let $GL((E))$ be the group of all linear homeomorphisms from (E) into (E). Then we have

Proposition 16. $\{G_\theta;\theta\in\mathbb{R}\}$ *is a regular*[12] *one parameter subgroup of* $GL((E))$ *with infinitesimal generator* $i(N+\Delta_G)$.

Let us consider some generalization. Suggested by N.Obata(personal communication) and Chung-Ji[1] we propose to define a class of infinite dimensional Fourier type transforms X_θ, $\theta\in\mathbb{R}$ as operator on $(E)^*$ whose U-functional is given by

$$(13)\qquad S(X_\theta\Phi)(\xi) = \langle\langle\Phi, \exp\Big(e^{\alpha\theta}\langle\cdot,\xi\rangle - \frac{1}{2}J(\alpha,\beta;\theta)|\xi|^2\Big)\rangle\rangle,\quad \xi\in E,\ \ \Phi\in(E)^*.$$

We set $J(\alpha,\beta;\theta) = e^{2\alpha\theta} - 2H(\alpha,\beta;\theta)$, with $H(\alpha,\beta;\theta) = \frac{\beta}{2\alpha}\cdot(e^{2\alpha\theta}-1)$, where α, β $\in\mathbb{C}$, $\alpha\neq 0$. We call it a *generalized F-type tramsform.* Then we denote the adjoint operator of X_θ by Z_θ.

Proposition 17. *The set* $\{Z_\theta;\theta\in\mathbb{R}\}$ *is a regular one parameter subgroup of* $GL((E))$ *with infinitesimal generator* $\alpha N+\beta\Delta_G$.

Theorem 18. *The generalized F-type transform* $\{X_\theta;\ \theta\in\mathbb{R}\}$ *is a differentiable one parameter subgroup of* $GL((E)^*)$ *with infinitesimal generator* $\alpha N+\beta\Delta_G^*$.

Remark 6. The above definition Eq.(13) of generalized F-type transform X_θ can be alternatively replaced by the following expression: $S(X_\theta\Phi)(\xi) = F(e^{\alpha\theta}\xi)\cdot\exp(H(\alpha,\beta;\theta)\cdot|\xi|^2)$, where F denotes the U-functional of Φ in $(E)^*$, i.e., $S\Phi = F$.

Remark 7. Especially when $\alpha=\beta=i(\in\mathbb{C})$, then the above-defined generalized F-type transforms X_θ are attributed to the PFM transforms Ψ_θ given by Eq.(6) in §2. And also for $\alpha=i$, $\beta=\frac{i}{2}$, simply $X_\theta=\mathcal{F}_\theta$.

Generally, in white noise analysis, any rotation invariant operator[7] in $\mathcal{L}((E),(E)^*)$ is generated by N, Δ_G, and Δ_G^*. In that sense we can say that our F-type transforms are chatacterized by its infinitesimal generator $\alpha N+\beta\Delta_G^*$. The generalized F-type transform is a highly interesting and stimulating object in the standpoint of infinite dimensional harmonic analysis.

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2.3 無限次元ラプラス作用素

数理物理への応用上の必要性から無限次元多様体を念頭に置き，外積空間値のホワイトノイズ汎関数を考え，その上のC^∞不変な性質の良いラプラス作用素を構成し，その作用の具体的表現も導出した。本研究の最も特徴的な点はホワイトノイズ解析における飛田微分の採用をその基礎にしたことにある。その結果として時間発展の記述の概念が定義自体の中に取り込まれ，因果性(Causality)を考慮の対象に含むprofoundな表現形式の理論が具現されている。またこのラプラス作用素の応用として，無限次元版ド・ラーム＝ホッジ＝小平型分解定理の飛田微分に付随したVersionをいくつか示すことができた。

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ON THE LAPLACIAN ON A SPACE OF WHITE NOISE FUNCTIONALS

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ON THE LAPLACIAN ON A SPACE OF WHITE NOISE FUNCTIONALS

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§ 1. Introduction.

We are greatly interested in the Laplacian on a space of white noise functionals. To have in mind aspects of application to mathematical physics, we can say that it is common in general to use the weak derivative D on a given basic Hilbert space, so as to define d_p which just corresponds to the de Rham exterior differential operator. In doing so, one of the remarkable characteristics of our work consists in adoption of the Hida differential ∂_t instead of D. This distinction from other related works does provide a framework of analysis equipped with the function for perception of the time t, with the result that it is converted into a more flexible and charming theory which enables us to treat time evolution directly. It can be said, therefore, that our work is successful in deepening works about the general theory done by Arai-Mitoma [2], not only on a qualitative basis but also from the applicatory point of view in direct description of operators in terms of time evolution.

The differential ∂_t has its adjoint operator ∂_t^* in Hida sense and it is called the Kubo operator. Indeed, ∂_t^* is realized by extending the functional space even into the widest one $(E)^*$, where a Gelfand triple $(E) \subsetneq (L^2) \subsetneq (E)^*$ is a fundamental setting in white noise analysis, in accordance with our more general choice of the basic Hilbert space H. On the contrary, we define the adjoint operator d_p^* of d_p associated with ∂_t without extending the space up to that much. Consequently the Laplacian Δ_p constructed in such an associated manner with d_p (so that, with ∂_t) is realized as an operator having analytically nice properties, such as C^∞-invariance, etc. On the other hand, when we take the Kubo operator as its adjoint, then the so-called Hida Laplacian Δ_H is naturally derived. It is, however, well-known that Δ_H is an operator which maps (S) into $(S)^*$, or in our general setting from (E) into $(E)^*$, which means that it

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transforms a smooth class not into itself, but into the widest class of generalized white noise functionals (or the so-called Hida distributions). On this account the following problem is highly interesting in the standpoint of operator theory or infinite-dimensional analysis.

Let us choose the Hida differential ∂_t as the starting point of the theory. Then if we assume that the Laplacian Δ constructed according to the de Rham theory should possesse a nice property such that it maps a smooth class into itself (i.e. $\Delta: C^\infty \to C^\infty$), what on earth would its adjoint d_p^* naturally corresponding to ∂_t (hence $d_p = d_p(\partial_t)$) be like? This is one of our motivations in this paper (cf. the beginning of § 5). The followings are in fact outstanding features of our work and what have been acquired in connection with the aforementioned problem: (1) in regard to the adjoint operator $d_p^* \equiv d_p^*(\partial_t)$ of $d_p \equiv d_p(\partial_t)$, we have as a matter of fact succeeded in constructing it in such a satisfactory manner as to fit into our requirement; (2) as a consequence the constructed Laplacian Δ_p, which is associated with ∂_t, enjoys extremely nicer properties on analytical basis, i.e., Δ_p is a C^∞-invariant operator on a space of white noise functionals (cf. Theorem 7.7); (3) moreover, peculiar ideas of generalized functions totally released from smearing with respect to time t produces the corresponding higher version of theory in operators on functionals, which allows us, despite its implicity, to draw the description of time evolution; (4) our Δ_p primarily settled with the Hida derivative ∂_t is a Laplace-Beltrami type operator getting possession of such a nice property, and it is completely distinct from other Laplacians in white noise analysis, such as the Lévy Laplacian Δ_L, the Gross Laplacian Δ_G, and the Volterra Laplacian Δ_V; (5) the Laplacian is in a sense successfully constructed in concrete and satisfactory manner, simply corresponding to our more general choice of the basic Hilbert space H, and the explicit form $\Delta_p \omega$ of the Laplacian on $\omega \in \mathscr{P}$ (the space of polynomials) is also obtained (cf. Proposition 6.3); (6) as one of applications in terms of our Laplacians, this paper includes several versions of the so-called de Rham-Hodge-Kodaira decomposition theorem associated with Hida derivative in white noise calculus or Hida calculus (cf. Theorem 5.3, Theorem 7.1, and Theorem 7.8). To comment upon the above (4) in addition, it is therefore expected in a quite natural way that Δ_p should play a remarkable and proper role in white noise analysis, which is entirely different from those of the other Laplacians. It remains to be stimulating object in relation with other works [21, 28 & 30] on Laplacians, and it is highly interesting as well.

This paper is organized as follows:

§ 1. Introduction.

§2. Notation and preliminaries.
§3. Hida differentiation.
§4. De Rham complex.
§5. Laplacians $\Delta_p(\Theta, \partial_t)$ of de Rham complex $\{\tilde{d}_p(\Theta, \partial_t)\}$.
§6. Explicit forms of the Laplacians $\Delta_p(\Theta, \partial_t)$.
§6. De Rham-Hodge-Kodaira decompositions associated with Hida derivative.
§8. Concluding remarks.

In §2 we shall introduce notations commonly used in this whole paper, and preliminary results are also stated in §2, some of which are generalizations [9-11] of the well-known results on basic and fundamental theorems in white noise analysis, having been obtained by many pioneers and forerunners [17, 24, 25 & 27]. §3 is devoted to general but brief explanations on the basic ideas, important concepts, and interpretations of Hida differentiation. This will be the key to understand the succeeding sections. There are contained some assertions, simply corresponding to our general setting (cf. [12-15]). §4-§7 are the main parts of our paper. In §4 we shall construct de Rham complexes. For a complex Hilbert space K, let $\Lambda^p K$ be the space of exterior product of order p. Consider a nonnegative selfadjoint operator A on a given normal Hilbert space H, and we denote by the symbol Θ the linear closed operator: $H_C \to K$, determined regarding A. Then the operator d_p from $\mathcal{P}(\Lambda^p K)$ into $\mathcal{P}(\Lambda^{p+1} K)$, depending on Θ, is able to be realized by making use of the Hida differential operator. In §5 we shall state a systematic construction of Laplacians Δ_p of $\{d_p\}$. The corresponding Laplace operator can be constructed theoretically and get into entity when we take advantage of the adjoint operator and have resort to functional analytical method (see Proposition 5.2). By virtue of closedness of the sequences of complexes we can obtain the de Rham-Hodge-Kodaira theorem (Theorem 5.3) in L^2-sense [16]. In §6 the explicit form of the Laplacian Δ_p will be obtained by a direct computation (see Proposition 6.3), where the leading idea is similar to [2], however, as stated before, the employed calculus and basic mathematical background are actually different, since we are totally based upon the white noise calculus or Hida calculus. In §7 we shall make mention of several versions of de Rham-Hodge-Kodaira type theorem associated with Hida derivative [8]. It is easy to see that such a type of decomposition holds for the space of smooth test functionals, induced by the Sobolev type space $H^{2,k}$ of functionals relative to the Laplacian (Theorem 7.1), namely,

$$H^{2,\infty}(\wedge_2^p(K))=\mathrm{Im}\,[\Delta_p(\Theta)\restriction H^{2,\infty}(\wedge_2^p(K))]\oplus\mathrm{Ker}\,\Delta_p(\Theta)\,.$$

On this account we may employ the Arai-Mitoma method (1991) to derive the similar decomposition theorem even for the category $(S)(\wedge^p K)$, just corresponding to the space of white noise test functionals (see Theorem 7.8). Basically, principal ideas for proofs are due to the spectral theory. However, some of statements include subtler precise estimates, for which we are definitely required to execute elaborate computation with some other results in orthodox probability theory and Malliavin calculus.

Finally it is quite interesting to note that this sort of result leads to the study of Dirac operators on the Boson-Fermion Fock space (cf. [1]), and also that our analysis could be another admissible key to the supersymmetric quantum field theory (e.g., [34]). We believe that this formalism proposed in this paper should be possibly regarded as a clue to open a new pass towards analysis of Dirac operators in quantum field theory through the framework of Hida calculus.

§ 2. Notation and preliminaries.

Let T be a separable topological space equipped with a σ-finite Borel measure $d\nu(t)$ on the topological Borel field $\mathscr{B}(T)$. Further suppose that ν be equivalent to the Lebesgue type measure dt. $H := L^2(T, d\nu; \boldsymbol{R})$ is the real separable Hilbert space of square integrable functions on T. Its norm and inner product will be denoted by $|\cdot|_0$ and $(\cdot, \cdot)_0$. Let A be a densely defined non-negative selfadjoint operator on H. We call A with domain $\mathrm{Dom}(A)$ *standard* if there exists a complete orthonormal basis $\{e_n\}_{n=0}^{\infty} \subset \mathrm{Dom}(A)$ such that

$$Ae_n = \lambda_n e_n \quad \text{for } \lambda_n \in \boldsymbol{R}, \tag{A.1}$$

$$1 < \lambda_0 \leqq \lambda_1 \leqq \cdots \longrightarrow \infty, \tag{A.2}$$

$$\sum_{n=0}^{\infty} \lambda_n^{-2} < \infty \quad \text{holds (cf. [9, 10])}. \tag{A.3}$$

Obviously, A^{-1} is extended to an operator of Hilbert-Schmidt class. Put

$$\rho := \lambda_0^{-1} = \|A^{-1}\|_{\mathrm{op}},$$

and

$$\delta := \left(\sum_{n=0}^{\infty} \lambda_n^{-2}\right)^{1/2} = \|A^{-1}\|_{\mathrm{HS}}$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm and $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. We also note the following apparent inequalities:

$$0 < \rho < 1, \qquad \rho < \delta.$$

For a complex separable Hilbert space K, we further assume that

(A.4) There exists a densely defined, closed linear operator Θ from H_C into K such that $A=\Theta^*\Theta$,

where we define the complexification $H_C=H+iH$ as usual way, and Θ^* means the adjoint of Θ.

Given such a standard operator A on H, we can construct a Gelfand triple in the standard manner (see [22, p. 259], [27]). For $p\geqq 0$ let E_p be the completion of $\mathrm{Dom}(A^p)$ with respect to the Hilbertian norm $|\xi|_p:=|A^p\xi|_0$, $\xi\in \mathrm{Dom}(A^p)$, where $\mathrm{Dom}(A^p)=H$ for $p<0$. Then E_p becomes a Hilbert space with the norm $|\cdot|_p$. We thus obtain a chain of Hilbert spaces:

$$\cdots\subset E_p\subset\cdots\subset E_q\subset\cdots\subset H\subset\cdots$$
$$\cdots\subset E_{-q}\subset\cdots\subset E_{-p}\subset\cdots$$

for $0\leqq q\leqq p$. Equipped with the Hilbertian norms $\{|\cdot|_p\}_{p\geq 0}$,

$$E:=\bigcap_{p\geq 0} E_p$$

becomes a nuclear Fréchet space. E is topologized by the projective limit of Hilbert spaces $\{E_p\}_{p\in Z}$ with inner products $(\xi,\eta)_p$ $(\xi,\eta\in E)$, and is called the space of test functions on T. The topological dual space E^* of E is obtained as

$$E^*:=\bigcup_{p\geq 0} E_{-p},$$

i.e., the dual space E^* of E is the inductive limit of E_{-p} as $p\to\infty$. E^* is equipped with the inductive limit convex topology (e.g. [10, Eq. (3.1), § III]). The triplet $E\subset H\subset E^*$ is called a rigged Hilbert space [3] or a Gelfand triple. Then note that the dual space $E_C{}^*=(E_C)^*$ is equivalent to $(E^*)_C=E^*+iE^*$. It is known that the strong dual topology of E^* coincides with the inductive limit topology in our setting (see [35]). Let μ be the Gaussian probability measure on the measurable space $(E^*, \mathcal{B})$ whose characteristic functional is uniquely determined, by virtue of the Bochner-Minlos theorem, by

$$\int_{E^*}\exp(i\langle x,\xi\rangle)\mu(dx)=\exp\left(-\frac{1}{2}|\xi|_0{}^2\right),\qquad \xi\in E, \tag{2.1}$$

where $\mathcal{B}$ is the σ-algebra containing cylinder sets. For simplicity we denote only by $\langle\cdot,\cdot\rangle$ the canonical bilinear forms between any dual pairs unless it causes any confusion in the context. For instance, when $\langle\cdot,\cdot\rangle$ is a bilinear form on $E^*\times E$, then it is naturally extended to a C-bilinear from on $E_C{}^*\times E_C$. We will denote the space $L^2(E^*,\mathcal{B},\mu;C)$ briefly by (L^2) according to the notation in [17]. Let $\|\cdot\|_0$ denote its norm. Note htat (L^2) is a complex Hilbert space. We them assume the following three conditions (cf. [9-11]) which are suggested by Kubo-Takenaka [24].

(A.5) For every $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T which coincides with ξ up to ν-null functions.

(A.6) For each $t \in T$ the evaluation map $\delta_t : \xi \to \xi(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$.

(A.7) The map $t \to \delta_t$ is continuous from T into E^*.

By virtue of (A.5) we agree then that E consists of continuous functions. The symbol $E_C{}^{\otimes n}$ denotes the n-fold tensor product of the complexification of E. For $f \in E^{\otimes n}$ and $p \in \boldsymbol{R}$, define $|f|_{p,\otimes n} := |(A^p)^{\otimes n} f|_0$. Let $(E_p)_C{}^{\hat{\otimes} n}$ be the n-fold symmetric tensor product of $(E_p)_C$. $E_C{}^{\hat{\otimes} n}$ denotes the projective limit of $(E_p)_C{}^{\hat{\otimes} n}$ and $(E_C{}^*)^{\hat{\otimes} n}$ the inductive limit of $(E_{-p})_C{}^{\hat{\otimes} n}$ as p tends to infinity. In the following we shall consider all the time the inductive limit space together with the inductive limit convex topology.

REMARK 2.1. Note that the measure ν is supposed to be rotation invariant in the setting of white noise calculus. T is often thought of as time parameter space. In the above we have in mind the harmonic oscillator Hamiltonian [19, p. 148] as a concrete model of A (cf. Example 2.1 given later in § 2), which is typical in Hida calculus (see [7, 27]).

By the Wiener-Itô decomposition theorem we have

$$(L^2) = \sum_{n=0}^{\infty} \oplus K_n, \tag{2.2}$$

where K_n is the space of n-fold Wiener integrals $I_n(f_n)$, $f_n \in H_C{}^{\hat{\otimes} n}$ (cf. [24, 1981] or [9, Remark 1.2, § I]). $H_C{}^{\hat{\otimes} n}$ is the n-fold symmetric completed Hilbert space tensor product of the complexification of H, hence $H_C{}^{\hat{\otimes} n}$ is again a Hilbert space. It is a fact that (L^2) is canonically isomorphic to the Fock space over H_C, that is,

$$(L^2) \cong \sum_{n=0}^{\infty} \oplus H_C{}^{\hat{\otimes} n}. \tag{2.3}$$

For each $\varphi \in (L^2)$ there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$, $f_n \in H_C{}^{\hat{\otimes} n}$ such that

$$\|\varphi\|_0{}^2 = \sum_{n=0}^{\infty} n! |f_n|_{0,\otimes n}^2, \tag{2.4}$$

and

$$\varphi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \qquad \mu\text{-a.e. } x \in E^*. \tag{2.5}$$

where the right hand side is an orthogonal direct sum of functions in (L^2) (e.g. [9, Theorem 2.3]; see also [20]). The symbol $:x^{\otimes n}:$ is the Wick ordering of the distribution $x^{\otimes n}\in(E^{\hat{\otimes}n})^*$, which is defined inductively as follows:

$$:x^{\otimes 0}:=1, \qquad :x^{\otimes 1}:=x,$$

$$:x^{\otimes n}:=x\hat{\otimes}:x^{\otimes(n-1)}:-(n-1)\tau\hat{\otimes}:x^{\otimes(n-2)}:, \quad (n\geqq 2)$$

where $\tau\in(E\hat{\otimes}E)^*$ is the distribution defined by

$$\langle\tau, \xi\otimes\eta\rangle=\langle\xi, \eta\rangle, \qquad \xi, \eta\in E. \tag{2.6}$$

Note that τ is also expressed as

$$\tau=\int_T \delta_t^{\otimes 2}d\nu(t)=\sum_{j=0}^{\infty}e_j\otimes e_j. \tag{2.7}$$

When we define S-transform as

$$S\varphi(\xi)=\int_{E^*}\varphi(x)F(\xi;x)\mu(dx), \tag{2.8}$$

then we have $(SI_n(f_n))(\xi)=\langle f_n, \xi^{\hat{\otimes}n}\rangle$, where

$$F(\xi;x)=:\exp\langle x, \xi\rangle:=\exp\left(\langle x, \xi\rangle-\frac{1}{2}|\xi|_0^2\right)$$

(see [24]; also [9, § I]). Based upon the result in (2.4) and (2.5) we may introduce a second quantized operator $\Gamma(A)$ on (L^2). Let $\mathrm{Dom}(\Gamma(A))$ be the subspace of $\varphi\in(L^2)$ given as in Eq. (2.5) such that

(i) $f_n=0$ except finitely many n;

(ii) $f_n\in\mathrm{Dom}(A)\otimes_{alg}\cdots\otimes_{alg}\mathrm{Dom}(A)$ (n-times).

Then for $\varphi\in\mathrm{Dom}(\Gamma(A))$ we put

$$(\Gamma(A)\varphi)(x)=\sum_{n=0}^{\infty}I_n(A^{\otimes n}f_n)(x). \tag{2.9}$$

Let (E_p) be the completion of $\mathrm{Dom}(\Gamma(A)^p)$ with respect to the Hilbertian norm

$$\|\varphi\|_p^2:=\|\Gamma(A)^p\varphi\|_0^2=\sum_{n=0}^{\infty}n!|f_n|_{p,\otimes n}^2$$

$$=\sum_{n=0}^{\infty}n!|(A^p)^{\otimes n}f_n|_{0,\otimes n}^2,$$

where $f_n\in(E_p)_C^{\hat{\otimes}n}$. Equipped with the norm $\{\|\cdot\|\}_{p\geq 0}$,

$$(E):=\bigcap_{p\geq 0}(E_p)$$

becomes a nuclear Fréchet space. Let $(E)^*$ be the dual space of (E). For any $\varphi\in(E)$, φ has a continuous version $\tilde{\varphi}$, and it is bounded on each bounded set of E^*, moreover the evaluation map $\delta_x:\varphi\to\tilde{\varphi}(x)$ is a continuous linear functional on (E), i.e., $\delta_x \in (E)^*$ for any $x \in E^*$ (cf. [25]; see also [10, 11]). By the above fact we always regard (E), as a space of continuous functions on E^*. An element in (E) (resp. $(E)^*$) is called a test (resp. generalized) white noise functional. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical $\boldsymbol{C}$-bilinear form on $(E)^* \times (E)$.

Lastly we introduce an example, which is enough to show that our general setting stated above is not unsubstantial.

EXAMPLE 2.1. When $T=\boldsymbol{R}$, $d\nu(t)=dt$, and when we choose $A=1+t^2-(d/dt)^2$, then Θ is given by $d/dt-t$ $(t \in \boldsymbol{R})$ with $H_C=K=L^2(\boldsymbol{R})$, and we have $(E)^*=(S)^*$, $(E)=(S)$ with Gelfand triple

$$(S) \subset (L^2) \subset (S)^* .$$

This is a typical model of white noise spaces in Hida calculus, originally introduced by T. Hida [17, 18] and developed by others [19, 22, 24 & 29] (see also [7, 26] for its applications).

§ 3. Hida differentiation.

We now introduce a differential operator ∂_t which plays a fundamental and important role in white noise calculus. We call ∂_t the Hida differential and $\partial_t \varphi(x)$ a Hida derivative. Originally the operator ∂_t is written as

$$\partial/\partial x(t) = \partial/\partial \dot{B}(t)$$

under the framework of choice $H=L^2(\boldsymbol{R}; dt)$, where $\dot{B}(t)$ indicates the formal time derivative of one-dimensional Brownian motion $B(t)$, $t \in \boldsymbol{R}$ (cf. [17, 18]). Because the causal calculus or Hida calculus is the analysis on white noise functionals and its basic idea is to take a white noise $\dot{B}(t)$ to be the system of variables of white noise functionals, it is quite natural to consider $\partial_t = \partial/\partial \dot{B}(t)$ as its coordinate differentiation. It is needless to say that T. Hida's original idea was a farsighted choice of coordinate system fitting for the causal calculus, if one sees its rapid exciting development and progress in white noise analysis (WNA) for the last few decades (cf. [19, 20 & 22]).

