研　究　成　果　報　告　書

多様体の諸構造と接分布の幾何学に関する研究

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はしがき

平成 15 年度，16 年度，17 年度の 3 年間にわたって研究代表者を中心にして研究課題名

「多様体の諸構造と接平面場にかかわる幾何学の研究」

のもとで科学研究費補助金（基盤研究（C)）の援助をうけて，研究を遂行した．ここにその研究成果を報告する．

研究組織

研究代表者 ： 水谷忠良 （埼玉大学理学部・教授）

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研究発表

学会誌等

当該研究期間において研究代表者及び研究分担者が学会誌等に発表した主な研究成果は以下のとおりである.

水谷忠良

[1] K. Mikami and T. Mizutani,
Foliations associated with Nambu-Jacobi structures,
Tokyo Journal of Mathematics Vol. 28, 33-54, (2005).

[2] K. Mikami and T. Mizutani,
Integrability of plane fields defined by 2-vector fields,
International Journal of Mathematics, 16(2), 197-212, (2005).

[3] K. Mikami and T. Mizutani,
Lie Algebroid Associated with an Almost Dirac Structure,
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[4] K. Mikami and T. Mizutani,
A Lie Algebroid and a Dirac structure associated to an Almost Dirac Structure,
(to appear).

[5] K. Mikami and T. Mizutani,
Lie algebroids associated with deformed Schouten bracket of 2-vector fields
(to appear)

阪本邦夫

[1] K.Sakamoto,
Variational problems of normal curvature tensorand concircular scalar fields,
Tohoku Math.Journal., Vol 55, 207-254, (2003).

[2] T. Ohkubo and K. Sakamoto,
CR Einstein-Weyl structures,
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長瀬正義

[1] M. Nagase,
Twistor spaces and the general adiabatic expansions,(preprint)

[2] M. Nagase,
On the trace and the infinitesimally deformed chiral anomaly of Dirac operators on twistor spaces and the change of metrics on the base spaces,(preprint)

[3] M. Nagase,
On the infinitesimally deformed super chiral anomaly of Dirac operators and the gauge transformation of twistor spaces, (preprint)

福井敏純

[1] T. Fukui and J.　Weyman,
Cohen-Macaulay properties of Thom-Boardman strata II: The defining ideals of Σ^{i,j},
Proceedings of London Mathematical Society,　87, 137-163, {2003}.

[2] T.. Fukui, K.. Kurdyka and L.. Paunescu,
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Geometric Singularity Theory (eds. Heisuke Hironaka, Stanislaw Janeczko, Stanislaw Lojasiewicz),Banach Center Publications, 65,　49-56, {2004}.

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酒井文雄

[1] M.Ohkouchi and F.Sakai,
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The gonality of singular plane curves,
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[3] Sakai,F. and Saleem,M,
Rational plane curves of type (d,d-2),
Saitama Math. J. 22, 11-34, {2004}.

下川航也

[1] M. Hachimori and K. Shimokawa,
Tangle sum and constructible spheres,
J. Knot Theory Ramifications, 13, 373-383, {2004}.

[2] M. Brittenham, C. Hayashi, M. Hirasawa, T. Kobayashi and K. Shimokawa,
Essential laminations and branched surfaces in the exteriors of links,
Japan. J. Math., 31, 25-96, {2005}.

口頭発表

当該研究期間において研究代表者及び研究分担者が行った研究経過，研究成果などの口頭によるおもな発表は以下のとおりである．

水谷忠良

[1] 講演題目 「Some properties of plane fields defined by 2-vector fields(joint work with Kentaro Mikami)」
（国際会議 Geometry and Foliations Kyoto 2003）（2003 年 9 月 18 日）

[2] 講演題目 「ポアソン幾何，Lie Algebroid に関する話題」
（Kwansai Seminar on Differential Analysis）（2004 年 1 月 5 日,6 日,7 日）

[3] 講演題目 「Lie Algebroid associated with 2-vecor fields」
（国際会議 Foliations 2005，Lodz Poland）（2005 年 6 月 18 日）

阪本邦夫

[1] 講演題目 「CR Einstein Weyl 構造について」
（研究集会「微分幾何とその応用}{2004 年 9 月 11 日-12 日}

[2] 講演題目 「法曲率テンソルに関する変分問題について」
（研究集会「曲線と曲面の非線形解析」）（2004 年 12 月 16 日-18 日）

酒井文雄

[1] 講演題目 「Lower bounds for the gonality of singular plane curves」
（日本数学会秋季総合分科会，岡山大学）（2005 年 9 月 22）

[2] 講演題目 「Lower bounds for the gonality of singular plane curves」
（シンポジウム 「代数曲線論」，中央大学）（2005 年 12 月 20 日）

[3] 講演題目 「Rational plane curves of type $(d,d-2)$」(M.Saleem, 酒井文雄)
(日本数学会年会, 筑波大学) (2004年3月29日)

[4] 講演題目 「The gonality of singular plane curves」
(談話会, 茨城大学) (2003年12月4日)

下川航也

[1] 講演題目 「Exceptional surgery and boundary slopes」
(研究集会 東北結び目セミナー) (2003年1月6日)

[2] 講演題目 「最近の Dehn surgery のいくつかの話題について」
(研究集会 結び目と多様体の幾何と代数 II) (2003年9月6日)

[3] 講演題目 「Tangle sum and constructible spheres」
(First KOOK Seminar International for Knot Theory and Related Topics)
(2004年7月10日)

[4] 講演題目 「Culler-Shalen theory and A-polynomials」
(A多項式サマーセミナー) (2004年8月6日)

研究成果による工業所有権の出願・取得状況

該当なし

研究成果

研究代表者及び研究分担者はそれぞれの研究分野での研究を行い，研究課題にかかわる共通のテーマに関して必要に応じて討論を重ねた。こでは，研究代表者が得た知見と新たな経過について触れるとともに，学会誌等に発表した論文，及び論文として完成し出版準備しているもの，または出版予定となっているものの主なものをうち掲載することにした。

研究代表者によるこの期間の研究経過の概要は以下のとおりである。

第一の研究期間では，ポアソン多様体におけるポアソン・コホモロジーに関してその幾何学的な意味あいを問題にした。

0 次元，1 次元コホモロジー H^0, H^1 の解釈は容易であるので 2 次元コホモロジー H^2 が最初に問題となるが，特に「ポアソン・テンソル自身が H^2 の元として消えるものにどういうものがあるか」を具体的な問題としてとりあげた。得られた成果の一つは，コンパクトな 3 次元多様体で上の条件を満たすものの中に，よく知られたリー群の商空間として得られるものと異なるものを構成したことである。このポアソン多様体のもつシンプレクティック葉層は以前，Hirsch 葉層として知られていたものになっている。この方向の研究にはいまだに位相幾何的あるいは幾何学的に興味深い問題が残されている。

第二の研究期間においては まず，南部ヤコビ多様体 がどのような 多重ベクトル場によって特徴付けられるのかを調べ，さらに自然に対応する葉層構造についての結果を得た。

q 次の南部ヤコビ多様体は 基本恒等式 {fundamental identity}と呼ばれる q 個の関数に対する 括弧積を持つものとして定義され，通常のヤコビ多様体ある方向への拡張である。関連して Leibniz algebroid に関しては Leibniz Algebras associated with foliations {Kodai Math.Journal}にその研究成果を発表した。

第三の研究期間では，一般の 2-ベクトル場 π があったとき，それの定義する接分布の完全積分可能性について調べることからはじめた。これに関しては，スカウテン・ブラケット $[\pi,\pi]$ と 1 次微分形式のブラケットに関する基本的な関係式を得て、議論を進め、研究成果を

Integrability of Plane Field Defined by 2-vector fields(International J. of Math.)

にまとめた。さらに、この結果を最近研究が盛んになった Dirac 構造の枠組みでとらえ almost Dirac 構造に付随する Lie algebroid 及び Dirac 構造に関して興味ある結果が得られた。これらの結果は収録論文として掲載してある。

収録論文

水谷忠良

1 Lie algebroids associated with deformed Schouten bracket of 2-vector fields. (by K. Mikami and T. Mizutani)

2 Lie Algebroid Associated with an Almost Dirac Structure. (by K. Mikami and T. Mizutani)

3 A Lie algebroid and a Dirac structure associated to an almost Dirac structure. (by K. Mikami and T. Mizutani)

阪本邦夫

1 CR EINSTEIN-WEYL STRUCTURE (by T. Ohkubo and K. Sakamoto)

長瀬正義

1 Twistor spaces and the general adiabatic expansions (by M. Nagase)

2 On the trace and the infinitesimally deformed chiaral anomaly of Dirac operators on twistor spaces and the change of metrics on the base spaces (by M. Nagase)

3 On the infinitesimally deformed super chiral anomaly of Dirac operators and the gauge transformation of twistor spaces (by M. Nagase)

福井敏純

1 ISOLATED SINGULARITIES OF BIANRY DIFFERENTIAL EQUATIONS OF DEGREE n (by T. Fukui and N. Ballesteros)

Lie algebroids associated with deformed Schouten bracket of 2-vector fields

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Abstract

Given a 2-vector field and a closed 1-form on a manifold, we consider the set of cotangent vectors which annihilate the deformed Schouten bracket of the 2-vector field by the closed 1-form. We show that if the space of cotangent vectors forms a vector bundle, it carries a structure of a Lie algebroid. We treat this theorem in the category of Lie algebroids. As a special case, this result contains the well known fact that the 1-jet bundle of functions of a contact manifold has a Lie algebroid structure.

1 Introduction

The Poisson bi-vector field on a Poisson manifold (M, π), defines a bundle morphism $\tilde{\pi}$: $\mathrm{T}^*(M) \to \mathrm{T}(M)$ which is given by $\alpha \mapsto \pi(\alpha, \cdot)$. The image of $\tilde{\pi}$ is called the characteristic distribution of the Poisson structure π. It is integrable and gives a generalized foliation of M consisting of leaves with symplectic structure. Moreover, $\mathrm{T}^*(M)$ has a structure of a Lie algebroid which leads to the Poisson cohomology. One can naturally ask the condition for a general 2-vector field π (not necessarily a Poisson), under which the image of $\tilde{\pi}$ is integrable and ask how special a Poisson bi-vector is. The condition for the integrability can well be seen from the formula (see Section 3)

$$[\pi(\alpha), \pi(\beta)] = \pi(\{\alpha, \beta\}) + \frac{1}{2}[\pi, \pi](\alpha, \beta) \quad \text{for 1-forms } \alpha \text{ and } \beta$$

where $\{\alpha, \beta\}$ is the bracket on $\Gamma(\mathrm{T}^*(M))$, and $\pi(\alpha)$ means $\tilde{\pi}(\alpha)$ in precise, but we often use both notations interchangeably. The formula above says, if the Schouten bracket $[\pi, \pi]$ is in the image of $\tilde{\pi}$, the Frobenius condition are satisfied and the distribution is integrable (while in the case of Poisson structure *a fortiori* $[\pi, \pi] = 0$ holds). In [5], the authors considered the condition that $[\pi, \pi]$ is a image of a closed 3-form under the induced map of $\tilde{\pi}$ and proved $\mathrm{T}^*(M)$ has a Lie algebroid structure which they call a twisted Poisson structure. Clearly, this condition implies the integrability of the image of $\tilde{\pi}$ by the above formula.

In our previous paper ([4]), we considered the space of cotangent vectors $\mathcal{A} = \{\alpha \mid [\pi, \pi](\alpha, \cdot, \cdot) = 0\}$ and proved $\mathcal{A}$ has a natural Lie algebroid structure (provided $\mathcal{A}$ is a vector bundle of constant rank). In this paper, we generalize the discussion to the case of deformed Schouten bracket $[\pi, \pi]^\phi$ and show that the same result is obtained in this case too (Theorem 3.4). Also, we introduce the definition of a Jacobi-Lie algebroid. It is nothing but a Lie algebroid equipped with a specified 1-cocycle. However, this definition is sometimes preferable when we treat such objects formally. For example, one can define the homomorphims between two Jacobi-Lie algebroids. In the next section, we recall some basics on the Lie algebroids and the Schouten-Jacobi bracket. In section 3, we prove our main theorem and give a computational example of the theorem.

2 Lie algebroids and Jacobi-Lie algebroids

In this section, we review some basic ingredients of Lie algebroids for later use and introduce the notion of a Jacobi-Lie algebroid. All manifolds and functions are assumed to be smooth (C^∞) throughout the paper.

Definition 2.1 A vector bundle $\mathcal{L}$ over a manifold M is a *Lie algebroid* if

(a) the space of sections $\Gamma(\mathcal{L})$ is endowed with a Lie algebra bracket $[\cdot, \cdot]$ over $\mathbb{R}$

(b) there is given a bundle map $a : \mathcal{L} \to \mathrm{T}(M)$ (called *anchor*) which induces a Lie algebra homomorphism $a : \Gamma(\mathcal{L}) \to \Gamma(\mathrm{T}(M))$, satisfying the condition

$$[X, fY] = \langle a(X), df \rangle Y + f[X, Y], \qquad X, Y \in \Gamma(\mathcal{L}),\ f \in C^\infty(M).$$

Thus a Lie algebroid is a triple $(\mathcal{L}, [\cdot,\cdot], a)$, however we often say $\mathcal{L}$ a Lie algebroid when the bracket and the anchor are understood. The most popular and important example of a Lie algebroid is the tangent bundle with usual Lie bracket of vector fields. The cotangent bundle of a Poisson manifold is another example of a Lie algebroid. There are many other examples of Lie algebroids which are useful in geometry (see [1]).

Let $\mathcal{L}^*$ be the dual vector bundle of $\mathcal{L}$. We note that the anchor of $\mathcal{L}$ induces a dual morphism $a^* : \mathrm{T}^*(M) \longrightarrow \mathcal{L}^*$.

The Lie algebra bracket on $\Gamma(\mathcal{L})$ and the action of $a(X)$ on $C^\infty(M)$ induces an 'exterior differential' $d_{\mathcal{L}}$ on $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ defined by a well-known formula;

$$\begin{aligned}(d_{\mathcal{L}}\omega)(X_0, X_1, \dots, X_r) := &\sum_{i=0}^{r}(-1)^i\langle d(\omega(\dots, \hat{X}_i, \dots)), a(X_i)\rangle \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots), \\ &\qquad \omega \in \Gamma(\Lambda^r\mathcal{L}^*),\ X_0, \dots, X_r \in \Gamma(\mathcal{L}) .\end{aligned}$$

For example,

$$\langle d_{\mathcal{L}}f, X\rangle = \langle df, a(X)\rangle = \langle a^*(df), X\rangle, \qquad f \in \Gamma(\Lambda^0\mathcal{L}^*) = C^\infty(M),\ X \in \Gamma(\mathcal{L}) .$$

$$\begin{aligned}(d_{\mathcal{L}}\beta)(X, Y) &= \langle d(\beta(Y)), a(X)\rangle - \langle d(\beta(X)), a(Y)\rangle - \langle \beta, [X, Y]\rangle \\ &= L_{a(X)}(\beta(Y)) - L_{a(Y)}(\beta(X)) - \langle \beta, [X, Y]\rangle, \quad \beta \in \Gamma(\Lambda^1\mathcal{L}^*),\ X, Y \in \Gamma(\mathcal{L}) .\end{aligned}$$

With this differential $d_{\mathcal{L}}$, $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ becomes a differential graded algebra and a^* induces a homomorphism of differential graded algebras $\Gamma(\Lambda^\bullet\mathrm{T}^*(M)) \to \Gamma(\Lambda^\bullet\mathcal{L}^*)$.

Conversely, the exterior differential $d_{\mathcal{L}}$ on $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ recovers the anchor and the Lie algebra bracket on $\mathcal{L}$, hence recovers the Lie algebroid structure of $\mathcal{L}$ by the formulas

(a') $\langle a(X), df\rangle := \langle X, d_{\mathcal{L}}f\rangle$,

(b') $\langle [X, Y], \beta\rangle := \langle X, d_{\mathcal{L}}(\beta(Y))\rangle - \langle Y, d_{\mathcal{L}}(\beta(X))\rangle - (d_{\mathcal{L}}\beta)(X, Y)$, $(\beta \in \Gamma(\mathcal{L}^*))$.

In [3], the authors introduced the *deformed exterior differential* and the *Schouten-Jacobi bracket* on $\Gamma(\Lambda^\bullet\mathcal{L})$ deformed by a 1-cocycle ϕ.

Definition 2.2 Let ϕ be a 1-cocycle in $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ with respect to $d_{\mathcal{L}}$, i.e., $\phi \in \Gamma(\mathcal{L}^*)$ and ϕ

satisfies

$$\phi([X,Y]) = L_{a(X)}(\phi(Y)) - L_{a(Y)}(\phi(X))$$

for $X, Y \in \Gamma(\mathcal{L})$. The deformed exterior differential is defined by

$$d_{\mathcal{L}}^{\phi}\alpha = d_{\mathcal{L}}\alpha + \phi \wedge \alpha, \qquad \alpha \in \Gamma(\Lambda^{\bullet}\mathcal{L}^*)\ . \tag{2.1}$$

The operator $d_{\mathcal{L}}^{\phi}$ satisfies

$$d_{\mathcal{L}}^{\phi} \circ d_{\mathcal{L}}^{\phi} = 0, \qquad d_{\mathcal{L}}^{\phi}(\alpha \wedge \beta) = d_{\mathcal{L}}^{\phi}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d_{\mathcal{L}}^{\phi}\beta - \phi \wedge \alpha \wedge \beta,$$

where $|\alpha|$ means the degree of α, namely, $\alpha \in \Gamma(\Lambda^{|\alpha|}\mathcal{L}^*)$. On the other hand, ($\phi$-deformed) Schouten-Jacobi bracket $[\cdot,\cdot]^{\phi}$ is defined by

$$[P,Q]^{\phi} = [P,Q] + (-1)^p P(\phi) \wedge (q-1)Q + (p-1)P \wedge Q(\phi), \tag{2.2}$$
$$P \in \Gamma(\Lambda^p\mathcal{L}), Q \in \Gamma(\Lambda^q\mathcal{L})\ .$$

Here and hereafter, $P(\phi)$ denotes the interior product $\iota_{\phi}P$ or $\phi \lrcorner P$ of ϕ and P. We use these notations interchangeably.

This bracket on $\Gamma(\Lambda^{\bullet}\mathcal{L})$ shares similar properties with the usual Schouten-Nijenhuis bracket. In our sign convention, formulas of calculation for $[\cdot,\cdot]^{\phi}$ are the following

(1) $[X,Y]^{\phi} = [X,Y]$ (Lie algebra bracket), for $X, Y \in \Gamma(\mathcal{L})$

(2) $[P,Q]^{\phi} = -(-1)^{(p-1)(q-1)}[Q,P]^{\phi}$

(3) $[P,[Q,R]^{\phi}]^{\phi} = [[P,Q]^{\phi},R]^{\phi} + (-1)^{(p-1)(q-1)}[Q,[P,R]^{\phi}]^{\phi}$ (super Jacobi identity)

(4) $[P,Q\wedge R]^{\phi} = [P,Q]^{\phi} \wedge R + (-1)^{(p-1)q}Q \wedge [P,R]^{\phi} + (-1)^p P(\phi) \wedge Q \wedge R$

(5) $[f,P]^{\phi} = -P(d_{\mathcal{L}}^{\phi}f), \quad f \in C^{\infty}(M)$

where $P \in \Gamma(\Lambda^p\mathcal{L}), Q \in \Gamma(\Lambda^q\mathcal{L}), R \in \Gamma(\Lambda^r\mathcal{L})$.

For $\phi = 0$, these are just the formulas for the Nijenhuis-Schouten bracket. The only difference is that the deformed one does not satisfy the Leibniz property for the wedge product (see (4) above). Since $d_{\mathcal{L}}^{\phi}f$ in (5) above is defined by $d_{\mathcal{L}}f + f\phi = a^*(df) + f\phi$ in $\Gamma(\mathcal{L}^*)$, the action of $X \in \Gamma(\mathcal{L})$ on $C^{\infty}(M)$ through $[\cdot,\cdot]^{\phi}$ is given by $X \cdot f := [X,f]^{\phi} = \langle a(X), df\rangle + f\langle X,\phi\rangle$ where $\langle a(X), df\rangle$ is the usual action of Lie algebroid through the anchor map. Putting $f = 1$, we see that the 1-cocycle ϕ is recovered from the bracket since $\phi(X) = [X, 1]^\phi = X \cdot 1$ holds.

