極小モデル理論 (森理論) 的手法による３次元アフィン代数多様体の構造研究

Analysis of the structure of affine algebraic threefolds from a point of view of Minimal Model Theory (Mori Theory)

By

岸本　崇（理学部数学科・助手）
Takashi KISHIMOTO (Department of Mathematics, Faculty of Science・Assistant Professor)

We shall explain the content of this report concerning the research project entitled " Analysis of the structure of affine algebraic threefolds from a point of view of Minimal Model Theory (Mori Theory) ". First of all, in §1, we list the information of the articles which are relevant to the research project. Hereafter, in §2 we shall explain what is our research project paying attention to the following matters:

(1) Motivation of the research.

(2) Methods of approach.

(3) Results (Explanations of the contents of articles listed in §1 a little bit in detail).

(4) Research plan in the (near) future.

§ 1 **List of the papers**

In what follows, we shall list the information of the articles which are relevant to the research project entitled " Analysis of the structure of affine algebraic threefolds from a point of view of Minimal Model Theory (Mori Theory) "

[1] Takashi Kishimoto, *On the compactifications of contractible affine threefolds and the Zariski Cancellation Problem*, Mathematische Zeitschrift, **247** (2004), pp. 149–181.

[2] Takashi Kishimoto, *Singularities on normal affine 3-folds containing* $\mathbb{A}^1$*-cylinderlike open subsets*, Contemporary Mathematics, **369** (2005), pp. 139–163.

[3] Takashi Kishimoto, *The explicit factorization of the Cremona transformation which is an extension of the Nagata automorphism into elementary links*, Mathematische Nachrichten, **278** No. 7–8 (2005), pp. 833–843.

[4] Takashi Kishimoto, *Compactifications of contractible affine 3-folds into smooth Fano 3-folds with* $B_2 = 2$, Mathematische Zeitschrift **(to appear)**.

[5] Takashi Kishimoto, *On the logarithmic Kodaira dimension of affine threefolds*, International Journal of Mathematics **(to appear)**.

[6] Takashi Kishimoto, *Compactifications of* $\mathbb{C}^3$ *into* $\mathbb{Q}$*-Fano 3-folds of Gorenstein index* 2 *and Fano index* $\frac{1}{2}$, Publ. Res. Inst. Math. Sci. **(submitted)**.

[7] Takashi Kishimoto, *Affine threefolds whose log canonical bundles are not numerically effective*, Journal of Pure and Applied Algebra **(submitted)**.

[8] Takashi Kishimoto, *Log Minimal Model Program and Affine Algebraic Threefolds*, RIMS-**1511 (preprint)**.

§ 2 Survey of the research

(1) Motivation of the research:

At first, we shall mention the motivation of the research. Our research filed is called *Algebraic Geometry*, the origin of which seems to be very old (but I do not know much about the origin). Although there are many divided research areas in Algebraic Geometry from a point of view of the current mathematics, the main object treated there lies in *Algebraic Varieties* absolutely. An algebraic varieties are obtained from *Affine Algebraic Varieties* by patching them together. This is why the consistent analysis of affine algebraic varieties is indispensable for the global analysis of general algebraic varieties.

For a given algebraic varieties, say X, we can define the *dimension* of X, which is denoted by $\dim X$ usually. Roughly speaking, $\dim X$ is the maximum of the number of axis contained in X. One-dimensional algebraic variety are called *curves*, and two-dimensional algebraic varieties are called *surfaces.* Curves have been investigated classically, and there exist many consistent theories about them. On the other hand, the framework for classification of projective surfaces were constructed substantially due to the works by Professor Kunihiko Kodaira around 1960's. By making use of the classification theory of projective surfaces, the research of the affine algebraic surfaces has been growing from 1980's up to now by the contributions of Professors Masayoshi Miyanishi, Fumio Sakai, Takao Fujita, R.V. Gurjar, M. Zaidenberg you name it. Indeed, we are able to look over these rapid development on the theory of affine algebraic surfaces in a lot of articles and books at the present time.

Meanwhile for the higher-dimensional case $\dim X \geq 3$, the situation makes involved drastically. The main difficulties for this seems to result from the existence of codimension two algebraic surgery called *flip*. However, the program to deal with higher-dimensional (projective) varieties so called *Minimal Model Program* (*Mori Theory*) has been raised and grown up by many of mathematician, especially due to the works of Professors Shigefumi Mori, Yujiro Kawamata, Shuichiro Tsunoda, J. Kollár and V.V. Shokurov. Around 1990, this program was at last established for the three-dimensional case, that is, there exists a framework to deal with three-dimensional projective varieties. (By abuse of language, we shall call three-dimensional algebraic varieties as *threefolds* in what follows.) In consideration of such a present condition, we intend to analyze the structure of affine algebraic threefolds by making use of three-dimensional Minimal Model Program.