For $\varphi \in (E)$ and $\delta_t \in E^*$ we put

$$\tilde{\partial}_t \varphi(x) := (D_{\delta_t} \varphi)(x) = \sum_{n=1}^{\infty} n \langle : x^{\otimes(n-1)} :, \delta_t * f_n \rangle , \tag{3.1}$$

where $f_n \in E_C^{\hat{\otimes} n}$. Note that $\tilde{\partial}_t = D_{\delta_t}$ is a continuous linear operator on (E) [12]. It is known that

$$(\tilde{\partial}_t\varphi)(x)=\lim_{\theta\to 0}\theta^{-1}\{\varphi(x+\theta\cdot\delta_t)-\varphi(x)\},$$

for $\varphi\in(E)$. For $\Phi\in(E)^*$, its generalized U-functional $U(\xi)=U_\Phi(\xi)$ is defined to be

$$U[\Phi](\xi):=\langle\!\langle\Phi,\ :e^{\langle\cdot,\xi\rangle}:\rangle\!\rangle,\qquad \xi\in E.$$

where $:\exp\langle\cdot,\xi\rangle:\ :=\exp\langle\cdot,\xi\rangle\times\exp(-(1/2)|\xi|_0^2)\in(E)$ (see [29] for its characterization). We can rephrase the above definition as follows: $(S\Phi)(\xi)=U[\Phi](\xi)$. In white noise calculus the collection $\{\dot{B}(t);t\in\boldsymbol{R}\}$ is taken as a coordinate system. Thus we need to define the coordinate differentiation with respect to this system. This can be done directly through the U-functional. Let Φ be in $(E)^*$. Suppose that the U-functional F of Φ has the Fréchet functional derivative $F'(\xi;u)\equiv\delta F(\xi)/\delta\xi(u)$. If the function $F'(\cdot;t)$ is a U-functional, then the Hida derivative $\partial_t\Phi$ of Φ is the element in $(E)^*$ with U-functional $F'(\cdot;t)$, i.e., $U[\partial_t\Phi](\xi)=F'(\xi;t)$. Note that in general $\partial_t\Phi$ is a distribution as a function of t. In other words, according to Kubo-Takenaka [24] we have

$$\partial_t\Phi(x)=S^{-1}\frac{\delta}{\delta\xi(t)}S\Phi(x), \tag{3.2}$$

(cf. [12-15]). Let $\mathcal{P}$ be the set of polynomials in E^*, and its element $P\in\mathcal{P}$ is expressed as

$$P(x)=\sum_{n=0}^{k}\langle :x^{\otimes n}:,\ f_n\rangle,\qquad f_n\in E_C^{\hat{\otimes}n}.$$

We know that, for $t\in T$, ∂_t and the Gâteaux derivative in direction δ_t coincide on $\mathcal{P}$ (see [14, Lemma 2.2]).

If $\varphi\in(E)$ has chaos expansion $\{f_n;n\in\boldsymbol{N}_0\}$, then denoting by $\tilde{\varphi}$ and $\tilde{f}_n$, $n\in\boldsymbol{N}_0$ their corresponding continuous versions (cf. [9, Remark 3.4], [10, Th. 3.1], and [11, Th. 2.1]), we have

$$\tilde{\partial}_t\tilde{\varphi}(x)=\sum_{n=1}^{\infty}n\langle :x^{\otimes(n-1)}:,\ \tilde{f}_n(t,\cdot)\rangle,\qquad t\in T,$$

where $\tilde{f}_n(t,\cdot)=\delta_t*\tilde{f}_n=\langle\delta_t,\tilde{f}_n\rangle$ (see [14, Remark 3.2]). We always identify $\varphi\in(E)$ with its continuous version on E^*, so that, in the following we shall suppress the distinction between them on a notational basis. The number operator N is defined by

$$N\Big(\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ f_n\rangle\Big)=\sum_{n=1}^{\infty}n\langle :x^{\otimes n}:,\ f_n\rangle. \tag{3.3}$$

By [14, Theorem 3.5], generally, for any $y\in E^*$, D_y extends from $\mathcal{P}$ to a continuous linear map from (E) into itself. In particular, (E) is infinitely Gâteaux differentiable in every direction of E^*, moreover, for any $\varphi\in(E)$ the function

$y \to D_y\varphi$ is strongly continuous from E^* into (E). Therefore, in particular, the function $t \to \partial_t\varphi$ is continuous from T to (E) (see also [12, 13]). The followings are verified by employing reflexiveness of (E) (Lemma 4.1 in [14]) with the celebrated Schwartz kernel theorem: namely, for $\varphi \in (E)$, $\nabla\varphi \in E \otimes (E)$ holds, and for every $y \in E^*$,

$$D_y\varphi = \langle y, \nabla\varphi \rangle, \qquad \mu\text{—a.e.,} \tag{3.4}$$

suggesting that $\nabla\varphi$ is the Fréchet derivative of $\varphi \in (E)$. In particular, if $h \in H$, then

$$D_h\varphi = \int_T h(t)\partial_t\varphi d\nu(t), \qquad \mu\text{—a.e.,} \tag{3.5}$$

where the integral should be interpreted as a Bochner integral. Furthermore, every $\varphi \in (E)$ is infinitely Fréchet differentiable and the k-th Féchet derivative of φ is given by $\nabla^k\varphi \in E^{\hat{\otimes} k} \otimes (E)$ (cf. Theorem 4.3 and Theorem 4.4 in [14]). Moreover, the gradient ∇ extends from $\mathcal{P}$ to a continuous linear operator from $\mathrm{Dom}(\sqrt{N})$ into $L^2(T \to (L^2); d\nu)$ (see [15]), where $(\nabla\varphi)(t, x) = \partial_t\varphi(x)$.

§ 4. De Rham complex.

First of all we start on a notation. $\mathcal{P}$ is the whole space of $\boldsymbol{C}$-valued polynomials on E^* as described in § 3. Note that $\mathcal{P}$ is dense in (L^2). For $p \in \boldsymbol{N}_+$, the p-fold exterior product space $\Lambda^p K$ is defined by $\Lambda^p K := \{\omega \in \otimes^p K : \sigma(\omega) = \mathrm{sgn}(\sigma)\cdot\omega, \ \forall \sigma \in \mathcal{G}_p\}$, where $\mathcal{G}_p$ is the symmetric group of order p. We introduce the following metric in $\wedge^p K$: i.e., for any $\omega, \gamma \in \wedge^p K$ such that $\omega = f_1 \wedge \cdots \wedge f_p$, $\gamma = g_1 \wedge \cdots \wedge g_p$, $f_k \in K$, $g_k \in K$ (for any $k = 1, 2, \cdots, p$), the inner product between ω and γ is given by

$$\langle \omega, \gamma \rangle^{\wedge^p K} := \sum_{\sigma \in \mathcal{G}_p} \mathrm{sgn}(\sigma) \cdot \prod_{k=1}^{p} \langle f_k, g_{\sigma(k)} \rangle_K .$$

$\Lambda^p K^c$ denotes the completion of $\Lambda^p K$ by the above metric $\langle \cdot, \cdot \rangle^{\wedge^p K}$, with $\Lambda^0 K^c = \boldsymbol{C}$. Its element is called a p-fold skew symmetric tensor, and A_p is an alternating operator from $\otimes^p K$ into $\Lambda^p K$. When $B := \Theta\Theta^*$, then $D^\infty(B) := \bigcap_{m \in N} \mathrm{Dom}(B^m)$. We denote by $\mathcal{P}(\Lambda^p K^c)$ the whole space of $\Lambda^p K^c$-valued polynomials on E^*, whose element is expressed by

$$\omega(x) = \sum_{n=1}^{k} \tilde{P}_n(x) \cdot \xi_n, \qquad x \in E^*, \tag{4.1}$$

where $\tilde{P}_n \in \mathcal{P}$, $\xi_n \in A_p(\otimes^p D^\infty(B)) \subset \Lambda^p K^c$. Notice that $\mathcal{P}(\Lambda^p K^c)$ is dense in $\Lambda_2^p(K)$, and $\Lambda_2^p(K)$ is defined to be $(L^2) \otimes \Lambda^p K^c$ which is identified with

$L^2(E^* \to \Lambda^p K^c ; d\mu)$ in a usual manner [32].

Now we will introduce a linear operator d_p from $\mathcal{P}(\Lambda^p K^c)$ into $\mathcal{P}(\Lambda^{p+1} K^c)$ for each $p \in \boldsymbol{N}_+$. Actually, for any $\omega \in \mathcal{P}(\Lambda^p K^c)$ especially of the form (4.1), the operator d_p $(\equiv d_p(\Theta) = d_p(\Theta, \partial_t))$ is defined as

$$(4.2) \qquad d_p \omega(x) \equiv (p+1) \sum_{n=1}^{k} A_{p+1}(\Theta \cdot \partial_t \tilde{P}_n(x) \otimes \xi_n),$$

where ∂_t is the Hida differential (see §3). We have $\tilde{P}_n$ in our standard representation of element in (L^2):

$$\tilde{P}_n(x) = \sum_{l=1}^{N(n)} \langle : x^{\otimes l} :, f_l \rangle,$$

where f_l is the element in $E_C{}^{\hat{\otimes} l}$ given by

$$f_l = \sum_{\alpha \in N^l} b_\alpha \eta_{\alpha_1, l} \hat{\otimes} \cdots \hat{\otimes} \eta_{\alpha_l, l}, \; b_\alpha \in \boldsymbol{C}, \; \eta_{\alpha_j, l} \in E_C ,$$

Note that all representations of $\tilde{P}_n$ are everywhere defined, continuous functions on E^*. Therefore, the U-functional of $d_p(\Theta)\omega$ is given by

$$(4.3) \qquad U[d_p(\Theta)\omega](\zeta)$$
$$= \sum_{n=1}^{k} \Big\{ \sum_{l=1}^{N(n)} \sum_{\alpha \in N^l} b_\alpha \sum_{m=1}^{l} (\eta_{\alpha_1, l}, \zeta) \cdots (\eta_{\alpha_m, l}{}^{\vee}, \zeta) \cdots (\eta_{\alpha_l, l}, \zeta) \Big\}$$
$$\cdot \Theta(\eta_{\alpha_\mu, l}(t)) \wedge w_1{}^{(n)} \wedge \cdots \wedge w_p{}^{(n)}, \qquad \zeta \in E,$$

where the symbol $\vee$ means omission of the term. For each $p \in \boldsymbol{N}_+$, $d_p(\Theta)$ is densely defined linear operator in $\Lambda_2{}^p(K)$, and, it is easy to see that

$$(4.4) \qquad d_{p+1}(\Theta, \partial_t) \circ d_p(\Theta, \partial_t) = 0 \qquad \text{on } \mathcal{P}(\Lambda^p K^c).$$

Its adjoint operator $d_p{}^*(\Theta) \equiv d_p{}^*(\Theta, \partial_t)$ from $\Lambda_2{}^{p+1}(K)$ into $\Lambda_2{}^p(K)$ is defined by

$$\langle d_p(\Theta)\omega, \gamma \rangle_{\Lambda_2^{p+1}(K)} = \langle \omega, d_p{}^*(\Theta)\gamma \rangle_{\Lambda_2^p(K)}$$

for $\omega \in \Lambda_2^p(K)$, $\gamma \in \Lambda_2^{p+1}(K)$.

REMARK 4.1. Note that the U-functional representation of $d_p{}^*(\Theta)\omega$ is given by

$$(4.5) \qquad U[d_p{}^*(\Theta)\omega](\zeta) = \sum_{n=1}^{k} \Big[\sum_{l=1}^{p+1} (-1)^{l-1} \Big\{ \sum_{j=1}^{N(n)} \sum_{\alpha \in N^j} b_\alpha \prod_{i=1}^{j} (\eta_{\alpha_i, j}, \zeta)_{H_C}$$
$$\times (\Theta^* \overline{w_1{}^{(n)}}, \zeta)_{H_C} - \Big(\sum_{j=1}^{N(n)} \sum_{\alpha \in N^j} b_\alpha \sum_{m=1}^{j} \eta_{\alpha_m, j}(t)$$
$$\cdot (\eta_{\alpha_1, j}, \zeta) \cdots (\eta_{\alpha_m, j}{}^{\vee}, \zeta) \cdots (\eta_{\alpha_j, j}, \zeta), \; \Theta^* w_1{}^{(n)} \Big)_{H_C} \Big\}$$
$$\times w_1{}^{(n)} \wedge \cdots \wedge \check{w}_1{}^{(n)} \wedge \cdots \wedge w_{p+1}{}^{(n)} \Big],$$

for $\zeta\in E$ (cf. Lemma 6.2). □

It follows immediately from (4.4) that

$$d_p^*(\Theta, \partial_t)\circ d_{p+1}^*(\Theta, \partial_t)=0 \qquad \text{on } \mathrm{Dom}\,(d_{p+1}^*(\Theta)). \tag{4.6}$$

It can be deduced from denseness and the adjoint argument that $d_p(\Theta)$ becomes closable for each $p\in \boldsymbol{N}_+$. We write its extension $\bar{d}_p$ of d_p, and we put $\Lambda_2^*:=\sum_{p=0}^{\infty}\Lambda_2{}^p(K)$. Then the sequence $(\Lambda_2^*(K), \{\bar{d}_p(\Theta, \partial_t)\})$ forms a de Rham complex.

REMARK 4.2. For $\zeta\in E$, $\omega\in\mathcal{P}(\Lambda^p K^c)$, we have

$$(S\omega)(\zeta)=\sum_{n=1}^{k}\Big(\sum_{l=1}^{N(n)}\sum_{\alpha\in N^l} b_\alpha\prod_{i=1}^{l}(\eta_{\alpha_i,l}, \zeta)_{H_C}\Big)w_1{}^{(n)}\wedge\cdots\wedge w_p{}^{(n)}.$$

Recall Eq. (3.2) in § 3, then (4.3) is obvious.

REMARK 4.3. In general, the operator $d_p(\Theta, \partial_t)$ constructed in such a way is not necessarily closable. The closability of $d_p(\Theta, \partial_t)$ depends on the structure of the measure μ on E^*. This is a very touchy problem indeed. However, fortunately in our case $d_p(\Theta, \partial_t)$ is well-defined for the Gaussian white noise measure μ defined in (2.1).

§ 5. Laplacians $\Delta_p(\Theta, \partial_t)$ of de Rham complex $\{\bar{d}_p(\Theta, \partial_t)\}$.

As we have stated in § 1, it is clear why we stick to the Hida differentiation, for we are aiming at opening a new pass toward analysis in mathematical physics through the framework of Hida calculus. On the other hand, when we say that an operator is called to be smooth if it transforms the space of smooth elements into itself, there is the fact that the Hida Laplacian (cf. § 1) is not smooth any longer in the above sense. That is why we would like to know what the desired Laplacian should be like, which is one of our motivations. One may find an answer to the matter in this section (see also Theorem 7.7 in § 7).

Thanks to the fact that $\mathrm{Im}\,(\bar{d}_{p-1}(\Theta))$ and $\mathrm{Im}\,(d_p^*(\Theta))$ are closed for $p\in\boldsymbol{N}_+$ in our case, by making use of the sesquilinear form and elaborate functional analysis methods we can define a unique nonnegative selfadjoint operator acting in $\Lambda_2{}^p(K)$. This is nothing but the desired Laplacian corresponding to the de Rham complex $\{\bar{d}_p(\Theta, \partial_t)\}$. In the last we shall give a primictive version of the de Rham-Hodge-Kodaira type decomposition for the p-forms in the L^2-sense.

We first consider the bilinear function J_p on $\mathrm{Dom}\,(J_p(\Theta)):=\mathrm{Dom}\,(\bar{d}_p(\Theta))\cap$

$\mathrm{Dom}\,(d_{p-1}{}^*(\Theta))$, which is dense in $\Lambda_2{}^p(K)$. For $p\in N_+$, $J_p(\Theta)\equiv J_p(\Theta,\partial_t)$ is defined to be

$$(5.1)\qquad J_p(\Theta)(\omega,\gamma):=\langle \bar{d}_p(\Theta)\omega,\ \bar{d}_p(\Theta)\gamma\rangle_{\Lambda_2{}^{p+1}(K)} + \langle d_{p-1}{}^*(\Theta)\omega,\ d_{p-1}{}^*(\Theta)\gamma\rangle_{\Lambda_2{}^{p-1}(K)}$$

for any ω, $\gamma\in\mathrm{Dom}\,(J_p(\Theta))$. This J_p turns to be a sesquilinear form on $\Lambda_2{}^p(K)\times\Lambda_2{}^p(K)$. Note that this formalism indicates the Laplacian Δ_p to be roughly given by $d_p{}^*\bar{d}_p+\bar{d}_{p-1}d_{p-1}{}^*$ as usual. As a matter of fact, it is easy to see that the form $J_p(\Theta)$ is a nonnegative, densely defined, closed form on $\mathrm{Dom}\,(J_p(\Theta))$. On this account, we obtain the following representation of Friedrichs type.

PROPOSITION 5.1 [16]. *Let $J_p(\Theta,\partial_t)$ be a nonnegative closed sesquilinear form with the dense domain* $\mathrm{Dom}\,(J_p(\Theta))$. *Then there exists a unique nonnegative selfadjoint operator $\Delta_p(\Theta)\equiv\Delta_p(\Theta,\partial_t)$ acting in $\Lambda_2{}^p(K)$ such that*

$$(5.2)\qquad \langle\omega,\ \Delta_p(\Theta)\gamma\rangle_{\Lambda_2{}^p(K)}=J_p(\Theta)(\omega,\gamma),$$

for $\omega\in\mathrm{Dom}\,(J_p(\Theta))$, $\gamma\in\mathrm{Dom}\,(\Delta_p(\Theta))$, $p\in N_+$.

REMARK 5.1. In the above assertion, $\mathrm{Dom}\,(\Delta_p(\Theta))$ is dense in $\mathrm{Dom}\,(J_p(\Theta))$ in the sense of $J_p(\Theta)$-form norm, as a consequence $\mathrm{Dom}\,(\Delta_p(\Theta))$ is also naturally dense in $\Lambda_2{}^p(K)$. For the proof, see Theorem 2.2 and § III in [16]. □

Proposition 5.1 and the second representation theorem [23, VI. 2] immediately gives:

PROPOSITION 5.2. *There exists a unique nonnegative selfadjoint operator $\Delta_p(\Theta,\partial_t)$ in $\Lambda_2{}^p(K)$ such that the equality*

$$(5.3)\qquad \langle\Delta_p{}^{1/2}(\Theta)\omega,\ \Delta_p{}^{1/2}(\Theta)\gamma\rangle_{\Lambda_2{}^p(K)}=J_p(\Theta)(\omega,\gamma)$$

holds for every ω, $\gamma\in\mathrm{Dom}\,(\Delta_p{}^{1/2}(\Theta))=\mathrm{Dom}\,(J_p(\Theta))$, $p\in N_+$ (see also [16, Theorem 2.3]).

REMARK 5.2. Proposition 5.1 is unsatisfactory in that it is not valid for all u, $v\in\mathrm{Dom}\,(J_p)$, which is furnished by Proposition 5.2. What is essential in (5.3) is that $\Delta_p{}^{1/2}(\Theta)$ is selfadjoint, nonnegative, $(\Delta_p{}^{1/2}(\Theta))^2=\Delta_p(\Theta,\partial_t)$, and that $\mathrm{Dom}\,(\Delta_p(\Theta))$ is a core of $\Delta_p{}^{1/2}(\Theta)$. □

For the case $p=0$, we need to define the operator $\Delta_0(\Theta)$ properly. The answer will be given by a version of the well-known von Neumann type theorem [32, II]. Hence we can define $\Delta_0(\Theta)\equiv\Delta_0(\Theta, \partial_t)$ by

$$\Delta_0(\Theta):=(d_0{}^*\tilde{d}_0)(\Theta)\,.$$

Thus we attain that $\{\Delta_p(\Theta)\}_{p=0}^{\infty}$ is the Laplacians associated with the de Rham complex $\{\tilde{d}_p(\Theta)\}_{p=0}^{\infty}$. Now we are in a position to state a decomposition theorem of de Rham-Hodge-Kodaira type for the sapace $\Lambda_2{}^p(K)$ in L^2-sense [16, Th. 2.5].

THEOREM 5.3 (Decomposition of de Rham-Hodge-Kodaira type for the space $\Lambda_2{}^p(K)$). *For all $p\in N_+$, the space $\Lambda_2{}^p(K)$ admits the following orthogonal decomposition*:

$$\Lambda_2{}^p(K)=\overline{\mathrm{Im}\,(\tilde{d}_{p-1}(\Theta))}\oplus\overline{\mathrm{Im}\,(d_p{}^*(\Theta))}\oplus\mathrm{Ker}\,\Delta_p(\Theta)\,. \tag{5.4}$$

N. B. Notice that the above decomposition assertion (5.4) is valid even for $p=0$ with $\tilde{d}_{-1}(\Theta)=0$ for convension.

§6. Explicit forms of the Laplacians $\Delta_p(\Theta, \partial_t)$.

Here we shall give an explicit form of the Laplacians $\{\Delta_p(\Theta, \partial_t)\}_p$ on $\{\mathcal{P}(\Lambda^p K^c)\}_p$, which is extremely important on a basis of the fundamental properties of our Laplacians. We first consider the element $\omega\in\mathcal{P}(\Lambda^p K^c)$ of the form:

$$\omega(x)=\sum_{n=1}^{k}\tilde{P}_n(x)\cdot\xi_n \qquad (x\in E^*,\ \tilde{P}_n\in\mathcal{P})$$

$$=\sum_{n=1}^{k}\langle :x^{\otimes n}:,\ f_n\rangle\cdot w_1^{(n)}\wedge\cdots\wedge w_p^{(n)}\,,$$

where $f_n\in E_C{}^{\hat{\otimes} n}$, $\xi_n\in A_p(\otimes^p D^\infty(B))$. Then, recalling Eq. (4.2) we have

$$d_p(\Theta)\omega(x)=\sum_{n=1}^{k}\sum_{\alpha\in N^n}b_\alpha\sum_{l=1}^{n}\langle :x^{\otimes(n-1)}:,\ \Xi^{\hat{\otimes}(n-1)}(\eta_*\,;\,l)\rangle \tag{6.1}$$

$$\times\Theta(\eta_{\alpha(l),n}(t))\wedge w_1{}^{(n)}\wedge\cdots\wedge w_p{}^{(n)}\,,$$

where we put

$$\Xi^{\hat{\otimes}(n-1)}(\eta_*\,;\,k):=\eta_{\alpha(1),n}\hat{\otimes}\cdots\overset{k}{\vee}\cdots\hat{\otimes}\eta_{\alpha(n),n}$$

and employed a formula for exterior products. Notice that

$$\delta_t{}^*f_n=\sum_{\alpha\in N^n}b_\alpha/n\cdot\sum_{k=1}^{n}\eta_{\alpha(k),n}(t)\cdot\Xi^{\hat{\otimes}(n-1)}(\eta_*\,;\,k)\,.$$

Then its U-functional (cf. (4.3)) is given by

$$U[d_p(\Theta)\omega](\xi)=\sum_{n=1}^{k}\sum_{\alpha\in N^n}b_\alpha\sum_{i=1}^{n}(\eta_{\alpha(1),n},\xi)$$

$$\cdots\overset{\vee}{(\eta_{\alpha(i),n},\xi)}\cdots(\eta_{\alpha(n),n},\xi)$$

$$\times\tilde{w}_1^{(n)}(i,t;\Theta)\wedge\tilde{w}_2^{(n)}\wedge\cdots\wedge\tilde{w}_{p+1}^{(n)},\qquad(\xi\in E),$$

where we set $\tilde{w}_j^{(n)}:=w_{j-1}^{(n)}$ for $j=2,3,\cdots,p+1$, and $\tilde{w}_1^{(n)}(i,t;\Theta):=\Theta(\eta_{\alpha(i),n}(t))$.