The difference of the action of X on $C^\infty(M)$ from the usual derivation leads to the different 'exterior differential' and 'Lie derivation'. The ϕ-Lie derivative operator L^ϕ for 'forms' and 'vectors' are defined by

$$L^\phi_X \alpha = (d^\phi_{\mathcal{L}} i_X + i_X d^\phi_{\mathcal{L}})\alpha = L_X\alpha + \phi(X)\alpha \tag{2.3}$$

$$L^\phi_X P = [X, P]^\phi = [X, P] - (p-1)\phi(X)P \tag{2.4}$$

respectively. Then we have the following list of formulas.

$$L^\phi_X(\alpha \wedge \beta) = L^\phi_X\alpha \wedge \beta + \alpha \wedge L^\phi_X\beta - \phi(X)\alpha \wedge \beta \tag{2.5}$$

$$L^\phi_X(P \wedge Q) = L^\phi_X P \wedge Q + P \wedge L^\phi_X Q - \phi(X)P \wedge Q \tag{2.6}$$

$$L^\phi_X(P(\alpha)) = (L^\phi_X P)(\alpha) + P(L^\phi_X\alpha) + (|\alpha| - 1)\phi(X)P(\alpha) \tag{2.7}$$

$$L^\phi_X(\alpha(P)) = \alpha(L^\phi_X P) + (L^\phi_X\alpha)(P) + (p-1)\phi(X)\alpha(P) \tag{2.8}$$

$$L^\phi_X[P, Q]^\phi = [L^\phi_X P, Q]^\phi + [P, L^\phi_X Q]^\phi \tag{2.9}$$

$$L^\phi_{fX} P = f L^\phi_X P - X \wedge P(d_{\mathcal{L}} f) \tag{2.10}$$

Note that (2.7) or (2.8) tells us that L^ϕ_X does not commute with the contraction in general, although L_X does.

Remark 2.1 Let ϕ be a usual closed 1-form on M. We can see a cue of defining of the ϕ-deformed Schouten-Jacobi bracket $[\cdot,\cdot]^\phi$ in the following observation when $\mathcal{L} = \mathrm{T}(M)$. Let $\phi = df$ locally where f is a function on M. For a p-vector field P, we put $\hat{P} = e^{-(p-1)f}P$. Note that this assignment $P \mapsto \hat{P}$ is injective and it is the identity transformation on the space of vector fields. If we compute $[\hat{P}, \hat{Q}]$, we have $e^{-(p+q-2)f}[P, Q]^\phi$

As we will see below, one of the advantage of introducing $[\cdot,\cdot]^\phi$ is that we can treat a Jacobi structure on M as if it were a Poisson structure on M with respect to $[\cdot,\cdot]^\phi$. It seems natural here to generalize the Lie algebroid slightly, and introduce the notion of *Jacobi-Lie algebroid.*

Let $\mathcal{T}^*M$ denote the bundle of 1-jets of functions on M. $\mathcal{T}^*M$ has a natural projection onto the bundle of 0-jets which is a trivial line bundle $\varepsilon \cong M \times \mathbb{R}$. The kernel of the projection is the cotangent bundle $T^*(M)$ and $\mathcal{T}^*M \cong T^*(M) \oplus \varepsilon$ by $j^1_x f \mapsto (df_x, f(x))$. Let $\mathcal{T}M$ denote the dual bundle and call it the *extended tangent bundle* of M. The sections of $\mathcal{T}M$ form the set of differential operators on $C^\infty(M)$ of order ≤ 1. Geometrically, $\mathcal{T}M$ can be identified with the tangent bundle $T(M \times \mathbb{R})$ restricted to $M \times \{0\}$ (or to any level $M \times \{t\}$). Then a section $\mathbf{X}$ of $\mathcal{T}M$ is expressed as

$$\mathbf{X} = X + \lambda \frac{\partial}{\partial \tau}$$

where X is a vector field on M lifted to $M \times \mathbb{R}$ and $\frac{\partial}{\partial \tau} = \left(\frac{\partial}{\partial t}\right)_0$ is the *tangent vector* of $\mathbb{R}$ at 0. From this view point, we may write 1-jet $j^1 f$ as $df + f d\tau$, where $d\tau$ is the dual of $\frac{\partial}{\partial \tau}$.

$\mathbf{X}$ acts on $C^\infty(M)$ as a first order differential operator by

$$\mathbf{X} \cdot f = \langle \mathbf{X}, j^1 f \rangle = L_X f + \lambda f.$$

The commutator bracket of $\mathbf{X} = X + \lambda \frac{\partial}{\partial \tau}$ and $\mathbf{Y} = Y + \mu \frac{\partial}{\partial \tau}$ in $\Gamma(\mathcal{T}M)$ as operators, is

$$[\![\mathbf{X}, \mathbf{Y}]\!] = [\![X + \lambda \frac{\partial}{\partial \tau}, Y + \mu \frac{\partial}{\partial \tau}]\!] = [X, Y] + (\langle X, d\mu \rangle - \langle Y, d\lambda \rangle) \frac{\partial}{\partial \tau}.$$

With this bracket on $\Gamma(\mathcal{T}M)$ and the natural projection $pr_1 : \mathcal{T}M \to T(M)$ as the anchor, $(\mathcal{T}M, [\![\cdot, \cdot]\!], pr_1)$ is a Lie algebroid, and the action of $\mathbf{X}$ on $f \in C^\infty(M)$ here, is through the vector field X. The difference between the two actions of $\mathbf{X}$ is the multiplication by λ. The map $\phi_0 : \mathbf{X} \mapsto \lambda = \mathbf{X} \cdot 1$ can be considered as a 1-*cocycle* of the Lie algebroid $\mathcal{T}M$. Indeed

$$(d\phi_0)(\mathbf{X}, \mathbf{Y}) = L_X \mu - L_Y \lambda - \phi_0([\![\mathbf{X}, \mathbf{Y}]\!]) = 0 \ .$$

We call this cocycle ϕ_0 of $\mathcal{T}M$ the *canonical 1-cocycle.*

Let $(\mathcal{L}, [\cdot, \cdot], a)$ be a Lie algebroid and ϕ any Lie algebroid-1-cocycle of $\mathcal{L}$. Then we have a bundle map $\bar{a} : \mathcal{L} \to \mathcal{T}M$ defined by $\bar{a}(X) = a(X) + \phi(X) \frac{\partial}{\partial \tau} \in T(M) \oplus \varepsilon = \mathcal{T}M$. Using this map, we formulate a Lie algebroid with specified 1-cocycle as follows.

Definition 2.3 A Jacobi-Lie algebroid over a manifold M is a triplet $(\mathcal{L}, [\cdot, \cdot], \bar{a})$ of a vector bundle $\mathcal{L}$, a Lie algebra structure $[\cdot, \cdot]$ on $\Gamma(\mathcal{L})$, and a bundle map $\bar{a}$ of $\mathcal{L}$ into $\mathcal{T}M$ (called also anchor), satisfying

(1) $(\mathcal{L}, [\cdot,\cdot], \mathrm{pr}_1 \circ \bar{a})$ is a Lie algebroid over M,

and

(2) $\bar{a}$ induces a Lie algebra homomorphism from $\Gamma(\mathcal{L})$ into $\Gamma(\mathcal{T}M)$.

Note that $\phi = \bar{a}^*(\phi_0)$ is a 1-cocycle of $\mathcal{L}$. Conversely, if a Lie algebroid $(\mathcal{L}, [\cdot,\cdot], a)$ has a 1-cocycle ϕ, then the map $\bar{a} : X \mapsto \bar{a}(X) = a(X) + \phi(X)\frac{\partial}{\partial\tau}$ is verified to be an anchor of Jacobi-Lie algebroid. Indeed, for $X, Y \in \Gamma(\mathcal{L})$, we have

$$\begin{aligned}\bar{a}([X,Y]) &= a([X,Y]) + \phi([X,Y])\frac{\partial}{\partial\tau}\\ &= [a(X), a(Y)] + (\langle a(X), d(\phi(Y))\rangle - \langle a(Y), d(\phi(X))\rangle)\frac{\partial}{\partial\tau}\\ &= [\![a(X) + \phi(X)\frac{\partial}{\partial\tau}, a(Y) + \phi(Y)\frac{\partial}{\partial\tau}]\!] = [\![\bar{a}(X), \bar{a}(Y)]\!],\end{aligned}$$

and

$$[X, fY] = f[X,Y] + \langle a(X), df\rangle Y.$$

Since $\mathcal{T}M \cong \mathrm{T}(M) \oplus \varepsilon$, we have an isomorphism $\Lambda^p\mathcal{T}M \cong \Lambda^p\mathrm{T}(M) \oplus \Lambda^{p-1}\mathrm{T}(M)$. Thus an element $\mathbf{P} \in \Gamma(\Lambda^p(\mathcal{T}M)$ is expressed also as a pair (P, P') of p-vector field and $(p-1)$-vector field. The correspondence is given by $\mathbf{P} = P + \frac{\partial}{\partial\tau} \wedge P' \leftrightarrow (P, P')$. Similarly, an element $\boldsymbol{\alpha} = \alpha + d\tau \wedge \alpha' \in \Gamma(\Lambda^p\mathcal{T}^*M)$ is given as a pair (α, α') consisting of p-form and $(p-1)$-form. Especially, the canonical 1-cocycle ϕ_0 is a pair $(0, 1)$ where 0 denotes the zero 1-form and 1 is a constant function. We sometimes adopt these notations.

Example 2.1 (Jacobi structure on M) Let $\boldsymbol{\pi} = (\pi, \xi)$ be an element in $\Gamma(\Lambda^2\mathcal{T}M)$. With the above notations, we have

$$\begin{aligned}[\boldsymbol{\pi}, \boldsymbol{\pi}]^{\phi_0} &= [(\pi,\xi),(\pi,\xi)]^{\phi_0} = [(\pi,\xi),(\pi,\xi)] + 2(i_{\phi_0}(\pi,\xi)) \wedge (\pi,\xi)\\ &= ([\pi,\pi], 2[\xi,\pi]) + (2\xi\wedge\pi, 0) = ([\pi,\pi] + 2\xi\wedge\pi, 2[\xi,\pi]) .\end{aligned}$$

Thus $[\boldsymbol{\pi}, \boldsymbol{\pi}]^{\phi_0} = 0$ is equivalent to that (π, ξ) is a Jacobi structure. The differential $d^{\phi_0}f$ is (df, f) and 'Hamiltonian vector field' $\boldsymbol{\pi}(d^{\phi_0}f)$ of f is a pair $\left(\pi(df) + f\xi, -\langle\xi, (df)\rangle\right)$. The bracket of functions f and g is given by

$$\{f,g\} = \pi(d^{\phi_0}f, d^{\phi_0}g) = L^{\phi_0}_{\pi(d^{\phi_0}f)}\ g = L_{(\pi(df)+f\xi,-\langle\xi,df\rangle)}\ g + \phi_0(\pi(d^{\phi_0}f))g$$
$$= \pi(df,dg) + f\langle\xi,dg\rangle - g\langle\xi,df\rangle\ .$$

In the case of contact structure, $\pi^n\wedge\xi$ is nowhere zero and the map $f\mapsto\pi(df)+f\xi$ is injective from $C^\infty(M)$ into $\Gamma(\mathrm{T}(M))$ and this vector field is called a contact Hamiltonian vector field.

3 Deformed bracket on 1-forms

Let $\mathcal{L}$ be a Lie algebroid over a manifold M whose anchor is $a:\mathcal{L}\to\mathrm{T}(M)$. We fix a 1-cocycle ϕ and consider ϕ-deformed exterior differential $d^\phi_{\mathcal{L}}$ and ϕ-deformed Schouten bracket $[\cdot,\cdot]^\phi$. By an abuse of language, we call $P\in\Gamma(\Lambda^p\mathcal{L})$ a *p-vector* field and $\alpha\in\Gamma(\Lambda^p\mathcal{L}^*)$ a *p-form*. In this section, we prove our main theorem. Namely, we show that $\left([\pi,\pi]^\phi\right)^0$ has a Lie algebroid structure (Theorem 3.4). (P^0 denotes the space of annihilating elements of P in $\mathcal{L}^*$.)

First we prove

Lemma 3.1 Let $P\in\Gamma(\Lambda^p\mathcal{L}), Q\in\Gamma(\Lambda^q\mathcal{L})$ be p-vector field and q-vector field, respectively. For a 1-form α, the following equality holds;

$$[P,Q]^\phi(\alpha) = [P(\alpha),Q]^\phi + (-1)^{p-1}[P,Q(\alpha)]^\phi$$
$$+ (-1)^p(P\wedge Q)(d^\phi_{\mathcal{L}}\alpha) + (-1)^{p-1}P(d^\phi_{\mathcal{L}}\alpha)\wedge Q + (-1)^{p-1}P\wedge Q(d^\phi_{\mathcal{L}}\alpha) \qquad (3.1)$$

where for $p\le 1$, we understand $P(d^\phi_{\mathcal{L}}\alpha)=0$ and similarly for $q\le 1$, $Q(d^\phi_{\mathcal{L}}\alpha)=0$.

This immediately shows the following

Corollary 3.2 For a 2-vector field π and a 1-form α, we have

$$[\pi(\alpha),\pi]^\phi = -\frac{1}{2}(\pi\wedge\pi)(d^\phi_{\mathcal{L}}\alpha) + \frac{1}{2}[\pi,\pi]^\phi(\alpha) + \pi(d^\phi_{\mathcal{L}}\alpha)\pi\ .$$

Proof of Lemma 3.1: In the case $\phi=0$, the proof is seen in [4]. For general ϕ, we recall the defining equation (2.1) $d^\phi_{\mathcal{L}}\alpha = d_{\mathcal{L}}\alpha + \phi\wedge\alpha$ of $d^\phi_{\mathcal{L}}$ and the equation (2.2) of $[\cdot,\cdot]^\phi$. Using these formulas, we can check that the terms containing ϕ are equal on both sides in (3.1). Consequently, the equality is valid for general Schouten-Jacobi bracket. $\square$

Given a 2-vector field $\pi \in \Gamma(\Lambda^2\mathcal{L})$ and a 1-cocycle ϕ, we define a bracket on 1-forms as follows.

$$\{\alpha,\beta\}_\pi^\phi := L^\phi_{\pi(\alpha)}\beta - L^\phi_{\pi(\beta)}\ \alpha - d^\phi_{\mathcal{L}}(\pi(\alpha,\beta)), \qquad \alpha,\beta \in \Gamma(\mathcal{L}^*)\ . \tag{3.2}$$

Since $d^\phi_{\mathcal{L}}(\pi(\alpha,\beta)) = L^\phi_{\pi(\alpha)}\beta - i_{\pi(\alpha)}d^\phi_{\mathcal{L}}\beta$, we have another expression

$$\{\alpha,\beta\}_\pi^\phi = i_{\pi(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\pi(\beta)}\alpha\ . \tag{3.3}$$

This bracket is not a Lie algebra bracket in general. The following formula is useful in our computations.

Lemma 3.3 For a 2-vector field π, the following equality holds:

$$[\pi(\alpha),\pi(\beta)]^\phi = \pi(\{\alpha,\beta\}_\pi^\phi) + \frac{1}{2}[\pi,\pi]^\phi(\alpha,\beta)\ . \tag{3.4}$$

Proof When $\mathcal{L} = \mathrm{T}(M)$ and $\phi = 0$, the above equation is already known in [4]. Since $\{\alpha,\beta\}_\pi^\phi = i_{\pi(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\pi(\beta)}\ \alpha$, we have

$$\begin{aligned}\pi(\{\alpha,\beta\}_\pi^\phi) &= \pi(i_{\pi(\alpha)}d^\phi_{\mathcal{L}}\beta) - \pi(L^\phi_{\pi(\beta)}\ \alpha) \\ &= \pi(i_{\pi(\alpha)}d^\phi_{\mathcal{L}}\beta) + [\pi(\alpha),\pi(\beta)]^\phi + [\pi(\beta),\pi]^\phi(\alpha)\ .\end{aligned} \tag{3.5}$$

Here, we used a general formula

$$L^\phi_X(P(\alpha)) = (L^\phi_X P)(\alpha) + P(L^\phi_X\ \alpha) + (|\alpha|-1)\phi(X)P(\alpha)$$

for $X = \pi(\beta)$ and $P = \pi$. By Corollary 3.2, (3.5) is followed by

$$\begin{aligned}\pi(i_{\pi(\alpha)}d^\phi_{\mathcal{L}}\beta) + [\pi(\alpha),\pi(\beta)]^\phi - \frac{1}{2}[\pi,\pi]^\phi(\alpha,\beta) - (\pi(\alpha)\wedge\pi)(d^\phi_{\mathcal{L}}\beta) + \pi(d^\phi_{\mathcal{L}}\beta)\pi(\alpha) \\ = [\pi(\alpha),\pi(\beta)]^\phi - \frac{1}{2}[\pi,\pi]^\phi(\alpha,\beta)\ .\end{aligned}$$

Here we used the identity

$$\pi(i_{\pi(\alpha)}d^{\phi}_{\mathcal{L}}\beta) - (\pi(\alpha)\wedge\pi)(d^{\phi}_{\mathcal{L}}\beta) + \pi(d^{\phi}_{\mathcal{L}}\beta)\pi(\alpha) = 0$$

which can be verified by putting $d^{\phi}_{\mathcal{L}}\beta = \theta_1 \wedge \theta_2$ if necessary, where $\theta_1, \theta_2 \in \Gamma(\mathcal{L})$. □

Remark 3.1 Since $[X,Y]^{\phi} = [X,Y]$ for each 1-vector fields, the Lemma above means, for a 2-vector field π, the following equality holds:

$$[\pi(\alpha), \pi(\beta)] = \pi(\{\alpha,\beta\}^{\phi}_{\pi}) + \frac{1}{2}[\pi,\pi]^{\phi}(\alpha,\beta)\ . \tag{3.6}$$

Theorem 3.4 Let $(\mathcal{L}, [\cdot,\cdot], a)$ be a Lie algebroid over a manifold M and ϕ be a 1-cocycle. That is, $\mathcal{L}$ has a Jacobi-Lie algebroid structure with anchor $\bar{a} : \mathcal{L} \to TM, X \mapsto a(X) + \phi(X)\frac{\partial}{\partial\tau}$. Let π be an arbitrary 2-field of $\mathcal{L}$, that is $\pi \in \Gamma(\Lambda^2\mathcal{L})$. Suppose that the rank of $[\pi,\pi]^{\phi}$ is constant. Then the sub-bundle $([\pi,\pi]^{\phi})^0$ is a Jacobi-Lie algebroid with respect to the bracket

$$\{\alpha,\beta\}^{\phi}_{\pi} = L^{\phi}_{\tilde{\pi}(\alpha)}\beta - L^{\phi}_{\tilde{\pi}(\beta)}\alpha - d^{\phi}_{\mathcal{L}}(\pi(\alpha,\beta))$$

and the anchor is given by the composition of $\bar{a}$ and $\tilde{\pi}$ restricted to $([\pi,\pi]^{\phi})^0$.

Corollary 3.5 $\mathcal{H} = a \circ \tilde{\pi}(([\pi,\pi]^{\phi})^0)$ is an integrable distribution.