(2) Methods of approach:

Let X be an affine algebraic threefold (with a certain kinds of mild singularities). In order to investigate X from a point of view of Minimal Model Program (=MMP, for short), we embed X into a suitable projective threefold, say V, with the boundary divisor D. Then, we apply MMP to the pair (V, D) to obtain the sequence:

$$(*) \quad \phi : (V, D) \overset{\phi^0}{\cdots\to} (V^1, D^1) \overset{\phi^1}{\cdots\to} \cdots\cdots \to (V^{s-1}, D^{s-1}) \overset{\phi^{s-1}}{\cdots\to} (V^s, D^s),$$

where the final object $(V', D') := (V^s, D^s)$ has a certain kind of distinguished properties (but we do not explain in detail here). The properties on (V', D') usually enables us to analyze the complement $X' := V' \backslash D'$ fairly explicitly. Since we want to investigate X itself, the problem then lies in how to recover the data on X from those on X'. But, this is very difficult and complicated in general. The main reason for this is summarized as follows:

OBSTACLE Each of the birational maps $\phi^i : (V^i, D^i) \cdots \to (V^{i+1}, D^{i+1})$ appearing in $(*)$ above is either a log-divisorial contraction or a log-flip. Let D^i denote the proper transform of D on V^i, and let $X^i := V^i \backslash D^i$ be the complement $(0 \le i \le s)$. Then:

(i) In the case where ϕ^i is of log-divisorial type and the exceptional divisor $E^i := \mathrm{Exc}(\phi^i)$ is NOT contained in the boundary D^i, then X^{i+1} is strictly smaller than the previous one X^i. If we were to describe the contraction ϕ^i and how E^i meets D^i explicitly, then we can recover the data on X^i from those on X^{i+1} in principle. But this seems to be hopeless in general.

(ii) In the case where ϕ^i is of log-flipping type, if we were to know that all the flipping curves (resp. flipped curves) are contained in the boundary D^i (resp. D^{i+1}), then there exists no difference between X^i and X^{i+1}. But, this expectation does not always hold true, namely, some of flipping curves or flipped curves may not be contained in the boundary. As a result, we can not compare X^i and X^{i+1} explicitly and, in addition to this, X^i (resp. X^{i+1}) may be no longer affine even if X^{i+1} (resp. X^i) is affine.

This is why there seems to be no clear principle to compare X^i with X^{i+1}. Hence, even if we can analyze X' concretely, it is impossible to recover the data on the original X from those on X' in general.

(3) Results:

In the series of articles listed in §1, we add cretain assumptions with respect to the embedding $X \hookrightarrow (V, D)$, and obtain the results in order to control the program $(*) : (V, D) \cdots \to (V', D')$ keeping the obstacles mentioned above in mind. The best result in this direction (obtained in [8]) guarantees that we can overcome the obstacles (i) and (ii) under the assumption that the complete linear system $|D|$ associated with the boundary divisor D contains a normal surface with at most Du Val singularities.

As (implicit) applications of the techniques used in [1], [2], [5], [8], we are able to classify compactifications of the complex affine 3-space $\mathbb{C}^3$ into *Fano threefolds* with certain kinds of mild singularities. These results are contained in [4] and [6]. Moreover, we consider the Cremona transformation on the three-dimensional projective space $\mathbb{P}^3$ obtained as a natural extension of the famous *Nagata automorphism* on $\mathbb{C}^3$. We apply the *Sarkisov Program* established by Professor A. Corti in 1995 to factorize this Cremona transformation explicitly into certain kinds of simple birational maps between Mori Fiber Spaces, so called *elementary links*.

(4) Research plan:

Our research plans in the (near) future consists of the following:

(a) Refinement and generalization of the results obtain in the series of articles [1], [2], [5], [8] to control the process $(*) : (V, D) \cdots \to (V', D')$.

(b) Application of results in the direction (a) to the classification of compactifications of $\mathbb{C}^3$ into $\mathbb{Q}$-Fano and weak $\mathbb{Q}$-Fano threefolds.

(c) Application of results in the direction (a) to 3-dimensional Zariski Cancellation Problem and the characterization of $\mathbb{C}^3$ from a point of view of Minimal Model Program.