LEMMA 6.1. *For any $\gamma\in\mathcal{P}(\Lambda^{p+1}K^c)$ with the form $\sum_{l=1}^{k}\tilde{Q}_l(x)\eta_l$, $d_p{}^*(\Theta)\gamma(x)$ is given by*

$$d_p{}^*(\Theta)\gamma(x)=\sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}\{\tilde{Q}_l(x)\cdot\langle x(t),\Theta^*(v_j^{(l)})\rangle \tag{6.2}$$

$$-(\overline{\Theta^*v_j{}^{(l)}},\partial_t\tilde{Q}_l(x))_{H_C}\}\cdot v_1{}^{(l)}\wedge\cdots\wedge\overset{\vee}{v_j{}^{(l)}}\wedge\cdots\wedge v_{p+1}{}^{(l)}.$$

PROOF. By the isomorphism in $\Lambda_2{}^{p+1}(K)$ we get

$$\langle\tilde{d}_p(\Theta)\omega,\gamma\rangle_{\Lambda_2{}^{p+1}(K)} \tag{6.3}$$

$$=\sum_{n=1}^{k}\sum_{l=1}^{k}\sum_{\sigma\in\mathcal{G}_{p+1}}\operatorname{sgn}(\sigma)\sum_{\alpha}b_\alpha\sum_{j=1}\prod_{i=1}^{p+1}\langle\tilde{w}_i{}^{(n)},v_{\sigma(i)}{}^{(l)}\rangle_K$$

$$\times\int_{E^*}\langle:x^{\otimes(n-1)}:,\Xi^{\hat{\otimes}(n-1)}(\eta_*;j)\rangle\langle:x^{\otimes l}:,g_l\rangle\mu(dx)$$

where note that only $\tilde{w}_1{}^{(n)}$ depends on the parameter j. By employing a direct result derived from the coordinate multiplication operator formula in WNA (cf. Remark 6.3 below), we may apply Lemma 2.2 [14] for (6.3) to obtain

$$\sum_{n=1}^{k}\sum_{l=1}^{k}\sum_{\sigma\in\mathcal{G}_{p+1}}\operatorname{sgn}(\sigma)\prod_{i=2}^{p+1}\langle\tilde{w}_i{}^{(n)},v_{\sigma(i)}{}^{(l)}\rangle_K$$

$$\times\int_{E^*}\langle x(t),\Theta^*v_{\sigma(1)}{}^{(l)}\rangle\cdot\langle:x^{\otimes n}:,f_n\rangle\cdot\langle:x^{\otimes l}:,g_l\rangle\mu(dx)$$

$$-\sum_{n=1}^{k}\sum_{l=1}^{k}\sum_{\sigma\in\mathcal{G}_{p+1}}\operatorname{sgn}(\sigma)\prod_{i=2}^{p+1}\langle\tilde{w}_i{}^{(n)},v_{\sigma(i)}{}^{(l)}\rangle_K$$

$$\times\int_{E^*}\langle:x^{\otimes n}:,f_n\rangle\cdot(\Theta^*v_{\sigma(1)}{}^{(l)},\partial_t\langle:x^{\otimes l}:,g_l\rangle)_{H_C}\mu(dx)$$

$$=:I_1+I_2,$$

because we used above the Fubini type theorem relative to $d\mu$ and $d\nu$. Note that the relation

$$(6.4) \qquad l_1\wedge\cdots\wedge l_n=\sum_{k=1}^{n}(-1)^{k-1}l_k\otimes(l_1\wedge\cdots\wedge\check{l}_k\wedge\cdots<l_n).$$

By making use of (6.4) we can rewrite

$$I_2=-\sum_{l=1}^{k}\sum_{j=1}^{p+1}(-1)^{j-1}\langle\omega,(\Theta^*v_j{}^{(l)},\partial_t\tilde{Q}_l(x))_{H_C}$$
$$\times v_1{}^{(l)}\wedge\cdots\wedge\check{v}_j{}^{(l)}\wedge\cdots\wedge v_{p+1}{}^{(l)}\rangle_{\Lambda_2{}^p(K)},$$

where

$$\partial_t\tilde{Q}_l(x)=\langle:x^{\otimes(l-1)}:,\sum_{\beta\in N^1}b_\beta\sum_{i=1}^{l}\eta_{\beta(i),l}(t)\cdot\tilde{\Xi}^{\hat{\otimes}(l-1)}(\eta_*;i)\rangle$$

when g_l is given by $\sum_{\beta\in N^1}b_\beta\cdot\eta_{\beta(1),l}\hat{\otimes}\cdots\hat{\otimes}\eta_{\beta(l),l}$. Likewise as to the I_1 term, we conclude the assertion. q.e.d.

REMARK 6.1. We need to explain how to interpret the term $\langle x(t),\Theta^*v_{\sigma(1)}{}^{(l)}\rangle$. The element $\Theta^*v_{\sigma(1)}{}^{(l)}$ in H_C is well approximated by a sequence $\{y_k\}_{k=1}^{\infty}\subset E_C$ under our abstract setting. So we can define it by a limiting procedure.

REMARK 6.2. As a technical merit of computation in white noise calculus (cf. Remark 1.1 and Lemma 2.4 in [9]), we have

$$:\langle x,f_1\rangle\cdots\langle x,f_n\rangle:=\prod_{i=1}^{n}\frac{d}{d\lambda_i}:e^{\langle x,\sum_j\lambda_jf_j\rangle}:|_{\lambda_1=\cdots=\lambda_n=0}. \qquad \square$$

In fact, the operation of $d_p{}^*$ on $\mathcal{P}(\Lambda^{p+1}K^c)$ is also described evidently by the U-functional (cf. Remark 4.1).

LEMMA 6.2. *The U-functional of $d_p{}^*(\Theta)\gamma(x)$ $(x\in E^*)$ is given by*

$$(6.5) \qquad U[d_p{}^*(\Theta)\gamma](\xi)=\sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}\Big\{\sum_\beta b_\beta\prod_{i=1}^{l}(\eta_{\beta(i),l},\xi)$$
$$\times(\Theta(v_j{}^{(l)},\xi)+\sum_\beta b_\beta\sum_{i=1}^{l-1}(\Theta^*(v_j{}^{(l)}),\eta_{\beta(i),l})$$
$$\times\prod_{\substack{k=1\\k\neq i}}^{l-1}(\eta_{\beta(k),l},\xi)-\sum_{\beta\in N^1}b_\beta\sum_{i=1}^{l}\sum_{\substack{k=1\\k\neq i}}(\eta_{\beta(k),l},\xi)$$
$$\times(\overline{\Theta^*v_j{}^{(l)}},\eta_{\beta(i),l}(t))\Big\}$$
$$\times v_1{}^{(l)}\wedge v_2{}^{(l)}\wedge\cdots\wedge\check{v}_j{}^{(l)}\wedge\cdots\wedge v_{p+1}{}^{(l)}, \quad (\xi\in E).$$

PROOF. By Lemma 6.1 we immediately obtain

$$(6.6)\qquad U[d_p^*(\Theta)\gamma](\xi)=S(d_p^*(\Theta)\gamma)(\xi)$$
$$=\sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}\{S(\tilde{Q}_l(x)\cdot\langle x(t),\ \Theta^*(v_j^{(l)})\rangle)(\xi)$$
$$-(\overline{\Theta^* v_j^{(l)}},\ S(\partial_t\tilde{Q}_l(x))(\xi))_{H_C}\}$$
$$\times v_1^{(l)}\wedge v_2^{(l)}\wedge\cdots\wedge \overset{\vee}{v_j^{(l)}}\wedge\cdots\wedge v_{p+1}^{(l)}\,.$$

While, we easily get

$$(6.7)\qquad S(\partial_t\tilde{Q}_l(\cdot))(\xi)=\frac{\delta}{\delta\xi(t)}S(\tilde{Q}_l)(\xi)$$
$$=\sum_{\beta\in N^1}b_\beta\sum_{i=1}^{l}\eta_{\beta(i),l}(t)\cdot(\eta_{\beta(1),l},\ \xi)_{H_C}\cdots$$
$$\cdots\overset{\vee}{(\eta_{\beta(i),l},\ \xi)_{H_C}}\cdots(\eta_{\beta(l),l},\ \xi)_{H_C}\,,\qquad \xi\in E\,.$$

To compute $S(\tilde{Q}_l(\cdot)\cdot\langle x,\ \Theta^*(v_j^{(l)})\rangle)(\xi)$ $(\xi\in E)$, we may utilyze similar type equalities as in Remark 6.2 (cf. Lemma 2.5 and §IV in [9]) to obtain

$$(6.8)\qquad S(:\prod_{i=1}^{l}\langle\cdot,\ \eta_{\beta(i),l}\rangle:\langle(\cdot)(t),\ \Theta^*(v_j^{(l)})\rangle)(\xi)$$
$$=\frac{1}{(l+1)!}\sum_{i=1}^{l+1}(-1)^{l-i+1}\sum_{j_1<j_2<\cdots<j_i}(\tilde{\eta}_{\beta(j_1),l}+\tilde{\eta}_{\beta(j_2),l}+\cdots+\tilde{\eta}_{\beta(j_k),l},\ \xi)^{l+1}$$
$$+\sum_{i=1}^{l-1}(\Theta^*(v_j^{(l)}),\ \eta_{\beta(i),l})\frac{1}{(l-1)!}\sum_{i=1}^{l-1}(-1)^{l-k-1}\sum_{j_1<j_2<\cdots<j_k}$$
$$\times(\eta_{\beta(j_1),l}+\cdots+\overset{\vee}{\eta_{\beta(j_i),l}}+\cdots+\eta_{\beta(j_k),l},\ \xi)^{l-1}$$

where we put

$$\tilde{\eta}_{\beta(k),l}:=\eta_{\beta(k),l},\qquad (\text{for } k=1,2,\cdots,l),$$
$$\tilde{\eta}_{\beta(l+1),l}:=\Theta^*(v_j^{(l)}),\qquad \text{for } k=l+1)\,.$$

In connection with Remark 6.1, commutativity between the S-transform and the limiting procedure with $k\to\infty$ is required in the above computation. However, it is verified with the Lebesgue type bounded convergence theorem with respect to the Gaussian white noise measure. To complete the proof it is sufficient to substitute (6.7) and (6.8) for (6.6), paying attention to the fact that

$$S(\langle :x^{\otimes l}:,\ g_l\rangle\cdot\langle x(t),\ \Theta^*(v_j^{(l)})\rangle)(\xi)$$
$$=\sum_{\beta\in N^1}b_\beta\cdot S\Big(:\prod_{k=1}^{l}\langle x,\ \eta_{\beta(k),l}\rangle:\langle x(t),\ \Theta^*(v_j^{(l)})\rangle\Big)(\xi)\,.$$

q.e.d.

REMARK 6.3. When we observe carefully the computation of the term

$\langle d_p(\Theta)\omega, \gamma\rangle_{\Lambda_2^{p+1}(K)}$ in the proof of Lemma 6.1, then we may regard that it is roughly equal to

$$C(p)A_{p+1}\Theta\langle\partial_t\tilde{P}_n, \tilde{Q}_l\rangle^\mu,$$

where $C(p)$ is some constant depending on $p\in N_+$. Then

$$\begin{aligned}\langle\partial_t\tilde{P}_n, \tilde{Q}_l\rangle^\mu &= \langle\tilde{P}_n, [x]\tilde{Q}_l\rangle^\mu - \langle\tilde{P}_n, \partial_t\tilde{Q}_l\rangle^\mu \\ &= \langle\tilde{P}_n, (\partial_t+\partial_t^*)\tilde{Q}_l\rangle^\mu - \langle\tilde{P}_n, \partial_t\tilde{Q}_l\rangle^\mu,\end{aligned}$$

where we used the significant discovery on the coordinate multiplication operator by $x(t)$ in WNA (cf. [26]). The above computation means roughly that the adjoint ∂_t^* is employed in order to determine $d_p^*(\Theta)$, but in Θ-dependent manner. It is interesting to note that our discussion in Lemma 6.1 and Lemma 6.2 provides a subtle framework to construct a nicer Laplacian $\Delta_p(\Theta, \partial_t)$ by making use of the operator Θ. We would be able to take much advantage of it to apply our theory later for the problems arizing in quantum physics (see §7 or [8]). □

Now we are in a positon to express the explicit form of our Laplacian $\Delta_p(\Theta, \partial_t)$ on $\mathscr{P}(\Lambda^p K^c)$. By the discussion in §5, we have only to compute $d_p^*(\Theta)\tilde{d}_p(\Theta)\omega(x)$ and $\tilde{d}_{p-1}d_{p-1}^*\omega(x)$ respectively. To take (6.1) and (6.2) into consideration, it is easily checked that

$$\begin{aligned}(6.9)\qquad d_p^*(\Theta)\tilde{d}_p(\Theta)\omega(x) = \sum_{n=1}\sum_{\alpha} b_\alpha \sum_{l}\Big[\sum_{j=1}^{p+1}(-1)^{j-1} \\ \cdot\{\langle :x^{\otimes(n-1)}:, \Xi^{\hat{\otimes}(n-1)}(\eta_*; l)\rangle\cdot\langle x(t), \Theta^*(\tilde{w}_j^{(n)})\rangle \\ -(\overline{\Theta^*\tilde{w}_j^{(n)}}, \partial_t\langle :x^{\otimes(n-1)}:, \Xi^{\hat{\otimes}(n-1)}(\eta_*; l)\rangle)_{H_C}\} \\ \times\tilde{w}_1^{(n)}(i, t; \Theta)\wedge\tilde{w}_2^{(n)}\wedge\cdots\wedge\check{\tilde{w}}_j^{(n)}\wedge\cdots\wedge\tilde{w}_{p1}^{(n)}\Big].\end{aligned}$$

Next we consider the other part: in fact,

$$\begin{aligned}\tilde{d}_{p-1}(\Theta)d_{p-1}^*(\Theta)\omega(x) &= \tilde{d}_{p-1}(\Theta)(d_{p-1}^*(\Theta)\omega(x)) \\ &= \sum_{j=1}^{p}(-1)^{j-1}\sum_{n=1}^{k}[d_{p-1}\{\tilde{P}_n(x)\langle x(t), \Theta^*(w_j^{(n)})\rangle \\ &\quad \times w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\}] \\ &\quad -\sum_{j=1}^{p}(-1)^{j-1}\sum_{n=1}^{k}[d_{p-1}\{(\overline{\Theta^* w_j^{(n)}}, \partial_t\tilde{P}_n(x))_{H_C} \\ &\quad \times w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\}] \\ &=: J_1+J_2.\end{aligned}$$

As to J_1-part computation, it is verified with ease that

$$d_{p-1}[\tilde{P}_n(x)\langle x(t),\ \Theta^*(w_j^{(n)})\rangle\cdot w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}]$$

$$=\sum_\alpha b_\alpha\sum_l\langle :x^{\otimes(n-1)}:,\ \Xi^{\hat{\otimes}(n-1)}(\eta_*;\ l)\rangle\cdot\langle x(t),\ \Theta^*(w_j^{(n)})\rangle$$

$$\times\Theta(\eta_{\alpha(l),n}(t))\wedge w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}$$

$$+\tilde{P}_n(x)\cdot\Theta[(\Theta^*w_j^{(n)})(t)]\wedge w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}.$$

As to J_2-part computation, it goes almost similarly. Indeed,

$$\tilde{d}_{p-1}\{(\Theta^*w_j^{(n)},\ \partial_t\tilde{P}_n(x))_{H_C}\cdot w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\}$$

$$=\sum_\alpha b_\alpha\sum_l^n\sum_{\substack{k\\k\neq l}}^n(\Theta^*w_j^{(n)},\ \eta_{\alpha(l),n}(t))_{H_C}\cdot$$

$$\cdot\langle :x^{\otimes(n-2)}:,\ \Xi^{\hat{\otimes}(n-2)}[\Xi^{\hat{\otimes}(n-1)}(\eta_*;\ l)](k)\rangle$$

$$\times\tilde{w}_1^{(n)}(k,\ t;\ \Theta)\wedge w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}.$$

Finally we attain the principal result in this section.

PROPOSITION 6.3. *For $p\in N_+$, we have*

$$\Delta_p(\Theta,\ \partial_t)\omega(x)=\sum_{n=1}^k\{\sum_\alpha b_\alpha\sum_m\langle x(t),\ \Theta^*\Theta(\eta_{\alpha(m),n}(t))\rangle$$

$$\cdot\langle :x^{\otimes(n-1)}:,\ \Xi^{\hat{\otimes}(n-1)}(\eta_*;\ m)\rangle-\sum_\alpha b_\alpha\sum_l\sum_{\substack{m\\m\neq 1}}$$

$$\cdot(\eta_{\alpha(m),n}(t),\ \Theta^*\Theta(\eta_{\alpha(l),n}(t)))_{H_C}$$

$$\cdot\langle :x^{\otimes(n-2)}:,\ \Xi^{\hat{\otimes}(n-2)}[\Xi^{\hat{\otimes}(n-1)}(\eta_*;\ l)](m)\rangle\}$$

$$\times w_1^{(n)}\wedge w_2^{(n)}\wedge\cdots\wedge w_p^{(n)}$$

$$+\sum_{j=1}^p\sum_{n=1}^k\tilde{P}_n(x)\cdot w_1^{(n)}\wedge w_2^{(n)}\wedge\cdots\wedge\Theta\Theta^*w_j^{(n)}\wedge\cdots\wedge w_p^{(n)}.$$

§ 7. De Rham-Hodge-Kodaira decompositions associated with Hida derivative.

The purpose of this section is to introduce two distinct decomposition theorems of de Rham-Hodge-Kodaira type [8] (R-H-K type for short). Similar results in infinite dimensional analysis or stochastic analysis may be found in [2] & [31]. It is quite natural to employ the weak derivative in some sense in order to define the exterior differentials on forms, instead we do adopt the Hida differential to realize it. This is only our unique point, compared with other related works. Our decompositions being supplying with interesting and stimulating objects in mathematical physics, namely, with those especially oriented to analysis of Dirac operators in quantum physics, are naturally derived as one of applications in terms of our Laplacians constructed in the previous sections, which can be said to be the R-H-K type theorems associated with Hida derivative in WNA.

For $p\in\boldsymbol{N}_+$ we define

$$D^\infty(\Delta_p(\Theta)) := \bigcap_{m\in N} \mathrm{Dom}\,(\Delta_p(\Theta)^m),$$

Moreover, for $\omega\in D^\infty(\Delta_p(\Theta))$, we define

$$\|\omega\|_k := \left\{\sum_{l=0}^{k} \|(I+\Delta_p(\Theta))^l\omega\|^2_{L^2(E^*\to\Lambda^p K^c;\,d\mu)}\right\}^{1/2}$$

and denote by $H^{2,k}(\Lambda_2{}^p(K))$ the completion of $D^\infty(\Delta_p(\Theta))$ with respect to the norm $\|\cdot\|_k$. When we set

$$H^{2,\infty}(\Lambda_2{}^p(K)) := \bigcap_{k=0}^{\infty} H^{2,k}(\Lambda_2{}^p(K)), \tag{7.1}$$

then $(H^{2,\infty}(\Lambda_2{}^p(K)), \|\cdot\|_k)$ is a complete, countably normed space. We denote the spectrum of operator A by the symbol $\sigma(A)$. The second quantization operator $d\Gamma_1(A)$ for a selfadjoint operator A in H_C is defined by

$$(d\Gamma_1(A)\omega)(x) = \sum_{k=1}^{n} \langle :x^{\otimes n}:,\ A^{\otimes I}[k]f_n\rangle, \qquad \omega\in\mathcal{P}$$

where $A^{\otimes I}[k] := I\otimes\cdots\otimes \underset{k}{\hat{A}}\otimes\cdots\otimes I$ $(k\leqq n)$. Then $d\Gamma_1(A)$ is a uniquely determined, selfadjoint operator acting in (L^2). We define the operator $d\Gamma_2(B)$ by

$$d\Gamma_2{}^{(p)}(B) := \sum_{k=1}^{p} B^{\otimes I}[k],$$

which is a nonnegative selfadjoint operator acting in $\Lambda^p K^c$. Recall that the operator B is given by $\Theta\Theta^*$ (cf. §4). So let us write the operator acting in $\Lambda_2{}^p(K)$ as

$$\mathcal{L}_p(\Theta) := d\Gamma_1(A)\otimes I_f + I_b\otimes d\Gamma_2{}^{(p)}(B) \tag{7.2}$$

with identities: $I_b := I_{(L^2)}$, $I_f := I_{\Lambda^p K^c}$. Further we define the unique nonnegative selfadjoint operator $\Gamma_1(A)$ acting in (L^2) by

$$\Gamma_1(A) := S^{-1}\Big(\sum_{n=0}^{\infty} A^{\otimes n}\Big)S,$$

where S is the S-transform (see (2.8)). Then it holds that

$$\Gamma_1(A)\omega(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:,\ A^{\otimes n}f_n\rangle$$

for $\omega\in(L^2)$, with $\Gamma_1(A)\mathbf{1}=\mathbf{1}$ (see (2.9)). The nonnegative selfadjoint operator

$\Gamma_2(B)$ in $\Lambda^p K^c$ is defined by

$$\Gamma_2^{(p)}(B) := \otimes^p B\,, \qquad (p \geqq 0)\,.$$

Let

$$\Gamma_p(\Theta) := \Gamma_1(A) \otimes \Gamma_2^{(p)}(B)$$

acting in $\Lambda_2^p(K)$. For $\omega \in \mathrm{Dom}\,(\Gamma_p(\Theta)^k)$ $(k \geqq 1)$, we define the norm

$$|||\omega|||_k := \|(I + \Gamma_p(\Theta))^k \omega\|_{\Lambda_2^p(K)}\,,$$

and denote by $(S)_k(\Lambda^p K)$ the completion of $\mathrm{Dom}\,(\Gamma_p(\Theta)^k)$ with respect to the inner product induced by the norm $|||\cdot|||_k$. Then $(S)_k(\Lambda^p K)$ becomes a Hilbert space. Set

$$(S)(\Lambda^p K) := \bigcap_{k=1}^{\infty} (S)_k(\Lambda^p K)\,. \tag{7.3}$$

$((S)(\Lambda^p K),\ |||\cdot|||_k)$ is a complete, countably normed space.

Now we shall state the first decomposition theorem:

THEOREM 7.1 ([8], 1992). *Suppose that* $\inf \sigma(\Theta^* \Theta) \setminus \{0\} > 0$. *Then the decomposition of R-H-K type*

$$H^{2,\infty}(\Lambda_2^p(K)) = \mathrm{Im}\,[\Delta_p(\Theta) \restriction H^{2,\infty}(\Lambda_2^p(K))] \oplus \mathrm{Ker}\,\Delta_p(\Theta) \tag{7.4}$$

holds for all $p \in \boldsymbol{N}_+$.

We need the following lemma:

LEMMA 7.2. *For all* $p \in \boldsymbol{N}_+$, *we have*

$$\Delta_p(\Theta) = \mathcal{L}_p(\Theta) \qquad (in\ \Lambda_2^p(K)) \tag{7.5}$$

holds in operator equality sense.

PROOF OF LEMMA 7.2. We put

$$\hat{\Xi}_m^{\hat{\otimes} n}[\eta_*\,; \Theta^*\Theta] := \eta_{\alpha(1),n} \hat{\otimes} \cdots \hat{\otimes} \Theta^*\Theta(\eta_{\alpha(m),n}(t)) \hat{\otimes} \cdots \hat{\otimes} \eta_{\alpha(n),n}\,.$$

A simple computation with Proposition 6.3 and the recursive relation of the Wick ordering (cf. §2) gives

$$\begin{aligned}\Delta_p(\Theta)\omega(x) = &\sum_{m=1}^{n}\Big(\sum_{n=1}^{k} \langle :x^{\otimes n}:\,,\ \sum_{\alpha} b_\alpha \cdot \hat{\Xi}_m^{\hat{\otimes} n}[\eta_*\,; \Theta^*\Theta]\rangle\Big)\\ &\cdot w_1^{(n)} \wedge \cdots \wedge w_p^{(n)}\\ &+ \sum_{j=1}^{p}\Big(\sum_{n=1}^{k} \tilde{P}_n(x) \cdot w_1^{(n)} \wedge \cdots \wedge \Theta\Theta^* w_j^{(n)} \wedge \cdots \wedge w_p^{(n)}\Big)\\ = &(d\Gamma_1(\Theta^*\Theta) \otimes I_f)\omega(x) + (I_b \otimes d\Gamma_2^{(p)}(\Theta\Theta^*))\omega(x)\,,\end{aligned}$$

which implies that (7.6) holds on $\mathcal{P}(\Lambda^p K^c)$. Clearly $\mathcal{L}_p(\Theta)$ is essentially selfadjoint on $\mathcal{P}(\Lambda^p K^c)$, since so is $d\Gamma_1(A)$ (resp. $d\Gamma_2^{(p)}(B)$) on $\mathcal{P}$ (resp. $\mathcal{P}(\Lambda^p K^c)$). Therefore the closedness verifies the assertion. q. e. d.