Proof of Theorem3.4: First we show the space of sections of $([\pi,\pi]^{\phi})^0$ is closed under the bracket $\{\ ,\ \}^{\phi}_{\pi}$. Let 1-forms α and β be sections of $([\pi,\pi]^{\phi})^0$ so that $\alpha \lrcorner [\pi,\pi]^{\phi} = \beta \lrcorner [\pi,\pi]^{\phi} = 0$. In order to prove $\{\alpha,\beta\}^{\phi}_{\pi} \lrcorner [\pi,\pi]^{\phi} = 0$, we use Corollary 3.2 again. It says

$$\frac{1}{2}\{\alpha,\beta\}^{\phi}_{\pi} \lrcorner [\pi,\pi]^{\phi} = [\tilde{\pi}(\{\alpha,\beta\}^{\phi}_{\pi}), \pi]^{\phi} + \frac{1}{2}(d^{\phi}_{\mathcal{L}}\{\alpha,\beta\}^{\phi}_{\pi}) \lrcorner (\pi\wedge\pi) - \pi(d^{\phi}_{\mathcal{L}}\{\alpha,\beta\}^{\phi}_{\pi})\pi$$

in general. By the same formula, α and β satisfy

$$[\tilde{\pi}(\alpha), \pi]^{\phi} + \frac{1}{2}(d^{\phi}_{\mathcal{L}}\alpha) \lrcorner (\pi\wedge\pi) - \pi(d^{\phi}_{\mathcal{L}}\alpha)\pi = 0$$

and

$$[\tilde{\pi}(\beta), \pi]^{\phi} + \frac{1}{2}(d^{\phi}_{\mathcal{L}}\beta) \lrcorner (\pi\wedge\pi) - \pi(d^{\phi}_{\mathcal{L}}\beta)\pi = 0\ .$$

Since $\tilde{\pi}(\{\alpha,\beta\}_\pi^\phi) = [\tilde{\pi}(\alpha),\tilde{\pi}(\beta)]^\phi$ and $\{\alpha,\beta\}_\pi^\phi = L^\phi_{\tilde{\pi}(\alpha)}\beta - L^\phi_{\tilde{\pi}(\beta)}\alpha - d^\phi_{\mathcal{L}}\pi(\alpha,\beta)$, we have

$$
\begin{aligned}
&\frac{1}{2}\{\alpha,\beta\}_\pi^\phi \lrcorner [\pi,\pi]^\phi \\
=&[[\tilde{\pi}(\alpha),\tilde{\pi}(\beta)]^\phi,\pi]^\phi + \frac{1}{2}\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\lrcorner\pi^2 - \pi\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\pi \\
=&[\tilde{\pi}(\alpha),[\tilde{\pi}(\beta),\pi]^\phi]^\phi + [[\tilde{\pi}(\alpha),\pi]^\phi,\tilde{\pi}(\beta)]^\phi \\
&+\frac{1}{2}\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\lrcorner\pi^2 - \pi\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\pi \\
=&L^\phi_{\tilde{\pi}(\alpha)}\left(-\frac{1}{2}(d^\phi_{\mathcal{L}}\beta)\lrcorner\pi^2 + \pi(d^\phi_{\mathcal{L}}\beta)\pi\right) - L^\phi_{\tilde{\pi}(\beta)}\left(-\frac{1}{2}(d^\phi_{\mathcal{L}}\alpha)\lrcorner\pi^2 + \pi(d^\phi_{\mathcal{L}}\alpha)\pi\right) \\
&+\frac{1}{2}\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\lrcorner\pi^2 - \pi\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\pi \\
=&-\frac{1}{2}\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta\right)\lrcorner\pi^2 - \frac{1}{2}d^\phi_{\mathcal{L}}\beta\lrcorner L^\phi_{\tilde{\pi}(\alpha)}\pi^2 - \frac{1}{2}\phi(\tilde{\pi}(\alpha))d^\phi_{\mathcal{L}}\beta\lrcorner\pi^2 \\
&+(L^\phi_{\tilde{\pi}(\alpha)}\pi)(d^\phi_{\mathcal{L}}\beta)\pi + \pi(L^\phi_{\tilde{\pi}(\alpha)}(d^\phi_{\mathcal{L}}\beta))\pi + \pi(d^\phi_{\mathcal{L}}\beta)L^\phi_{\tilde{\pi}(\alpha)}\pi \\
&+\frac{1}{2}\left(L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\lrcorner\pi^2 + \frac{1}{2}d^\phi_{\mathcal{L}}\alpha\lrcorner L^\phi_{\tilde{\pi}(\beta)}\pi^2 + \frac{1}{2}\phi(\tilde{\pi}(\beta))d^\phi_{\mathcal{L}}\alpha\lrcorner\pi^2 \\
&-(L^\phi_{\tilde{\pi}(\beta)}\pi)(d^\phi_{\mathcal{L}}\alpha)\pi - \pi(L^\phi_{\tilde{\pi}(\beta)}(d^\phi_{\mathcal{L}}\alpha))\pi - \pi(d^\phi_{\mathcal{L}}\alpha)L^\phi_{\tilde{\pi}(\beta)}\pi \\
&+\frac{1}{2}\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\lrcorner\pi^2 - \pi\left(L^\phi_{\tilde{\pi}(\alpha)}d^\phi_{\mathcal{L}}\beta - L^\phi_{\tilde{\pi}(\beta)}d^\phi_{\mathcal{L}}\alpha\right)\pi \\
=&-\frac{1}{2}d^\phi_{\mathcal{L}}\beta\lrcorner L^\phi_{\tilde{\pi}(\alpha)}\pi^2 + (L^\phi_{\tilde{\pi}(\alpha)}\pi)(d^\phi_{\mathcal{L}}\beta)\pi + \pi(d^\phi_{\mathcal{L}}\beta)L^\phi_{\tilde{\pi}(\alpha)}\pi \\
&+\frac{1}{2}d^\phi_{\mathcal{L}}\alpha\lrcorner L^\phi_{\tilde{\pi}(\beta)}\pi^2 - (L^\phi_{\tilde{\pi}(\beta)}\pi)(d^\phi_{\mathcal{L}}\alpha)\pi - \pi(d^\phi_{\mathcal{L}}\alpha)L^\phi_{\tilde{\pi}(\beta)}\pi \\
&-\frac{1}{2}\phi(\tilde{\pi}(\alpha))d^\phi_{\mathcal{L}}\beta\lrcorner\pi^2 + \frac{1}{2}\phi(\tilde{\pi}(\beta))d^\phi_{\mathcal{L}}\alpha\lrcorner\pi^2 .
\end{aligned}
$$

The sum of the 2nd and 5th term of the very right hand side of the equations above is zero as we see from the assumption

$$
\begin{aligned}
&(L^\phi_{\tilde{\pi}(\alpha)}\pi)(d^\phi_{\mathcal{L}}\beta)\pi - (L^\phi_{\tilde{\pi}(\beta)}\pi)(d^\phi_{\mathcal{L}}\alpha)\pi \\
=&(-\frac{1}{2}d^\phi_{\mathcal{L}}\beta\lrcorner d^\phi_{\mathcal{L}}\alpha\lrcorner\pi^2 + \pi(d^\phi_{\mathcal{L}}\alpha)\pi(d^\phi_{\mathcal{L}}\beta))\pi - (-\frac{1}{2}d^\phi_{\mathcal{L}}\alpha\lrcorner d^\phi_{\mathcal{L}}\beta\lrcorner\pi^2 + \pi(d^\phi_{\mathcal{L}}\beta)\pi(d^\phi_{\mathcal{L}}\alpha))\pi \\
=&0
\end{aligned}
$$

and also from the assumption the sum of 3rd term and 6th term becomes

$$
-\frac{1}{2}\pi(d^\phi_{\mathcal{L}}\beta)d^\phi_{\mathcal{L}}\alpha\lrcorner\pi^2 + \frac{1}{2}\pi(d^\phi_{\mathcal{L}}\alpha)d^\phi_{\mathcal{L}}\beta\lrcorner\pi^2 .
$$

Thus,

$$
\begin{aligned}
&\{\alpha,\beta\}_\pi^\phi \lrcorner [\pi,\pi]^\phi \\
=&-d_{\mathcal{L}}^\phi\beta \lrcorner (2[\tilde{\pi}(\alpha),\pi]^\phi \wedge \pi - \phi(\tilde{\pi}(\alpha))\pi^2) + d_{\mathcal{L}}^\phi\alpha \lrcorner (2[\tilde{\pi}(\beta),\pi]^\phi \wedge \pi - \phi(\tilde{\pi}(\beta))\pi^2) \\
&-\pi(d_{\mathcal{L}}^\phi\beta)d_{\mathcal{L}}^\phi\alpha \lrcorner \pi^2 + \pi(d_{\mathcal{L}}^\phi\alpha)d_{\mathcal{L}}^\phi\beta \lrcorner \pi^2 - \phi(\tilde{\pi}(\alpha))d_{\mathcal{L}}^\phi\beta \lrcorner \pi^2 + \phi(\tilde{\pi}(\beta))d_{\mathcal{L}}^\phi\alpha \lrcorner \pi^2 \\
=&d_{\mathcal{L}}^\phi\beta \lrcorner \Big((d_{\mathcal{L}}^\phi\alpha \lrcorner \pi^2)\wedge\pi - 2\pi(d_{\mathcal{L}}^\phi\alpha)\pi^2\Big) - d_{\mathcal{L}}^\phi\alpha \lrcorner \Big((d_{\mathcal{L}}^\phi\beta \lrcorner \pi^2)\wedge\pi - 2\pi(d_{\mathcal{L}}^\phi\beta)\pi^2\Big) \\
&-\pi(d_{\mathcal{L}}^\phi\beta)d_{\mathcal{L}}^\phi\alpha \lrcorner \pi^2 + \pi(d_{\mathcal{L}}^\phi\alpha)d_{\mathcal{L}}^\phi\beta \lrcorner \pi^2 \\
=&d_{\mathcal{L}}^\phi\beta \lrcorner \Big((d_{\mathcal{L}}^\phi\alpha \lrcorner \pi^2)\wedge\pi\Big) - \pi(d_{\mathcal{L}}^\phi\alpha)(d_{\mathcal{L}}^\phi\beta \lrcorner \pi^2) - d_{\mathcal{L}}^\phi\alpha \lrcorner \Big((d_{\mathcal{L}}^\phi\beta \lrcorner \pi^2)\wedge\pi\Big) \\
&+\pi(d_{\mathcal{L}}^\phi\beta)(d_{\mathcal{L}}^\phi\alpha \lrcorner \pi^2)\ .
\end{aligned}
$$

We claim that the above is identically zero. To prove this, it suffices to verify in the case when $d_{\mathcal{L}}^\phi\alpha = \theta_1 \wedge \theta_2$ and $d_{\mathcal{L}}^\phi\beta = \eta_1 \wedge \eta_2$. By a direct and lengthy computation, we can verify that the above actually vanishes.

Proof of the Jacobi identity: Let $\alpha,\beta,\gamma \in ([\pi,\pi]^\phi)^0$. Using the definition of the bracket, we see that

$$
\begin{aligned}
\{\alpha,\{\beta,\gamma\}_\pi^\phi\}_\pi^\phi &= \tilde{\pi}(\alpha) \lrcorner d_{\mathcal{L}}^\phi\{\beta,\gamma\}_\pi^\phi - L^\phi_{\tilde{\pi}(\{\beta,\gamma\}_\pi^\phi)}\alpha \\
&= L^\phi_{\tilde{\pi}(\alpha)}\{\beta,\gamma\}_\pi^\phi - d_{\mathcal{L}}^\phi(\tilde{\pi}(\alpha) \lrcorner \{\beta,\gamma\}_\pi^\phi) - L^\phi_{\tilde{\pi}(\{\beta,\gamma\}_\pi^\phi)}\alpha
\end{aligned}
$$

using Lemma 3.3

$$
\begin{aligned}
&= L^{\phi}_{\tilde{\pi}(\alpha)}\left(\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma - L^{\phi}_{\tilde{\pi}(\gamma)}\beta\right) - d^{\phi}_{\mathcal{L}}\left(\tilde{\pi}(\alpha)\lrcorner(\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma - L^{\phi}_{\tilde{\pi}(\beta)}\gamma)\right) \\
&\quad - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha \\
&= L^{\phi}_{\tilde{\pi}(\alpha)}\left(L^{\phi}_{\tilde{\pi}(\beta)}\gamma - d^{\phi}_{\mathcal{L}}(\tilde{\pi}(\beta)\lrcorner\gamma)\right) - L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta \\
&\quad - d^{\phi}_{\mathcal{L}}\left(\tilde{\pi}(\alpha)\lrcorner\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma - \tilde{\pi}(\alpha)\lrcorner L^{\phi}_{\tilde{\pi}(\beta)}\gamma\right) - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha \\
&= L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma - L^{\phi}_{\tilde{\pi}(\alpha)}d^{\phi}_{\mathcal{L}}(\pi(\beta,\gamma)) - L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta \\
&\quad - d^{\phi}_{\mathcal{L}}\left(\tilde{\pi}(\alpha)\lrcorner\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma - \tilde{\pi}(\alpha)\lrcorner L^{\phi}_{\tilde{\pi}(\beta)}\gamma\right) - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha \\
&= L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma - L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha \\
&\quad - d^{\phi}_{\mathcal{L}}\left(L^{\phi}_{\tilde{\pi}(\alpha)}(\pi(\beta,\gamma)) + \tilde{\pi}(\alpha)\lrcorner\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma - \tilde{\pi}(\alpha)\lrcorner L^{\phi}_{\tilde{\pi}(\beta)}\gamma\right) \\
&= L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma - L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha \\
&\quad - d^{\phi}_{\mathcal{L}}\Big(\tilde{\pi}(\alpha)\lrcorner d^{\phi}_{\mathcal{L}}(\pi(\beta,\gamma)) + \tilde{\pi}(\alpha)\lrcorner\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma \\
&\qquad - \tilde{\pi}(\alpha)\lrcorner\tilde{\pi}(\beta)\lrcorner d^{\phi}_{\mathcal{L}}\gamma - \tilde{\pi}(\alpha)\lrcorner d^{\phi}_{\mathcal{L}}(\tilde{\pi}(\beta)\lrcorner\gamma)\Big) \\
&= L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma - L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha
\end{aligned}
$$

and so we have

$$\underset{\alpha,\beta,\gamma}{\mathfrak{S}}\{\alpha,\{\beta,\gamma\}^{\phi}_{\pi}\}^{\phi}_{\pi} = \underset{\alpha,\beta,\gamma}{\mathfrak{S}}\left((L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma - L^{\phi}_{\tilde{\pi}(\beta)}L^{\phi}_{\tilde{\pi}(\alpha)}\gamma) - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha\right) = 0$$

using $L^{\phi}_X \circ L^{\phi}_Y - L^{\phi}_Y \circ L^{\phi}_X = L^{\phi}_{[X,Y]^{\phi}}$ on $\Gamma(\Lambda^{\bullet}\mathcal{L}^*)$ for each vector fields X and Y, which is true by virtue of the closedness of ϕ.

The anchor for Lie algebroid: Since

$$
\begin{aligned}
L^{\phi}_X(f\beta) &= (L^{\phi}_X f)\beta + fL^{\phi}_X\beta - \langle\phi, X\rangle f\beta \\
L^{\phi}_{fX}\alpha &= fL^{\phi}_X\alpha + (X\lrcorner\alpha)d_{\mathcal{L}}f
\end{aligned}
$$

we have

$$
\begin{aligned}
\{\alpha, f\beta\}^{\phi}_{\pi} =& (L^{\phi}_{\tilde{\pi}(\alpha)}f)\beta + fL^{\phi}_{\tilde{\pi}(\alpha)}\beta - \langle\phi,\tilde{\pi}(\alpha)\rangle f\beta - (fL^{\phi}_{\tilde{\pi}(\beta)}\alpha + (\tilde{\pi}(\beta))\lrcorner\alpha)d_{\mathcal{L}}f \\
&- (fd_{\mathcal{L}}(\pi(\alpha,\beta)) + \pi(\alpha,\beta)d_{\mathcal{L}}f - f\pi(\alpha,\beta)\phi) \\
=& f\{\alpha,\beta\}^{\phi}_{\pi} + \text{Rest},
\end{aligned}
$$

where

$$\begin{aligned}\text{Rest} =&(L^{\phi}_{\tilde{\pi}(\alpha)}f)\beta - \langle\phi,\tilde{\pi}(\alpha)\rangle f\beta - (\tilde{\pi}(\beta) \lrcorner \alpha)d_{\mathcal{L}}f - (\pi(\alpha,\beta)d_{\mathcal{L}}f - f\pi(\alpha,\beta)\phi)\\ =&\langle\tilde{\pi}(\alpha), d_{\mathcal{L}}f\rangle\beta = \langle a(\tilde{\pi}(\alpha)), df\rangle\beta\ .\end{aligned}$$

Thus, we have

$$\{\alpha, f\beta\}^{\phi}_{\pi} = f\{\alpha,\beta\}^{\phi}_{\pi} + \langle a(\tilde{\pi}(\alpha)), df\rangle\beta\ .$$

This shows $a \circ \tilde{\pi}$ is the anchor for Lie algebroid $(([\pi,\pi]^{\phi})^0, \{\cdot,\cdot\}^{\phi}_{\pi})$.

Corresponding 1-cocycle: We will verify $\phi \circ \tilde{\pi}$ is a 1-cocycle on $(([\pi,\pi]^{\phi})^0, \{\cdot,\cdot\}^{\phi}_{\pi}, a\circ\tilde{\pi})$. Put here $\phi\circ\tilde{\pi}$ by φ. We have to show

$$\varphi(\{\alpha,\beta\}^{\phi}_{\pi}) = L_{b(\alpha)}(\varphi(\beta)) - L_{b(\beta)}(\varphi(\alpha)) \qquad \text{for each } \alpha,\beta \in ([\pi,\pi]^{\phi})^0\ .$$

The right hand side is reduced as

$$\text{RHS} = L_{a(\tilde{\pi}(\alpha))}(\phi\tilde{\pi}(\beta)) - L_{a(\tilde{\pi}(\beta))}(\phi\tilde{\pi}(\alpha)) = \phi([\tilde{\pi}(\alpha),\tilde{\pi}(\beta)])$$

because of ϕ being closed. Concerning to the left hand side, we have

$$\begin{aligned}\text{LHS} =&(\phi\circ\tilde{\pi})\{\alpha,\beta\}^{\phi}_{\pi}\\ =&\phi([\tilde{\pi}(\alpha),\tilde{\pi}(\beta)] - \frac{1}{2}[\pi,\pi]^{\phi}(\alpha,\beta)) \qquad \text{using (3.6)}\\ =&\phi([\tilde{\pi}(\alpha),\tilde{\pi}(\beta)]\end{aligned}$$

because of $\alpha,\beta \in ([\pi,\pi]^{\phi})^0$. Thus we have checked the equality of the both sides, and $\bar{a}\circ\tilde{\pi}$ is the anchor for the Jacobi-Lie algebroid. □

Remark 3.2 In the proof above, we see that if ϕ is exact, then the corresponding 1-cocycle is also exact. In fact, assume $\phi = d_{\mathcal{L}}f$ for some f, i.e., $\langle\phi, X\rangle = \langle d_{\mathcal{L}}f, X\rangle = \langle df, a(X)\rangle$ for each $X \in \Gamma(\mathcal{L})$. Then, we have $\langle\varphi,\alpha\rangle = \langle\phi\tilde{\pi},\alpha\rangle = \langle\phi,\tilde{\pi}(\alpha)\rangle = \langle df, a(\tilde{\pi}(\alpha))\rangle$.

3.1 An example

We show an example on the 5-dimensional Euclidean space $\mathbb{R}^5$ with the Cartesian coordinates $(x^1, \ldots, x^5)$, which tell us some difference between ordinary bracket and the deformed one. Since the space is simply-connected, every closed 1-form is exact, and every closed 1-form ϕ is of form $\phi = df = \sum_{j=1}^{5} \frac{\partial f}{\partial x^j} dx^j = \sum_{j=1}^{5} f_j dx^j$ for some function f, where $f_j = \frac{\partial f}{\partial x^j}$.

Take the frame field $\{Z_1, \ldots, Z_5\}$ defined by

$$
\begin{aligned}
Z_1 &= \frac{\partial}{\partial x^1} - \frac{x^2}{2}\frac{\partial}{\partial x^5}, \quad Z_2 = \frac{\partial}{\partial x^2} + \frac{x^1}{2}\frac{\partial}{\partial x^5}, \\
Z_3 &= \frac{\partial}{\partial x^3} - \frac{x^4}{2}\frac{\partial}{\partial x^5}, \quad Z_4 = \frac{\partial}{\partial x^4} + \frac{x^3}{2}\frac{\partial}{\partial x^5}, \quad Z_5 = \frac{\partial}{\partial x^5}.
\end{aligned}
$$

Then, Z_5 is a central element and the bracket relations are given by

$$
[Z_1, Z_2] = -[Z_2, Z_1] = [Z_3, Z_4] = -[Z_4, Z_3] = Z_5
$$

and all the other brackets vanish.

