A STUDY OF AUTOMORHIC FORMS

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§1 **Notation and genus characters.** We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. We fix an totally real number field F of degree n with class number one and denote by $\mathfrak{o}, d_F$, and $\mathfrak{d}$, the maximal order of F, the discriminant of F and the different of F relative to $\mathbb{Q}$, respectively. We denote by E the unit group of F. Let $\tau_1, \cdots, \tau_n$ be the isomorphisms of F to $\mathbb{R}$. For each $\alpha \in F$, we put $\alpha^{(\nu)} = \tau_\nu(\alpha)$ $(1 \leq \nu \leq n)$. We assume that $[E : E^+] = 2^n$ with $E^+ = \{\epsilon \in E | \epsilon \gg 0\}$. For an element N of $\mathfrak{o}$ satisfying $N \gg 0$, put $\tilde{\Gamma}_0(N) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{o}) \Big| N | c \text{ and } \det \gamma \gg 0\right\}$. Put $\mathfrak{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$. We define two actions of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o})$ on $\mathfrak{H}^n$ and $\mathfrak{H}^n \times \mathbb{C}^n$ by

(1-1)

$$z \to \gamma(z) = \Big(\frac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}}, \cdots, \frac{a^{(n)}z_n + b^{(n)}}{c^{(n)}z_n + d^{(n)}}\Big) \quad \text{for every } z = (z_1, \cdots, z_n) \in \mathfrak{H}^n$$

and

$$(\tau, z) \to \gamma(\tau, z) = \Big(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \cdots, \frac{a^{(n)}\tau_n + b^{(n)}}{c^{(n)}\tau_n + d^{(n)}}; \frac{z_1}{c^{(1)}\tau_1 + d^{(1)}}, \cdots, \frac{z_n}{c^{(n)}\tau_n + d^{(n)}}\Big)$$

for every $(\tau, z) = (\tau_1, \cdots, \tau_n; z_1, \cdots, z_n) \in \mathfrak{H}^n \times \mathbb{C}^n$. We also define an action of $(\lambda, \mu) \in \mathfrak{o}^2$ on $\mathfrak{H}^n \times \mathbb{C}^n$ by

$$(\tau, z) \to (\lambda, \mu)(\tau, z) = (\tau_1, \cdots, \tau_n; z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}, \cdots, z_n + \lambda^{(n)}\tau_n + \mu^{(n)})$$

for every $(\tau, z) \in \mathfrak{H}^n \times \mathbb{C}^n$. Let N and $k = (k_1, \cdots, k_n)$ be elements such that $N \in \mathfrak{o}, k \in \mathbb{Z}^n$. We consider a holomorphic function $\phi(\tau, z)$ on $\mathfrak{H}^n \times \mathbb{C}^n$ satisfying the conditions:

$$\text{(1-2)} \quad (i) \ \phi(\gamma(\tau, z)) = (c\tau + d)^k e[\mathrm{tr}\Big(\frac{N}{\delta}\Big(\frac{cz^2}{c\tau + d}\Big)\Big)]\phi(\tau, z) \text{ for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\in SL_2(\mathfrak{o})$,

(ii) $\phi((\lambda,\mu)(\tau,z)) = e[-\mathrm{tr}\Big(\frac{N}{\delta}(\lambda^2+2\lambda z)\Big)]\phi(\tau,z)$ for every$(\lambda,\mu)\in\mathfrak{o}^2$ and

(iii) $\phi(\tau,z) = \sum_{n,r\in\mathfrak{o}^2, 4Nn-r^2\gg 0} c(n,r)e[\mathrm{tr}(\frac{n}{\delta}\tau+\frac{r}{\delta}z],$