PROOF. By virtue of the spectral property of the second quantization operators and the Deift theorem [4] for commutation formulae of operators, it follows immediately from Lemma 7.2 that $\inf \sigma(\Delta_p(\Theta))\setminus\{0\}>0$. Obviously we have

$$\Lambda_2{}^p(K)=\mathrm{Im}\,(\Delta_p(\Theta))\oplus\mathrm{Ker}\,\Delta_p(\Theta)\,.$$

Roughly speaking, the matter is whether $\Lambda_2{}^p(K)$ should be replaced with $H^{2,\infty}(\Lambda_2{}^p(K))$ when we put restriction on the domain of $\Delta_p(\Theta)$ to $H^{2,\infty}(\Lambda_2{}^p(K))$ in the right hand side. However, clearly this turns to be true. An application of the spectral representation theorem leads to

$$D^\infty(\Delta_p(\Theta))=\mathrm{Im}\,[\Delta_p(\Theta)\restriction D^\infty(\Delta_p(\Theta))]\otimes\mathrm{Ker}\,\Delta_p(\Theta)\,.$$

To complete the proof we have only to note that $H^{2,\infty}(\Lambda_2{}^p(K))$ is isomorphic to $D^\infty(\Delta_p(\Theta))$ as a vector space. q. e. d.

REMARK 7.1. In Lemma 7.2, when $p=0$ then we have $\mathcal{L}_0(\Theta)=d_0{}^*(\Theta)\cdot\bar{d}_0(\Theta)$, which is, of course, a nonnegative and selfadjoint operator. This is due to von Neumann theorem.

REMARK 7.2. It is generally right that the heat equation method is even effective for the proof of decomposition theorem on the space of the type like $H^{2,\infty}(\Lambda_2{}^p(K))$. In fact, similar works on R-H-K type decompositions by Shigekawa [31] and Arai-Mitoma [2] are greatly due to the heat equation method. □

Finally we shall introduce our second decomposition theorem for the space $(S)(\Lambda^p K)$ (see Theorem 7.8). However, since the structure of $(S)(\Lambda^p K)$ is different from that of $H^{2,\infty}(\Lambda_2{}^p(K))$, the heat equation method is not applicable any more to the case. So necessity will occur that we have resort to the Arai-Mitoma method. Their method is principally due to a comparison theorem, which is derived by a series of finer estimates based on precise computation of weighted norms. There the spectral theory plays again an essential role in reduction of the problem, representation of the operators, and precise estimates. Before mentioning the decomposition theorem we need to prepare for the basic estimates whereby the nice property of our Laplacians reveals itself, namely, our Laplacians do serve as desired operators which map the space of smooth p-forms into itself (see Theorem 7.7 below).

LEMMA 7.3. *Suppose that*

$$\Theta^*\Theta \geqq (1+\varepsilon) I_{H_C} \tag{7.6}$$

holds with a positive constant ε. *Then for each* $s>0$, *all* $p \in \boldsymbol{N}_+$ *and* $k \in \boldsymbol{N}_+$, *there exists a positive constant* $C_0(\varepsilon, k)$ *and there can be found a proper positive integer* k_0 *such that the inequality*

$$\|\tilde{T}_s^{-1}(I+\Delta_p(\Theta))^k \omega\|_{\Lambda_2{}^p(K)} \leqq C_0(\varepsilon, k) \cdot \|(I+\Gamma_p(\Theta))^{k_0} \omega\|_{\Lambda_2{}^p(K)} \tag{7.7}$$

holds for every $\omega \in \mathrm{Dom}\,(\Gamma_p(\Theta)^{k_0})$, *where* $\tilde{T}_s^{-1} := \Gamma_1(\mathrm{e}^s) \otimes I_f$. □

The proof is an easy exercise. It follows from the spectral theory and the fundamental properties of Ornstein-Uhlenbeck semigroups.

REMARK 7.3. We write the Ornstein-Uhlenbeck semigroup (e.g. [33]) on (L^2) as $T_s := \Gamma_1(\mathrm{e}^{-s})$, $s \geqq 0$. There exists its inverse operator T_s^{-1} being self-adjoint, which is given qy $T_s^{-1} = \Gamma_1(\mathrm{e}^s)$, $s \geqq 0$. Moreover, its natural extension to $\Lambda_2{}^p(K)$ is written as $\tilde{T}_s^{-1} := \Gamma_1(\mathrm{e}^s) \otimes I_f$, which appeared in the above (7.7). □

As a direct corollary of Lemma 7.3 we readily obtain

LEMMA 7.4. *Under the assumption* (7.6), *for all* $p \in \boldsymbol{N}_+$ *and* $k \in \boldsymbol{N}$ *there exists a positive constant* $C_1(\varepsilon, k)$ *such that the inequality*

$$\|(I+\Delta_p(\Theta))^k \omega\|_{\Lambda_2{}^p(K)} \leqq C_1(\varepsilon, k) \cdot \|(I+\Gamma_p(\Theta))^k \omega\|_{\Lambda_2{}^p(K)}$$

holds for every $\omega \in \mathrm{Dom}\,(\Gamma_p(\Theta)^k)$. □

Therefore, by repeating the reduction to the subspace $\mathcal{K}_n{}^p := K_n \otimes \Lambda^p K^c$ and employing the limiting proceeding for the acquired relative to $\mathcal{P}(\Lambda^p K^c)$, we can easily see that

LEMMA 7.5. *Under the assumption* (7.6) *we have*

$$\mathrm{Dom}\,(\Gamma_p(\Theta)^k) \subset \mathrm{Dom}\,(\Delta_p(\Theta)^k)$$

for all $k \in \boldsymbol{N}$ *and* $p \in \boldsymbol{N}_+$. □

The next proposition is a comparison theorem for the spaces $H^{2,\infty}(\Lambda_2{}^p(K))$ and $(S)(\Lambda^p K)$, whereby our second decomposition can be derived according to the Arai-Mitoma theory. One may find some of familiar techniques and methods useful and effective in this argument as well, and those have been used well in the Malliavin calculus [33].

PROPOSITION 7.6. *Suppose* (7.6). *Then the inclusion*

$$(S)(\Lambda^p K)\subset H^{2,\infty}(\Lambda_2{}^p(K)) \tag{7.8}$$

holds for all $p\in \boldsymbol{N}_+$.

As to the proof it is sufficient to show that $\|\gamma\|_k\leqq \tilde{C}|||\gamma|||_N$, $(\gamma\in\mathcal{P}(\Lambda^p K^c))$, for any $N>k$ (N, $k\in \boldsymbol{N}$), each $p\in \boldsymbol{N}_+$, and some positive constant $\tilde{C}$. In fact, an application of Khinchin's inequalities yields the assertion by virtue of hypercontractivity of T_s. The next assertion indicates that our Laplacians have such a nice property as stated in § 1.

THEOREM 7.7. *Under the assumption* (7.6) *we have*

$$\Delta_p(\Theta)[(S)(\Lambda^p K)]\subset(S)(\Lambda^p K) \tag{7.9}$$

for all $p\in \boldsymbol{N}_+$.

It is sufficient to prove

$$\Delta_p{}^n(\Theta)\omega\in \mathrm{Dom}\,(\Gamma_p{}^n(\Theta)^k)\,,$$

for $\omega\in(S)_k(\Lambda^p K)\cap\mathcal{K}_n{}^p$, all $k\in \boldsymbol{N}$, and each $p\in \boldsymbol{N}_+$. It is easy, hence omitted. Ultimately, we are now in a position to state our R-H-K type decomposition theorem for $(S)(\Lambda^p K)$.

THEOREM 7.8 ([8], 1992). *Assume the condition* (7.6). *Then the space* $(S)(\Lambda^p K)$ *admits the decomposition*

$$(S)(\Lambda^p K)=\mathrm{Im}\,[\Delta_p(\Theta)\restriction(S)(\Lambda^p K)]\oplus\mathrm{Ker}\,\Delta_p(\Theta) \tag{7.10}$$

for all $p\in \boldsymbol{N}_+$.

PROOF. According to Theorem 7.1 and Proposition 7.6 the element ω of $(S)(\Lambda^p K)$ is decomposed into

$$\omega=\omega_1+\omega_2=\Delta_p(\Theta)\eta+\omega_2\,,$$

with $\omega_1\in\mathrm{Im}\,[\Delta_p(\Theta)\restriction H^{2,\infty}(\Lambda_2{}^p(K))]$, $\omega_2\in\mathrm{Ker}\,\Delta_p(\Theta)$, and

$$\eta=Q_p(\Theta)\omega=\int_0^\infty(K_s(p\,;\,\Theta)\omega-\omega_2)ds\in H^{2,\infty}(\Lambda_2{}^p(K))\,,$$

where $K_s(p\,;\,\Theta):=\int_0^\infty e^{-s\lambda}dE_p(\Theta\,;\,\lambda)$, $(s\geqq 0)$ and $\{E_p(\Theta\,;\,\lambda)\,;\,\lambda\in \boldsymbol{R}\}$ is a family of spectral measures associated with the operator $\Delta_p(\Theta)$. Because of (7.9), it results from the following lemma:

LEMMA 7.9. *Under the condition* (7.6) *we have*

$$Q_p(\Theta)\omega \in (S)_k(\Lambda^p K), \qquad (\omega \in \mathrm{Dom}(\Gamma_p(\Theta)^k))$$

for all $k \geqq 1$, *each* $p \in N_+$. *q.e.d.* □

§ 8. Concluding remarks.

After having finished writing this paper, the author learned that H.-H. Kuo, J. Potthoff, and J.-A. Jan had obtained very useful and important results in "Continuity of affine transformations of white noise test functionals and applications", Stochastic Processes and their Applications **43** (1992), 85-98. They succeeded in obtaining a direct simple proof of the fact that the space of white noise test functionals is infinitely differentiable in Fréchet sense, which is closely related to our results in § 3. We found it very interesting and suggestive, and stimulating as well.

In addition, we were informed of the publication of H.-H. Kuo's paper entitled "Lectures on white noise analysis", which appeared as Special Invited Paper in Soochow J. Math. **18** (1992), 229-300. There can be found at pp. 251-266 very interesting and remarkable descriptions about a variety of differential operators in white noise analysis, which are deeply connected with the contents of § 3 and § 5 in our paper (cf. [12-15]). Especially so excellent are his works on the characteristics of various sorts of Laplacians (pp. 279-249) via an infinite dimensional version of the Fourier transform which is compatible with Hida calculus (see [7], [26]).

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2.4　確率変分と確率変分方程式および数理物理への応用

理論物理の場の理論における朝永＝シュヴィンガー超多時間理論に現れる朝永＝シュヴィンガー方程式と，ホワイトノイズ解析の２次の確率変分方程式とを対比しながら論じた。また確率変分解析に出てくる変換の具体例や，その定義の意味を無限次元の調和解析的立場から論じた。

また量子重力理論におけるアシュテカ形式の数学的枠組みについても考察した。カイラル分解とアシュテカ変数の関係を幾何学的立場から論じ，アシュテカ理論では分解から得られる自己共役接続が重要な役割を果たしていること，また微分幾何の接続をその基本変数に採用するなど，表面的には従来の量子重力の理論と著しく異なるが，実は一般相対論に対する正準理論の変種に他ならないことを指摘した。加えて，アシュテカ形式によって難解なWheeler-DeWitt方程式が非常にシンプルになる数理的メカニズムについて考察を行った。

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Tomonaga-Schwinger Supermany-Time Formalism and Hida's Stochastic Causal Calculus

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Abstract

We give a rough sketch of the idea on Tomonaga-Schwinger supermany-time formalism in Quantum Field Theory, which is originally based upon P.A.M. Dirac's many-time formalism in Quantum Mechanics. It is well-known that the so-called Tomonaga-Schwinger equation plays an important role in supermany-time theory. And besides we introduce only a core part of Hida's stochastic causal calculus, namely, stochastic variational calculus. We can say that its mathematical essence is very peculiar in White Noise Analysis. Especially one of the most remarkable characteristics consists in Hida's theory of stochastic variational equations, which is originally suggested by P. Lévy's infinitesimal equation, and is chiefly motivated by the above-mentioned Tomonaga-Schwinger equation as well. Lastly we discuss a physical perspective for stochastic variational formulation of the problem in connection with other analytical tools provided in White Noise Analysis.

§ 1. Introduction

The Pauli-Heisenberg Theory is a kind of Relativistic Theory, i.e., in the sense that the magnetic field is contained in a dynamical system, and of course, the electron itself is there expressed in a form of quantization of wave. It is an idea of elaborate treatment of both of light field and electron field in the same level as objects of Quantum Mechanics [14]. On the contrary, Dirac proposed the many-time formalism [1]. He thinks it is the electron itself that should be the object of Quantum Mechanics. He considers basically a many-body problem in Quantum Mechanics with its own time to each particle. Clearly it is a philosophy that the electromagnetic field is nothing but a measures to observe states of electron.

Generally speaking, the Heisenberg representation is extremely useful in theoretical consideration on quantum field theory, but the field operator is too complicated in expression and we cannot say that it is suitable for concrete computation of S matrices [9]. On the other hand, although the Schrödinger representation does not have time in the field operator and is, on this account, very simple (implying that it is expressed in a simple manner), we cannot use it for relativistic covariant theory because it allows us to regard time as a special quantity. Thus it is required to seek for an intermediate representation in which we can take advantage of its covariant treatment and the field

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operator is described in a simple form as well. It had been thought for years and has been found now ; we call it an *interaction picture.*. In the interaction picture, the field operator satisfies a four dimensional commutation relation (cf. § 2), and the state vector or probability amplitude may possibly vary only in the part of interaction.

This paper is organized as follows. § 2 is devoted to a rough explanation of Tomonaga -Schwinger supermany-time formalism, where a basic idea of the supermany-time theory will be stated from a point of view of mathematical physics. In § 3 we introduce Hida's stochastic variational equations in accourdance with his causal calculus. There can be found that his theory on variational equations in white noise analysis is deeply motivated by the idea of Tomonaga -Schwinger equations which comes from the supermany-time formalism. In § 4 we discuss a physical perspective for stochastic variational formulation in line with white noise analysis.

§ 2. Tomonaga-Schwinger Supermany-Time Formalism

In this section we shall review a physical perspective for famous and celebrated Tomonaga -Schwinger Supermany-Time formalism [18]. First of all we shall give a description of quantum field theory in terms of canonical form (e.g. see [14], [16]). Let $\varphi(x)$ be a free neutral scalar field with mass μ, where a terminology "*neutral*" means that $\varphi(x)$ should be Hermitian. The lagrangian density of $\varphi(x)$ is given by

$$\tilde{L}(x)=\frac{1}{2}[\partial^{\mu}\varphi(x)\cdot\partial_{\mu}\varphi(x)-\mu^{2}\varphi^{2}(x)],$$

and the action integral is given by

$$S=\int d^{4}x\ \tilde{L}(x).$$

On the other hand, we regard $\varphi(x)$as a canonical variable including parameter in the quantum field theory via canonical form. When $\partial_{0}\varphi(x)\equiv\dot{\varphi}(x)$, its canonical conjugate quantity $\pi(x)$ is denoted by

$$\pi(x)\equiv\frac{\partial\ \tilde{L}(x)}{\partial\ \dot{\varphi}(x)}=\dot{\varphi}(x).$$

Then the Hamiltonian H is as follows :

$$H\equiv\int d^{3}x\ [\pi(x)\ \dot{\varphi}(x)-\tilde{L}(x)]=\frac{1}{2}\int d^{3}x\ \{\pi^{2}(x)+\sum_{k}[\partial_{k}\ \varphi(x)]^{2}+\mu^{2}\ \varphi^{2}(x)\}.$$

We set $\omega_{p}\equiv\sqrt{\mu^{2}+\mathbf{p}^{2}}$ as usual. As is well known, $\varphi(x)$ can be expanded as

$$\varphi(x)=\frac{1}{(2\pi)^{3/2}}\int d^{3}p\ \frac{1}{\sqrt{2\ \omega_{p}}}\times[a(\mathbf{p})e^{-ipx}+a^{\perp}(\mathbf{p})e^{+ipx}]$$

with $\mathrm{p}_{0}=\omega_{p}$, $\mathbf{p}x=p^{\mu}\cdot x_{\mu}$. $\varphi^{(+)}(x)$ (resp. $\varphi^{(-)}(x)$) denotes respectively the positive (resp. negative) frequency part of $\varphi(x)$. Hence it follows that

$$a(\mathbf{p})=\frac{1}{(2\pi)^{3/2}}\cdot\frac{1}{\sqrt{2\ \omega_{p}}}\int d^{3}x\ e^{-ipx}\times[\omega_{p}\cdot\varphi(0,\ x)+i\ \pi(0,\ x)],$$

$$a^{\perp}(\mathbf{p})=\frac{1}{(2\pi)^{3/2}}\cdot\frac{1}{\sqrt{2\ \omega_{p}}}\int d^{3}x\ e^{+ipx}\times[\omega_{p}\cdot\varphi(0,\ x)-i\ \pi(0,\ x)],$$

Thus we readily obtain the commutation relation

$[a(\mathbf{p}),\ a^{\perp}(\mathbf{p})]=\delta(\mathbf{p}-\mathbf{q}),$

and $[a(\mathbf{p}),\ a(\mathbf{q})]=[a^{\perp}(\mathbf{p}),\ a^{\perp}(\mathbf{q})]=0.$

On this account, the four dimensional commutation relation is given by

$$[\varphi(x),\ \varphi(y)]=[\varphi^{(+)}(x),\ \varphi^{(-)}(y)]+[\varphi^{(-)}(x),\ \varphi^{(+)}(y)]=\frac{1}{(2\pi)^3}\int d^3p\ \frac{1}{2\ \omega_p}(e^{-ip(x-y)}-e^{+ip(x-y)}).$$

So much for preliminaries (cf. [9]). We are in a position to state the supermany-time formalism in quantum field theory, which will be prerequistites for better understanding of Hida's stochastic variational formulation in his causal calculus to be introduced in the following section.

Distinguished from the case in Heisenberg picture, the state vector changes itself as time t does in the interaction picture. However, it is contrary to the principle of relativistic covariance that we may regard the state vector simply as a function of t. For instance, it is impossible to describe a relativistic many-body problem in a covariant manner when we consider the common time t for each space coordinates $x^{(1)}$, $x^{(2)}$, $\cdots$, $x^{(n)}$ corresponding to n particles in the many-body problem. In order to satisfy the covariance requirement, it is necessary to introduce times $x_0^{(1)}$, $\cdots$, $x_0^{(n)}$ corresponding to each particle. If fortunately

$$(x^{(j)}-x^{(k)})^2<0 \tag{2.1}$$

holds for all distinct pair j, k, then thanks to the Einstein causality, we can define without contradiction a concept of the state being in possession of many time like the above. We call it many-time formalism [1]. Supermany-time formalism [18] is the one obtained by extending this many-time formalism so that we may gear it to quantum field theory [14]. Now we consider introducing a concept of spacelike hypersurface σ (or$=C$) instead of a finite number of times. The hypersurface is the totality of points which are expressed as

$$x_0=f_\sigma(\mathbf{x}), \tag{2.2}$$

where f_σ is a continuous function. We assume that any two points on Eq. (2.2) consist in a mutually spacelike position as indicated in (2.1). Generally speaking, the function form of f_σ may change by a Lorentz transformation, but there would not be any change in the result that the transformed one should still lie in a spacelike hypersurface. We denote a state vector by the symbol $|\sigma>$ in supermany-time formalism. The time evolution in supermany-time formalism is to see how $|\sigma>$ varies in accordance with the change of σ. Let σ_x be a spacelike hypersurface which is distinct from σ only in the neighborhood of a point x_μ on σ, and ω_x denotes a four dimensional volume of the infinitesimal part which is surrounded between σ_x and σ. The sign of ω_x will be defined as follows for convention: it is plus (resp. minus) when σ_x consists in the future (resp. past) side of σ. That is to say,

$$\omega_x\equiv\int d^3x\ [f_{\sigma_x}(\mathbf{x})-f_\sigma(\mathbf{x})]. \tag{2.3}$$

Then the variation $\frac{\delta}{\delta\ \sigma(x)}\ |\ \sigma>$ with respect to σ is defined by

$$\frac{\delta}{\delta\ \sigma(x)}\ |\ \sigma>\equiv\lim_{\omega_x\to 0}\frac{|\ \sigma_x>-\ |\ \sigma>}{\omega_x}, \tag{2.4}$$

where the above definition should be interpreted not only as an extension of time derivative but also as an operation of taking its density even spatially.

In the interaction picture the Hamiltonian H is given by

$H = H_0 + H_I,$

where H_0 is a Hamiltonian of the free field, and H_I is a Hamiltonian of the interaction. Then we have the following Schrödinger equation

$$i \frac{\partial}{\partial t} \mid t>_S = H \mid t>_S,$$

where $\mid t>_S$ denotes the state at time t in the Schrödinger representation. The state vector $\mid t>_I$ in the interaction picutre is given by

$$\mid t>_I \equiv e^{iH_0 t} \mid t>_S.$$

Thus it follows immediately that

$$i \frac{\partial}{\partial t} \mid t>_I = H_I(t) \mid t>_I \tag{2.5}$$

with $H_I(t) \equiv \exp\{i\ H_0\ t\} \cdot H_I \exp\{-i\ H_0\ t\}$. If any derivative coupling is not contained in the interaction, then H_I is obtained by integrating a polynomial of several field operators φ_j over a three dimensional space. Since those operators are independent of time in Schrödinger representation, writing them as $\varphi_j(\mathbf{x})$ $(j=1, 2, \cdots, \mathrm{n})$, we have the following expression

$$H_I = H_I[\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \cdots, \varphi_n(\mathbf{x})].$$

On this account $H_I(t)$ can be rewritten into

$$H_I(t) \equiv H_I[\varphi_1(t, \mathbf{x}), \cdots, \varphi_n(t, \mathbf{x})].$$

Essentially, $\varphi_j(t, \mathbf{x})$ is nothing but an operator $\varphi_j(\mathbf{x})$ of the free field, so that, we can say that $H_I(t)$ is an interaction Hamiltonian by which the free field operator is described. Furthermore, in the case without derivative coupling, it turns out to be that

$$\tilde{H}_I(x) = -\tilde{L}_I(x),$$

where $\tilde{H}_I$ is an interaction Hamiltonian density, and $\tilde{L}_I$ is an interaction Lagrangian density. Then we may employ the above to rewrite Eq. (2.5) into

$$i \frac{\delta}{\delta\ \sigma(x)} \mid \sigma>_I = \tilde{H}_I(x) \mid \sigma>_I. \tag{2.6}$$

Notice that $\tilde{H}_I$ in Eq. (2.6) is a polynomial of the free field $\varphi_j(\mathbf{x})$ $(j=1, \cdots, n)$. We call (2.6) a Tomonaga-Schwinger equation. It is interesting to note that Eq. (2.6) is itself a covariant equation. The Tomonaga-Schwinger equation Eq. (2.6) looks like an infinite simultaneous partial differential equations, so the necessary and sufficient condition in order that the solution of Eq. (2.6) may exist, is as follows:

$$\frac{\delta^2}{\delta\sigma(x)\ \delta\sigma(y)} \mid \sigma>_I = \frac{\delta^2}{\delta\sigma(y)\ \delta\sigma(x)} \mid \sigma>_I. \tag{2.7}$$

holds for any $x_\mu, y_\mu \in \tilde{C}$, where $\tilde{C}$ is the totality of points such that $x_\mu - y_\mu$ is spacelike. While, an application of Eq. (2.6) enables us to rewrite the condition Eq. (2.7). As a matter of fact, inorder for Eq. (2.7) to hold for any $\mid \sigma>_I$, it must be that

$$[\tilde{H}_I(x), \tilde{H}_I(y)] = 0, \tag{2.8}$$

as far as $(x-y)^2 < 0$. We call it the integrability condition.

Remark. If there exists a derivative coupling in the interaction, then the relation $\tilde{H}_I(x) = -\tilde{L}_I(x)$ fails to hold, with the result that the Tomonaga-Schwinger equation Eq. (2.6) can never be covariant any longer in relativistic sense.

§ 3. Hida's Stochastic Variational Calculus.

We introduce briefly stochastic variational calculus for a random field $\{X(C)\}$, C being a surface in a Euclidean space, which lives in the space of generalized white noise functionals [10].