Let us consider a 2-vector field π:

$$
\pi = a^{12} Z_1 \wedge Z_2 + a^{13} Z_1 \wedge Z_3 + a^{15} Z_1 \wedge Z_5 + a^{23} Z_2 \wedge Z_3 + a^{25} Z_2 \wedge Z_5 + a^{35} Z_3 \wedge Z_5
$$

where $\{a^{ij}\}$ are constant. The rank of π is 4 if and only if $\Delta := a^{12}a^{35} - a^{13}a^{25} + a^{15}a^{23} \neq 0$. Hereafter, we assume that π is of rank 4. We have the following calculation:

$$
\begin{aligned}
[\pi, \pi] &= 2a^{12}\left(a^{12} Z_1 \wedge Z_2 + a^{13} Z_1 \wedge Z_3 + a^{23} Z_2 \wedge Z_3\right) \wedge Z_5 \\
&= 2a^{12}\left(a^{12}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + a^{13}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + a^{23}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}\right) \wedge \frac{\partial}{\partial x^5}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}[\pi, \pi]^\phi &= \frac{1}{2}[\pi, \pi] + \tilde{\pi}(\phi) \wedge \pi \\
&= -f_5 \Delta \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + \left(a^{12}a^{12} + f_3\Delta\right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^5} \\
&\quad + \left(a^{12}a^{13} - f_2\Delta\right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^5} + \left(a^{12}a^{23} + f_1\Delta\right) \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^5}.
\end{aligned}
$$

These equations above imply that $[\pi, \pi] = 0$ if and only if $a^{12} = 0$, and $[\pi, \pi]^\phi = [\pi, \pi]$ if and only if $\phi = df$ with $f_1 = f_2 = f_3 = f_5 = 0$ for some function f, and $[\pi,\pi]^\phi = 0$ if and only if

$$f_1 = -\frac{1}{\Delta}a^{12}a^{23}\ , \quad f_2 = \frac{1}{\Delta}a^{12}a^{13}\ , \quad f_3 = -\frac{1}{\Delta}a^{12}a^{12}\ , \quad f_5 = 0\ . \tag{3.7}$$

Now, we consider the following special cases.

(Case 1) If $a^{12} = 0$ and $\Delta \neq 0$, then $[\pi,\pi] = 0$, and so $[\pi,\pi]^0$ is the whole cotangent bundle of $\mathbb{R}^5$ and $\dim \tilde{\pi}([\pi,\pi]^0) = 4$, $\tilde{\pi}([\pi,\pi]^0) = \mathrm{Im}\tilde{\pi}$. Choose $\phi = df$ with $f_1 \neq 0$ and $f_2 = f_3 = 0$. Then $([\pi,\pi]^\phi)^0$ is spanned by ϕ and dx^4. $\tilde{\pi}(([\pi,\pi]^\phi)^0)$ is spanned by

$$\begin{aligned}\tilde{\pi}(df) =& f_5(\frac{x^4}{2}a^{13} - a^{15})\frac{\partial}{\partial x^1} + f_5(\frac{x^4}{2}a^{23} - a^{25})\frac{\partial}{\partial x^2} \\ &+ \left(f_1 a^{13} + f_5(\frac{x^1}{2}a^{23} - \frac{x^2}{2}a^{13} - a^{35})\right)\frac{\partial}{\partial x^3} - f_1(\frac{x^4}{2}a^{13} - a^{15})\frac{\partial}{\partial x^5}\end{aligned}$$

and we see that this does never vanish from the assumption $\Delta \neq 0$. Thus, $\tilde{\pi}(([\pi,\pi]^\phi)^0)$ is of dimension 1.

(Case 2) Assume $a^{12} \neq 0$ and $\Delta \neq 0$. For example, choose $a^{12} = a^{35} = 1, a^{13} = a^{23} = 0$. Then $[\pi,\pi] = 2\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^5}$ and so $[\pi,\pi]^0$ is spanned by dx^3 and dx^4. Thus, $\tilde{\pi}([\pi,\pi]^0)$ is spanned by $\frac{\partial}{\partial x^5}$ and $\dim \tilde{\pi}([\pi,\pi]^0) = 1$. According to the condition (3.7), if we choose $f_1 = f_2 = 0$, $f_3 = 1$ and $f_5 = 0$ then $[\pi,\pi]^\phi = 0$ and so $([\pi,\pi]^\phi)^0$ is the whole cotangent bundle and $\tilde{\pi}(([\pi,\pi]^\phi)^0) = \mathrm{Im}\tilde{\pi}$ is of dimension 4, which is spanned by Z_1, Z_2, Z_3, Z_5.

If we choose $f_1 \neq -a^{12}a^{23}/\Delta$, (i.e., $f \neq 0$ right now), $f_2 = f_5 = 0$, and $f_3 = 1$, then $[\pi,\pi]^\phi = 2f_1\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^5} \neq 0$. $([\pi,\pi]^\phi)^0$ is spanned by dx^1 and dx^4. Since $\tilde{\pi}(dx^1) = \frac{\partial}{\partial x^2} + (a^{15} + \frac{x^1}{2})\frac{\partial}{\partial x^5}$ and $\tilde{\pi}(dx^4) = 0$, we see that $\tilde{\pi}(([\pi,\pi]^\phi)^0)$ is 1-dimensional.

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Lie Algebroid Associated with an Almost Dirac Structure

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Abstract

We show that to an almost Dirac structure of a manifold, there associates a Lie algebroid. In the case of a Poisson manifold, this Lie algebroid coincides with the usual cotangent Lie algebroid with Lie algebra bracket on the space of one-forms.

1 Introduction

Let π be an arbitrary 2-vector field on M, i.e. a smooth section of $\wedge^2(\mathrm{T}M)$. We denote by $\tilde{\pi}$, the bundle homomorphism $\mathrm{T}^*M \to \mathrm{T}M$ defined by $\alpha_x \mapsto \pi(\alpha_x, \cdot)$ $(x \in M)$. By an abuse of notations, we denote by the same letter $\tilde{\pi}$, the homomorphism $\Gamma(\mathrm{T}^*M) \to \Gamma(\mathrm{T}M)$ between sections. For the Schouten bracket $[\pi, \pi]$ of π, which is a 3-vector field, we define $\ker[\pi,\pi] = \{\alpha \in \mathrm{T}^*M \mid [\pi,\pi](\alpha,\cdot,\cdot) = 0\}$. If $\ker[\pi,\pi]$ forms a bundle of constant rank, it was proved in [7] that $\ker[\pi,\pi]$ becomes a Lie algebroid with respect to the bracket $L_{\tilde{\pi}(\alpha)}\beta - L_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha,\beta))$ and the anchor $\rho(\alpha) = \tilde{\pi}(\alpha)$. Clearly, it coincides with the usual Lie algebroid structure of T^*M of a Poisson manifold (M, π), where $[\pi,\pi] = 0$. On the other hand, the graph of $\tilde{\pi} : \mathrm{T}^*M \to \mathrm{T}M$ defines a sub-bundle of $\mathrm{T}M \oplus \mathrm{T}^*M$, which is an almost Dirac structure (see Section 1), and $\ker[\pi,\pi]$ can be identified with a subset of this almost Dirac structure. The aim of this paper is, generalizing the above result, to show that a certain sub-bundle $\mathcal{L}_0$ of an almost Dirac structure is a Lie algebroid with respect to the bracket and the anchor, which are naturally defined on the almost Dirac structure (Theorem 2.1). The sub-bundle $\mathcal{L}_0$ is given as the kernel of the 3-tensor field T restricted to the almost Dirac structure, introduced in [1] (see Definition 2.2).

In Section 1, we review some basic facts on Dirac structures and prove that $\mathcal{L}_0$ is a Lie algebroid. In Section 2, in order to clarify the conditions under which an element belongs to $\ker T$, we use the description of an almost Dirac structure by means of a "2-vector field on a sub-bundle of T^*M". In Section 3, we give a 'dual' description of Dirac structures in which we use "2-forms" defined on a sub-bundle of $\mathrm{T}M$. We also give simple examples.

It is possible to generalize our result in the case of deformed bracket in [4] or [3], and also seems highly possible in the case of the twisted Poisson structures [8]. However, we restricted ourselves to the case of the ordinary Dirac structures in order to make the arguments and the computations clear. We hope interesting examples will come about from the further generalizations.

2 Dirac Structures

Let $\mathrm{T}(M)$ and $\mathrm{T}^*(M)$ be the tangent and the cotangent bundle of M, respectively. Let $\langle\cdot,\cdot\rangle_+$ be the symmetric pairing on $\mathrm{T}(M)\oplus\mathrm{T}^*(M)$ defined by

$$\langle(X_x,\alpha_x),(Y_x,\beta_x)\rangle_+ = \alpha_x(Y_x)+\beta_x(Y_x),\quad (X_x,\alpha_x),(Y_x,\beta_x)\in \mathrm{T}_xM\oplus\mathrm{T}^*_xM.$$

Definition 2.1 (T. Courant). A smooth sub-bundle $\mathcal{L}\subset \mathrm{T}(M)\oplus\mathrm{T}^*(M)$ is an almost Dirac structure if $\mathcal{L}$ is maximally isotropic with respect to the pairing $\langle\cdot,\cdot\rangle_+$. This means $\mathcal{L}$ is a sub-bundle of rank $n(=\dim M)$ and the restriction of $\langle\cdot,\cdot\rangle_+$ to $\mathcal{L}\times\mathcal{L}$ vanishes identically.

Remark 2.1. In [1], an almost Dirac structure is called a Dirac structure, however we use the word *Dirac structure* to mean the one which was called an *integrable* Dirac structure in [1].

On $\Gamma(\mathrm{T}(M)\oplus\mathrm{T}^*(M))$, we have a bracket defined by
(2.1)
$$[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!] = \left([X_1,X_2]\,,\ L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \frac{1}{2}d\left(\alpha_1(X_2)-\alpha_2(X_1)\right)\right)$$

where $[X_1,X_2]$ is the usual Lie bracket of vector fields and $L_X\alpha$ is the Lie derivative of 1-form α with respect to the vector field X.

The bracket $[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!]$ is skew-symmetric but does not satisfy the Jacobi identity. Indeed, let (J_1,J_2) denote the Jacobiator

$$(J_1,J_2) = [\![[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!],(X_3,\alpha_3)]\!] + c.p.\ .$$

Clearly $J_1=0$. As for J_2, however, we have

Proposition 2.1. *The second component J_2 of the above Jacobiator is given by*

$$J_2 = \frac{1}{4}d\left(2\alpha_1([X_2,X_3]) + L_{X_1}\left(\alpha_2(X_3)-\alpha_3(X_2)\right)\right) + c.p.\ .$$

Especially, the restriction of J_2 to an almost Dirac structure $\mathcal{L}$ is

$$\frac{1}{2}d\big(\alpha_1([X_2,X_3]) + L_{X_1}(\alpha_2(X_3))\big) + c.p.\ .$$

Proof. This is shown directly from the definitions of $\langle\cdot,\cdot\rangle_+$ and $[\![\cdot,\cdot]\!]$. □

Definition 2.2. An almost Dirac structure $\mathcal{L}$ is called a(n) (*integrable*) *Dirac structure* if $\Gamma(\mathcal{L})$ is closed under the bracket $[\![\cdot,\cdot]\!]$.

In [1], Courant introduced the $\boldsymbol{R}$-tri-linear map on $\mathrm{T}(M) \oplus \mathrm{T}^*(M)$ to $\boldsymbol{R}$ defined by $T((X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3)) = \langle [\![(X_1, \alpha_1), (X_2, \alpha_2)]\!], (X_3, \alpha_3) \rangle_+$ for $(X_i, \alpha_i) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M)$ $(i = 1, 2, 3)$, and showed that an almost Dirac structure is integrable if and only if T restricted to $\mathcal{L}$ vanishes: $T|_{\mathcal{L}} \equiv 0$. We note that the restriction $T|_{\mathcal{L}}$ has the tensor property. That is $T|_{\mathcal{L}}$ is tri-linear over $C^\infty(M)$.

Proposition 2.2. *Let $\mathcal{L}$ be an almost Dirac structure. Then $T|_{\mathcal{L}}$, T restricted to $\mathcal{L}$, is computed as*

$$T|_{\mathcal{L}}((X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3)) = \big(\alpha_1([X_2, X_3]) + L_{X_1}(\alpha_2(X_3))\big) + c.p.$$

and

$$J_2|_{\mathcal{L}}((X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3)) = \frac{1}{2} d\left(T|_{\mathcal{L}}((X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3))\right).$$

Proof. On $\mathcal{L}$, we have

$$\begin{aligned}
&\langle [\![(X_1, \alpha_1), (X_2, \alpha_2)]\!], (X_3, \alpha_3) \rangle_+ \\
&\quad = (L_{X_1}\alpha_2)(X_3) - (L_{X_2}\alpha_1)(X_3) - \frac{1}{2} L_{X_3}(\alpha_2(X_1) - \alpha_1(X_2)) + \alpha_3([X_1, X_2]) \\
&\quad = L_{X_1}(\alpha_2(X_3)) - \alpha_2([X_1, X_3]) - L_{X_2}(\alpha_1(X_3)) + \alpha_1([X_2, X_3]) \\
&\qquad\qquad - L_{X_3}(\alpha_2(X_1)) + \alpha_3([X_1, X_2]).
\end{aligned}$$

This together with Proposition 2.1 shows Proposition 2.2. □

Let $\mathcal{L}$ be an almost Dirac structure. We consider the 'sub-bundle' $\mathcal{L}_0$ of $\mathcal{L}$ consisting of the elements in $\ker T|_{\mathcal{L}}$. More precisely, we put

$$\mathcal{L}_0 = \{ e = (Z, \gamma) \in \mathcal{L} \mid T(e_1, e_2, e) = 0, \ e_1, e_2 \in \mathcal{L} \}.$$

Since T restricted to $\mathcal{L}$, is skew-symmetric with respect to all the arguments, $\mathcal{L}_0$ can be considered as the kernel of the bundle map $T : \mathcal{L} \to \wedge^2 \mathcal{L}^*$, $e \mapsto T(\cdot, \cdot, e)$. Since the fiber dimension of $\mathcal{L}_0$ may change from point to point, to get a Lie algebroid, we have to restrict $\mathcal{L}_0$ to a submanifold of M where $\mathcal{L}_0$ is of constant rank. Hereafter, for simplicity, *we assume that $\mathcal{L}_0$ is a bundle of constant rank* on whole M. The following proposition is obvious from Proposition 2.2.

Proposition 2.3. *If one of e_1, e_2, e_3 in $\Gamma(\mathcal{L})$ is an element in $\Gamma(\mathcal{L}_0)$, we have the Jacobi identity:*

$$[\![[\![e_1, e_2]\!], e_3]\!] + [\![[\![e_2, e_3]\!], e_1]\!] + [\![[\![e_3, e_1]\!], e_2]\!] = 0.$$

The following proposition is used to show that $\mathcal{L}_0$ is closed under the bracket $[\![\cdot, \cdot]\!]$.

Proposition 2.4. *For* $e = (Z, \gamma) \in \Gamma(\mathcal{L}_0)$ *and* $e_1 = (Y, \beta) \in \Gamma(\mathcal{L})$, *we have* $[\![e, e_1]\!] \in \Gamma(\mathcal{L})$.

Proof. Since T restricted to $\mathcal{L}$ is skew symmetric, we have $\langle [\![e, e_1]\!], e_2\rangle_+ = T(e, e_1, e_2) = T(e_1, e_2, e) = 0$, for any $e_1, e_2 \in \Gamma(\mathcal{L})$. By the maximality of $\mathcal{L}$, we can conclude $[\![e, e_1]\!]$ is in $\Gamma(\mathcal{L})$. □

By the above propositions, we obtain the following theorem.

Theorem 2.1. *Let* $\mathcal{L}$ *be an almost Dirac structure and* $\mathcal{L}_0$ *the kernel of* T, *which we assume a sub-bundle of* $\mathcal{L}$. *Then* $\mathcal{L}_0$ *is a Lie algebroid with respect to the bracket* $[\![\cdot,\cdot]\!]$ *and the anchor* $\rho_{\mathcal{L}_0}$, *which is the natural projection* $\rho : \mathrm{T}(M) \oplus \mathrm{T}^*(M) \to \mathrm{T}(M)$ *restricted to* $\mathcal{L}_0$.

Proof. Let e_1, e_2 be two elements of $\Gamma(\mathcal{L}_0)$. Then for any e_3 and e_4 in $\Gamma(\mathcal{L})$, we have

$$\begin{aligned} T([\![e_1, e_2]\!], e_3, e_4) &= \langle [\![[\![e_1, e_2]\!], e_3]\!], e_4\rangle_+ = \langle [\![[\![e_1, e_3]\!], e_2]\!] + [\![e_1, [\![e_2, e_3]\!]]\!], e_4\rangle_+ \\ &= T([\![e_1, e_3]\!], e_2, e_4) + T(e_1, [\![e_2, e_3]\!], e_4) = 0. \end{aligned}$$

The second equality holds because of the Jacobi identity (Proposition 2.3) for e_1, e_2, e_3 and the last one is true because $[\![e_1, e_3]\!], [\![e_2, e_3]\!]$ are both in $\Gamma(\mathcal{L})$ by Proposition 2.4. This shows that $\Gamma(\mathcal{L}_0)$ is closed under the bracket. Since the Jacobi identity is obvious for the elements in $\Gamma(\mathcal{L}_0)$(Proposition 2.3), $[\![\cdot,\cdot]\!]$ is a Lie algebra bracket on $\Gamma(\mathcal{L}_0)$. That $\rho_{\mathcal{L}_0}$ satisfies the condition of an anchor map is also verified directly from the definition (2.1) of $[\![\cdot,\cdot]\!]$. □

3 An alternative description of a Dirac structure

In this section and the next, we give alternative descriptions of an almost Dirac structure and give more explicit conditions for an element in $\mathcal{L}$ to be in the kernel of 3-tensor $T|_{\mathcal{L}}$.

Let $\mathcal{L}$ be an almost Dirac structure on M and $\rho_{\mathcal{L}}$ and $\rho^*_{\mathcal{L}}$ denote the restriction of the natural projections $\mathrm{T}(M) \oplus \mathrm{T}^*(M) \to \mathrm{T}(M)$ and $\mathrm{T}(M) \oplus \mathrm{T}^*(M) \to \mathrm{T}^*(M)$ to $\mathcal{L}$, respectively. We put $\mathcal{E} = \mathrm{Im}\rho_{\mathcal{L}}$ and $\mathcal{A} = \mathrm{Im}\rho^*_{\mathcal{L}}$. As was remarked before, the fiber rank of $\mathcal{E}$ as well as $\mathcal{A}$, may not be constant. To justify our computations we only treat the case when $\mathcal{E}$ and $\mathcal{A}$ are bunldes of constant rank.