where $(c\tau+d)^k = \prod_{i=1}^n (c^{(i)}\tau_i+d^{(i)})^{k_i}$, $e[\mathrm{tr}\Big(\frac{N}{\delta}(\frac{cz^2}{c\tau+d})\Big)] = e[\sum_{i=1}^N \frac{N^{(i)}}{\delta^{(i)}}\Big(\frac{c^{(i)}z_i^2}{c^{(i)}\tau_i+d^{(i)}}\Big)]$, $e[-\mathrm{tr}\Big(\frac{N}{\delta}(\lambda^2\tau+2\lambda z)\Big)] = e[-\sum_{i=1}^n\Big(\frac{N^{(i)}}{\delta^{(i)}}((\lambda^{(i)})^2\tau_i+2\lambda^{(i)}z_i)\Big)$ and $e[\mathrm{tr}(\frac{n}{\delta}\tau+\frac{r}{\delta}z)]$ $= e[\sum_{i=1}^n(\frac{n^{(i)}}{\delta^{(i)}}\tau_i+\frac{r^{(i)}}{\delta^{(i)}}z_i)]$. We denote by $J_{k,N}^{\mathrm{cusp}}$ the set of all such functions ϕ. We call such a ϕ a Jacobi cusp forms of index N and of weight k over F. We introduce the Jacobi group $\Gamma(1)^J = \{(\gamma,(\lambda,\mu)|\gamma\in SL_2(\mathfrak{o}),\lambda,\mu\in\mathfrak{o}\}$ determined by the group law $(\gamma,(\lambda,\mu))\cdot(\gamma',(\lambda',\mu')) = (\gamma\gamma',(\lambda,\mu)\gamma'+(\lambda',\mu'))$ for every $\gamma,\gamma'\in SL_2(\mathfrak{o})$, (λ,μ) and $(\lambda',\mu')\in\mathfrak{o}^2$. We define an action of $(\gamma,(\lambda,\mu))\in\Gamma(1)^J$ on $\mathfrak{H}^n\times\mathbb{C}^n$ by

(1-3)

$$(\tau,z)\to(\gamma,(\lambda,\mu))(\tau,z) = \Big(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\Big) \text{ for every } (\tau,z)\in\mathfrak{H}^n\times\mathbb{C}^n.$$

For a function ϕ on $\mathfrak{H}^n\times\mathbb{C}^n$ and $(\gamma,(\lambda,\mu))\in\Gamma(1)^J$, define a function $\phi|_{k,N}(\gamma,(\lambda,\mu))$ on $\mathfrak{H}^n\times\mathbb{C}^n$ by

(1-4)

$$\phi|_{k,N}(\gamma,(\lambda,\mu))(\tau,z) = (c\tau+d)^{-k}e[\mathrm{tr}\Big(\frac{N}{\delta}\Big(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\Big)\Big)]$$

$\times\phi(\gamma,(\lambda,\mu))(\tau,z)$ for every $(\tau,z)\in\mathfrak{H}^n\times\mathbb{C}^n$.

For α and β in $\mathfrak{o}$ satisfying $(2,\beta)=1$, define a symbol $(\frac{\alpha}{\beta})$ by

$$\Big(\frac{\alpha}{\beta}\Big) = \prod_{i=1}^s\Big(\frac{\alpha}{\mathfrak{p}_i}\Big)^{e_i} \text{ and } \Big(\frac{\alpha}{\mathfrak{p}_i}\Big) = \#\{x\in\mathfrak{o}/\mathfrak{p}_i|x^2\equiv\alpha \pmod{\mathfrak{p}_i}\}-1,$$

where $(\beta) = \prod_{i=1}^s\mathfrak{p}_i^{e_i}$ with an odd prime ideal $\mathfrak{p}_i$ $(1\leqq i\leqq s)$. Let ρ and Δ be elements satisfying the conditions that

(1-5) $\rho\in\mathfrak{o}/2N\mathfrak{o}, \Delta\in\mathfrak{o}, \Delta\gg 0$ and $\Delta\equiv\rho^2 \pmod{4N}$.

We consider a set of quadratic forms $L_{N,\Delta,\rho}$ defined by

(1-6)

$L_{N,\Delta,\rho} =$

$$\Big\{Q=[Na,b,c]=\begin{pmatrix}Na & b/2\\ b/2 & c\end{pmatrix}\Big|a,b,c\in\mathfrak{o}, b^2-4Nac=\Delta \text{ and } b\equiv\rho \pmod{2N}\Big\}.$$

Assume that D_0 is an element of $\mathfrak{o}$ such that $D_0|\Delta$ and Δ/D_0 is square modulo $4N$. Moreover, we impose the following condition: $D_0\ll 0, (D_0,4N)=1$, the finite part of the conductor of the abelian extension

(1-7) $F(\sqrt{D_0})$ over F equals (D_0) and $D_0=\pi_1^*\cdots\pi_l^*$ with distinct primary odd prime elements π_i^* of F $(1 \leqq i \leqq l)$. We define a genus character $\chi_{D_0}(Q)$ by