Let us consider a random field $\{X(C)\}$ indexed by a manifold C which is supposed to run in the parameter space $\mathbf{R}^d$. The important thing is that we assume through this paper that $X(C)$ be a generalized white noise functionals. In other words, $X(C)\in(S)^*$ [3] (see also [5], [7], [8]). Actually an admissible parameter class $\mathbf{C}$ in Hida's theory [12] is taken to be the totality of simple closed convex manifolds C which is diffeomorphic to the sphere S^{d-1}. Based upon the basic Hilbert space $L^2(C, d\sigma(s))$ with the surface element $d\sigma(s)$ over C, we get a Gelfand triple

$$E(C)\subset L^2(C, d\sigma(s))\subset E^*(C)$$

relative to the white noise space as usual [8]. The white noise measure μ_C is able to be defined uniquely on the above-mentioned white noise space. Suggested by Lévy's well-known stochastic infinitesimal equation [15], T. Hida proposed the analysis of random fields $\{X(C)\,;\ C\in\mathbf{C}\}$ in line with the white noise analysis, and discussed in [12] stochastic variational equations of the form

$$\delta X(C)=\Phi(X(Q),\ (Q)\subset(C),\ Y(u(s)),\ u(s)\in C,\ C,\ \delta C), \tag{3.1}$$

with a functional X(C) of the special form

$$X(C)=\Phi\Big(\int_{(C)} F(C,\ x(u),\ u)du\Big), \tag{3.2}$$

for $x\in E^*$, where (C) denotes the open domain enclosed by C. While, Si Si [17] discussed the existence of unique solution to equations in the form

$$\delta U(\eta)=\int_C f(\eta,\ U,\ s)\ \xi_C(s)\ \delta\eta(s)\ d\sigma(s)$$

with the initial condition $U_0=U(\eta_0)$, under the assumtion on a continuous function f with stronger additional conditions, where $U(\eta)$ is the S-transform [3] of $X(C)$. An elaborate example of the Schrödinger type stochastic variational equation was also given by T. Hida [12], which is closely related to the so-called Tomonaga-Schwinger equations (cf. Eq. (2.6) in § 2) appearing in the relativistic quantum field theory.

Let us consider the class $\mathbf{H}$ of formally selfadjoint operators acting on $(S)^*$, whose element is a polynomial of ∂_t and ∂_t^* ($t\in\mathbf{R}^d$) with degree 2 at most, where ∂_t is the Hida differential [4] (see also [6]) in white noise analysis and ∂_t^* is the Kubo operator [6] [8]. ∂_t just corresponds to the annihilation operator and ∂_t^* to the creation operator. When we set

$$H(C)=\iint_{(C)^2} F(u,\ v)\{\partial_u\partial_v+\partial_u^*\partial_v^*\}dudv, \tag{3.3}$$

then $H(C)\in\mathbf{H}$, and a requirement of the operator $\exp\{i\ H(C)\}$ to be $(S)^*$-invariant provides with the restriction that it should be at most of order 2, because of the Potthoff-Streit characterization theorem [10] (cf. [8]). On that occasion the kernel F is supposed to be symmetric as a generalized function. It is interesting to note that the objects to be investigated are restricted to only the case where the integrand in $H(C)$ is free from dependency on $C\in\mathbf{C}$. We define

$$X(C)=:\exp\{i\ H(C)\}:\ X(C_0), \tag{3.4}$$

where $C,\ C_0\in\mathbf{C}$, and $(C_0)\subset(C)$, and notice that C_0 is fixed. When we consider the equation

$$\frac{\delta X(C)}{\delta C}(s)=i\,\frac{\delta X(C)}{\delta C}(s)\;X(C), \tag{3.5}$$

or equivalently,

$$\frac{\delta X(C)}{\delta C}(s)=i:\{\partial_s\int_C F(s,\,u)\;\partial_u\,du+\partial_s{}^*\int_C F(s,\,u)\;\partial_u{}^*\,du\}H(C):\;X(C) \tag{3.6}$$

with the initial condition $X(C_0)=X_0$, then the following assertion was obtained by T. Hida [12], namely

Theorem 3.1. *The random field* $X(C)$ *defined by* (3.4) *satisfies the stochastic variational equation* (3.5). *Conversely, the solution to the Cauchy problem* (3.5) *is given by* (3.4).

Generally speaking, it is a task of extreme difficulty to solve a stochastic variational equation directly. For this purpose, it seems that a kind of deformation from C into $C+\delta C$ allows us to obtain the stochastic variation $\delta X(C)$ in Hida sense, only by making use of a one parameter family of diffeomorphisms which is deeply connected with an infinite dimensional rotation subgroup named *whisker* [13] (see also § 4 below).

Theorem 3.2. [11] *When* δ_j *denotes the variations of* C *in Hida sense by the one parameter family of diffeomorphisms in* $\mathbf{R}^d$, *then an integrability condition for the equation of the type* (3.5) *is given by*

$$\frac{\delta_1\;\delta_2\;X(C)}{\delta C\;\delta Q}=\frac{\delta_2\;\delta_1\;X(C)}{\delta Q\;\delta C}. \tag{3.7}$$

Remark. The existence theorem for a much generaler form of stochastic variational equation will be fund in [12]. T. Hida discussed in [11] the problem of integrability conditions for a wide class of variation equations. Moreover, it is in [12], we have to refer, that a similar sort of theorem as the above Theorem 3.1 is proved for the case of high level where the Lévy Laplacian $\Delta_L(C)$ is contained in $H(C)$.

§ 4. A Physical Perspective for Stochastic Variational Formulation.

Let $\mathbf{R}^d$ be the parameter space in this section. We define the space $D_0(\mathbf{R}^d)$ as the totality of $\xi\in C^\infty(\mathbf{R}^d)$ such that

$$\xi\Big(\frac{u}{|u|^2}\Big)\cdot|u|^{-d}\in C^\infty(\mathbf{R}^d). \tag{4.1}$$

Then it follows immediately that

Proposition 4.1. $D_0(\mathbf{R}^d)$ *is a nuclear space.*

We call it the Basic Nuclear Space, and set $E:=D_0(\mathbf{R}^d)$ in the following. Let us define the characteristic functional $C(\xi)$ on E as

$$C(\xi)=\exp\Big(-\frac{1}{2}\,\|\xi\|^2\Big),\;\xi\in E, \tag{4.2}$$

where $\|\cdot\|$ means a usual L^2-norm. The Bochner-Minlos theorem allows us to obtain a unique probability measure μ, called the white noise measure, with parameter space $\mathbf{R}^d$, on the dual space E^* of E.

Definition 4.1 g is said to be a rotation of E if (i) g is a transformation on E, and is also a linear isomorphism of E ; (ii) g is orthogonal in the sense that $\|g\xi\|=\|\xi\|$ holds for any $\xi\in E$.

The symbol O(E) denotes the collection of such g's as described in the above. Then we readily get

Proposition 4.2. *$O(E)$ forms a group under the usual product operation.*

On this account, $O(E)$ (or O_∞) is called a rotation group of E, or an infinite dimensional rotation group when E is not explicitly mentioned. $O(E)$ is topologized by the compact open topology.

Definition 4.2 The adjoint g^* of $g \in O(E)$ is defined by

$$\langle g^* x, \xi\rangle = \langle x, g\,\xi\rangle, \text{ for } x \in E^*, \ \xi \in E.$$

Hence we set $O^*(E^*) = \{g^* ;\ g \in O(E)\}$. We may employ the correspondence between g^* and g^{-1}, to establish

Proposition 4.3. $O^*(E^*) \cong O(E)$.

The following theorem will give a connection between the probability measue μ and the rotation group $O(E)$.

Theorem 4.4 (cf. Theorem 2 in [19]). *For any $g \in O(E)$, the relation $g^*\mu = \mu$ holds.*

Now we consider a subgroups of $O(E)$. The subgroup consists of whiskers, and a whisker means a one parameter subgroup of $O(E)$, that comes from automorphisms of the parameter space, that is to say, we are supposed to have a time change in mind. More precisely we have:

Definition 4.3. $\{g_t ;\ t \in \mathbf{R}\}$ is a whisker if each g_t acts on E, in such a manner that

$$(g_t\,\xi)(u) \equiv \xi(\psi_t(u)) \cdot \left|\frac{\partial}{\partial u}\psi_t(u)\right|^{1/2},$$

with $\psi_t(u) \in \mathbf{R}^d$, and a Jacobian $\frac{\partial}{\partial u}\psi_t(u)$, and if $g_t \cdot g_s = g_{t+s}$ holds, or equivalently the relation $\psi_t(\psi_s(u)) = \psi_{t+s}(u)$ holds [19].

We shall give some examples of whisker, which are themselves very important in Probability Theory.

Example 4.1 (T. Hida (1985) [19]). We consider shifts $\{S_t^j ;\ t \in \mathbf{R}\}$. S_t^j is defined by

$$(S_t^j\,\xi)\,(u_1, \cdots, u_d) = \xi(u_1, \cdots, u_j - t, \cdots, u_d).$$

Then $\{(S_t^j)^* ;\ t \in \mathbf{R}\}$ is a flow on (E^*, μ). It is quite interesting to note that S_t^j can be regarded as the time shift whereby the flows may change as the time varies and they may express the so-called random phenomena that are realized generally in the space of Hida distributions.

Example 4.2 (T. Hida (1985) [19]). Let us consider dilation $\{T_t ;\ t \in \mathbf{R}\}$. This comes from isotropic dilations of the time variable, and is defined by

$$(T_t\,\xi)(u) \equiv \xi(e^t\,u)e^{td/2}.$$

As for the dilation, the flow $\{(T_t)^* ;\ t \in \mathbf{R}\}$ can provide us with an Ornstein-Uhlenbeck process.

Remark 4.1. One may wonder whether there is any relation between the aforementioned two kinds of whiskers. Actually, the shifts are always transversal to the dilation in terms of a theory of dynamical systems. And besides shifts are mutually commutative (cf. § 4 in [19]).

There is another approach to stochastic variational equations (cf. p. 55, § 2 in [12]). According to T. Hida (1989): Proc. 24th Karpacz WSTP report, the variation of $X(C)$ will be assumed to be gained by the action of the conformal group $C(d)$. Of course, $C(d)$ is one of the most important and interesting subgroups of O_∞. It is well known [20] that a white noise enjoys the conformal invariant property. So that, we may use the infinitesimal generators A_j of the actions U_t^j that is determined by one parameter subgroups $\{g_t^j\}$ of $C(d)$, in order to find the stochastic variational equation, where $U_t^j\,\varphi(x) = \varphi((g_t^j)^* x)$, for $\varphi \in (S)^*$, and any j.

Remark 4.2. It seems reasonable for the present to restrict the operators acting on the random field $X(C)$ to the class of operators that come from the above-mentioned conformal group $C(d)$.

Remark 4.3. The conformal subgroup $C(d)$ is not only of a different type from the Lévy group [11], but also distinct from those which are obtained by the limit operation to finite dimensional rotation groups (see also [12], [13]).

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Mathematical Aspects of Complex Canonical Quantization in Quantum Gravity

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Abstract

We consider the Ashtekar formulation in quantum gravity, because it is believed that the Ashtekar theory should enable us to rewrite the general relativity into a new form much more amenable and lead to a desirable consistent quantum gravity theory. Mathematical aspects of its formalism is in particular discussed for applicational interest.

§ 1. Introduction

The Kaluza-Klein theory (cf. § 19.7, pp. 650-652 in [9]) is aiming at construction of the Unification Theory by extending the Einstein theory to the space-time of higher dimension than four dimensional one and taking in Gauge theory within the gravitational theory. While there is another idea to connect Gauge theory with the gravitational theory by rewriting the four dimensional Einstein theory itself into a diverse formalism that is close to Gauge theory. As a matter of fact, A. Ashtekar proposed in 1986 [2] (see also [3], [5]) a new theory in line with the above latter idea. The theory is nothing but a canonical theory of general relativity, however it is completely different from the usual canonical thóry [12] (cf. [1], [9], [10]). Its characteristics are as follows: 1) the basic variable of the usual general relativity is a metric, on the contrary, a connection is regarded as its basic variable; 2) so that, a Gauge theoretical method is naturally taken in; 3) a complex momentum variable is adopted.

The purpose of Quantum Gravity Theory is to construct a consistent theory to unite the quantum theory with the general theory of relativity (cf. Chapter 19, pp. 633-661 in [9]). So many trials have been made, but any of them has not succeeded yet in its construction despite such a long history. The main mathematical tool in Quantum Cosmology nowadays, namely, Cosmology in terms of Quantum Gravity, is the Wheeler-DeWitt equation [6]. However, as is weel known, there have been left a plenty of unsolved problems on a technical basis as well as on an interpretational one. A remarkable progress has been made by Ahtekar's formalism [4]. That is to say, the difficult Wheeler-DeWitt equation will be simplified in its form and will be changed into an amenable one [7], [8] (see also [11], [13]), when are employed the new variables which Ashtekar proposed. Therefore it is expected that Ashtekar formulation should be valid, useful, and powerful especially in the field of Quantum Theory.

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§ 2. Einstein Formalism

In Einstein's general relativity, the rate of deviation of a space-time, namely, the curvature denotes the strength of gravitational field. In order to describe it, the space-time metric $g_{\mu\nu}$ is adopted as elementary variables of gravitational field. The Einstein equation is usually a field equation which is satisfied by the gravitational field and material field. Its expression is very convenient for checking a locally causal structure and a general covariance, but it cannot be suitable for description of time evolution of a global space-time structure. As a consequence, usually the formulation to regard the Einstein equation as an evolution equation for the dynamics with infinite degree of freedom is used so as to investigate a global dynamics of the space-time.

Let us consider a (3+1) decomposition of the space-time. This is a sort of formulation that a four dimensional physical quantity decomposes into a physical quantity on a three dimensional hypersurface with time parameter, by regarding a four dimensional space-time as a one dimensional time series $\Sigma(t)$ of sapcelike three dimensional hypersurface. Let $\Sigma(t)$ be a time-constant surface, and n a unit normal vector of $\Sigma(t)$. We denote a lapse function by the symbol N, then $N\ dt$ denotes a distance along the normal vector between two hypersurfaces $\Sigma(t)$ and $\Sigma(t+dt)$. And besides $N^j dt$ means a shift vector, indicating how far the point of intersection made by dropping a normal vector on $\Sigma(t+dt)$ from a point on $\Sigma(t)$ is shifted away from the curve $x^j = const.$ with a constant space coordinate. Then we readily get

Lemma 2.1. *When we set* $n=\frac{1}{N}(\partial_t - N^j \partial_j)$, *then* $N\ dt\ n=(dt, -N^j dt)$ *under an ordinary representation by components.*

Lemma 2.2. *When we write the metric tensor of a three dimensional hypersurface* $\Sigma(t)$ *by* $q_{jk}(t, x^j)$, *then the corresponding four dimensional metric in question is given by*
$ds^2 = -N^2\ dt^2 + q_{jk}(dx^j + N^j\ dt)\ (dx^k + N^k\ dt).$

Proof. The assertion is a direct result from an orthogonal decomposition $(dx^\mu) = N\ dt\ n + (0, dx^j + N^j\ dt)$, where x^μ is a point on $\Sigma(t)$, and dx^μ is a vector the starting point x^μ and the terminal point on $\Sigma(t+dt)$.

q.e.d.

An interpretation of the general relativity as a theory of dynamical system means to regard $q_{jk}(t, x^j)$ as a dynamical variable of infinite components with parameter x^j and to express the Einstein equation as a time evolution equation. In fact the equation can be easily derived by the corresponding variational equation. On that occasion, needless to say, the Einstein-Hilbert action for gravity

$$S_G = \int dt\ L_G\ ;\ L_G = \int d^3x \frac{1}{2\kappa^2}\sqrt{-g}\ R,$$

and the action for material field

$$S_m = \int dt\ L_m\ ;\ L_m = \int d^3x\ \mathrm{L}_m$$

play essential roles in derivation. By virtue of Dirac's general theory for canonical formulation [12] (see also [9]), the Einstein theory can be easily converted into the following canonical theory with constraints. Let φ be a complex scalar field, and let A_μ be the electromagnetic field with interaction. We regard them as material fields. p_{jk} is a qeneralized momentum being conjugate to q_{jk}, and $F_{\mu\nu}$ is given by $\partial_\mu A_\nu - \partial_\nu A_\mu$. We denote a complex conjugate of φ by φ^*, and D_μ is a differential operator, i.e., $D_\mu \equiv \partial_\mu - i\ e\ A_\mu$. E^j (resp, π, π $*$) denotes respectively a conjugate momentum of A_j (resp. φ, φ^*). When we set $B_j \equiv \frac{1}{2}\ \varepsilon_{jkl}\ F^{kl}$, then

$$\mathrm{H}_{m0} = \frac{\pi\ \pi^*}{\sqrt{q}} + \sqrt{q}\ \{(\mathrm{D}^j\ \varphi)^*\ (\mathrm{D}_j\ \varphi) + V\} + \frac{1}{2\sqrt{q}} E^j E_j + \frac{\sqrt{q}}{2} B_j B^j,$$

$$\mathrm{H}_{mj} = \pi \mathrm{D}_j\ \varphi + (\pi\ \mathrm{D}_j\ \varphi)^* + \varepsilon_{jkl} E^k B^l,$$

$$\mathrm{C}_A = i\ e\ (\pi\varphi - \pi^* \varphi^*) - \partial_j E^j.$$

Theorem 2.3. *The Lagrangian of canonical form for the gravitational field which has interaction with a complex scalar field and an electromagnetic field can be expressed as*

(2.1) $L = \int d^3x(-\dot{p}^{jk} q_{jk} + \pi\ \dot{\varphi} + \pi^* \dot{\varphi}^* + E^j \dot{A}_j) - H,\ H = \int d^3x(A_0 C_A + N^\mu\ \mathrm{H}_\mu + \mathrm{H}_\infty),$

where $\mathrm{H}_\infty = \partial_j\ [2N_k p^{kj} - pN^j + \kappa^{-2} \sqrt{q}\ D^j N + A_0 E^j]$.

Remark 2.1. The constraint $\mathrm{H}_\mu = 0$ obtained from the above Lagrangian is conservative under time ovolution, and so is the constraint for C_A. So that, no other constraints appear except these primary constraints.

Proposition 2.4. *The following Poisson bracket relations hold :*

$$\{\langle f\ \mathrm{H}_0 \rangle, \langle g\ \mathrm{H}_0 \rangle\} = \langle f\ D^j g - g\ D^j f)\,\mathrm{H}_j \rangle,\ \{\langle f^j\ \mathrm{H}_j \rangle, \langle g\ \mathrm{H}_0 \rangle\} = \langle (f^j\ \partial_j g)\ \mathrm{H}_0 - q^{-1/2}\ g\ f^j\ E_j\ C_A \rangle,$$

$$\{\langle f^j\ \mathrm{H}_j \rangle, \langle g^j\ \mathrm{H}_j \rangle\} = \langle ([f,\ g]^j\ \mathrm{H}_j + f^j\ g^k\ F_{jk}\ C_A \rangle,\ \{\mathrm{H}_\mu(\mathrm{x}),\ C_A(\mathrm{y})\} = \{C_A(\mathrm{x}),\ C_A(\mathrm{y})\} = 0,$$

where $\langle X \rangle = \int d^3x\ X(\mathrm{x})$, *and* $[f,\ g]$ *is a bracket for bector fields* f^j, g^j.

As for the proof of Proposition 2.4, it is simply due to a direct computation. It is easy but rather tiresome, hence omitted.

§ 3. Wheeler–DeWitt Equation

First of all, let us recall the Dirac quantizatin. It is well known [9] that the gauge field theory or the gravitational theory may be rewritten into the canonical theory with constraints of the first kind, as far as the classical theory is concerned:

$$\{Q^I,\ P_J\} = \delta^I_J\ ;\ \dot{F} = \{F,\ H\}\ ;\ F = F(Q,\ P)\ ;\ C_\alpha = 0\ ;\ \{C_\alpha,\ C_\beta\} = f^\gamma_{\alpha\beta}\ C_\gamma.$$

According to the canonical quantization procedure, these equations can be exchanged into the commutation relation between operators corresponding to the canonical variables and the Heisenberg equation of motion:

$$[\hat{Q}^I,\ \hat{P}_J] = i\ \delta^I_J\ ;\ \dot{\hat{F}} = i\ [\hat{H},\ \hat{F}].$$

Notice that in the Dirac quantization we need exchange the constraints for the condition to state vectors, namely.

(3.1) $$\hat{C}_\alpha \mid \Psi \rangle = 0,$$

since the operator $\hat{C}_\alpha$ corresponding to the constraint function C_α is not commutative to the fundamental canonical variables. By taking it into consideration that the classical constraints are of the first kind, we can readily get the commutation relation

$$[\hat{C}_\alpha,\ \hat{C}_\beta] = i\,\hat{f}^\gamma_{\alpha\beta}\ \hat{C}_\gamma$$

if we ignore the order problem of operator product, with the result that the condition Eq. (3.1) becomes consistent with the commutation relation.

It is easy to apply the aforementioned for the canonical theory of gravity. As a matter of fact, we obtain firstly

Lemma 3.1. *The momentum constraint is given by*

(3.2) $$\hat{H}_j\Psi=\{2i\ q_{jl}(\mathbf{x})\ D_k\frac{\delta}{\delta\ q_{jk}(\mathbf{x})}+\sqrt{q}\ \hat{T}_{nj}\}\ \Psi[q,\ \varphi]=0,$$

where $\hat{T}_{nj}=-i\ D_j\ \varphi(\mathbf{x})\ \frac{\delta}{\delta\ \varphi(\mathbf{x})}$.

Proof. For the operators corresponding to the canonical variables, we have only to adopt the representation in which q_{jk} and φ are diagonalized:

$$|\ \Psi>\to\Psi[q,\ \varphi]\ ;\ \hat{q}_{jk}(\mathbf{x})\to q_{jk}(\mathbf{x})\ ;\ \hat{p}^{jk}(\mathbf{x})\to -i\frac{\delta}{\delta\ q_{jk}(\mathbf{x})};\ \hat{\varphi}(\mathbf{x})\to\varphi(\mathbf{x})\ ;\ \hat{\pi}(\mathbf{x})\to -\mathrm{i}\frac{\delta}{\delta\ \varphi(\mathbf{x})}.$$

q.e.d.

Lemma 3.2. *The wave function* $\Psi[q,\ \varphi]$ *is invariant under the space coordinate transformation.*

Proof. Because the wave function has an expression

$$\delta\ \Psi[q,\ \varphi]=\int d^3x\ \sqrt{q}\ \{\delta\ q_{jk}\frac{\delta\ \Psi}{\delta\ q_{jk}(\mathbf{x})}+\delta\ \varphi\frac{\delta\ \Psi}{\delta\ \varphi}\}=-i\int d^3x\ L^j\ \hat{H}_j\ \Psi,$$

Eq. (3.2) in Lemma 3.1 indicates the assertion. q.e.d.

Definition 3.1 The symbol C denotes the set of equivalence classes generated by the identification of such elements in the whole set of configulation $(q,\ \varphi)$ as can be transformed mutually by the space coordinate transformations. We call the set C a *superspace.*

Remark 3.1. By virtue of Lemma 3.2 together with Lemma 3.1 we can regard the wave function satisfying the momentum constraint as a function on the set C.

Lemma 3.3. *The Hamiltonian constraint is given by the following functional differential equation of second order for the wave functions* :

(3.3) $$\hat{H}_0\Psi=\{-\frac{\kappa_2}{\sqrt{q}}\ G_{jklm}\frac{\delta^2}{\delta\ q_{jk}(\mathbf{x})\ \ \delta\ q_{lm}(\mathbf{x})}-\frac{\sqrt{q}}{\kappa_2}{}^3R+\sqrt{q}\ \hat{T}_{nn}\}\ \Psi[q,\ \varphi]=0,$$

where $G_{jklm}=q_{jl}\ q_{km}+q_{jm}\ q_{kl}-q_{jk}\ q_{lm},$ *and* $\sqrt{q}\ \hat{T}_{nn}=-\frac{1}{2\sqrt{q}}\frac{\delta^2}{\delta\ \varphi(\mathbf{x})^2}+\sqrt{q}\{\frac{1}{2}\ q^{jk}\ \partial_j\ \varphi\ \partial_k\ \varphi+V(\varphi)\}$.