Take an element $\alpha \in \mathcal{A}_x$ (the fiber over $x \in M$), then for some $X \in \mathrm{T}(M)$, (X, α) lies in $\mathcal{L}$. That $\mathcal{L}$ is isotropic implies that the restriction $X|_{\mathcal{A}_x}$ of X considered as an element in $\mathcal{A}^*_x$ (dual space) depends only on α and we obtain an well-defined fiber map $\pi : \mathcal{A} \to \mathcal{A}^*$ (see [1]). We may consider π as a '2-vector field' defined on (each fiber of) $\mathcal{A}$. It is skew-symmetric since for $\alpha, \beta \in \mathcal{A}$, we have

$$\pi(\alpha, \beta) = X(\beta) = -Y(\alpha) = -\pi(\beta, \alpha), \quad \text{where } (Y, \beta) \in \mathcal{L}.$$

From the sub-bundle $\mathcal{A} = \mathrm{Im}\rho^*_{\mathcal{L}}$ and a skew-symmetric 2-field π on $\mathcal{A}$, we can recover $\mathcal{L}$ as a bundle given by

$$\mathcal{L}' = \{(X, \theta) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M) \mid \theta \in \mathcal{A},\ \tilde{\pi}(\theta) = X|_{\mathcal{A}}\}.$$

Indeed it is easy to see that $\mathcal{L}'$ is a vector bundle of rank $n(= \dim M)$. That $\mathcal{L}'$ is isotropic with respect to $\langle\cdot,\cdot\rangle_+$ follows from the skewness of π. If $(X', \alpha') \in \mathcal{L}$, we see

$$\begin{aligned}\langle(X', \alpha'), (X, \theta)\rangle_+ =&\alpha'(X) + \theta(X') = \pi(\theta, \alpha') + \theta(X') \\ =&-\pi(\alpha', \theta) + \theta(X') = -X'(\theta) + \theta(X') = 0\end{aligned}$$

for $(X, \theta) \in \mathcal{L}'$. This together with the maximality of $\mathcal{L}$ implies $\mathcal{L} = \mathcal{L}'$.

Now we are going to characterize the element of $\mathcal{L}_0$ in terms of π and $\mathcal{A}$, where $\mathcal{L}_0$ is the sub-bundle $\ker T|_{\mathcal{L}}$ of $\mathcal{L}$. First, we observe that $\mathcal{A}^*$ is a quotient bundle of $\mathrm{T}(M)$ by the sub-bundle $\mathcal{A}^\circ$, where $\mathcal{A}^\circ$ is the bundle consisting of the annihilators of $\mathcal{A}$. π is an element $\wedge^2\mathcal{A}^*$, however we choose and fix a splitting to the projection $\mathrm{T}(M) \to \mathcal{A}^*$, and consider $\mathcal{A}^*$ as a direct summand of $\mathrm{T}(M)$, obtaining a 2-vector field which extends π. This is possible since we are assuming $\mathcal{A}$ is of constant rank. We denote this extended 2-vector field by the same letter π, since we hope this will not cause any confusion. Then $\mathcal{L}$ is given by

$$\text{(3.1)}\quad \mathcal{L} = \{(X, \alpha) \mid \alpha \in \mathcal{A}, \tilde{\pi}(\alpha) = X|_{\mathcal{A}}\} = \{(\tilde{\pi}(\alpha) + \bar{X}, \alpha) \mid \alpha \in \mathcal{A}, \bar{X} \in \mathcal{A}^\circ\}.$$

For $e_1 = (X, \alpha), e_2 = (Y, \beta)$ and $e_3 = (Z, \gamma)$ in $\mathcal{L}$, we look for the condition on e_3 under which $T(e_1, e_2, e_3) = 0$ holds for all $e_1, e_2 \in \mathcal{L}$. We can write $e_1 = (\tilde{\pi}(\alpha) + \bar{X}, \alpha), e_2 = (\tilde{\pi}(\beta) + \bar{Y}, \beta)$ and $e_3 = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$, respectively, where $\bar{X}, \bar{Y}, \bar{Z} \in \mathcal{A}^\circ$. With these notations, we have

$$\begin{aligned}[\![(X, \alpha), (Y, \beta)]\!] = \Big(&[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)] + [\tilde{\pi}(\alpha), \bar{Y}] + [\bar{X}, \tilde{\pi}(\beta)] + [\bar{X}, \bar{Y}], \\ &L_{\tilde{\pi}(\alpha)}\beta + L_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta)) + L_{\bar{X}}\beta - L_{\bar{Y}}\alpha\Big).\end{aligned}$$

Writing $\{\alpha, \beta\}_\pi$ for $L_{\tilde{\pi}(\alpha)}\beta - L_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta))$ and making the pairing $\langle\cdot,\cdot\rangle_+$ of the above element and $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$, we obtain

$$\text{(3.2)}\quad \begin{aligned}&[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)](\gamma) + [\tilde{\pi}(\alpha), \bar{Y}](\gamma) + [\bar{X}, \tilde{\pi}(\beta)](\gamma) + [\bar{X}, \bar{Y}](\gamma) \\ &+ \pi(\gamma, \{\alpha, \beta\}_\pi) + \bar{Z}(\{\alpha, \beta\}_\pi) + \pi(\gamma, L_{\bar{X}}\beta - L_{\bar{Y}}\alpha) + \bar{Z}(L_{\bar{X}}\beta - L_{\bar{Y}}\alpha),\end{aligned}$$

which is nothing but $T(e_1, e_2, e_3)$.

If we choose $\bar{X} = \bar{Y} = 0$, then (3.2) gives

$$\text{(3.3)}\quad [\tilde{\pi}(\alpha), \tilde{\pi}(\beta)](\gamma) + \pi(\gamma, \{\alpha, \beta\}_\pi) + \bar{Z}(\{\alpha, \beta\}_\pi) = 0,$$

for $\alpha, \beta \in \mathcal{A}$. We put $\bar{Y} = 0$ and $\alpha = 0$ in (3.2), we obtain

$$\text{(3.4)}\quad [\bar{X}, \tilde{\pi}(\beta)](\gamma) + \pi(\gamma, L_{\bar{X}}\beta) + \bar{Z}(L_{\bar{X}}\beta) = 0, \quad \bar{X} \in \mathcal{A}^\circ, \beta \in \mathcal{A}.$$

If we put $\alpha = \beta = 0$ into (3.2), we get $\gamma([\bar{X}, \bar{Y}]) = 0$ ($\bar{X}, \bar{Y} \in \mathcal{A}^\circ$). It is easy to see that this is equivalent to

$$L_{\bar{X}}\gamma \in \mathcal{A}, \quad \bar{X} \in \mathcal{A}^\circ. \tag{3.5}$$

Conversely, it can also be seen that if $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$ satisfies conditions (3.3), (3.4) and (3.5) then (3.2) vanishes identically.

In the following, we will simplify the conditions (3.3) and (3.4). First, we note (3.4) is equivalent to the following:

$$(L_{\bar{X}}\pi)(\beta, \gamma) + \pi(L_{\bar{X}}\beta, \gamma) + \pi(\gamma, L_{\bar{X}}\beta) + L_{\bar{X}}(\bar{Z}(\beta)) - [\bar{X}, \bar{Z}](\beta) = 0.$$

Since $\bar{Z}(\beta) = 0$, this means

$$(L_{\bar{X}}\pi)(\gamma) + L_{\bar{X}}\bar{Z} = 0, \quad \text{on } \mathcal{A}. \tag{3.6}$$

To simplify the condition (3.3), we use the following

Lemma 3.1. *For $\bar{Z} \in \mathcal{A}^\circ$ and $\alpha, \beta \in \mathcal{A}$, we have*

$$[\bar{Z}, \pi](\alpha, \beta) = \bar{Z}(\{\alpha, \beta\}_\pi) .$$

Proof. By the definition of $\{\alpha, \beta\}_\pi$, we have

$$\begin{aligned}
\bar{Z}(\{\alpha, \beta\}_\pi) =&(L_{\tilde{\pi}(\alpha)}\beta)(\bar{Z}) - (L_{\tilde{\pi}(\beta)}\alpha)(\bar{Z}) - L_{\bar{Z}}(\pi(\alpha, \beta)) \\
=&L_{\tilde{\pi}(\alpha)}(\beta(\bar{Z})) - \beta(L_{\tilde{\pi}(\alpha)}\bar{Z}) - L_{\tilde{\pi}(\beta)}(\alpha(\bar{Z})) \\
&+ \alpha(L_{\tilde{\pi}(\beta)}\bar{Z}) - L_{\bar{Z}}(\pi(\alpha, \beta)) \\
\text{(since } \alpha(\bar{Z}) = \beta(\bar{Z}) = 0) & \\
=& - \alpha(L_{\bar{Z}}(\tilde{\pi}(\beta))) - \tilde{\pi}(\alpha)(L_{\bar{Z}}\beta) \\
=& - \alpha([\bar{Z}, \tilde{\pi}])(\beta)) = [\bar{Z}, \pi](\alpha, \beta) .
\end{aligned}$$

□

Lemma 3.2. *The condition* (3.3) *for* $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$ *can be replaced by the next equality:*

$$[\tilde{\pi}(\gamma), \tilde{\pi}(\beta)] + (L_{\bar{Z}}\tilde{\pi})(\beta) - \tilde{\pi}(\{\gamma, \beta\}_\pi) = 0, \quad \beta \in \mathcal{A},$$

or equivalently by

$$\frac{1}{2}[\pi, \pi](\gamma) + L_{\bar{Z}}\pi = 0 \quad \textit{on} \quad \mathcal{A} .$$

Proof. By Lemma 3.1, (3.3) can be replaced by

$$[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)](\gamma) + \pi(\gamma, \{\alpha, \beta\}_\pi) + [\bar{Z}, \pi](\alpha, \beta) = 0 \quad \alpha, \beta \in \mathcal{A}. \tag{3.7}$$

Using the general formula for a 2-vector field (see [7], [9])

$$[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)] = \tilde{\pi}(\{\alpha, \beta\}_\pi) + \frac{1}{2}[\pi, \pi](\alpha, \beta) \;, \tag{3.8}$$

we thus rewrite (3.7) as

$$\frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma) + (L_{\bar{Z}}\pi)(\alpha, \beta) = 0 \quad \text{i.e.,} \quad \frac{1}{2}[\pi, \pi](\gamma) + L_{\bar{Z}}\pi = 0 \;.$$

□

From the above lemmas, we can summarize the conditions on $\mathcal{L}_0$ as follows.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{A}^\circ$ be as before and π a skew symmetric bilinear form on $\mathcal{A}$. Let*

$$\mathcal{L} = \{(X, \alpha) \mid \alpha \in \mathcal{A}, \pi(\alpha) = X|_{\mathcal{A}}\} = \{(\tilde{\pi}(\alpha) + \bar{X}, \alpha) \mid \alpha \in \mathcal{A}, \bar{X} \in \mathcal{A}^\circ\}$$

be an almost Dirac structure defined by π. We put

$$\mathcal{L}_0 = \{e = (Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma) \in \mathcal{L} \mid T(e_1, e_2, e) = 0, \; e_1, e_2 \in \mathcal{L}\}.$$

Then $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma) \in \mathcal{L}$ belongs to $\mathcal{L}_0$ if and only if the following conditions (C1), (C2) *and* (C3) *are satisfied:*

(C1) $L_{\bar{X}}\gamma \in \mathcal{A}$, *for all* $\bar{X} \in \mathcal{A}^\circ$,

(C2) $(L_{\bar{X}}\pi)(\gamma) + L_{\bar{X}}\bar{Z} = 0$ *on* $\mathcal{A}$, *for all* $\bar{X} \in \mathcal{A}^\circ$,

(C3) $\frac{1}{2}[\pi, \pi](\gamma) + [\bar{Z}, \pi] = 0$ *on* $\mathcal{A}$.

Example 3.1. Let $\mathcal{A}$ be an arbitrary Pfaffian system. We consider the case when $\pi \equiv 0$. Then

$$\mathcal{L} = \{(X, \alpha) \mid \alpha \in \mathcal{A}, X \in \mathcal{A}^\circ\} \;.$$

(C1) means $L_X\gamma \in \mathcal{A}$ for any $X \in \mathcal{A}^\circ$, and (C2) mean $[X, Z] \in \mathcal{A}^\circ$ for any $X \in \mathcal{A}^\circ$. Clearly, (C3) is vacuous in this case. Thus $\mathcal{L}_0 = \mathrm{Char}(\mathcal{A}) \times \mathcal{A}_1$, where $\mathrm{Char}(\mathcal{A})$ is the Cauchy characteristic of $\mathcal{A}$ and $\mathcal{A}_1$ is the first derived (Pfaffian) system of $\mathcal{A}$, respectively. In particular, if $\mathcal{A}$ is completely integrable and hence $\mathcal{A}$ is the tangent bundle of a foliation $\mathcal{F}$, $\mathcal{L}_0$ is just the product $T\mathcal{F} \times (T\mathcal{F})^\circ$. The bracket in $\mathcal{L}_0$ is given by

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], L_X\beta - L_Y\alpha).$$

Example 3.2 ([7]). We consider the case when $\mathcal{A} = \mathrm{T}^*(M)$ and $\pi : \mathrm{T}^*(M) \to \mathrm{T}(M)$ is an arbitrary 2-vector field. Since $\mathcal{A}^\circ = \{0\}$, the conditions (C1) and (C2) are trivial. (C3) implies $[\pi, \pi](\alpha, \gamma, \cdot) = 0$ for any $\alpha \in \mathrm{T}^*(M)$. Thus, $\mathcal{L}_0 = \{(\tilde{\pi}(\gamma), \gamma) \mid \gamma \in \ker[\pi, \pi]\}$ and $\ker[\pi, \pi]$ is a Lie algebroid with respect to $\{\cdot, \cdot\}_\pi$. This is our previous result in [7].

4 Description by 2-forms

In this section, we describe an almost Dirac structure by a '2-form' on $\mathcal{E} = \rho_{\mathcal{L}}(\mathcal{L}) \subset \mathrm{T}(M)$ and find the conditions which characterize $\mathcal{L}_0$. To justify the compution, we assume $\mathcal{E}$ is of constant rank again.

Let $\omega : \mathcal{E} \to \mathcal{E}^*$ be a skew symmetric bundle homomorphism as before. The almost Dirac structure is given by

$$\mathcal{L} = \{(X, \alpha) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M) \mid i_X\omega = \alpha|_{\mathcal{E}},\ X \in \mathcal{E},\ \alpha \in \mathrm{T}^*(M)\}.$$

The bracket on $\Gamma(\mathcal{L})$ is given by

$$[\![(X_1, \alpha_1), (X_2, \alpha_2)]\!] = ([X_1, X_2]\ ,\ L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + d\,(\omega(X_1, X_2)))\,.$$

Let $e_1 = (X, \alpha), e_2 = (Y, \beta), e_3 = (Z, \gamma)$ be three elements in $\Gamma(\mathcal{L})$. We look for the conditions on $e_3 = (Z, \gamma)$, so that $T(e_1, e_2, e_3) = 0$ holds for all $e_1, e_2 \in \Gamma(\mathcal{L})$. We choose a section s of the natural projection $i^* : \mathrm{T}^*(M) \to \mathcal{E}^*$ and consider the map $s \circ \omega : \mathcal{E} \to \mathrm{T}^*(M)$. Extending $s \circ \omega$ to a map from $\mathrm{T}(M)$ to $\mathrm{T}^*(M)$, we obtain a 2-form $\tilde{\omega} \in \wedge^2(\mathrm{T}^*(M))$ satisfying $\tilde{\omega}(e_1, e_2) = \omega(e_1, e_2)$, for $e_1, e_2 \in \mathcal{E}$. We write an element (X, α) in $\mathcal{L}$ as $(X, i_X\tilde{\omega} + \bar{\alpha})$, where $\bar{\alpha} \in \mathcal{E}^\circ$ (= the annihilators of $\mathcal{E}$). We compute $T(e_1, e_2, e_3)$ using the formula in Proposition 2.2:

$$\begin{aligned}
&T((X, \alpha), (Y, \beta), (Z, \gamma))\\
=&\alpha([Y, Z]) + \beta([Z, X]) + \gamma([X, Y]) + L_X(\beta(Z)) + L_Y(\gamma(X)) + L_Z(\alpha(Y))\\
=&-d\alpha(Y, Z) - d\beta(Z, X) - d\gamma(X, Y) + L_Y(\alpha(Z)) + L_Z(\beta(X)) + L_X(\gamma(Y)).
\end{aligned}$$

Making use of

$$\begin{aligned}
d\alpha =& di_X\tilde{\omega} + d\bar{\alpha} = L_X\tilde{\omega} - i_Xd\tilde{\omega} + d\bar{\alpha},\\
d\alpha(Y, Z) =& (L_X\tilde{\omega})(Y, Z) - (d\tilde{\omega})(X, Y, Z) + (d\bar{\alpha})(Y, Z),\\
L_Y(\alpha(Z)) =& L_Y(\tilde{\omega}(X, Z) + \bar{\alpha}(Z)) = L_Y(\tilde{\omega}(X, Z)),
\end{aligned}$$

we see the above $T(e_1, e_2, e_3)$ is equal to

$$d\tilde{\omega}(X, Y, Z) + \bar{\alpha}([Y, Z]) + \bar{\beta}([Z, X]) + \bar{\gamma}([X, Y]).$$

From this, we obtain the following conditions (4.1) and (4.2) on $e_3 = (Z, \gamma)$ which assure $T((X, \alpha), (Y, \beta), (Z, \gamma)) = 0$ for all $(X, \alpha), (Y, \beta) \in \mathcal{L}$.

$$d\tilde{\omega}(X, Y, Z) + \bar{\gamma}([X, Y]) = 0 \quad \text{for } X, Y \in \mathcal{E}, \tag{4.1}$$

$$\bar{\beta}([Z, X]) = 0 \quad \text{for } X \in \mathcal{E},\ \bar{\beta} \in \mathcal{E}^\circ. \tag{4.2}$$

Now, (4.1) is equivalent to that $(d\tilde{\omega})(Z) - d\bar{\gamma} = 0$ on $\mathcal{E}$ and from $\gamma = i_Z\tilde{\omega} + \bar{\gamma}$, this is equivalent to $L_Z\tilde{\omega} - d\gamma = 0$ (on $\mathcal{E}$). Similarly, (4.2) is equivalent to that $L_Z\mathcal{E} \subset \mathcal{E}$. Thus $\mathcal{L}_0$ is given by the following:

$$\mathcal{L}_0 = \{(Z, \gamma) \in \mathcal{L} \mid L_Z\mathcal{E} \subset \mathcal{E},\ L_Z\tilde{\omega} - d\gamma = 0 \text{ on } \mathcal{E}\}.$$

We note that $L_Z\omega$ is well-defined since the right-hand side of

$$i_X(L_Z\tilde{\omega}) = L_Z(i_X\tilde{\omega}) - i_{[Z,X]}\tilde{\omega}$$

is independent of the choice of $\tilde{\omega}$. The bracket (in $\mathcal{L}$) is given by

$$[\![(Z,\gamma),(W,\delta)]\!] = ([Z,W], L_Z\delta - L_W\gamma + d(\gamma(W)))\ .$$

Since $L_{[Z,W]}\mathcal{E} = L_Z(L_W\mathcal{E}) - L_W(L_Z\mathcal{E}) \subset \mathcal{E}$ and

$$\begin{aligned} L_{[Z,W]}\tilde{\omega} =& L_Z(L_W\tilde{\omega}) - L_W(L_Z\tilde{\omega}) = L_Z(d\delta) - L_W(d\gamma) \\ =& d(L_Z\delta - L_W\gamma + d(\gamma(W))), \text{ on } \mathcal{E}, \end{aligned}$$

that $[\![(Z,\gamma),(W,\delta)]\!] \in \mathcal{L}_0$ is verified.

Example 4.1. Consider the case where $\mathcal{E} = \mathrm{T}(M)$ and ω is an arbitrary 2-form. Then

$$\mathcal{L} = \{(X,\alpha) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M) \ \mid\ i_X\omega = \alpha\} = \{(X, i_X\omega)\ \mid\ X \in \mathrm{T}(M)\}.$$

It is easy to see $\mathcal{L}_0 = \{(Z, i_Z\omega)\ \mid\ Z \in \ker d\omega\}$. In particular, if ω is closed, $\mathcal{L}_0$ is a Dirac structure given by the presymplectic structure on M.

Example 4.2. Let $\mathcal{E}$ be a contact distribution with its contact 1-form θ, and let $\omega = d\theta$. The only vector field in $\mathcal{E}$ satisfying $L_Z\mathcal{E} \subset \mathcal{E}$ is the zero vector field. Thus $\Gamma(\mathcal{L}_0) = \{(0, f\theta) \mid f \in C^\infty(M)\}$ with trivial bracket. Similar situations occur with distributions whose Cauchy characteristic is trivial, since the condition $L_Z\mathcal{E} \subset \mathcal{E}$ means that Z is contained in the Cauchy characteristic of $\mathcal{E}$. With such distributions, it is appropriate to consider the ϕ-deformed bracket ([3],[4]).