(1-8) $$\chi_{D_0}(Q) = \begin{cases} \left(\dfrac{m}{D_0}\right) & \text{if } (a,b,c,D_0) = 1, \\ 0 & \text{otherwise} \end{cases}$$

for every $Q = [Na, b, c] \in L_{N,\Delta,\rho}$, where m is an element of $\mathfrak{o}$ such that $(m, D_0) = 1$ and $m = aN_1x^2 + bxy + cN_2y^2$ for some N_1, N_2, x and $y \in \mathfrak{o}$ with $N = N_1N_2$ and $N_1 \gg 0, N_2 \gg 0$. Let r_0, n_0, r, n' and $b \in \mathfrak{o}$ denote elements such that

(1-9) $$D_0 = r_0^2 - 4Nn_0, D = \Delta/D_0 = r^2 - 4Nn' \text{ and } b \equiv r_0 r \pmod{2N}.$$

Given an integral ideal (a) in F, we define a sum F_a by

(1-10) $$F_a = F_a(N, r_0, n_0, r, s, n') = N(a)^{-1} \sum_{\lambda(a)^*} \sum_{x,y \in (a)} e[\mathrm{tr}(\frac{\lambda F(x,y)}{a\delta})]$$

with $F(x,y) = Nx^2 + r_0xy + n_0y^2 + rx + sy + n'$ with $s = (r_0r - b)/2N$.

Proposition 1.1. *Suppose that D_0 satisfies the condition (1-10). Then*

(1-11) $$N(a)^{-1} \sum_{(d)|a, d \gg 0} \left(\frac{d}{D_0}\right) F_{a/d} = \begin{cases} \chi_{D_0}([Na, b, \frac{b^2-\Delta}{4Na}]) & \text{if } a | \frac{b^2-\Delta}{4N}, \\ 0 & \text{otherwise.} \end{cases}$$

§2 A correspondence from Jacobi forms to Hilbert modular forms. Let $k = (k_1, \cdots, k_n)$ be an element of $\mathbb{Z}^n$ with $k_i > 1$ $(1 \leqq i \leqq n)$. We denote by $S_{2k}(\tilde{\Gamma}_0(N))$ the space of cusp forms of weight $2k$ with respect to $\tilde{\Gamma}_0(N)$. Given Δ and D_0 satisfying (1-5) and (1-7), we define a function $f_{k,N,\Delta,\rho,D_0}(z)$ on $\mathfrak{H}^n$ by

(2-1) $$f_{k,N,\Delta,\rho,D_0}(z) = \sum_{Q \in L_{N,\Delta,\rho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \quad \text{for every } z = (z_1, \cdots, z_n) \in \mathfrak{H}^n,$$

where $Q(z,1)^k = \prod_{i=1}^n (N^{(i)}a^{(i)}z_i^2 + b^{(i)}z_i + c^{(i)})^{k_i}$ with $Q = [Na, b, c]$. We see that $f_{k,N,\Delta,\rho,D_0}(z)$ belongs to the space

(2-2) $$M_{2k}(N)^{\mathrm{sgn}\, D_0} = \{f \in S_{2k}(\tilde{\Gamma}_0(N)) | f(-\frac{1}{Nz}) = (-Nz^2)^k (\mathrm{sgn}\, D_0) f(z)\}$$

with $\mathrm{sgn}\, D_0 = \prod_{i=1}^n \mathrm{sgn}\, D_0^{(i)}$. Given a $(n,r) \in \mathfrak{o}^2$ satisfying $r^2 - 4Nn \ll 0$, we define a function $P_{k,N,(n,r)}(\tau, z)$ on $\mathfrak{H}^n \times \mathbb{C}^n$ by

(2-3) $$P_{k,N,(n,r)}(\tau, z) = \sum_{\gamma \in \Gamma_\infty^J(1) \backslash \Gamma^J(1)} e^{n,r}|_{k,N}\gamma(\tau, z),$$

where $\Gamma_\infty^J(1) = \left\{\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu)\right) \middle| n, \mu \in \mathfrak{o}\right\}$ and $e^{n,r}(\tau, z) = e[\mathrm{tr}(\frac{n}{\delta}\tau + \frac{r}{\delta}z)]$. Define a function $\Omega_{k,N,D_0,r_0}(w; \tau, z)$ on $\mathfrak{H}^n \times (\mathfrak{H}^n \times \mathbb{C}^n)$ by