We call this functional differential equation (3.3) *a Wheeler−DeWitt equation.*

Proof. It goes almost similarly as the proof in Lemma 3.1.

q.e.d.

Proposition 3.4. *Generally we have*

$[\langle f^j\ \hat{H}_j\rangle,\ \langle g\ \hat{H}_0\rangle],\ =i\ \langle f^j\ \partial_j\ g)\ \mathrm{H}_0\rangle.$

Proof. The assertion follows immediately from the commutation relations for constraints (cf. Proposition 2.4 in § 2).

q.e.d.

Seemingly the momentum constraint (3.2) and the Wheeler-DeWitt equation (3.3) consist of an infinite number of simultaneous equations, just corresponding to the degree of freedom for the space coordinates. However, because of the general relation (cf. Proposition 3.4), if another condition

$$\langle f\ \mathrm{H}_0\rangle|\Psi>=0$$

is satisfied together with the momentum constraint for the function f such that $\partial_j f$ never attains null except a set of discrete points, then it follows automatically that all the Hamiltonian constraints are satisfied. This implies that there exists one independent Wheeler-DeWitt equation under the momentum constraints.

Remark 3.2. The above-mentioned result does not mean that apparently the Wheeler-DeWitt equation can be expressed as one equation on the superspace, since $\langle f\ H_0\rangle|\Psi\rangle$ does not always satisfy the momentum constraint even if the state vector $|\Psi\rangle$ satisfies the same condition.

It is quite interesting to note that the Wheeler-DeWitt equation possesses a hyperbolic structure in the sense of differential equation. Essentially it is greatly due to the property of G_{jklm}.

Lemma 3.5. *When we regard G_{jklm} as an innear product of the six dimensional linear space that is generated by three dimensional symmetric tensor of order 2, then the eigenvalue has the following sign*: $[-, +, +, +, +, +]$.

Proof. Let X_{IJ} be a three dimensional symmetric tensor of order 2, and $\langle X, Y\rangle$ be a scalar product defined by $\langle X, Y\rangle = X_{IJ}\ Y_{IJ}$.

We denote by V a six dimensional linear space generated by those tensors X_{IJ} with the above-defined inner product, and f^{α}_{IJ} denotes the orthonormal basis of V. In particular we take

$$f^1_{IJ}=2^{-1/2}\ (\delta_{I2}\ \delta_{J3}+\delta_{I3}\ \delta_{J2}),$$
$$f^2_{IJ}=2^{-1/2}\ (\delta_{I3}\ \delta_{J1}+\delta_{I1}\ \delta_{J3}),$$
$$f^3_{IJ}=2^{-1/2}\ (\delta_{I1}\ \delta_{J2}+\delta_{I2}\ \delta_{J1}),$$
$$f^4_{IJ}=2^{-1/2}\ (\delta_{I1}-\delta_{I2})\delta_{IJ},$$
$$f^5_{IJ}=6^{-1/2}\ (\delta_{I1}+\delta_{I2}-2\delta_{I3})\delta_{IJ},$$

and $f^0_{IJ}=3^{-1/2}\delta_{IJ}$.

We need the following lemma.

Lemma 3.6. (cf. Proposition 7.3.1, pp. 225-226 in [13])

$$\sum_{\alpha=0}^{5} f^{\alpha}_{IJ}\ f^{\alpha}_{KL}=\frac{1}{2}(\delta_{IK}\ \delta_{JL}+\delta_{IL}\ \delta_{JK})$$

holds. □

By making use of Lemma 3.6 and decomposing q^{ik} into the dreibein e^j_I like $q^{ik}=e^j_I\ e^k_I$, we can deduce that

$$G_{JKLM}=2\sum_{\alpha=1}^{5} f^{\alpha}_{JK}\ f^{\alpha}_{LM}-f^0_{JK}\ f^0_{LM},$$

because we set $G_{JKLM}=e^j_J\ e^k_K\ e^l_L\ e^m_M\ G_{jklm}$. This implies that $G_{JKLM}/2$ may be diagonalize as $[-1/2, +1, +1, +1, +1, +1]$.

q.e.d.

Theorm 3.7. *The Wheeler-DeWitt equation* (3.3) *is of hyperbolic type for each point* **x** *of the space.*

Proof. Let θ^I_j be the dual basis of e^j_I, and we put

$$D_{\alpha}=\sqrt{2}\ \theta^I_j\ \theta^J_k\ f^{\alpha}_{IJ}\frac{\delta}{\delta\ q_{jk}}\ (\alpha\neq 0),$$

and

$$D_0=\frac{1}{\sqrt{3}}\,q_{jk}\frac{\delta}{\delta\ q_{jk}}.$$

By virtue of the argument in the proof of Lemma 3.5 we readily obtain

$$G_{jklm}\frac{\delta^2}{\delta\ q_{jk}\ \ \delta\ q_{lm}}=-D_0{}^2+\sum_{\alpha=1}^{5} D_{\alpha}{}^2+(\text{the first order differential term}).$$

Moreover we know that the second order part of functional differentials for the material field is positive difinite. On this account, it is easy to verify the hyperbolicity of the Wheeler–DeWitt equation.

q.e.d.

Example. (Quantum Bianchi Model) For the class A Bianchi model except the type VI_0 (cf. p227 in [13]), the Wheeler–DeWitt equation on a four dimensional mini–superspace is given by the following partial differential equation

$$\{\frac{\partial^2}{\partial \alpha^2}-\frac{\partial^2}{\partial \beta_+^2}-\frac{\partial^2}{\partial \beta_-^2}-\frac{\partial^2}{\partial \varphi^2}+2U\}\Psi=0.$$

It is certain that this equation should be of hyperbolic type with the conformal degree of freedom α as its time direction.

§ 4. Chiral Decomposition

When we choose the metric as its fundamental variable, then the Einstein equation is derived from the Einstein–Hilbert action integral (cf. § 2) which contains differentials up to second order with respect to the metric. In fact, it is possible to rewrite such an action integral of order 2 into an action integral with at most first order differentials relative to the fundamental variable, by employing proper auxiliary variables. It is well known that there exist various sorts of rewriting systems, and the Ashtekar theory is nothing but one of them, where the vierbein and connection form are used as fundamental variables. The principle idea of the Ashtekar theory consists in the fact that a complexification of the connection form provides with a complete change of the situation and also that the formalism of making use of the vierbein would rather be simplified than the theory of employing only the metric. Furthermore the key point of the Ashtekar theory lies in the novelty that only the selfdual connection is used which is obtained by a chiral decomposition of the spin connection with respect to the tensor indices.

The principle of equivalence asserts that the effect of gravity can be cancelled out for each point, in other words, that we area able to select a local inertial system in a proper way. Roughly speaking, it allows us to think of the situation such that a flat Minkowski space M_P is stuck down as a tangent space $T_P\ M$ [17], [18] for each space-time point $p \in M$. Let e_u be the vierbein, i.e., an orthonormal basis of the vector field, and let θ^a be its dual basis [17]. When A is a linear connection: $\Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$ [14], then we write as $A^a{}_b \in M$ (3.1) the connection form of A with respect to the basis $e_a \in \Gamma(E \uparrow U)$ [14], [15]. The metric g_{ij} gives a distance of the space-time in the general coordinates, and $ds^2 = g_{ij}\, dx^i\, dx^j$ [16]. While, we have $ds^2 = \eta_{ij}\, dX^i\, dX^j$ in the local Minkowski space, where η_{ij} is a Lorentz metric.

Lemma 4.1. *Then the vierbein e_a gives a relation*

$$dX^i = e^i{}_j\, dx^j$$

between the Minkowski space and the space-time vector, namely,

$$g_{ij} = \eta_{ab}\, e^a{}_i\, e^b{}_j. \qquad \square$$

For a Lorentz transformation $\Lambda \in SO(3,1)$, we have

$$e_a' = e_b\ (\Lambda^{-1})^b{}_a,\ \theta'^a = \Lambda^a{}_b\ \theta^b.$$

If we introduce a spin connection $\omega^{ab}{}_i$ as the gauge field for a group $SO(3,1)$ of local Lorentz transformation, then we can describe the general relativity as a gauge theory, where the curvature of the space-time is expressed by the strength of field of spin connection. The spin connection $\omega^{ab}{}_i$ will play an important role to control how to displace a vector of the local Minkowski space parallel to a point in the infinitesimal neighborhood.

Remark 4.1. Especially when the connection form $A^a{}_b$ corresponds to the Riemannian connection for the vierbein e_a, then the scalar curvature [16]

$$s(p)\equiv\sum_j \mathrm{Ric}(e_j,\ e_j)$$

is expressed as $e^{ai}\ e^{bj}\ F_{abij}$ by using the corresponding curvature form

$$F^a{}_b\equiv d\ A^a{}_b+A^a{}_c\wedge A^c{}_b,$$

where Ric is a Ricci tensor and $\mathrm{Ric}\in\Gamma(T^*M\otimes T^*M)$ and (e_j) is an orthonormal basis of T_pM, and $s\in C^\infty(M)$ (cf. [14], [15], [17]). □

Definition 4.1. The dual transformation $*p$ is defined by

$$*p\ \omega^{ab}=\frac{1}{2}\ \varepsilon^{ab}{}_{cd}\ \omega^{cd}$$

where ε^{abcd} is a Levi-Civita complete antisymmetric pseudo-tensor. □

Let us consider the Lagrangian L and action integral S defined by

$$\begin{aligned}\mathrm{L}:&=\frac{1}{2\kappa^2}e^{ai}\ e^{bj}\ F_{abij}\ |\theta|\ d^4x\\ &=\frac{1}{2\kappa^2}*(\theta^a\wedge\theta^b)\wedge F_{ab},\end{aligned}$$

and $S=\int \mathrm{L}$ for a general connection form $A^a{}_b$ (cf. Remark 4.1), where notice that $|\theta|=\det\theta^\alpha_\mu=\sqrt{-g}$, and $*$ means the Hodge $*$ operator (e.g. [14]).

Lemma 4.2. *When Θ^α is a torsion form*

$$\Theta^\alpha=d\theta^\alpha+A^a{}_b\wedge\theta^b$$

for the connection form $A^a{}_b$, then the variation of L *with respect to A is given by*

$$2\kappa^2\ \delta_A\ \mathrm{L}=d(\theta^a\wedge\theta^b\wedge *^p\ \delta\ A_{ab})-2\Theta^a\wedge\theta^b\wedge *^p\ \delta\ A_{ab},$$

where $*^p\ A_{ab}=\frac{1}{2}\ \varepsilon_{ab}{}^{cd}\ A_{cd}$.

Proposition 4.3. *The condition δ_A $\mathrm{S}=0$ is equivalent to the fact that the linear connection*

$$A\in L(\Gamma(E);\ \Gamma(T^*M\otimes E))$$

coincides with the Riemannian connection ∇ (cf. [15], [16]).

Proof. By virtue of Lemma 4.2, clearly the condition that the variation of the action integral relative to A is equal to zero is equivalent to the condition

$$\Theta[^a\wedge\theta^b]=0,$$

where [ ] means antisymmetrization with respect to the indices indicated by the symbol [ ]. And besides it its easy to show that the above expression is equivalent to the condition $\Theta^a=0$, which implies the assertion.

q.e.d.

If we define

$$(4.1)\qquad {}^\pm\omega^{ab}\equiv\frac{1}{2}(\omega^{ab}\mp i\ *^p\ \omega^{ab}),$$

then it turns out to be that ${}^{+}\omega^{ab}$ (resp. ${}^{-}\omega^{ab}$) represents respectively a proper state corresponding to the eigenvalue$+i$ (resp. $-i$) of the dual transformation.

Definition 4.2. We call ${}^{+}\omega$ (resp. ${}^{-}\omega$) a selfdual (resp. anti-selfdual) connection respectively. □

Then we get

Proposition 4.4. (cf. [7], [8]) *The spin connection* ω^{ab} *allows the following unique decomposition :*

$$\omega^{ab} = {}^{+}\omega^{ab} \oplus {}^{-}\omega^{ab}, \tag{4.2}$$

We call it the chiral decomposition.

Remark 4.2. The reason why we call the above decomposition (4.2) a chiral one is due to the following fact: when we adopt its spinor representation, then each decomposed part always combine only with either a right chiral spinor or a left chiral spinor.

Remark 4.3 The above (4.2) is an orthogonal decomposition. As a consequence, the curvature itself can be uniquely decomosed in a corresponding manner (see the theorem below). □

Theorem 4.5. [13] *When we define*

$$ {}^{\pm}F_{ab} \equiv \frac{1}{2}\,(F_{ab} \pm i *^{p} F_{ab}),$$

then the curvature form $F^{a}{}_{b}$ *allows the following unique orthogonal decomposition*:

$$F_{ab} = {}^{+}F_{ab} \oplus {}^{-}F_{ab}.$$ □

§ 5. Ashtekar Variables and Ashtekar Theory

Let $S[g]$ be the action integral whereby the Einstein equation can be derived. Then

$$S[g] = \int d^{4}x\,\sqrt{-g}\;R(g),$$

where g is the metric. Although the metric is a fundamental variable in the above action integral, we can rewrite it into another form $S[e, \omega]$ in which the vierbein and the spin connection behave themselves as fundamental variables. In fact

Lemma 5.1. $S[e, \omega] = \int \frac{1}{2}\,\varepsilon_{abcd}\; e^{a} \wedge e^{b} \wedge R^{cd}(\omega),$

where $R^{ab}(\omega)$ *describes the strength of gauge field* ω^{ab} *and is given by* $R^{ab}(\omega) \equiv d\,\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}$.

Let $S[e, {}^{+}\omega]$ be the action integral that is obtained by adoption of the selfdual connection (cf. Def. 4.2 in § 4) instead of the spin connection in $S[e, \omega]$.

Remark 5.1. This $S[e, {}^{+}\omega]$ is also an equivalent action integral by which we can derive the Einstein equation. □

It can be said that the selfdual action integral just corresponds to a covariant form in the Ashtekar theory. The Ashtekar variables are nothing but canonical variables in the case when we apply a (3+1) decomposition for the action integral to transform it into a canonical form.

Indeed, a wise choice

$$e^{0} = N\,dt, \qquad e^{I} = e^{I}_{i}\,(dx^{i} + N^{i}\,dt)$$

gives a (3+1) decomposition of the vierbein. e^{I}_{i} forms the dreibein being the orthonormal basis of space vectors. For a metric q_{ij} of the three dimensional space, we have

$q_{ij}=e^I_i\ e^I_j,\ \delta^{IJ}=q^{ij}\ e^I_i\ e^I_j.$

Notice that its dual basis E^i_I is a candidate of Ashtekar variables, and satisfies the relations

$E^i_I\ e^J_i=\delta^J_I,\ E^i_I\ e^I_j=\delta^i_j.$

It is quite interesting to note that ${}^+\omega^{ab}$ has only three independent vectors because of its selfduality. We shall write them as $A^I\equiv 2\ {}^+\omega^{0I}$. Recall that originally ω^{ab} is a gauge field for a group $SO(3, 1)$ of Lorentz transformations, but apparently A^I seems to be a gauge field for a group $SO(3)$. Taking its complexification into account, we get

$$A^I=2\ {}^+\omega^{0I}=\ \omega^{0I}-i\ \frac{1}{2}\ \varepsilon^{IJK}\ \omega^{JK}.$$

Remark 5.2. In the above expression, the Lorentz boost is embedded in its real part and the part for spatial rotation $SO(3)$ is embedded in its imaginary part. Hence we get aware that there is no loss in the degree of freedom of the gauge field for a group of Lorentz transformations.

Remark 5.3. Since only the Lie algebra for the gauge group does matter, we may regard A^I as a gauge field for a complex group $SU(2)$.

In the canonical formalism, a spatial component A^I_i of the gauge field for a complex group $SO(3)$ is a dynamical variable, and its conjugate momentum is given by the dreibein $\tilde{E}^i_I$. More precisely, it is a density of the dreibein; $\tilde{E}^i_I=\sqrt{q}\ E^i_I$. Thus it is the canonical variable $(A^I_i,\ \tilde{E}^i_I)$ *that are* Ashtekar variables.

Proposition 5.2. *When we take advantage of the Ashtekar variables, then the constraints* $C(E, A)$ *of the canonical formalism are given by*

$$C^G_I\equiv D_i\ \tilde{E}^i_I\ ;\ C^D_k\equiv \tilde{E}^j_I\ F^I_{jk}\ ;\ C^H\equiv \varepsilon^{IJK}\ \tilde{E}^j_J\ \tilde{E}^k_K\ F^I_{jk},$$

where D_i *(resp.* F^I_{jk}*) denotes respectively the covariant differential with respect to the gauge field* A^I_j *(resp. the strength of field).*

Note that $F^I_{jk}=\partial_j\ A^I_k-\partial_k\ A^I_j-i\ \varepsilon^{IJK}\ A^J_j\ A^K_k$. C^G_I is called a gauge constraint and is a condition for requirement of gauge invariance. C^D_k is called a diffeomorphism constraint and is a condition for requirement of invariance under three dimensional diffeomorphisms. Lastly C^H just corresponds to the Hamiltonian constraint. Summing up remarkable characteristics of the Ashtekar theory, we obtain the followings:

1) various expressions appearing in the theory are all simple differential polynomials of the fundamental variables;

2) the theory is given by a form of $SC(3, \mathbf{C})$ gauge theory, which means that not only the methods used in the ordinary gauge theory but also the results obtained there are quite useful for the investigation of gravitational theory;

3) the space-time is restricted to the case of four dimensional, because it plays an essential role in the theory that a chiral decomposition, consequently a $*^p$ operator transforms tensors of order 2 into itself relative to the internal degree of freedom:

4) the dynamical variable takes values in the complex number.

§ 6. Quantum Gravity via Ashtekar Formalism

Let us consider the canonical quantization. It is sufficient to adopt the operators $(\hat{A},\ \hat{E})$ acting on the space of state functions instead of the canonical variables (A, E), and we have only to change the constraints $C=0$ into another constraints $\hat{C}\ \Psi=0$ for state functions. As a matter of fact, since we habe $(\hat{A}, \hat{E})=(A, \frac{\delta}{i\,\delta A})$ in the A representation, it is easy to see the following.

Proposition 6.1. *The quantum constraints are given by*

$$\hat{C}^G_I\ \Psi[A]\equiv D_i\frac{\delta}{\delta A^I_j}\Psi[A]=0,$$

$$\hat{C}^D_k\ \Psi[A]\equiv F^I_{jk}\frac{\delta}{\delta A^I_j}\Psi[A]=0,$$

$$\hat{C}^H\ \Psi[A]\equiv \varepsilon^{IJK}\ F^I_{jk}\frac{\delta}{\delta A^J_j}\frac{\delta}{\delta A^K_k}\Psi[A]=0. \qquad \square$$

Now we begin with the gauge constraint. The gauge constraint requires invariance under the gauge transformation, so that it is automatically satisfied if we take $\Psi[A]$ as a gauge invariant functional.

Definition 6.1. The Wilson loop $W[A, \gamma]$ is defined by

$$W[A, \gamma]\equiv \mathrm{Tr}\ P\ \exp\ (i\oint_\gamma A_i\ dx^i)=\mathrm{Tr}\lim_{N\to\infty}\prod_{k=0}^{N}(1+i\ A_i(k)\ \triangle x^i(k)), \tag{6.1}$$

where P means to take a path order product. $\square$

Lemma 6.2. *The Wilson loop $W[A, \gamma]$ is a functional of connection A, taking values in a closed loop γ, and is invariant under the gauge transformation for connection A.*

Remark 6.1. Because the above $W[A, \gamma]$ is a family of infinite dimension with loop γ as its parameter, we may think that it forms a basis of the solution for gauge constraint.

Lemma 6.3. *The Wilson loop satisfied the Hamiltonian constraint.*

Lemma 6.4. *The Wilson loop does not satisfy the diffeomorphism constraint.*

Proof. It is impossible for the Wilson loop to keep invariant since the loop is deformed by the action of coordinate transformation.

q.e.d.

By the above discussion, the Wilson loop $W[A, \gamma]$ fails to become a solution for all the quantum constraints. Then the following question naturally arises: whether it is possible to sum up the Wilson loop relative to γ so that the summed may satisfy the diffeomorphism constraint.

Remark 6.2. In connection with the above-mentioned problem, there is an idea of investigating it by transforming the wave function $\Psi[A]$ of connection representation into $\Psi[\gamma]$ of loop representation. Actually analogous to Fourier transform, the integral $\Psi[\gamma]=\int[dA]\ W[A, \gamma]\ \Psi[A]$ is proposed, however we are ignorant of how to define the integral mesure in questin.

Lemma 6.5. *In the case where the cosmological term Λ exists, only the Hamiltonian constraint is changed into*

$$\hat{C}^H=\varepsilon^{IJK}\frac{\delta}{\delta A^J_j}\frac{\delta}{\delta A^K_k}(F^I_{jk}+i\frac{\Lambda}{3}\varepsilon_{ijk}\frac{\delta}{\delta A^I_i}).$$

In fact, H. Kodama (1990) found the solution satisfying all the constraints in the above case.

Theorem 6.6. (cf. H. Kodama, Phys. Rev. D42 (1990), 2548)
In the case where the cosmological term Λ exists, the solution satisfying all the quantum constraints

is given by

$$\Psi[A]=\exp(-i\,\frac{3}{2\Lambda}\,S_{SC}[A]), \tag{6.2}$$

where $S_{SC}[A]$ *is a Chrn-Simon term.* □

It is well known that the Chrn-Simon term is a three dimensional topological invariant, and is defined by

$$S_{SC}[A]=\int_{\Sigma} d^3x\ \varepsilon_{ijk}\,(A^I_i\ \partial_j\ A^I_k-\frac{i}{3}\,\varepsilon^{IJK}\ A^I_i\ A^J_j\ A^K_k).$$

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2.5 ホワイトノイズ解析の確率境界値問題への応用

汎関数過程の概念を導入し，それに基づいて確率境界値問題を考察し，その漸近解の収束性について論じた。問題の定式化はホワイトノイズ解析の手法に基づいて行った。その結果，与えられた確率問題はエルミート変換を通して対応する通常の偏微分方程式の境界値問題に帰着される。ここで採用したHLOUZ-Formalism(1993)の著しい特徴は，伊藤型の確率積分を解釈し直して，ホワイトノイズ汎関数とのWick積を被積分関数とするLebesgue型積分と見なせる点にある。この研究では飛田の超汎関数のクラスより広いクラスを解の存在域に設定し，解をKondratiev空間に値をとる一般化汎関数過程と見なして，変換された系に対するランダムな漸近解がマルチンゲール項により駆動された確率偏微分方程式を満たすことを導いた。さらに対応する解過程に関する確率的極限定理を証明した。

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短 期 共 同 研 究

量子確率解析とその周辺

京 都 大 学 数 理 解 析 研 究 所

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WHITE NOISE ANALYSIS AND THE BOUNDARY VALUE PROBLEM IN THE SPACE OF STOCHASTIC DISTRIBUTIONS

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ABSTRACT

We introduce the concept of functional process and consider the stochastic boundary value problem and discuss the convergence of its asymptotic solution process. The formulation of the problem is totally based upon the white noise analysis. In particular the so-called Hermite transform does play an essential role in derivation of the corresponding partial differential equation. One of the peculiar features under adoption of HLOUZ formalism (1993) consists in interpretation of the stochastic integral term as an integral of the Wick product of white noise functionals. We regard the solution of the problem as a Kondratiev space valued functional process, and the corresponding asymptotic solution satisfies some stochastic partial differential equation with a martingale term.