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A Lie algebroid and a Dirac structure associated to an almost Dirac structure

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Abstract

We show that to an almost Dirac structure there associates a Lie algebroid. From this Lie algebroid, we obtain a Dirac structure. Thus, to an almost Dirac structure, there associates a Dirac structure. We apply these results in the case of a deformed bracket.

1 Introduction

The Lie algebroid structure of $\mathrm{T}^*(M)$ of a Poisson manifold (M, π) is one of the basic tools in Poisson geometry and it comes from the condition $[\pi, \pi] = 0$. If we use the Schouten-Jacobi bracket, this is generalized to the existence of Lie algebroid structure of $\mathcal{T}^*(M) = \mathrm{T}^*(M) \times \mathbf{R}$ of a Jacobi manifold (M, π, ξ). For an arbitrary 2-vector field π on a manifold, we proved that $\ker[\pi, \pi]$ has a natural(in a sense) Lie algebroid structure provided $\ker[\pi, \pi]$ is a sub-bundle of $\mathrm{T}^*(M)$ of constant rank ([7]). This result is generalized further to the case of the deformed bracket by a 1-form ([8]). To prove these results in a unified framework, it is relevant to utilize an almost Dirac structure of a Courant algebroid. An almost Dirac structure is just a maximally isotropic sub-bundle of a Courant algebroid ([9]). In this paper, we consider the Lie algebroid associated to an almost Dirac structure including the case of twisted bracket. In particular, from an arbitrary 2-vector field and a closed 3-form, we obtain a certain 3-vector field whose kernel has a Lie algebroid structure (Theorem 2). We also show that to an almost Dirac structure, there associates a Dirac structure (Theorem 5).

In section 2, we review some basic notions related with Lie algebroids. Here, we define a Courant algebroid starting from a Lie algebroid. Then, we introduce an almost Dirac structure and a 3-tensor T on it and give a proof of our fundamental result to the effect that the kernel of T forms a Lie algebroid(Theorem 1). In section 3, we compute the tensor T in the case where the almost Dirac structure is given as a graph of a 2-vector field. As a result, we obtain a description of our Lie algebroid in terms of the 2-vector field and its Schouten bracket. In section 4, we discuss the process to get a Dirac structure from the Lie algebroid which was obtained from an almost Dirac structure. In section 5, we take an opportunity to deal with a Jacobi manifold in a framework of deformed Schouten bracket and give a computational example for a result in the previous section.

Throughout the paper, we work in the C^∞ category.

2 Courant algebroid of a Lie algebroid

Let A be a Lie algebroid over a C^∞ manifold M with the anchor $a : A \to \mathrm{T}(M)$. Namely,

(a) A is a C^∞ vector bundle over M, whose space of sections $\Gamma(A)$ has a Lie algebra bracket $[\cdot,\cdot]_A$ over $\mathbf{R}$

(b) $a : A \to \mathrm{T}(M)$ is a bundle map which induces a Lie algebra homomorphism $a : \Gamma(A) \to \Gamma(\mathrm{T}(M))$, satisfying the condition

$$[v_1, fv_2]_A = \langle a(v_1), df\rangle v_2 + f[v_1, v_2]_A, \quad v_1, v_2 \in \Gamma(A),\ f \in C^\infty(M).$$

We will use the same letter a to denote both the bundle map and the induced homomorphism of sections. The Lie algebra bracket on $\Gamma(A)$ and the action of $a(v)$ on $C^\infty(M)$ induce an 'exterior differential' d_A on $\Gamma(\bigwedge^\bullet A^*)$ defined by a well-known formula. For example,

$$(d_A\theta)(v_1, v_2) = L_{a(v_1)}(\theta(v_2)) - L_{a(v_2)}(\theta(v_1)) - \langle\theta, [v_1, v_2]_A\rangle,$$
$$\theta \in \Gamma(\textstyle\bigwedge^1 A^*), \; v_1, v_2 \in \Gamma(A) .$$

We usually write L_v in stead of $L_{a(v)}$ which denotes the Lie derivative operator with respect to the vector field $a(v)$. From a Lie algebroid we get a Courant algebroid in the following way. First, we recall the definition.

Definition 1 *A Courant algebroid is a vector bundle E over a manifold M equipped with*

(a) a (usually non-skew) bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$,

(b) a non-degenerate symmetric bi-linear form $\langle\cdot, \cdot\rangle_+$ on E,

(c) a bundle map $\rho : E \to \mathrm{T}M$ (also called anchor) which induces a homomorphism $\rho : \Gamma(E) \to \Gamma(\mathrm{T}M)$, satisfying the conditions

(1) $[e_1, [e_2, e_3]_E]_E = [[e_1, e_2]_E, e_3]_E + [e_2, [e_1, e_3]_E]_E$

(2) $\rho(e)\langle e_1, e_2\rangle_+ = \langle e, [e_1, e_2]_E + [e_2, e_1]_E\rangle_+$

(3) $\rho(e)\langle e_1, e_2\rangle_+ = \langle [e, e_1]_E, e_2\rangle_+ + \langle e_1, [e, e_2]_E\rangle_+$

It can be shown that $[e_1, fe_2]_E = f[e_1, e_2]_E + (L_{\rho(e_1)}f)e_2$ and $\rho([e_1, e_2]_E) = [\rho(e_1), \rho(e_2)]$ hold ([5]). Thus a Courant algebroid is a Leibniz algebroid with additional conditions (b), (c2) and (c3).
From a Lie algebroid A, we construct a Courant algebroid as follows. Let $E_A = A \oplus A^*$. On E_A, we define the symmetric bi-linear form $\langle\cdot, \cdot\rangle_+$ and the bracket $[\cdot, \cdot]_{E_A}$ by

$$\langle(v_1, \theta_1), (v_2, \theta_2)\rangle_+ = \theta_1(v_2) + \theta_2(v_1) \tag{1}$$
$$[(v_1, \theta_1), (v_2, \theta_2)]_{E_A} = ([v_1, v_2]_A, L_{v_1}\theta_2 - \iota_{v_2} d_A\theta_1) , \tag{2}$$
$$(v_i, \theta_i) \in \Gamma(E) \quad (i = 1, 2),$$

where, ι_v denotes the interior product. The anchor $\rho : E_A \to \mathrm{T}(M)$ is given by $\rho = a \circ \mathrm{pr}_1$, where pr_1 is the projection to the first factor $A \oplus A^* \to A$ and $a : A \to \mathrm{T}(M)$ is the anchor of A as a Lie algebroid.

Then, it is a standard calculation to verify E_A is a Courant algebroid with this bracket, the bi-linear form and ρ as the anchor. We abbreviate this Courant algebroid $(E_A, \rho, [\cdot,\cdot]_{E_A}, \langle\cdot,\cdot\rangle_+)$ to E_A.

Besides E_A, we have another class of Courant brackets on $A \oplus A^*$. They are deformed brackets. To define a deformed bracket, we choose a closed 1-form (precisely, a Lie algebroid 1-cocycle) of A, namely an element $\phi \in \Gamma(A^*)$ satisfying

$$\phi([v_1, v_2]_A) = L_{v_1}(\phi(v_2)) - L_{v_2}(\phi(v_1)).$$

Then we have a ϕ-deformed exterior differential operator d_A^ϕ and ϕ-deformed Lie differentiation operator L_v^ϕ defined respectively by

$$\begin{aligned} d_A^\phi \alpha &= d_A\alpha + \phi \wedge \alpha, \\ L_{v_1}^\phi v_2 &= [v_1, v_2]_A, \\ L_v^\phi \alpha &= d_A^\phi \iota_v \alpha + \iota_v d_A^\phi \alpha, \quad \text{for } v, v_1, v_2 \in \Gamma(A), \alpha \in \Gamma(\textstyle\bigwedge^\bullet A^*). \end{aligned} \tag{3}$$

The operator L_v^ϕ does not commute with contraction but it satisfies the formula

$$L_v^\phi(\alpha(P)) = \alpha(L_v^\phi P) + (L_v^\phi \alpha)(P) + (p-1)\phi(v)\alpha(P)$$

for a 'p-vector field' P and a 'form' α .

Fortunately, we have the following familiar formulas in this case, too.

$$d_A^\phi \circ L_v^\phi = L_v^\phi \circ d_A^\phi, \quad [L_{v_1}^\phi, L_{v_2}^\phi] = L_{[v_1,v_2]_A}^\phi, \tag{4}$$

$$L_{v_1}^\phi \circ \iota_{v_2} - \iota_{v_2} \circ L_{v_1}^\phi = \iota_{[v_1,v_2]_A}, \tag{5}$$

$$L_{v_1}^\phi(\alpha(v_2)) = (L_{v_1}^\phi \alpha)(v_2) + \alpha(L_{v_1}^\phi v_2), \ (\alpha : \text{ '1-form'}). \tag{6}$$

Using d_A^ϕ and L_v^ϕ, we will define a new bracket $[\cdot,\cdot]_{E_A}^\phi$ on $\Gamma(E_A)$, $(E_A = A \oplus A^*)$, by simply replacing d_A by d_A^ϕ and L_v by L_v^ϕ;

$$[(v_1, \theta_1), (v_2, \theta_2)]_{E_A}^\phi = ([v_1, v_2]_A, L_{v_1}^\phi \theta_2 - \iota_{v_2} d_A^\phi \theta_1).$$

Since we have the formulas (4), (5), (6), we can verify the axioms for $(E_A, \rho, [\cdot,\cdot]_{E_A}^\phi, \langle\cdot,\cdot\rangle_+)$ to be a Courant algebroid, where the anchor ρ is the same as before. We denote this Courant algebroid by E_A^ϕ.

Now, we define a(n) (almost) Dirac structure which we will concern.

Definition 2 (*Dirac structure*) *Let* $(E, \rho, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_+)$ *be any Courant algebroid. A smooth sub-bundle* $\mathcal{L} \subset E$ *is an almost Dirac structure if* $\mathcal{L}$ *is maximally isotropic with respect to the symmetric pairing* $\langle\cdot,\cdot\rangle_+$. *If, moreover,* $\mathcal{L}$ *is closed under the bracket* $[\cdot,\cdot]_E$, *it is called a Dirac structure.*

To state our results, we also need a map defined by

$$\begin{aligned} T : \Gamma(E) \times \Gamma(E) \times \Gamma(E) &\to C^\infty(M) \\ T(e_1, e_2, e_3) &= \langle [e_1, e_2]_E, e_3\rangle_+. \end{aligned} \tag{7}$$

T is not skew-symmetric in general. Note that however, on the bundle $\mathcal{L}$, it is skew symmetric and $C^\infty(M)$ tri-linear, by (c2) and (c3) in the definition of a Courant algebroid (Definition 1).

Now, we have the following general result.

Theorem 1 ([9]) *For an almost Dirac structure* $\mathcal{L} \subset E$, *we put*

$$\mathcal{L}_0 = \{e \in \mathcal{L} | T(e, e_2, e_3) = 0, \quad \forall e_2, \forall e_3 \in \mathcal{L}\} = \ker T.$$

Assume that $\mathcal{L}_0$ *is a* C^∞ *sub-bundle of* $\mathcal{L}$ *of constant rank. Then* $\mathcal{L}_0$ *is a Lie algebroid with respect to the bracket* $[\cdot,\cdot]_E$ *and the natural projection* $\rho_{|\mathcal{L}_0} : \mathcal{L}_0 \to \mathrm{T}(M)$ *as anchor.*

Proof. Since the bracket restricted to $\Gamma(\mathcal{L}_0)$ is skew symmetric, as we remarked above, what we have to see is only that $\Gamma(\mathcal{L}_0)$ is closed under the bracket. The Jacobi identity is automatic. We can see

(1) If $e_1 \in \Gamma(\mathcal{L}_0)$ and $e_2 \in \Gamma(\mathcal{L})$ then $0 = T(e_1, e_2, e_3) = \langle [e_1, e_2]_E, e_3\rangle_+$ for any $e_3 \in \Gamma(\mathcal{L})$. Thus the maximality of $\mathcal{L}$ means $[e_1, e_2]_E \in \Gamma(\mathcal{L})$.

(2) Let e_1, e_2 be two elements of $\Gamma(\mathcal{L}_0)$. Then for any e_3 and e_4 in $\Gamma(\mathcal{L})$, we have

$$\begin{aligned} T([e_1, e_2]_E, e_3, e_4) &= \langle [[e_1, e_2]_E, e_3]_E, e_4\rangle_+ \\ &= \langle [e_1, [e_2, e_3]_E]_E, e_4\rangle_+ - \langle [e_2, [e_1, e_3]_E]_E, e_4\rangle_+ \\ &= T(e_1, [e_2, e_3]_E, e_4) - T(e_2, [e_1, e_3]_E, e_4) = 0. \end{aligned}$$

This shows $[e_1, e_2]_E$ is in $\Gamma(\mathcal{L}_0)$ and $\Gamma(\mathcal{L}_0)$ is closed under bracket $[\cdot,\cdot]_E$. □

A further example of a Courant bracket on E_A is obtained by choosing a 'd_A^ϕ-closed 3-form'Φ in $\bigwedge^3(A^*)$ ([10]). This bracket is given by the following

$$[(v_1,\theta_1),(v_2,\theta_2)]_\Phi^\phi = ([v_1,v_2]_A, L_{v_1}^\phi\theta_2 - \iota_{v_2}d_A^\phi\theta_1 + \Phi(v_1,v_2,\cdot)).$$

That this new bracket defines a Courant algebroid on E_A is verified just in the same way as in ([10]) since we have formulas (4),(5),(6) and so on, for ϕ-deformed differentiations. In the next section, we will consider the Dirac structure defined by a 2-vector field and examine the Lie algebroid $\mathcal{L}_0$ in the relationship with the 2-vector field.

3 Computation of the kernel of T

In this section, we consider the almost Dirac structure given as a graph of '2-vector field' π. Then we express the map T in terms of π and apply Theorem 1. We use the same notations as in the previous section.

For elements $e_i = (v_i,\theta_i)$ $(i=1,2)$, $e=(v,\theta)$ in $\Gamma(E_A)$ and the bracket $[\cdot,\cdot]_\Phi^\phi$, we have

$$\begin{aligned}
T(e_1,e_2,e) &= \langle [e_1,e_2]_\Phi^\phi, e\rangle_+ \\
&= \theta([v_1,v_2]_A) + (L_{v_1}^\phi\theta_2)(v) - (i_{v_2}d_A^\phi\theta_1)(v) + \Phi(v_1,v_2,v) \\
&= (L_{v_1}^\phi\theta_2)(v) - (L_{v_2}^\phi\theta_1)(v) + \theta([v_1,v_2]_A) + L_v^\phi(\theta_1(v_2)) + \Phi(v,v_1,v_2) \qquad (8)
\end{aligned}$$

We consider the case where $\mathcal{L}$ is given as a graph of a '2-vector field' $\pi\in\Gamma(\bigwedge^2 A)$:

$$\mathcal{L} = \{(\tilde\pi(\theta),\theta)\in A\oplus A^* \mid \theta\in A^*\},$$

where, $\tilde\pi$ is π considered as a map $A^*\to A$ (We often use π to denote $\tilde\pi$ when there is no danger of confusion). Then, (8) above is calculated to be

$$\begin{aligned}
&(L_{\tilde\pi(\theta_1)}^\phi\theta_2 - L_{\tilde\pi(\theta_2)}^\phi\theta_1)(\tilde\pi(\theta)) - L_{\tilde\pi(\theta)}^\phi(\pi(\theta_1,\theta_2)) \\
&\qquad\qquad + \theta([\tilde\pi(\theta_1),\tilde\pi(\theta_2)]_A) + \Phi(\tilde\pi(\theta),\tilde\pi(\theta_1),\tilde\pi(\theta_2)) \\
&= \pi(\theta, L_{\tilde\pi(\theta_1)}^\phi\theta_2 - L_{\tilde\pi(\theta_2)}^\phi\theta_1) - L_{\tilde\pi(\theta)}^\phi(\pi(\theta_1,\theta_2)) + \pi(\{\theta_1,\theta_2\}_\pi^\phi,\theta) \\
&\qquad\qquad + \frac{1}{2}[\pi,\pi]_A^\phi(\theta_1,\theta_2,\theta) + (\tilde\pi_*\Phi)(\theta,\theta_1,\theta_2) \\
&= (\frac{1}{2}[\pi,\pi]_A^\phi + \tilde\pi_*\Phi)(\theta,\theta_1,\theta_2).
\end{aligned}$$

In the above, $\tilde{\pi}_*$ denotes the map $\bigwedge^3 A^* \to \bigwedge^3 A$ induced by $\tilde{\pi}$ and we used the formula $[\tilde{\pi}(\theta_1), \tilde{\pi}(\theta_2)]_A^\phi = \pi(\{\theta_1, \theta_2\}_\pi^\phi) + \frac{1}{2}[\pi, \pi]_A^\phi(\theta_1, \theta_2)$ ([8] Lemma 3.3). From this, we see $(\tilde{\pi}(\theta), \theta) \in \mathcal{L}_0$ is equivalent to that $[\pi, \pi]_A^\phi(\theta) + 2(\tilde{\pi}_*\Phi)(\theta) = 0$. Applying Theorem 1, we obtain

Theorem 2 *Let π be an arbitrary '2-vector field' of a Lie algebroid A. Let ϕ and Φ be a 1-cocycle and a 3-cocycle of A , respectively. If $\ker([\pi, \pi]_A^\phi + 2\tilde{\pi}_*\Phi)$ forms a sub-bundle of A^* of constant rank, then it is a Lie algebroid with respect to the bracket*

$$[\theta_1, \theta_2]_\Phi^\phi = L_{\tilde{\pi}(\theta_1)}^\phi \theta_2 - i_{\tilde{\pi}(\theta_2)} d_A^\phi \theta_1 + \Phi(\tilde{\pi}(\theta_1), \tilde{\pi}(\theta_2), \cdot)$$

and the anchor is the composition of maps

$$\ker([\pi, \pi]_A^\phi + 2\tilde{\pi}_*\Phi) \hookrightarrow A^* \xrightarrow{a \circ \tilde{\pi}} \mathrm{T}(M).$$

An example is found in the last section.

Similarly, we treat the case where $\mathcal{L}$ is given as a graph of a '2-form' ω;

$$\mathcal{L} = \{(v, \omega(v)) \in E_A = A \oplus A^* | v \in A\}.$$

We put $e = (v, \omega(v))$, $e_1 = (v_1, \omega(v_1))$, $e_2 = (v_2, \omega(v_2))$ and compute $T(e, e_1, e_2)$ After a short calculation, we obtain $T(e, e_1, e_2) = (d_A^\phi \omega + \Phi)(v, v_1, v_2)$. Thus by Theorem 1, we have

Theorem 3 *Let ω be an arbitrary '2-form' of a Lie algebroid A, that is, $\omega \in \Gamma(\bigwedge^2 A^*)$. Let ϕ and Φ be a 1-cocycle and a 3-cocycle of A, respectively. If $\ker(d_A^\phi \omega + \Phi) \subset A$ forms a vector sub-bundle of A of constant rank, then it is a sub-Lie algebroid of A.*

Example 1 (*d^ϕ-closedness*) Let $A = \mathrm{T}(M)$ be a tangent bundle with usual bracket. We choose ϕ to be a closed 1-form on M and $\Phi \equiv 0$. Then the condition $d^\phi \omega = 0$ for a 2-form ω means that ω is expressed locally as a multiple of a closed 2-form by a positive function. Indeed, writing locally $\phi = df$, we have

$$d(e^f \omega) = e^f d\omega + e^f df \wedge \omega = e^f (d\omega + df \wedge \omega) = e^f d^\phi \omega = 0.$$

Thus, ω is called a locally conformally presymplectic form on M.

4 Presymplectic structure of $\mathcal{L}_0$

The image of the anchor of a Dirac structure is an integrable distribution of the base manifold. It is a generalized foliation each leaf of which has a presymplectic structure (A presymplectic structure on a manifold is just a closed 2-form on it). In this section, we prove that $\mathcal{L}_0$ defines a fiberwise 'presymplectic structure'μ . This structure in turn gives a Dirac structure. In this way, starting from an almost Dirac structure we obtain a Dirac structure.