(2-4)
$$\Omega_{k,N,D_0,r_0}(w; \tau, z) = c_{k,N,D_0} \times \sum_{(n,r)} (4Nn - r^2)^{k-1/2} f_{k,N,D_0(r^2-4Nn),r_0r,D_0}(w) e[\mathrm{tr}(\frac{n}{\delta}\tau + \frac{r}{\delta}z)]$$

with $c_{k,N,D_0} = (-2i)^{k-1}2^{n-1}\delta^{-3/2}N^{1-k}\pi^{-k}|D_0|^{k-1/2}$, where (n,r) runs over $c(N) = \{(n,r) \in \mathfrak{o}^2 | 4Nn - r^2 \gg 0\}$. We may deduce the following theorem.

Theorem 2.1(A basic identity). *Suppose that $D_0 = r_0^2 - 4Nn_0$ satisfies the assumption (1-7). Then*

(2-4)

$$\Omega_{k,N,D_0,r_0}(w;\tau,z) = c_{k,N,D_0}(2\pi)^k i^{k-1}(k-1)!^{-1}\delta^{-k+1/2}$$
$$\times \sum_{m\in\mathfrak{o},m\gg 0} m^{k-1}\Big(\sum_{dd'=m,d\in\mathfrak{o}^+/E^+}\left(\frac{d}{D_0}\right)(d')^k P_{k+1,N,(n_0(d')^2,r_0d')}(\tau,z)\Big)e[tr(\frac{mw}{\delta})].$$

Define a mapping $\Psi_{D_0,r_0}(\phi) : J^{\text{cus}}_{k+1,N} \to M_{2k}(N)^{\text{sgn}\ D_0}$ by

(2-5) $\qquad \Psi_{D_0,r_0}(\phi)(w) = \langle \phi, \Omega_{k,N,D_0,r_0}(-\overline{w};*)\rangle$ for every $\phi \in J^{\text{cus}}_{k+1,N}$.

By virtue of Theorem 2.1, we may deduce that

(2-6)

$$\Psi_{D_0,r_0}(\phi)(w)$$
$$= \sum_{m\in\mathfrak{o},m\gg 0}\Big(\sum_{(d)|m,d\gg 0}\left(\frac{d}{D_0}\right)d^{k-1}c((n/d)^2n_0,(n/d)r_0)\Big)e[\text{tr}\frac{mw}{\delta}]$$

with $\phi(\tau,z) = \sum_{(n,r)\in C(N)} c(n,r)e[\text{tr}(\frac{n}{\delta}\tau + \frac{r}{\delta}z)]$. We may define Hecke operators on $J^{\text{cusp}}_{k+1,N}$. We may derive that this mapping Ψ_{D_0,r_0} is commutative the action the Hecke operators.

§3 Fourier coefficients of Jacobi forms and the critical values of the zeta function associated with Hilbert modular forms.

Let $\phi(\tau,z) = \sum_{(n,r)\in C(N)} c(n,r)e[\text{tr}(\frac{n}{\delta}\tau + \frac{r}{\delta}z)]$ be an element of $J^{\text{cusp}}_{k+1,N}$. We deduce the following theorem.

Theorem 3.1. *Let D_0 be an element satisfying the condition (1-7). Suppose that ϕ is an eigenfunction of Hecke operators on $J^{\text{cusp}}_{k+1,N}$ and satisfies the assumption about multiplicity one theorem concerning Hecke operators. Then*

(3-2) $$|c(n_0,r_0)|^2 = \frac{\langle\phi,\phi\rangle}{\langle f,f\rangle}(k-1)!\frac{\delta^{k-3/2}|D_0|^{k-1/2}}{2^{2k-1}N^{k-1}\pi^k}(E^+ : E_0)D(k_0,\chi,\left(\frac{*}{D_0}\right)),$$

where $D(s,\chi,(\frac{*}{D_0}))$ is the zeta function attached to the eigenvalues χ of f twisted by a character $(\frac{*}{D_0})$ in the sense of Shimura, $k_0 = \max\{k_1,\cdots,k_n\}$, $E_0 = \{\epsilon^2|\epsilon \in E, \epsilon^2 \equiv 1(mod.D_0)\}$ and f is the primitive form associated with $\Psi_{D_0,r_0}(\phi)$.