1. Preliminaries

1.1 White Noise Probability Space

Let $d \in \mathbb{N}$ fixed, and it indicates the parameter dimension. $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ denotes a Schwartz space on $\mathbb{R}^d$. $\mathcal{S}$ is a Fréchet space under a family of seminorms $\|\cdot\|_{k,\alpha}$, where

$$\|f\|_{k,\alpha} = \sup_{x\in\mathbb{R}^d} (1+|x|^k)|\partial^\alpha f(x)|, \qquad k \geq 0,$$

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $\partial^\alpha f = \partial^{|\alpha|} f / \partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_d} x_d$. $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ is a dual of $\mathcal{S}$, equipped with weak-* topology. It is called the space of tempered distributions. We denote by $\mathcal{B} = \mathcal{B}(\mathcal{S}')$ the family of Borel subsets of $\mathcal{S}'$. By the Bochner–Minlos theorem, there exists a unique Gaussian probability measure (called a white noise measure) on $\mathcal{B}$ such that

$$\int_{\mathcal{S}'} \mathrm{e}^{i\langle x,\varphi\rangle} d\mu(x) = \mathrm{e}^{-\frac{1}{2}|\varphi|_2^2}, \qquad \forall\varphi \in \mathcal{S},$$

where $|\cdot|_2$ is a $L^2(\mathbb{R}^d)$-norm. We call the triplet $(\mathcal{S}', \mathcal{B}, \mu)$ a white noise probability space. The canonical biliear form $\langle x, \varphi\rangle$, for $x \in \mathcal{S}$, $\varphi \in L^2(\mathbb{R}^d)$ is defined as follows: for $\forall\varphi \in L^2(\mathbb{R}^d)$; $\exists$ $\{\varphi_k\} \subset \mathcal{S}$ such that $\varphi_k \to \varphi$ in $L^2(\mathbb{R}^d)$ as k approaches to infinity, and define $\langle x, \varphi\rangle := L^2\text{-}\lim_{k\to\infty} \langle x, \varphi_k\rangle$. In particular, when we define

$$\tilde{B}_t(x) := \langle x, \chi_{[0,t_1]\times\cdots\times[0,t_d]}\rangle, \qquad \text{for } t_k \geq 0, \quad t = (t_1, \cdots, t_d),$$

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then it is well-known that there exists a t-continuous version B_t of $\tilde{B}_t$, and we call it a d-parameter Brownian motion, where χ_A denotes an indicator of the set A. Next we introduce a d-parameter white noise process (WN process for short) $W \equiv W_\varphi$, which can be expressed in terms of Itô integral with respect to d-parameter Brownian motion $B = (B_t(x))$, $t \in \mathbb{R}^d$; i.e., the white noise process is a mapping W: $\mathcal{S} \times \mathcal{S}' \to \mathbb{R}$, given by

$$W(\varphi, x) = W_\varphi(x) = \langle x, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(t) dB_t(x), \qquad x \in \mathcal{S}', \quad \varphi \in \mathcal{S}. \tag{1}$$

1.2 The Space (L^2) and its Representations

Let $\tilde{L}^2$ be the totality of square integrable measurable functions on $\mathcal{S}'$ with respect to the white noise measure μ. We denote by the symbol $(L^2) = L^2(\mathcal{S}', \mu)$ the quotient space of $\tilde{L}^2$ by the equivalence class, namely, the equivalent relation $f \sim g$ is given by $\|f - g\|_2 = 0$. The Wiener-Itô expansion theorem gives the following decomposition of the space (L^2): indeed, $(L^2) = L^2(\mathcal{S}', \mu) = \sum_{n=0}^{\infty} \bigoplus K_n$, where each K_n is the totality of multiple Wiener integrals $I_n(f_n)$ of order n, and f_n is an element of the symmetric L^2-space $\hat{L}^2((\mathbb{R}^d)^n)$. For $\forall\ F \in (L^2)$ we have the expression:

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^{dn}} f_n(u) dB_u^{\otimes n}(x) \qquad f_n \in \hat{L}^2((\mathbb{R}^d)^n) \\ &= \sum_{n=0}^{\infty} \int \cdots \int_{\mathbb{R}^{dn}} f_n(u_1, \cdots, u_n) dB^{\otimes n}(u_1, \cdots, u_n)(x), \quad u_k \in \mathbb{R}^d. \end{aligned}$$

For the norm $\|\cdot\|$ (or $\equiv \|\cdot\|_2$) of the Hilbert space (L^2), we have

$$\|F\|^2 = \sum_{n=0}^{\infty} n! |f_n|_2^2,$$

for $f_n \in \hat{L}^2((\mathbb{R}^d)^n)$.

We consider an alternative representation of the element of (L^2). Let $h_n(y)$, $n = 0, 1, 2, \cdots$, be Hermite polynomials defined by

$$h_n(y) := (-1)^n e^{\frac{y^2}{2}} \frac{d^n}{dy^n}(e^{-\frac{y^2}{2}}), \qquad y \in \mathbb{R}.$$

Then it is well-known that the Hermite functions $\xi_n(y)$ are defined, by employing the Hermite polynomials, as

$$\xi_n(y) = \pi^{-\frac{1}{4}} \{(n-1)!\}^{-\frac{1}{2}} e^{-\frac{y^2}{2}} h_{n-1}(\sqrt{2} y), \qquad n \geq 1.$$

Note that $\{\xi_n(y)\}_{n=1}^{\infty}$ forms an orthonormal basis of $L^2(\mathbb{R})$ for the case $d=1$. Let $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbb{Z}_+^d$ be a multi-index. Then there is always a proper ordering so that we may rearrange the elements numerically and make it countable in the following manner:

$$\{\beta = (\beta_1, \cdots, \beta_d)\} = \{\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \cdots\}, \quad \text{and} \quad \beta^{(n)} = (\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_d^{(n)}).$$

Therefore we can define $e_n \equiv e_{\beta^{(n)}} := \xi_{\beta_1^{(n)}} \otimes \xi_{\beta_2^{(n)}} \otimes \cdots \otimes \xi_{\beta_d^{(n)}}$. Note that $e_k \in \mathcal{S}(\mathbb{R}^d)$ for each k. Thus we obtain an orthonormal basis $\{e_n\}_n = \{e_1, e_2, e_3, \cdots\}$ $(\subset \mathcal{S})$ for $L^2(\mathbb{R}^d)$. Set

$$\theta_j(x) := W_{e_j}(x) = \int_{\mathbb{R}^d} e_j(t) dB_t(x) = \langle x, e_j \rangle, \quad \text{for} \quad j = 1, 2, \cdots$$

For every multi-index $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}_+^m$, we define $h_\alpha\ (u_1, \cdots, u_m) := h_{\alpha_1}(u_1) \cdot h_{\alpha_2}(u_2) \cdots\ h_{\alpha_m}(u_m)$, and set

$$H_\alpha(x) := h_\alpha(\theta_1(x), \cdots, \theta_m(x)) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j(x)) = \prod_{j=1}^{m} h_{\alpha_j}(\langle x, e_j \rangle).$$

It hence follows that with $|\alpha| = n = \alpha_1 + \cdots + \alpha_m$,

$$\begin{aligned} (2) \qquad \int_{(\mathbb{R}^d)^n} e^{\otimes \alpha} dB^{\otimes |\alpha|} &\equiv \int_{(\mathbb{R}^d)^n} e_1^{\hat{\otimes} \alpha_1} \hat{\otimes} \cdots \hat{\otimes} e_m^{\hat{\otimes} \alpha_m} dB_t^{\hat{\otimes} n} \qquad (t \in \mathbb{R}^d) \\ &= \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) = H_\alpha(x). \end{aligned}$$

Theorem 1. (i) $\{H_\alpha(\cdot); \alpha \in \mathbb{N}^m : m = 0, 1, 2, \cdots\}$ *forms an orthonormal basis of the Hilbert space* (L^2).
(ii) $\mathbb{E}[H_\alpha^2] = \|H_\alpha\|^2 = \alpha!$, *where* $\alpha! = \prod_{j=1}^{m} \alpha_j!$, $\alpha = (\alpha_1, \cdots, \alpha_m)$.

On this account, an arbitrary element F of (L^2) can be expressed as

$$(3) \qquad F(x) = \sum_\alpha c_\alpha \cdot H_\alpha(x), \qquad c_\alpha \in \mathbb{R}, \quad \alpha \in \mathbb{Z}^m, \quad \forall m.$$

Moreover, the equality $\|F\|^2 = \sum_\alpha \alpha! c_\alpha^2$ holds.

Example 1. (White Noise Process) Recall the white noise process W_ψ (cf. Eq.(1)), which was introduced in the end of the section 1.1. For $\psi \in \mathcal{S}$, $x \in \mathcal{S}'$,

$$W_\psi(x) = \langle x, \psi \rangle = \int_{\mathbb{R}^d} \psi(t) dB_t(x) \equiv \int \cdots \int_{\mathbb{R}^d} \psi(t_1, \cdots, t_d) dB_{t_1 \cdots t_d}(x).$$

Since we have $\psi(t) = \sum_{k=1}^{\infty} (\psi, e_k) e_k \in \mathcal{S}$ by making use of the orthonormal basis $\{e_k\}$ for $L^2(\mathbb{R}^d)$, it is easy to see that

$$(4) \qquad W_\psi(x) = \sum_{k=1}^{\infty} (\psi, e_k) \int_{(\mathbb{R}^d)^{\times 1}} e^{\hat{\otimes} \varepsilon(k)} dB^{\otimes |\varepsilon(k)|} = \sum_{k=1}^{\infty} (\psi, e_k) H_{\varepsilon(k)}(x),$$

where we used Eq.(2) and $\alpha = \varepsilon_k = \varepsilon(k) = (0, \cdots, 0, \overset{k}{1}, 0, \cdots, 0) \in \mathbb{Z}_+^m$.

1.3 Stochastic Distributions

Recall that we have a Gelfand triple: $\mathcal{S} \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'$. It is possible to construct a similar structure in functional level (i.e. infinite dimensional case), which is modelled on the above-mentioned Gelfand triple in function level (i.e. finite dimensional case). Actually the second quantized operator $\Gamma(A)$ plays an essential role in its construction (see e.g. [HKPS]), where A is a positive selfadjoint operator in $L^2(\mathbb{R}^d)$ with Hilbert-Schmidt inverse. The standard construction (cf. pp.33-35,[OB] or [D5]) gives a Gelfand triple $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$, where $(\mathcal{S})$ is the space of test white noise functionals and $(\mathcal{S})^*$ is the space of generalized white noise functionals. And besides the latter may be called the space of Hida distributions. The Potthoff-Streit characterization theorem (cf. pp.123-134, [HKPS]) for those spaces are based on the S-transform in white noise calculus. In line with this characterization, a generalization of Hida distributions has been established ([OB],[D7]). However, in fact there is another characterization based on the so-called chaos expansion of functionals, whose basic concept is nothing but the alternative representation given by Eq.(3) in the previous section. For near-future application's sake, we will go to the other way, different from the standard setting in white noise analysis. For $(L^2) \ni F$, we have the chaos expansion $F(x) = \sum_\alpha c_\alpha H_\alpha(x)$. We are now in a position to state the characterization of the white noise test functionals and Hida distributions in terms of the coefficients of their Hermite transforms (see the next section) due to Zhang [Z].

Theorem 2. (i) $F \in (\mathcal{S})$ *if and only if the condition*

$$\sup_\alpha c_\alpha^2 \cdot \alpha!(2\mathbb{N})^{\alpha k} < \infty$$

holds for any $k < \infty$, $k \in \mathbb{N}$, *where* $(2\mathbb{N})^\alpha := \prod_{j=1}^m (2^d \cdot \beta_1^{(j)}\beta_2^{(j)} \cdots \beta_d^{(j)})^{\alpha(j)}$ *if* $\alpha = (\alpha_1, \cdots, \alpha_m)$ *with* $\alpha_j = \alpha(j)$ *for simplicity.*
(ii) $G \in (\mathcal{S})^*$, $G = \sum_\alpha b_\alpha H_\alpha$ *(formal series) if and only if the condition*

$$\sup_\alpha b_\alpha^2 \cdot \alpha!(2\mathbb{N})^{-\alpha q} < \infty$$

holds for some $q > 0$.

It is interesting to note that the action of G on F is given by

$$\langle G, F\rangle = \sum_\alpha \alpha! b_\alpha \cdot c_\alpha \tag{5}$$

if $G \in (\mathcal{S})^*$ such that $G = \sum_\alpha b_\alpha H_\alpha$ and $F \in (\mathcal{S})$ such that $F = \sum_\alpha \alpha! c_\alpha H_\alpha$.

Next we shall introduce the Kondratiev spaces [KSW].

Definition 1. (a) *Let* $0 \leq \rho \leq 1$. *We say* $f \in (\mathcal{S})^{\rho}$ *if* $f = \sum_{\alpha} c_{\alpha} \cdot H_{\alpha} \in (L^2)$ *such that*

$$\|f\|^2_{\rho,k} := \sum_{\alpha} c^2_{\alpha} \cdot (\alpha)^{1+\rho}(2\mathbb{N})^{\alpha k} < \infty, \quad (\forall k < \infty). \tag{6}$$

We call this $(\mathcal{S})^{\rho}$ *the Kondratiev space of stochastic test functions.*
(b) *Let* $0 \leq \rho \leq 1$. *We say* $F \in (\mathcal{S})^{-\rho}$ *if* $F = \sum_{\alpha} b_{\alpha} \cdot H_{\alpha}$ *such that*

$$\sum_{\alpha} b^2_{\alpha} \cdot (\alpha!)^{1-\rho}(2\mathbb{N})^{-\alpha q} < \infty, \qquad (\exists q < \infty), \tag{7}$$

where q *need to be large enough (i.e.* $q \gg 1$*).* $(\mathcal{S})^{-\rho}$ *is called the Kondratiev space of stochastic distributions.*

The family of seminorms $\|f\|^2_{\rho,k}$ $(k = 1, 2, \cdots)$ gives rise to a topology on the space $(\mathcal{S})^{\rho}$. In fact, the space $(\mathcal{S})^{-\rho}$ can be regarded as a dual of $(\mathcal{S})^{\rho}$ by the action $\langle F, f\rangle = \sum_{\alpha} b_{\alpha}c_{\alpha} \cdot \alpha!$ if $F = \sum_{\alpha} b_{\alpha}H_{\alpha} \in (\mathcal{S})^{-\rho}$ and $f = \sum_{\alpha} c_{\alpha}H_{\alpha} \in (\mathcal{S})^{\rho}$. It follows therefore that

$$(\mathcal{S})^1 \subset (\mathcal{S})^{\rho} \subset (\mathcal{S})^0 = (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* = (\mathcal{S})^{-0} \subset (\mathcal{S})^{-\rho} \subset (\mathcal{S})^{-1}. \tag{8}$$

2. Elementary Wick Calculus

2.1 Wick Product $\diamondsuit$

The purpose of this section consists in definition of the Wick product and its extension for application to stochastic equations. We shall introduce first the primitive definition of the Wick product, and later on try to extend it to the largest space, namely the Kondratiev space.

N.B. We already know that there exist much larger spaces of generalized functionals in white noise calculus, such as the Meyer-Yan space $\mathcal{M}^*$ (cf. *LNM* **1485** (1991)), and the Carmona-Yan space $\tilde{\mathcal{M}}^*$ (cf. *Prog. Probab.* **36** (1995)). We have the following inclusion:

$$(L^2) \subset (\mathcal{S})^* \subset (\mathcal{S})^{-\beta} \subset \mathcal{M}^* \subset \tilde{\mathcal{M}}^*.$$

Moreover there are continuous embeddings: $\tilde{\mathcal{M}} \hookrightarrow \mathcal{M} \hookrightarrow (L^2) \hookrightarrow \mathcal{M}^* \hookrightarrow \tilde{\mathcal{M}}^*$. In addition, $\tilde{\mathcal{M}}$ is a nuclear Fréchet space which is stable under Wick and Wiener products. While, $\tilde{\mathcal{M}}^*$ is the topological dual of the locally convex topological vector space $\tilde{\mathcal{M}}$. However, we need not use those spaces in this paper. The Kondratiev space is large enough to discuss the stochastic problem here in question.

In accordance with [HLOUZ1], [HLOUZ2], we define the Wick product of X and Y as

$$X \diamondsuit Y := \iint_{(\mathbb{R}^d)^2} \varphi \otimes \psi dB^{\otimes 2} \tag{9}$$

if $X = \langle x, \varphi \rangle = \int_{\mathbb{R}^d} \varphi dB$, (for $x \in \mathcal{S}'$, $\varphi \in \mathcal{S}$) and $Y = \langle x, \psi \rangle = \int_{\mathbb{R}^d} \psi dB$, (for $x \in \mathcal{S}'$, $\psi \in \mathcal{S}$). We can extend it with ease to (L^2) by making use of the expression:

$$(L^2) \ni F(x) = \sum_{n=0}^{\infty} \int \cdots \int_{(\mathbb{R}^d)^n} f_n(u_1, \cdots, u_n) dB_u^{\otimes n}, \quad (f_n \in \hat{L}^2(\mathbb{R}^{dn})).$$

Definition 2. *(Representation by Expansion) If X and Y are elements of (L^2) such that $X = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} f_n dB^{\otimes n}$ and $Y = \sum_{m=0}^{\infty} \int_{(\mathbb{R}^d)^m} g_m dB^{\otimes m}$, then the Wick product of X and Y is defined by*

$$X \diamondsuit Y = \sum_{n,m=0}^{\infty} \int \cdots \int_{(\mathbb{R}^d)^{n+m}} f_n \otimes g_m dB^{\otimes (n+m)},$$

where the right hand side is considered as convengence in $L^1(\mathcal{S}', \mu)$.

Next let us consider the alternative definition corresponding to the representation Eq.(3).

Definition 3. *If X and Y are elements of (L^2) such that $X = \sum_\alpha a_\alpha H_\alpha$, and $Y = \sum_\beta b_\beta H_\beta$, then*

$$X \diamondsuit Y = \sum_{\alpha,\beta} a_\alpha b_\beta \cdot H_{\alpha+\beta},$$

where we consider the right hand side as convergence in $L^1(\mathcal{S}', \mu)$ as far as it exists.

Needless to say, the above two definitions are equivalent. A direct computation leads to the equivalence. As a matter of fact, by taking Eq.(2) into account we can easily get

$$\begin{aligned} H_\alpha \diamondsuit H_\beta &= \Big(\prod_{j=1}^{m} h_{\alpha_j}(\theta_j)\Big) \diamondsuit \Big(\prod_{i=1}^{k} h_{\beta_i}(\theta_i)\Big) = \Big(\int_{(\mathbb{R}^d)^n} e^{\otimes \alpha} dB^{\otimes |\alpha|}\Big) \Big(\int_{(\mathbb{R}^d)^l} e^{\otimes \beta} dB^{\otimes |\beta|}\Big) \\ &= \int_{(\mathbb{R}^d)^{|\alpha+\beta|}} e^{(\alpha+\beta)} dB^{\otimes |\alpha+\beta|} = H_{\alpha+\beta}(x), \end{aligned}$$

with $n = |\alpha| = \alpha_1 + \cdots + \alpha_m$ and $l = |\beta| = \beta_1 + \cdots + \beta_k$. Note that the Wick product $X \diamondsuit Y \equiv \sum_{\alpha,\beta} a_\alpha b_\beta \cdot H_{\alpha+\beta}$ which we have defined is independent of the choice of the base $\{e_k\}$ of $L^2(\mathbb{R}^d)$.

Example 2. (Wick Product and Stochastic Integral: cf. p.398, [HLOUZ1]) If Y_t is an adapted bounded stochastic process defined on the white noise probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{S}', \mathcal{B}, \mu)$, then we have the following equality:

$$\int_0^T Y_t(x)dB_t(x) = \int_0^T Y_t \Diamond W_t(x)dt. \tag{10}$$

2.2 Wick Product of Distributions and Wick Exponential

Likewise, we can define the Wick product even for Hida distributions. In general, the spaces of stochastic distributions are stable under the Wick product. However, some smaller spaces are not always stable. Actually the followings are verified:
(a) If $F = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^*$, and if $G = \sum_\beta b_\beta H_\beta \in (\mathcal{S})^*$, then $F \Diamond G = \sum_{\alpha,\beta} a_\alpha b_\beta \cdot H_{\alpha+\beta}$ holds.
(b) If $f, g \in (\mathcal{S})$, then $f \Diamond g \in (\mathcal{S})$.
(c) However, for $F, G \in (L^2)$, $F \Diamond G \notin (L^2)$ (not always!).
(d) For $X, Y \in L^1(\mathcal{S}', \mu)$, suppose that there are $X_n, Y_n \in (L^2)$ such that $X_n \to X$ in $L^1(\mathcal{S}', \mu)$, and $Y_n \to Y$ in $L^1(\mathcal{S}', \mu)$ (as $n \to \infty$). If $\exists\, Z := \lim_{n\to\infty} X_n \Diamond Y_n \in L^1(\mathcal{S}', \mu)$, then we define $X \Diamond Y = Z$.

It is interesting to note that the discussion in $L^1(\mathcal{S}', \mu)$ is very delicate, because the space $L^1(\mathcal{S}', \mu)$ is not necessarily contained in the space $(\mathcal{S})^*$ of Hida distributions [HLOUZ1]. Next we shall introduce the Wick exponential, which is one of the most important tools in Wick calculus applied to stochastic differential equations in the standpoint of how to solve the problem. If X belongs to $L^1(\mathcal{S}', \mu)$, then we define the Wick exponential

$$\mathrm{Exp}X := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\Diamond n}. \tag{11}$$

Of course, this definition is well-defined if there exists the Wick powers of X, namely, $\exists X^{\Diamond n}$ for any n, and if the series is convergent in $L^1(\mathcal{S}', \mu)$. Furthermore, we obtain the exponential rule:

$$\mathrm{Exp}(X+Y) = \mathrm{Exp}X \Diamond \mathrm{Exp}Y. \tag{12}$$

Example 3. ($\mathrm{Exp}W_\psi$: the Wick exponential of WN process) Since we have $\sum_{n=0}^{\infty} h_n(x)t^n/n! = \exp\{tx - t^2/2\}$, it is easy to see that the WN process satisfies the relation

$$\mathrm{Exp}W_\psi = \exp\Big(W_\psi - \frac{1}{2}|\psi|_2^2\Big).$$

Let $\mathcal{A}$ be the algebra generated by $\exp(W_\psi)$. Since $\mathcal{A}$ is dense in $(\mathcal{S})$, immediately $\mathrm{Exp}W_\psi \in (\mathcal{S})$. Thus it follows that $\mathrm{Exp}W_\psi \in L^p(\mathcal{S}',\mu)$, for any $p \in [1,\infty)$.

For the elements of the Kondratiev space, we define

$$F\diamondsuit G := \sum_{\alpha,\beta} a_\alpha b_\beta \cdot H_{\alpha+\beta}, \tag{13}$$

if $F = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^{-1}$ and $G = \sum_\beta b_\beta \, H_\beta \in (\mathcal{S})^{-1}$. The well-definedness above is guaranteed by the following lemma.

Lemma 1. (i) *$f,g \in (\mathcal{S})^1$, then $f\diamondsuit g \in (\mathcal{S})^1$.*
(ii) *$F,G \in (\mathcal{S})^{-1}$, then $F\diamondsuit G \in (\mathcal{S})^{-1}$.*

2.3 Hermite Transform

We shall introduce the Hermite transform, which is a powerful tool in white noise calculus, especially when it is used for the study of stochastic differential equations.

Definition 4. *(Hermite Transform $\mathcal{H}$) For $\forall F \in (L^2)$ (resp. $(\mathcal{S})^*$, $(\mathcal{S})^{-1}$) such that $\exists$ its chaos expansion $F = \sum_\alpha c_\alpha \, H_\alpha$, the Hermite transform $\mathcal{H}$ of F is defined respectively as*

$$\mathcal{H}F \equiv \tilde{F} := \sum_\alpha c_\alpha z^\alpha, \tag{14}$$

where $z = (z_1, z_2, \cdots) \in \mathbb{C}^{\mathbb{N}}$.

Note that, in the above, if $\alpha = (\alpha_1, \cdots, \alpha_m)$ then $z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$.

Proposition 3 [LOU]. (i) *If $X = \sum_\alpha c_\alpha H_\alpha \in (L^2)$, then for each M $(< \infty)$, each $n \in \mathbb{N}$, its Hermite transform $\tilde{X}(z) = \sum_\alpha c_\alpha z^\alpha$ converges absolutely for $z = (z_1, z_2, \cdots, z_n, 0, 0, \cdots, 0)$, $|z_k| \le M$ $(\forall k)$.*
(ii) *(Therefore) for each n,*

$$\tilde{X}^{(n)}(z_1, \cdots, z_n) \equiv \tilde{X}(z_1, \cdots, z_n, 0, \cdots, 0)$$

is analytic on $\mathbb{C}^n$.

Theorem 4 [LOU]. *Suppose that $X, Y \in (L^2)$ satisfying $X\diamondsuit Y \in (L^2)$. Then*

$$\mathcal{H}(X\diamondsuit Y) = \mathcal{H}(X) \cdot \mathcal{H}(Y)$$

holds, where "$\cdot$" indicates the usual complex product.