To fix the arguments, we discuss the case of Courant algebroid $E_A = A \oplus A^*$ with the bracket $[\cdot,\cdot]_{E_A}$. The arguments are also valid in the cases of $[\cdot,\cdot]^{\phi}_A$ and $[\cdot,\cdot]^{\phi}_{\Phi}$ with suitable modifications.

Let $(E_A, \rho, [\cdot,\cdot]_{E_A}, \langle\cdot,\cdot\rangle_+)$ be our Courant algebroid over M. Let $\mathrm{pr}_1 : E_A \to A$ be the projection and $K = \mathrm{pr}_1(\mathcal{L}_0)$. On each fibre K_x over $x \in M$, there exists a '2-form' $\tilde{\mu}_x$ defined by $\tilde{\mu}_x(v) = \theta_{|K_x}$, where, $(v,\theta) \in \mathcal{L}_0$ and $\theta_{|K_x}$ denotes the restriction. The totality $\tilde{\mu}$ of $\tilde{\mu}_x, (x \in M)$ gives rise to a well-defined map $K \to K^*$ and it satisfies

$$\tilde{\mu}(v_1)(v_2) = \theta_1(v_2) = -\theta_2(v_1) = -\tilde{\mu}(v_2)(v_1) \qquad (v_1,\theta_1),(v_2,\theta_2) \in \mathcal{L}_0.$$

This shows that $\mu(v_1,v_2) = \tilde{\mu}(v_1)(v_2)$ is a skew-symmetric 2-form on fibers of K. Now, we prove μ is d_A-closed. Indeed, for $v_1, v_2, v_3 \in \Gamma(K)$, we have

$$\begin{aligned}(d_A\mu)(v_1,v_2,v_3) &= (i_{v_1}d_A\mu)(v_2,v_3)\\ &= (L_{v_1}\mu - d_A(i_{v_1}\mu))(v_2,v_3)\\ &= (i_{v_2}L_{v_1}\mu)(v_3) - (d_A\theta_1)(v_2,v_3).\end{aligned}$$

That is,

$$\begin{aligned}(d_A\mu)(v_1,v_2,\cdot) &= L_{v_1}i_{v_2}\mu - i_{[v_1,v_2]_A}\mu - i_{v_2}d_A\theta_1\\ &= L_{v_1}\theta_2 - i_{v_2}d_A\theta_1 - \mu([v_1,v_2]_A). \qquad (9)\end{aligned}$$

Since $\mathcal{L}_0$ is closed under the bracket $[\cdot,\cdot]_{E_A}$, $([v_1,v_2]_A, L_{v_1}\theta_2 - i_{v_2}d_A\theta_1) \in \mathcal{L}_0$, and the above should be 0 on K, hence μ is d_A-closed. In this way, μ is considered as a kind of presymplectic structure of M. In the case where $A = \mathrm{T}(M)$, μ is actually a usual presymplectic structure of each leaf of a foliation.

In the case of a general Lie groupoid A, we discuss as follows. Put $D = \mathrm{Im}\rho_{|\mathcal{L}_0}$, $(\rho = a \circ \mathrm{pr}_1$, a: anchor of A). As is well-known, D is an integrable distribution and defines a generalized foliation.

Proposition 4 *Assume K is a smooth bundle and* $\ker a_{|K} \subset \ker \tilde{\mu}$. *Then each leaf of D has a presymplectic structure.*

Proof. For an element $u \in D$, we choose any element $v \in K$ such that $u = a(v)$. The ambiguity of the choice of v is in $\ker a_{|K}$. Because of the assumption $\ker a_{|K} \subset \ker \tilde{\mu}$, as elements in K^*, $\tilde{\mu}(v)$ is determined by u. For an element w in $\ker a_{|K}$, we have $\tilde{\mu}(v)(w) = -\tilde{\mu}(w)(v) = 0$. If we look at the exact sequence

$$0 \to D^* \to K^* \to (\ker a_{|K})^* \to 0$$

this shows that $\tilde{\mu}(v)$ is in D^*. Thus, we obtain a well-defined map $\bar{\mu} : D \to D^*$ which is skew symmetric as easily seen and $\bar{\mu}$ may be regarded as a leafwise 2-form. To prove $\bar{\mu}$ is leafwise closed, we note that the sections of $\ker a_{|K}$ form a Lie ideal in $\Gamma(K)$ and that in our notation the Lie derivations L_v and $L_{a(v)}$ are the same. Then, we can see from the usual formula of exterior differential, $(d\bar{\mu})(u_1, u_2, u_3)$ is equal to $(d_A\mu)(v_1, v_2, v_3)$, $(a(v_i) = u_i, i = 1, 2, 3)$ which is equal to zero. □

Now, we will show that we have a Dirac structure associated with $\mathcal{L}_0$. For this, we put

$$\hat{\mathcal{L}}_0 = \{(v, \theta) \in K \times A^* \mid \tilde{\mu}(v) = \theta_{|K}\} \tag{10}$$

where, $K = \mathrm{pr}_1(\mathcal{L}_0)$ and μ is a 2-form on K.

Since $\theta + K^\circ$ ($^\circ$ denotes annihilator) defines a single element$\theta_{|K}$, we know that at each point of M, the fiber dimension of $\hat{\mathcal{L}}_0$ is equal to the fiber dimension of A.

Theorem 5 *Assume that $\hat{\mathcal{L}}_0$ is a smooth bundle, then it is a Dirac structure with respect to the bracket* $[\cdot,\cdot]_{E_A}$

Proof. For $(v_1, \theta_1), (v_2, \theta_2) \in \hat{\mathcal{L}}_0$, we have

$$\begin{aligned}\langle (v_1, \theta_1), (v_2, \theta_2) \rangle_+ &= \theta_1(v_2) + \theta_2(v_1) \\ &= \mu(v_1, v_2) + \mu(v_2, v_1) = 0.\end{aligned}$$

Since the fiber dimension is equal to that of A, as remarked above, we may conclude that $\hat{\mathcal{L}}_0$ is maximally isotropic with respect to $\langle \cdot, \cdot \rangle_+$. Next, we show $\Gamma(\hat{\mathcal{L}}_0)$ is closed under the bracket $[\cdot,\cdot]_{E_A}$. Since

$$[(v_1, \theta_1), (v_2, \theta_2)]_{E_A} = ([v_1, v_2]_A, L_{v_1}\theta_2 - i_{v_2} d_A \theta_1),$$

what we have to show is $\tilde{\mu}([v_1, v_2]_A) = (L_{v_1}\theta_2 - \iota_{v_2} d_A \theta_1)_{|K}$. From that K is closed under the bracket $[\cdot,\cdot]_A$, we see that $(L_{v_1}\theta_2)_{|K} = L_{v_1}(\theta_{2|K})$ and $(\iota_{v_2} d_A \theta_1)_{|K} = \iota_{v_2} d_A(\theta_{1|K})$. By this we have,

$$\begin{aligned}(L_{v_1}\theta_2 - \iota_{v_2} d_A\theta_1)_{|K} &= L_{v_1}(\tilde{\mu}(v_2)) - \iota_{v_2} d_A(\tilde{\mu}(v_1)) \\ &= \mu([v_1, v_2]_A), \qquad \text{(see (9) above)}.\end{aligned}$$

Thus, we have proved $\hat{\mathcal{L}}_0$ is a Dirac structure. □

What we have shown above is that for an almost Dirac structure $\mathcal{L}$ in $(E_A, \rho, [\cdot,\cdot]_{E_A}, \langle\cdot,\cdot\rangle_+)$ there corresponds a Dirac structure $\hat{\mathcal{L}}_0$ provided $\hat{\mathcal{L}}_0$ is a smooth vector bundle.

5 Jacobi Structure and an example

In this section, we recall a formulation of Jacobi structure expressed in terms of a deformed bracket and give a simple computational example of the preceding result.

Let $\mathcal{T}(M)$ be the extended tangent bundle of M. Namely, $\mathcal{T}(M)$ is the tangent bundle of $M \times \mathbf{R}$ restricted over $M \times \{0\}$. A section of $\mathcal{T}(M)$ is written as $X + a\frac{\partial}{\partial\tau}$, where X is a vector field of M, $a \in C^\infty(M)$ and $\frac{\partial}{\partial\tau}$ is a canonical vector field along $M \times \{0\}$ in the direction of $\mathbf{R}$. $\mathcal{T}(M)$ has a Lie algebroid structure whose bracket is given by

$$[X + a\frac{\partial}{\partial\tau}, Y + b\frac{\partial}{\partial\tau}] = [X, Y] + (L_X b - L_Y a)\frac{\partial}{\partial\tau} \tag{11}$$

and the anchor is $\rho(X + a\frac{\partial}{\partial\tau}) = X$. Let $\phi = d\tau$ be the dual to $\frac{\partial}{\partial\tau}$. Then ϕ is a Lie algebroid cocycle since we have

$$\begin{aligned}d\phi(\mathbf{X}, \mathbf{Y}) &= L_{\rho(\mathbf{X})}\phi(\mathbf{Y}) - L_{\rho(\mathbf{Y})}\phi(\mathbf{X}) - \phi([\mathbf{X}, \mathbf{Y}]) \\ &= L_X b - L_Y a - (L_X b - L_Y a) = 0,\end{aligned}$$

here, $\mathbf{X}$ and $\mathbf{Y}$ denote $X + a\frac{\partial}{\partial\tau}$ and $Y + b\frac{\partial}{\partial\tau}$, respectively.

A Jacobi structure on M is given by a 2-vector field $\boldsymbol{\pi} \in \Gamma(\bigwedge^2 \mathcal{T}M)$ which satisfies $[\boldsymbol{\pi}, \boldsymbol{\pi}]^{d\tau} = 0$, where $[\cdot,\cdot]^{d\tau}$ is a Schouten-Jacobi bracket with 1-cocycle $d\tau$ (or a $d\tau$ deformed bracket). Writing $\boldsymbol{\pi} = \pi + \frac{\partial}{\partial\tau} \wedge \xi$, where

$\pi \in \Gamma(\bigwedge^2 \mathrm{T}(M))$ and $\xi \in \Gamma(\mathrm{T}(M))$, by the formulas of Schouten-Jacobi bracket ([8]), we have

$$[\boldsymbol{\pi}, \boldsymbol{\pi}]^{d\tau} = [\pi, \pi] + 2\xi \wedge \pi + 2\frac{\partial}{\partial \tau} \wedge [\xi, \pi].$$

Thus, the condition $[\boldsymbol{\pi}, \boldsymbol{\pi}]^{d\tau} = 0$ is equivalent to that

$$[\pi, \pi] + 2\xi \wedge \pi = 0 \quad \text{and} \quad [\xi, \pi] = 0,$$

which is the condition often used as the definition of a Jacobi structure (π, ξ).

A contact manifold is a special case of Jacobi manifold where $\boldsymbol{\pi} = \pi + \frac{\partial}{\partial \tau} \wedge \xi$ is non-degenerate. The 1-form θ determined uniquely by $\theta(\xi) = 1$ and $\pi(\theta, \cdot) = 0$ is a contact 1-form with ξ as the Reeb vector field. Conversely, from a contact 1-form one can obtain (π, ξ). On a Jacobi manifold, clearly $\mathcal{T}^*M = \ker[\boldsymbol{\pi}, \boldsymbol{\pi}]^{d\tau}$ holds. Therefore, by our results in preceding sections, $\mathcal{T}^*M$ which is the 1-jet bundle of functions on M, is a Lie algebroid with respect to the bracket

$$\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\} = L^{d\tau}_{\boldsymbol{\pi}(\boldsymbol{\theta}_1)} \boldsymbol{\theta}_2 - \iota_{\boldsymbol{\theta}_2} d^{d\tau} \boldsymbol{\theta}_1. \tag{12}$$

As an example, we take a contact form

$$\theta = dy - \sum_{i=1}^{n} z_i dx_i$$

which is the canonical form on $M = J^1(\mathbf{R}^n, \mathbf{R}^1)$ or just a contact form on $\mathbf{R}^{2n+1}$. The Reeb vector field θ is easily seen to be $\frac{\partial}{\partial y}$. The corresponding Jacobi structure (π, ξ) is

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \wedge (\frac{\partial}{\partial x_i} + z_i \frac{\partial}{\partial y}), \qquad \xi = \frac{\partial}{\partial y}.$$

Thus, we obtain the extended 2-vector field

$$\begin{aligned} \boldsymbol{\pi} &= \pi + \frac{\partial}{\partial \tau} \wedge \xi \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \wedge (\frac{\partial}{\partial x_i} + z_i \frac{\partial}{\partial y}) + \frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial y}. \end{aligned}$$

The almost Dirac structure we consider is the Dirac structure $\mathcal{L}$ which is the graph of $\pi : \mathcal{T}^*M \to \mathcal{T}M$. It is spanned by

$$\left(-\frac{\partial}{\partial z_1}, dx_1\right), \dots, \left(-\frac{\partial}{\partial z_n}, dx_n\right),$$
$$\left(\frac{\partial}{\partial x_1} + z_1\frac{\partial}{\partial y}, dz_1\right), \dots, \left(\frac{\partial}{\partial x_n} + z_n\frac{\partial}{\partial y}, dz_n\right), \left(\frac{\partial}{\partial y}, d\tau\right), \left(-\frac{\partial}{\partial \tau}, dy\right).$$

We choose $d\tau$ as a 1-cocycle and $d\tau \wedge dx_1 \wedge dz_1$ for 3-cocycle Φ and compute $\mathcal{L}_0$ with respect to the Courant bracket $[\cdot,\cdot]_\Phi^{d\tau}$. As we have seen, $\mathcal{L}_0$ is the graph of the map π restricted to $\ker([\pi,\pi]^{d\tau} + 2\pi_*\Phi) = \ker \pi_*\Phi$. We have $\pi_*\Phi = -\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z_1}$ and $\ker(\pi_*\Phi)$ is spanned by 1-forms

$$dx_2, \dots, dx_n, dz_2, \dots, dz_n, d\tau.$$

Therefore, $\mathcal{L}_0$ is spanned in $\mathcal{T} \oplus \mathcal{T}^*$ by elements

$$\left(-\frac{\partial}{\partial z_2}, dx_2\right), \dots, \left(-\frac{\partial}{\partial z_n}, dx_n\right),$$
$$\left(\frac{\partial}{\partial x_2} + z_2\frac{\partial}{\partial y}, dz_2\right), \dots, \left(\frac{\partial}{\partial x_n} + z_n\frac{\partial}{\partial y}, dz_n\right), \left(\frac{\partial}{\partial y}, d\tau\right).$$

From this, we have the '2-form' μ defined on K, which is given by

$$\mu = \sum_{i=2}^{n} dx_i \wedge (dz_i - z_i d\tau) + dy \wedge d\tau.$$

Finally, Dirac structure $\hat{\mathcal{L}_0}$ is spanned by $2n+2$ elements

$$(0, dx_1), \left(-\frac{\partial}{\partial z_2}, dx_2\right), \dots, \left(-\frac{\partial}{\partial z_n}, dx_n\right),$$
$$(0, dz_1), \left(\frac{\partial}{\partial x_2} + z_2\frac{\partial}{\partial y}, dz_2\right), \dots, \left(\frac{\partial}{\partial x_n} + z_n\frac{\partial}{\partial y}, dz_n\right), (0, d\tau), \left(\frac{\partial}{\partial y}, d\tau\right).$$

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CR EINSTEIN-WEYL STRUCTURES

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ABSTRACT. An Einstein-Weyl structure is a natural generalization of an Einstein structure within the framework of conformal geometry. We are interested in considering an Einstein-Weyl structure on a CR manifold. A CR manifold has a conformal structure only on its hyperdistribution. In this paper, on a CR manifold we naturally define an Einstein-Weyl structure closely related to the conformal structure on the hyperdistribution.

0. INTRODUCTION

A conformal structure on a differentiable manifold is a conformal equivalence class of Riemannian metrics (or pseudo-Riemannian metrics) on the manifold. On a conformal manifold, the objects which are invariant for every Riemannian metric included in the conformal class are important, or more strictly, the object except for them does not have significance. Weyl conformal curvature tensor is representative one of them. It is interesting to consider whether the results obtained in conformal geometry also hold in CR geometry. In this paper, we study an analogy of Weyl structure in CR geometry. A CR structure on an odd dimensional manifold is a pair $(\mathscr{D}, J)$ of a 1-codimensional subbundle $\mathscr{D}$ of the tangent bundle and a complex structure J on $\mathscr{D}$ with a certain integrability condition. Assuming the nondegenerate property for $\mathscr{D}$, we have a conformal class of fiber metrics on $\mathscr{D}$. It is well-known that Bochner curvature tensor is one of the objects which are invariant for this conformal class on CR manifolds.

In this paper, we discuss a structure analogous to Einstein-Weyl structure on a conformal manifold and especially consider whether we can comfortably define this structure for a conformal class on $\mathscr{D}$. An Einstein-Weyl structure is a natural generalization of an Einstein structure within the framework of conformal geometry. Strictly speaking, Einstein-Weyl structure is a pair of $([g], D)$ of a Riemannian metric class $[g]$ and a linear connection D, preserving $[g]$, whose Ricci tensor satisfies an equation that the symmetric part is proportional to g pointwise. On a CR manifold there are naturally almost contact structures (ϕ, ξ, θ) which determine a conformal class on $\mathscr{D}$. Therefore almost contact structures (ϕ, ξ, θ) associated with $(\mathscr{D}, J)$ correspond to Riemannian structures in conformal geometry. Furthermore, a connection corresponding to Levi-Civita connection is defined by Tanaka [11], which is called Tanaka connection. We need to define a connection which preserves the conformal class on $\mathscr{D}$. Such connection corresponds to the Weyl connection D

In Section 1, we recall the definition of Einstein-Weyl structure and relation between a Weyl connection D and Levi-Civita connection ∇ of a Riemannian metirc included in a given conformal structure (cf. [7], [8]). This section will be useful to understand the analogy mentioned above. In Section 2, we recall the definition of CR structure, results obtained in [9] and certain cochain complex $\{C^{p,q}(M), d''\}$ defined by Tanaka

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[11]. In Section 3, we define CR Weyl connection and study the relation between CR Weyl connection D and Tanaka connection ∇, where Tanaka connection ∇ is a unique linear connection associated with almost contact structure (ϕ, ξ, θ) introduced in Section 2. In Section 4, we see a CR Weyl connection from the standpoint of the frame bundle. Section 5 is devoted to the study of curvature tensor of a CR Weyl connection. In Section 6 we study the relation between the curvature tensor of a CR Weyl connection and that of a Tanaka connection. In fact, we obtain an equation including these two tensors, which is similar to the equation appearing in [2]. Using this equation, we define a CR Einstein-Weyl structure in a natural fashion. In the last section, we introduce an example of a CR Einstein-Weyl manifold. In fact, we see that $SO(3)$-bundle over a quaternion Kähler manifold admits a CR Einstein-Weyl structure.

1. Einstein-Weyl structures

Let M be an n-dimensional manifold with a conformal class $[g]$. A *Weyl connection* on M is a torsion-free linear connection which satisfies the following condition:

$$Dg = -2p \otimes g \tag{1.1}$$

for some 1-form p. If we choose $g' = e^{2\mu}g$ for a smooth function μ in the conformal class $[g]$, we have a 1-form $p' = p - d\mu$ instead of p for the equation (1.1). From this we can say that a Weyl connection D *preserves* the conformal class $[g]$. Let $([g], D)$ be a pair of a conformal class $[g]$ and a Weyl connection preserving it. A pair $([g], D)$ is called a *Weyl structure* on M and if M admits a Weyl structure, then $(M, [g], D)$ is called a *Weyl manifold*. We can also say that a Weyl connection is a torsion-free linear connection which is reducible to a connection in $CO(M)$ corresponding to the conformal class $[g]$, where $CO(M)$ is a subbundle in the frame bundle $F(M)$ with a structure group $CO(n)$.