Example 4. (a) (WN process W_φ) Recall that $W_\varphi(x) = \sum_k (\varphi, e_k) H_{\varepsilon(k)}(x) = \sum_k (\varphi, e_k) h_1(\theta_k)$ for $x \in \mathcal{S}'$, $\varphi \in \mathcal{S}$ (see Example 1). Then we have

$$\mathcal{H}(W_\varphi) = \tilde{W}_\varphi(z) = \sum_{k=1}^{\infty} (\varphi, e_k) \cdot z_k.$$

(b) (The Square of WN process: $W_\varphi^{\Diamond 2} = W_\varphi \Diamond W_\varphi$) We have

$$\mathcal{H}(W_\varphi^{\Diamond 2}) = \sum_{k,j=1}^{\infty} (\varphi . e_k)(\varphi, e_j) z_k \cdot z_j.$$

For Hida distributions, the same assertion as Theroem 4 holds; indeed, for $F, G \in (\mathcal{S})^*$, $\mathcal{H}(F \Diamond G) = \mathcal{H}F \cdot \mathcal{H}G$. What about the Kondratiev space? Is the same assertion valid for the elements of $(\mathcal{S})^{-\rho}$?

Remark 1. If F lies in $(\mathcal{S})^{-\rho}$ for $\rho < 1$, then it is easy to see that $\mathcal{H}F(z_1, z_2, \cdots)$ converges for any finite sequence $Z = (z_1, z_2, \cdots, z_m)$ of complex numbers for each $m \in \mathbb{N}$.

Remark 2. If F is an element of $(\mathcal{S})^{-1}$, then we can only obtain the convergence of $\mathcal{H}F(z_1, z_2, \cdots)$ in a neighborhood of the origin. Actually we have $\mathcal{H} = \tilde{F} = \sum_\alpha c_\alpha \cdot z^\alpha$ for $F = \sum_\alpha c_\alpha H_\alpha$. So that, we get

$$\sum_\alpha |c_\alpha| \cdot |z^\alpha| \leq \left\{ \sum_\alpha c_\alpha^2 \cdot (2\mathbb{N})^{-\alpha q} \right\}^{1/2} \cdot \left\{ \sum_\alpha |z^\alpha|^2 \cdot (2\mathbb{N})^{\alpha q} \right\}^{1/2}. \tag{15}$$

The first term of the right hand side in Eq.(15) clearly converges for $q \gg 1$ (large enough), because $F \in (\mathcal{S})^{-1}$. For such a value of q ($\gg 1$), the second factor is convergent if z is taken from the set

$$\mathbb{B}_q(\delta) := \left\{ \zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbb{C}^{\mathbb{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 \cdot (2\mathbb{N})^{\alpha q} < \delta^2 \right\} \tag{16}$$

for some $\delta < \infty$ (cf. [HLOUZ2]).

Proposition 5. *If $F, G \in (\mathcal{S})^{-1}$, then*

$$\mathcal{H}(F \Diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

holds for any $z \in \mathbb{C}^{\mathbb{N}}$ so that both $\mathcal{H}F$ and $\mathcal{H}G$ may exist.

The next assertion is of importance in applicational basis, especially when we apply the Hermite transform to rewrite the stochastic equation into an ordinary one and discuss the convergence of its approximate solutions. The topology on $(\mathcal{S})^1$ can conveniently be expressed in terms of Hermite transforms as follows.

Proposition 6. *The following two convergences are equivalent:*
(i) $X_n \to X$ *in* $(\mathcal{S})^{-1}$.
(ii) $\exists \delta > 0$, $q < \infty$, $M < \infty$ *such that*

$$\mathcal{H}X_n(z) \to \mathcal{H}X(z) \quad (as\ n \to \infty) \quad for\ z \in \mathbb{B}_q(\delta)$$

and $|\mathcal{H}X_n(z)| \leq M$ *for all* $n = 1, 2, \cdots$, $\forall z \in \mathbb{B}_q(\delta)$.

Theorem 7. *(Characterization for the Kondratiev Space) Suppose that* $g(z_1, z_2, \cdots)$ *be a bounded analytic function on* $\mathbb{B}_q(\delta)$ *(* $\exists \delta > 0$, $q < \infty$ *). Then there exists an element* X *in* $(\mathcal{S})^{-1}$ *such that* $\mathcal{H}X = g$ *holds.*

Corollary 8. *Suppose that* $g = \mathcal{H}X$ *(* $\exists X \in (\mathcal{S})^{-1}$ *). Let* f *be an analytic function in the neighborhood of* $g(0)$ *in* $\mathbb{C}$. *Then there exists an element* Y *in* $(\mathcal{S})^{-1}$ *such that* $\mathcal{H}Y = f \circ g$.

Example 5. Let $X \in (\mathcal{S})^{-1}$. Then $X \diamond X = X^{\diamond 2} \in (\mathcal{S})^{-1}$ is always true by (ii) of Lemma 1. More generally, $X^{\diamond n} \in (\mathcal{S})^{-1}$ holds for $\forall n \in \mathbb{N}$. Hence we attain that

$$\mathrm{Exp} X \equiv \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n} \in (\mathcal{S})^{-1}$$

by applying Corollary 8 with $f(z) = \exp(z)$.

Remark 3. The Hermite transform $\mathcal{H}$ and the S-transform in white noise analysis are closely connected. As a matter of fact, the following relation holds.

$$\mathcal{H}F(z_1, z_2, \cdots, z_m) = SF(z_1 e_1 + z_2 e_2 + \cdots + z_m e_m)$$

for any $z = (z_1, z_2, \cdots, z_m) \in \mathbb{C}^m$, $(\exists m \in \mathbb{N})$.

Theorem 9. *(Interchangeability of Integration and Wick Product) Assume that* $F(\cdot, \cdot) \in L^2(\mathcal{S}' \times \mathcal{S}', \mu \otimes \mu)$. *For any* $G \in (\mathcal{S})^*$,

$$\int_{\mathcal{S}'} F(\eta, x) \diamond G(x) d\mu(\eta) = \int_{\mathcal{S}'} F(\eta, x) d\mu(\eta) \diamond G(x).$$

Theorem 10. *Assume that* $Y \in (L^2)$, *and* $\psi \in C_0^\infty(\mathbb{R})$ *such that* supp $\psi \subset [a, b]$. *If* $\psi(s) Y(\omega)$ *is Skorohod integrable, then*

$$Y(\omega) \diamond W_\psi(\omega) = \int_a^b \psi(s) \cdot Y(\omega) \delta B_s(\omega)$$

holds, where the right hand side means the Hitsuda-Skorohod integral (cf. [HKPS]).

3. Functional Process

3.1 (L^p)-Functional Process

We write $L^p(\mathcal{S}',\mu)$ as (L^p). When X is an (L^p)-functional process, we write $X \in \mathcal{L}^p$.

Definition 5. *((L^2)-Functional Process) We say $X \in \mathcal{L}^2$ if $X = X(\varphi, t, x)$ is a mapping $: \mathcal{S} \times \mathbb{R}^d \times \mathcal{S}' \to \mathbb{R}$ such that*

$$X(\varphi,t,x) = \sum_{\alpha} c_\alpha(\varphi,t) \cdot H_\alpha(x),$$

where $c_\alpha(\cdot,\cdot)$ is a mapping $: \mathcal{S} \times \mathbb{R}^d \to \mathbb{R}$ for $|\alpha| \geq 1$, and for each $\varphi \in \mathcal{S}$, the mapping $: \mathbb{R}^d \ni t \mapsto c_\alpha(\varphi,t)$ is Borel measurable, and if $\alpha = 0$, $c_0(\cdot)$ is just a measurable function on $\mathbb{R}^d$, independent of φ. Moreover,

$$\mathbb{E}[X(\varphi,t,\cdot)^2] = \sum_{\alpha} c_\alpha^2(\varphi,t) \cdot \alpha! < \infty$$

for any $\varphi \in \mathcal{S}$, and any $t \in \mathbb{R}^d$.

Definition 6. *((L^p)-Functional Process) We say $X \in \mathcal{L}^p$ if $X = X(\varphi,t,x) : \mathcal{S} \times \mathbb{R}^d \times \mathcal{S}' \to \mathbb{R}$ such that*
(a) a mapping $: \mathbb{R}^d \ni t \mapsto X(\varphi,t,x)$ is Borel measurable for any $\varphi \in \mathcal{S}$, μ-a.e. $x \in \mathcal{S}'$; and
(b) a mapping $: \mathcal{S}' \ni x \mapsto X(\varphi,t,x) \in (L^p)$ for any $\varphi \in \mathcal{S}$, any $t \in \mathbb{R}^d$.

The functional process $X(\varphi,t,x)$ is called *positive* or *a positive noise* if $X(\varphi,t,x) \geq 0$ holds μ-a.e. $x \in \mathcal{S}'$ for any $\varphi \in \mathcal{S}$, any $t \in \mathbb{R}^d$.

Example 6. (cf. [LOU]) Let $X = X(\varphi,t,x)$, $Y = Y(\varphi,t,x)$ be positive (L^2)-functional processes such that

$$X_\varphi(x) = \sum_{\alpha} a_\alpha(\varphi^{\otimes n}) \cdot H_\alpha(x), \qquad Y_\varphi(x) = \sum_{\beta} b_\beta(\varphi^{\otimes n}) \cdot H_\beta(x).$$

Then the Wick product $X \Diamond Y$ is also positive.

Theorem 11 [LOU]. *(Characterization of Positive Functional Process) Let $X \in (L^2)$. Then X is positive (μ-a.e. $x \in \mathcal{S}'$) if and only if $M^n(y) \equiv \tilde{X}^{(n)}(iy) \cdot \exp(-\frac{1}{2}|y|^2)$ is positive definite as a matrix of $M(n \times n)$ for any $n \in \mathbb{N}$, $y \in \mathbb{R}^n$, where $\tilde{X}^{(n)}(z) \equiv \tilde{X}(z_1, z_2, \cdots, z_n, 0, 0, \cdots, 0)$.*

Let us consider the WN process. We shall introduce an interesting and important fact that the WN process provides a typical example of (L^p)-functional process, which very often can be found useful in applications to stochastic partial differential equations[B], [D8], [HLOUZ1]. Set $W(\varphi,t,x) \equiv W_{\varphi(t)}(x)$, and define $\varphi_t(u) = \varphi(t)(u) = \varphi(u-t)$. Actually the WN process

$$W_{\varphi(t)}(x) = \langle x, \varphi_t \rangle = \int_{\mathbb{R}^d} \varphi_t(u) dB_u(x)$$

is naturally regarded as an (L^p)-functional process, i.e. $W_{\varphi(t)} \in \mathcal{L}^p$.

3.2 The Kondratiev Space Valued Process

Definition 7. *(Stochastic Distribution Valued Process)*

$$\Phi \equiv \Phi(t,p,\cdot) : \mathbb{R} \times \mathbb{R}^n \ni (t,p) \mapsto \Phi(t,p)(\cdot) \in (\mathcal{S})^{-1}$$

is regarded as a stochastic distribution valued process. We call such a function a $(\mathcal{S})^{-1}$-process.

Let us consider the derivative of $(\mathcal{S})^{-1}$-process. Let $F(t)$ be a $(\mathcal{S})^{-1}$-process: namely,

$$F(t,\cdot) : \mathbb{R} \ni t \mapsto F(t,\cdot) \in (\mathcal{S})^{-1}.$$

Definition 8. $\Xi \equiv \Xi(t_0) \in (\mathcal{S})^{-1}$ *is said to be a derivative of $(\mathcal{S})^{-1}$-process $F(t)$ with respect to t at $t = t_0$ if there exists an element Ξ in $(\mathcal{S})^{-1}$ such that*

$$\frac{F(t_0+h) - F(t_0)}{h} \to \Xi \quad in \quad (\mathcal{S})^{-1} \quad (as \ \ h \to 0).$$

When the above limit exists, we write $\Xi(t_0) \equiv \frac{dF}{dt}(t_0)$ ($\in (\mathcal{S})^{-1}$).

We set $\mathcal{H}F(t) = \tilde{F}(t_0;z)$ and $\mathcal{H}\Xi(t_0) = \tilde{\Xi}(t_0;z)$. By virtue of the characterization of topology of $(\mathcal{S})^{-1}$ (see Proposition 6 in §2.3), the aforementioned definition is equivalent to the following:

$$\text{(17)} \qquad \frac{\tilde{F}(t_0+h;z) - \tilde{F}(t_0;z)}{h} \to \tilde{\Xi}(t_0;z) \qquad \text{as} \quad h \to 0$$

holds *pointwise, boundedly* for any $z \in \mathbb{B}_q(\delta)$ ($\exists q < \infty$, $\delta > 0$). If the mapping : $t \mapsto \frac{d}{dt}\tilde{F}(t;z) = \frac{d}{dt}\mathcal{H}F(t)$ is continuous in t, and uniformly bounded for any $z \in \mathbb{B}_q(\delta)$, and any t in the neighborhood of t_0, then instead of the condition (17), the condition

$$\text{(18)} \qquad \text{“}\frac{d}{dt}\tilde{F}(t;z) = \tilde{\Xi}(t;z) \quad \text{for } t = t_0, \text{ pointwise for each } z \in \mathbb{B}_q(\delta)\text{”}$$

is just sufficient. Because, if Eq. (18) holds, we can write it as

$$\frac{\tilde{F}(t_0+h;z) - \tilde{F}(t_0;z)}{h} = \frac{1}{h}\int_{t_0}^{t_0+h} \frac{s}{ds}\tilde{F}(s;z)ds \quad \text{for small } h,$$

and therefore, this expression turns out to be uniformly bounded for $z \in \mathbb{B}_q(\delta)$ as h tends toward zero. If $\frac{d}{dt}F$ exists and is t-continuous, then it follows that $(\mathcal{S})^{-1}$-process $F(t) \in C^1$.

4. The Stochastic Boundary Value Problem

4.1 Formulation

We consider the following stochastic boundary value problem:

$$
\begin{aligned}
& du(t,r) = \{\Delta u(t,r) + R(u(t,r))\}dt + h(t,r)u(t,r)dB_t, \\
(19) \qquad & 0 \le t \le T, \quad r \in [0,1], \\
& u(t,0) = u(t,1), \quad u(0,r) = u_0(r),
\end{aligned}
$$

where Δ is the Laplacian and $R(y)$ is a polynomial of $y \in \mathbb{R}$. B_t denotes a one dimensional Brownian motion. h, u_0 are non random functions being continuous. In addition, assume $u_0 \in C^3$.

Definition 9. *(Functional Process Solution) $u \equiv u(\varphi, t, r, x)$ is said to be a $(\mathcal{S})^{-1}$ functional process solution of Eq. (19) if*

$$u : C_0^\infty(\mathbb{R}) \times [0,T] \times \mathbb{R} \to (\mathcal{S})^{-1}$$

is a Kondratiev space valued functional process and satisfies

$$(20) \qquad u(t) = u_0(r) + \int_0^t \Delta_r u(s)ds + \int_0^t R^\diamond(u(s))ds + \int_0^t h(s,r)u(s)\diamond W_{\varphi(s)}(x)ds,$$

for $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi_s(t) = \varphi(t-s)$ with boundry condition.

We resort to the asymptotic solution theory. We shall say that u_k is an asymptotic solution for the problem (20) if there exists $u_k = u_k(t,r)$ solving the reduced, modified or simplified equation, satisfying

$$(21) \qquad u_k(t,r) \to u(t,r) \quad \text{in} \quad (\mathcal{S})^{-1}.$$

Let $u_k = u_k(\varphi, t, r, x, \omega)$ satisfies the following stochastic partial differential equation (SPDE for short):

$$
\begin{aligned}
(22) \qquad u_k(t) = u_{0k}(r) &+ \int_0^t \Delta_k u_k(s)ds + \int_0^t R^\diamond(u_k(s))ds \\
&+ \int_0^t h_k(s,r)u_k(s)\diamond W_{\varphi(s)}(x)ds + M_k(t,r,\omega),
\end{aligned}
$$

with boundary condition, where ω is an element of some proper probability space on which a martingale M_k is realized. We propose that the asymptotic problem for our case is to show that

$$\sup_t \|X_k(t) - \tilde{u}(t)\|_\infty \to 0 \quad (k \to \infty),$$

for $T > 0$, if we take Eq.(21) into consideration with characterization of topology in $(\mathcal{S})^{-1}$ in accordance with Holden-Lindstrøm-Øksendal-Ubøe-Zhang formalism (cf. Proposition 6 in §2; see also [HLOUZ1], [HLOUZ2]).

$\tilde{u}$ is a solution solving

$$\frac{\partial}{\partial t}\tilde{u}(t) = (\Delta_r + c(t,r))\tilde{u}(t) + R(\tilde{u}(t)), \tag{23}$$

with the initial and boundary conditions, where we put $c = h \cdot \tilde{W}_\varphi$. The corresponding model for asymptotic solution is described as

$$\begin{aligned} dX_k(t) &= (\Delta_k + c_k)X_k(t)dt + R(X_k(t))dt + dM_k(t), \\ &\text{with} \quad X_k(t,0) = X_k(t,1), \quad X_k(0,r) = u_{0k}(r). \end{aligned} \tag{24}$$

If we assume boundedness for R and the initial value, then the problem (23) has a continuous bounded solution by virtue of the implicit approximation scheme. Under further assumptions on R there exists a unique solution X_k for the problem (24). In fact we can construct it by employing the classical probability theory related to some jump type Markov processes with suitable conditions.

Theorem 12. *Under the assumption of convergence $\|X_k(0)-\tilde{u}(0)\|_\infty \to 0$ in probability, then we get*

$$\lim_{k\to 0} \mathbb{P}\Big(\sup_t \|X_k(t) - \tilde{u}(t)\|_\infty > \varepsilon\Big) = 0, \tag{25}$$

as far as $z \in \mathbb{B}_q(\delta)$, for some positive δ, q.

4.2 The Probabilistic Model

Let us consider the totality of real valued step functions on $[0,1]$, and we extend those functions periodically with period 1. We denote the extension by H_k. For $f \in H_k$, we define

$$\Delta_k f(r) = k^2 \left\{ f\left(r + \frac{1}{k}\right) - 2f(r) + f\left(r - \frac{1}{k}\right) \right\}.$$

We shall now introduce the discretized problem of Eq.(23), i.e.,

$$\frac{\partial}{\partial t}\tilde{u}_k(t,r) = (\Delta_k + c_k)\tilde{u}_k(t,r) + R(\tilde{u}_k(t,r)), \tag{26}$$

with the corresponding initial and boundary conditions. Then we have the bounded solution $\tilde{u}_k(t)$ for all t, and

$$\sup_{t\in[0,T]} \|\tilde{u}_k(t) - \tilde{u}(t)\|_\infty \le C(T,R,u_0) \cdot C'(k) \quad \text{for} \quad T > 0,$$

with $C'(k) = O(k^{-1})$, ($k \to \infty$).

While we consider the following SPDE driven by a martingale term M:

$$dX(t,r) = \{\Delta_r + c(t,r)\}X(t,r)dt + R(X(t,r))dt + dM_t. \tag{27}$$

We follow the standard notation in stochastic analysis (e.g. [IW]). Let M be a continuous square integrable local martingale on $(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{F}_t)$. If the quadratic variation process of M is given by an integral of $G(s,\omega)^2$ relative to s over $[0,t]$ where $G(\neq 0)$ is a $(\mathcal{F}_t)$-predictable process and belongs to $L^2([0,T])$ with probability one, then the representation theorem for martingales(p.90, [IW]) guarantees that there exists an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ with $\mathcal{F}_t'$ and there exists an $(\mathcal{F}_t')$-Brownian motion such that $M(t) = \int_0^t G(s)\ dB(s)$. So we assume that Eq.(27) has a solution (X, B) on $(\Omega', \mathcal{F}', \mathbb{P}')$. Define an $\mathcal{A}^{1,1}$ process $\gamma(t,X) = -c(t,r)\ X(t,r)\ G(t)^{-1}$. Further suppose that

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^t |\gamma(s,X)|^2 ds\right) < \infty, \qquad \forall t > 0, \tag{28}$$

$$\Gamma \exp\left\{\int_0^t \gamma(s,X)dB(s) - \frac{1}{2}\int_0^t |\gamma(s,X)|^2 ds\right\} \text{ is a } (\mathcal{F}_t') - \text{martingale.} \tag{29}$$

Put $\hat{\mathbb{P}} = \Gamma \cdot \mathbb{P}'$ and $\hat{B}(t) = B(t) - \int_0^t \gamma(s,X)ds$. An application of the Girsanov theorem[G] allows that $\hat{B}(t)$ becomes a $(\mathcal{F}_t')$-Brownian motion on $(\Omega', \mathcal{F}', \hat{\mathbb{P}})$. Therefore $(X, \hat{B})$ on $(\Omega', \mathcal{F}', \hat{P})$ solves the stochastic equation:

$$dX(t,r) = \Delta_r X(t,r)dt + R(X(t,r))dt + d\hat{M}_t, \tag{30}$$

with $\hat{M}(t) = \int_0^t G(s)d\hat{B}(s)$. On the other hand, we consider the stochastic process $U(t)$ describing a density dependent birth and death process. In fact, let $U(t) = (U_1(t), \cdots, U_k(t))$ be a $\mathbb{N}^k$-valued jump type Markov process whose Markovian particle may diffuse on the circle in accordance with simple random walk with jump rate $2k^2$, and besides with birth rate $pR_1(U_i/p)$ and with death rate $pR_2(U_i/p)$ where p is a given parameter and $R = R_1 - R_2$. We can construct such a process $U(t)$ by classical probability theory and realize it as a cadlag process on some suitable probability space. $\mathcal{F}_t^p$ denotes the completed σ-field of $\sigma(U(s); s \le t)$. Let $T(\omega)$ be an $\mathcal{F}_t^p$ stopping time satisfying

$$\{\omega \in \Omega; T(\omega) \le t\} \in \mathcal{F}_t^p \quad \text{for } \forall t, \quad \text{and} \quad \sup_t\{U(t \wedge T(\omega)) \cdot I_{T(\omega)>0}(\omega)\} < \infty.$$

Then by martingale theory [LS] it follows that

$$U_i(t \wedge T(\omega)) - \int_0^{t\wedge T(\omega)} \Phi(U, R, p, i; s)ds$$

is an $\mathcal{F}_t^p$-martingale [BL], where we set $\Phi(U,R,p,i;s) = pR(U_i(s)/p) + k^2\{U_{i+1}(s) + U_{i-1}(s) - 2U_i(s)\}$. Define

$$X_k(t,r) := U_i(t)/p \quad \text{for} \quad r \in [i/k,(i+1)/k), \quad i = 1,2,\cdots,k-1.$$

Thus we attain that the H_k valued Markov process X_k satisfies the discretized version of Eq.(30):

$$dX_k(t,r) = \Delta_k X_k(t,r)dt + R(X_k(t,r))dt + d\hat{M}_k(t). \tag{31}$$

4.3 Law of Large Numbers for the Stochastic Problem

In order to prove Eq.(25) it is sufficient to show that

$$\mathbb{P}\left\{\sup_t \|X_k(t) - \tilde{u}_k(t)\|_\infty > \varepsilon\right\}$$

converges to zero as k tends toward infinity. Set $T_t = \exp(t\Delta_k)$ and

$$Y_k(t) = \int_0^t T_{t-s} d\hat{M}_k(s \wedge T(\omega)).$$

Moreover, a simple calculation leads to $\|\delta X_k(t \wedge T(\omega))\|_\infty = O(p^{-1})$ with precise estimates. On this account, the problem can be attributed finally to computation of the term $\sup_t \|Y_k(t)\|_\infty$. In fact we need to estimate

$$\sup_{t\in[a,b]} \|Y_k(t)\|_\infty \le C_1\|Y_k(a)\|_\infty + C_2 \sup_{t\in[a,b]} \|M_k(t\wedge T(\omega)) - M_k(a \wedge T(\omega))\|_\infty.$$

By making use of Gronwall's inequality, Markov' inquality and Doob's inequality, we deduce that

$$\mathbb{P}\left\{C_3(T) \sup_{t\in[c,d]} \|Y_k(t)\|_\infty > \varepsilon\right\} \le C_4(k,p,\varepsilon),$$

because we applied martingale theory. For the final estimate, we need Lemma 4.4, p.135 [BL].

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