Now let ∇ be the Levi-Civita connection of g on a Weyl manifold M. We can write $D = \nabla + H$ where H is a tensor field of type $(1, 2)$. Then we have from (1.1)

$$H(X, Y) = p(X)Y + p(Y)X - g(X, Y)P \tag{1.2}$$

for $X, Y \in \mathfrak{X}(M)$, where P is the dual vector field of p with respect to g. Conversely if we define D with (1.2) for an arbitrary pair (p, g), D satisfies the equation (1.1). Therefore we see that an arbitrary pair (p, g) determines a Weyl structure on M.

Now let r^D be the Ricci tensor of a Weyl connection D. Note that as D is not a metric connection, r^D is not necessarily symmetric. A Weyl structure $([g], D)$ is called an *Einstein-Weyl structure* if the symmetric part of r^D is proportional to g pointwise. Note that the proportional factor may be non-constant. If M admits an Einstein-Weyl structure $([g], D)$, then M is called an *Einstein-Weyl manifold*.

Now if we let r^∇ be the Ricci tensor of the connection ∇, then r^D and r^∇ are related by the following equation (cf. [7], [8]):

$$\begin{aligned} r^D(X, Y) = {} & (1-n)(\nabla_X p)(Y) + (\nabla_Y p)(X) + (n-2)p(X)p(Y) \\ & + g(X, Y)(\delta p + (n-2)g(P, P)) + r^\nabla(X, Y) \end{aligned} \tag{1.3}$$

for $X, Y \in \mathfrak{X}(M)$, where δp denotes the codifferential with respect to g.

We have the following local characterization of Einstein-Weyl structures (cf. [7], [8]):

Proposition 1.1. *Let (p, g) be a Weyl structure on M. Then (p, g) is an Einstein-Weyl structure if and only if there exists a smooth function Λ on M satisfying the equation*

$$\frac{2-n}{2}((\nabla_X p)(Y) + (\nabla_Y p)(X) - 2p(X)p(Y)) + r^{\nabla}(X, Y) = \Lambda g(X, Y)$$

for every $X, Y \in \mathfrak{X}(M)$.

2. CR structure and Tanaka connection

Let M be a connected differentiable manifold of dimension $2n+1$ $(n \geq 1)$. An *almost contact structure* on M is a triplet of a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form θ satisfying

$$\theta(\xi) = 1, \qquad \phi^2 = -I + \theta \otimes \xi \tag{2.1}$$

which imply

$$\phi\xi = 0, \quad \theta \circ \phi = 0 \quad \text{and} \quad \operatorname{rank} \phi = 2n, \tag{2.2}$$

where I denotes the identity transformation. An almost contact structure (ϕ, ξ, θ) naturally corresponds to a reduced bundle in the frame bundle $F(M)$ with structure group

$$\left\{ \begin{pmatrix} 1 & {}^t0 \\ 0 & C \end{pmatrix} \middle| \, C \in GL(n; \mathbb{C}) \right\}.$$

Now let $\mathscr{D}$ denote a 1-codimensional subbundle of the tangent bundle TM, which is called a hyperdistribution. A cross section J of the bundle $\mathscr{D} \otimes \mathscr{D}^*$ satisfying $J^2 = -I$ is called a complex structure on $\mathscr{D}$, where $\mathscr{D}^*$ is the dual bundle of $\mathscr{D}$.

If M admits a pair $(\mathscr{D}, J)$, there is always a locally defined almost contact structure (ϕ, ξ, θ) satisfying that the 1-form θ annihilates $\mathscr{D}$ and the restriction of ϕ to $\mathscr{D}$ coincides with J. In fact, since there always exists a 1-form θ annihilating $\mathscr{D}$ in each coordinate neighborhood U of M, we have a vector field ξ on U in such a way that $\theta(\xi) = 1$. Then we can define, on U, a $(1,1)$ tensor field ϕ by

$$\phi(V) = J(V - \theta(V)\xi)$$

for $V \in \mathfrak{X}(U)$ because $V - \theta(V)\xi$ belongs to $\mathscr{D}$. We shall denote $V - \theta(V)\xi$ by $V_{\mathscr{D}_\xi}$ and call $\mathscr{D}$-component of V with respect to ξ. Then a straightforward calculation shows that (ϕ, ξ, θ) is an almost contact structure on U. An almost contact structre (ϕ, ξ, θ) such that the 1-form θ annihilates $\mathscr{D}$ and the restriction of ϕ to $\mathscr{D}$ coincides with J is said that the almost contact structure (ϕ, ξ, θ) *belongs* to the pair $(\mathscr{D}, J)$. In addition, if M is orientable, there are globally defined almost contact structures (ϕ, ξ, θ) belonging to $(\mathscr{D}, J)$. A 1-form θ annihilating $\mathscr{D}$ is determined up to a non-vanishing smooth function. Moreover we have

$$d(f\theta)(X, Y) = f d\theta(X, Y)$$

for every $X, Y \in \Gamma(\mathscr{D})$ and smooth function f, where $\Gamma(\mathscr{D})$ denotes the set of cross sections of the vector bundle $\mathscr{D}$ on M. Therefore, in virtue of this fact, the following definition is well-defined. If $d\theta$ is nondegenerate on $\mathscr{D}$, then $(\mathscr{D}, J)$ is said to be *nondegenerate.*

A pair $(\mathscr{D}, J)$ is called a *CR structure* if the following two conditions hold:

$$[JX, JY] - [X, Y] \in \Gamma(\mathscr{D}) \tag{C.1}$$

$$[JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0 \tag{C.2}$$

for every $X, Y \in \Gamma(\mathscr{D})$. If M admits a CR structure $(\mathscr{D}, J)$, then M is called a *CR manifold*. In the sequel, $(\mathscr{D}, J)$ will be a nondegenerate CR structure.

Now let M be a connected orientable manifold furnished with a CR structure $(\mathscr{D}, J)$ and (ϕ, ξ, θ) an almost contact structure belonging to $(\mathscr{D}, J)$. Define ω by

$$\omega = -2d\theta. \tag{2.3}$$

Then ω satisfies

$$\omega(JX, JY) = \omega(X, Y) \tag{2.4}$$

for every $X, Y \in \Gamma(\mathscr{D})$ because of the condition (C.1). Moreover define $g : \mathscr{D} \times \mathscr{D} \to \mathbb{R}$ by

$$g(X, Y) = \omega(JX, Y), \tag{2.5}$$

which satisfies the equations

$$g(X, Y) = g(Y, X), \qquad g(JX, JY) = g(X, Y) \tag{2.6}$$

for every $X, Y \in \Gamma(\mathscr{D})$. Therefore g is symmetric, Hermitian and nondegenerate, which is called *Levi metric*.

From a given almost contact structure belonging to $(\mathscr{D}, J)$ we can always make an almost contact structure which belongs to the same $(\mathscr{D}, J)$ and satisfies the following condition

$$[\xi, \Gamma(\mathscr{D})] \subset \Gamma(\mathscr{D}) \tag{$*$}$$

(cf. [9]). This condition $(*)$ is equivalent to

$$\mathscr{L}_\xi \theta = 0 \qquad \text{or} \qquad \omega(\xi, X) = 0 \tag{2.7}$$

for $X \in \mathscr{D}$, where $\mathscr{L}_\xi$ denotes the Lie differentiation with respect to ξ. Such an almost contact structure is denoted by $(\phi, \xi, \theta)^*$ and we call it a *$\mathscr{D}$-preserving almost contact structure.* We shall restrict our attention to the family of $\mathscr{D}$-preserving almost contact structures which belong to CR structure $(\mathscr{D}, J)$. The following result is proved in [9]:

Lemma 2.1. *If $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ belong to $(\mathscr{D}, J)$, then they are related by*

$$\theta' = \varepsilon e^{2\mu}\theta, \quad \xi' = \varepsilon e^{-2\mu}(\xi - 2Q^*), \quad \phi' = \phi - 2\theta \otimes P^* \tag{2.8}$$

where $\varepsilon = \pm 1$, μ is a smooth function, $P^* \in \Gamma(\mathscr{D})$ is defined by $g(P^*, X) = d\mu(X)$ for $X \in \Gamma(\mathscr{D})$ and $Q^* = JP^*$.

Next we shall explain Tanaka connection associated with $(\phi, \xi, \theta)^*$ and how the connection changes under (2.8). We don't have to assume the condition (C.2) so far, but we need to assume the condition (C.2) for the next lemma (cf. [9], [12]).

Lemma 2.2. *Let $(\phi, \xi, \theta)^*$ be a $\mathscr{D}$-preserving almost contact structure. Then there exists uniquely a linear connection ∇ such that $\nabla\phi = 0, \nabla\xi = 0, \nabla\theta = 0$, $\nabla^\circ g = 0$, $T_{\mathscr{D}_\xi} = 0$ and $T(\xi, X) = -1/2\phi(\mathscr{L}_\xi\phi)X$, where ∇° denotes the induced connection on the hyperdistribution $\mathscr{D}$ and $T_{\mathscr{D}_\xi}(X, Y)$ the $\mathscr{D}$-component of the torsion tensor $T(X, Y)$ of ∇ with respect to ξ for $X, Y \in \Gamma(\mathscr{D})$.*

Remark. We put $FV = T(\xi, V)$ for $V \in TM$. Note that F is symmetric with respect to g and anticommutes with J (cf. [9]).

The linear connection stated in the above lemma is called *Tanaka connection* associated with $(\phi, \xi, \theta)^*$. We give the following (cf. [9]).

Lemma 2.3. *Let $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ be two $\mathscr{D}$-preserving almost contact structures which belong to the CR structure $(\mathscr{D}, J)$. Let ∇ and ∇' be Tanaka connections associated with $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ respectively. Define the difference H between ∇ and ∇' by*

$$H(V, W) = \nabla'_V W - \nabla_V W, \qquad V, W \in \mathfrak{X}(M).$$

Then we have

$$(2.9)\quad H(X, Y) = p^*(X)Y + p^*(Y)X - g(X, Y)P^* + q^*(X)JY + q^*(Y)JX - g(JX, Y)Q^*,$$

$$(2.10)\qquad H(\xi, X) = \nabla_{JX}P^* + \nabla_X Q^* - 2q^*(X)P^* + 2p^*(X)Q^* + 2g(P^*, P^*)JX,$$

for every $X, Y \in \Gamma(\mathscr{D})$, where $p^* = d\mu$ and $q^* = -p^* \circ \phi$.

Remark. We have $g(P^*, X) = p^*(X)$ and $g(Q^*, X) = q^*(X)$ for every $X \in \Gamma(\mathscr{D})$.

Next we shall introduce a cochain complex $\{C^{p,q}, d''\}$ of a CR manifold M with complex coefficients, which corresponds to that in the case of a complex manifold (cf. [11]). We shall use the following fact in Section 6.

Let $(\mathscr{D}, J)$ be a nondegenerate CR structure of a $(2n+1)$-dimensional orientable manifold M. Then the complexification $\mathbb{C}TM$ of the tangent bundle TM is decomposed as $\mathbb{C}TM = \mathbb{C}\mathscr{D} \oplus \mathscr{L}$ where $\mathbb{C}\mathscr{D}$ is the complexification of $\mathscr{D}$ and $\mathscr{L}$ is a trivial line bundle isomorphic with $\mathbb{C}TM/\mathbb{C}\mathscr{D}$. The complex structure J on $\mathscr{D}$ can be uniquely extended to a complex linear endomorphism of $\mathbb{C}\mathscr{D}$ and the extended endomorphism will be also denoted by J. Let $\mathscr{D}^{1,0}$ (resp. $\mathscr{D}^{0,1}$) be a subbundle of $\mathbb{C}\mathscr{D}$ composed of the eigenvectors corresponding to i (resp. $-i$) of the endomorphism J. Note that $\mathscr{D}^{0,1} = \bar{\mathscr{D}}^{1,0}$, where the notation "bar" denotes the conjugate operator. It is clear that conditions (C.1) and (C.2) are equivalent to

$$(2.11)\qquad [\Gamma(\mathscr{D}^{1,0}), \Gamma(\mathscr{D}^{1,0})] \subset \Gamma(\mathscr{D}^{1,0}).$$

Now we put $A^k(M) = \Gamma(\Lambda^k(\mathbb{C}TM))$ and denote by $F^p(\Lambda^k(\mathbb{C}TM))$ the subbundle of $\Lambda^k(\mathbb{C}TM)$ consisting of all $\psi \in \Lambda^k(\mathbb{C}TM)$ which satisfy the equality:

$$(2.12)\qquad \psi(X_1, \dots, X_{p-1}, \bar{Y}_1, \dots, \bar{Y}_{k-p+1}) = 0$$

for all $X_1, \dots, X_{p-1} \in \mathbb{C}TM$ and $Y_1, \dots, Y_{k-p+1} \in \mathscr{D}^{1,0}$. Note that we define $F^0(\Lambda^k(\mathbb{C}TM)) = \Lambda^k(\mathbb{C}TM)$. Then we have

$$(2.13)\qquad F^{p+1}(\Lambda^k(\mathbb{C}TM)) \subset F^p(\Lambda^k(\mathbb{C}TM)), \qquad F^{p+1}(\Lambda^p(\mathbb{C}TM)) = 0.$$

Furthermore putting $A^{p,q}(M) = \Gamma(F^p(\Lambda^{p+q}(\mathbb{C}TM)))$, we easily find that

$$(2.14)\qquad d\,A^{p,q}(M) \subset A^{p,q+1}(M),$$

because of (2.11). Moreover putting $C^{p,q}(M) = A^{p,q}(M)/A^{p+1,q-1}(M)$, then we have the well-defined operator $d'' : C^{p,q}(M) \to C^{p,q+1}(M)$ which is naturally induced from the operator d satisfying (2.14). And we obtain the cochain complex

$$(2.15)\qquad 0 \to \Omega^p \to C^{p,0}(M) \to C^{p,1}(M) \to C^{p,2}(M) \to \cdots,$$

where Ω^p denotes the kernel of $C^{p,0}(M) \to C^{p,1}(M)$, whose element is called a *holomorphic p-form* in the mean of CR geometry. Since $A^{p,q}(M) = A^{p+1,q-1}(M) \oplus C^{p,q}(M)$, we have the decompositon:

$$A^{p,q}(M) = \bigoplus_{i=0}^{q} C^{p+q-i,i}(M).$$

Now for $\psi \in C^{p,q}(M)$ we have $d\psi \in A^{p,q+1}(M)$ or more precisely the following fact is well-known (cf. [11]):

(2.16) $$d\psi \in C^{p+2,q-1}(M) \oplus C^{p+1,q}(M) \oplus C^{p,q+1}(M).$$

Consequently $d\psi$ can be written uniquely in the form:

$$d\psi = A\psi + d'\psi + d''\psi,$$

where $A\psi \in C^{p+2,q-1}(M)$ and $d'\psi \in C^{p+1,q}(M)$. For any $\psi \in C^{p,q}(M)$, $A\psi$, $d'\psi$ and $d''\psi$ are described as follows:

(2.17) $$\begin{aligned}&(A\psi)(X_1, \ldots, X_{p+2}, \bar{Y}_1, \ldots, \bar{Y}_{q-1})\\ &= \frac{1}{p+q+1}\sum_{\lambda<\mu}(-1)^{\lambda+\mu+1}\psi(T(X_\lambda, X_\mu), X_1, \ldots, \hat{X}_\lambda,\\ &\qquad\qquad \ldots, \hat{X}_\mu, \ldots, X_{p+1}, \bar{Y}_1, \ldots, \bar{Y}_{q-1})\end{aligned}$$

(2.18) $$\begin{aligned}&(d'\psi)(X_1, \ldots, X_{p+1}, \bar{Y}_1, \ldots, \bar{Y}_q)\\ &= \frac{1}{p+q+1}\sum_{\lambda}(-1)^{\lambda+1}(\nabla_{X_\lambda}\psi)(X_1, \ldots, \hat{X}_\lambda, \ldots, X_{p+1}, \bar{Y}_1, \ldots, \bar{Y}_q),\end{aligned}$$

(2.19) $$\begin{aligned}&(d''\psi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_{q+1})\\ &= \frac{(-1)^p}{p+q+1}\Big\{\sum_{\lambda}(-1)^{\lambda+1}(\nabla_{\bar{Y}_\lambda}\psi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \hat{\bar{Y}}_\lambda, \ldots, \bar{Y}_{q+1}),\\ &\quad + \sum_{\lambda,\mu}(-1)^{\lambda+\mu+1}\psi(T(X_\lambda, \bar{Y}_\mu), X_1, \ldots, \hat{X}_\lambda,\\ &\qquad\qquad \ldots, X_p, \bar{Y}_1, \ldots, \hat{\bar{Y}}_\mu, \ldots, \bar{Y}_{q+1})\Big\}\end{aligned}$$

for $Y_\lambda \in \mathscr{D}^{1,0}$ and $X_\lambda \in \mathscr{D}^{1,0} \oplus \mathscr{L}$, where ∇ is a Tanaka connection associated with some $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ and T is the torsion tensor of ∇. Note that $\mathscr{L} = \mathbb{C} \otimes \operatorname{span}\{\xi\}$.

3. CR Weyl structures

Let $(M, \mathscr{D}, J)$ be a connected orientable $(2n+1)$-dimensional manifold furnished with a nondegenerate CR structure $(\mathscr{D}, J)$. Under the notation of lemma 2.1, if g and g' are the Levi-metrics made from θ and θ' respectively, we have

(3.1) $$g' = \varepsilon e^{2\mu} g$$

Therefore the family of $\mathscr{D}$-preserving almost contact structures which belong to the CR structure $(\mathscr{D}, J)$ induces pseudo conformal geometry only on the hyperdistribution $\mathscr{D}$. We shall naturally define a certain Weyl structure with respect to this pseudo conformal geometry. The word " naturally " of the above sentence means that the relation between a CR Weyl connection of the CR structure $(\mathscr{D}, J)$ and a Tanaka connection of a $\mathscr{D}$-preserving almost contact structure belonging to $(\mathscr{D}, J)$ is analogous to that between a

Weyl connection of a conformal class and Levi-Civita connection of a Riemannian metric in the conformal class.

Definition. Let $(\phi, \xi, \theta)^*$ be an arbitrary $\mathscr{D}$-preserving almost contact structure belonging to $(\mathscr{D}, J)$. A linear connection D on M is a *CR Weyl connection* if, for every $V \in \mathfrak{X}(M)$, $X, Y \in \Gamma(\mathscr{D})$ and for some 1-form p on M, the following conditions are satisfied:

$$\text{(a)} \qquad D_V\theta = -2p(V)\theta$$

$$\text{(b)} \qquad D_V\xi_p = 2p(V)\xi_p$$

$$\text{(c)} \qquad D^\circ_V J = 0$$

$$\text{(d)} \qquad D^\circ_V g = -2p(V)g$$

$$\text{(e)} \qquad T(X, Y) = -\omega(X, Y)\xi_p$$

$$\text{(f)} \qquad T(\xi_p, X) = -\frac{1}{2}\phi_p(\mathscr{L}_{\xi_p}\phi_p)X,$$

where D° denotes the induced connection on the hyperdistribution $\mathscr{D}$, T the torsion tensor of D, $\xi_p = \xi - 2Q$, $\phi_p = \phi - 2\theta \otimes P$, P the cross-section of $\mathscr{D}$ such that $g(P, X) = p(X)$ for every $X \in \Gamma(\mathscr{D})$ and $Q = JP$.

Remark. If D is a CR Weyl connection, we can show that

$$\text{(3.2)} \qquad D_V\phi_p = 0\,, \qquad (D_VT)(X, Y) = 0$$

for every $V \in \mathfrak{X}(M)$ and $X, Y \in \Gamma(\mathscr{D})$ by direct calculation. In addition, we note that (ϕ_p, ξ_p, θ) is also an almost contact structure belonging to $(\mathscr{D}, J)$ which may not satisfy condition $(*)$.

The family of almost contact structures belonging to $(\mathscr{D}, J)$ and satisfying $(*)$ is smaller than that of all almost contact structures belonging to $(\mathscr{D}, J)$. However, we can always obtain an almost contact structure satisfying $(*)$ from almost contact structure belonging to the same $(\mathscr{D}, J)$ if it is nondegenerate (cf. [9]). Therefore we may deal with only $\mathscr{D}$-preserving almost contact structures. The following propositoin allows us to call D a CR Weyl connection. By direct computation, we obtain

Proposition 3.1. *The CR Weyl connection D is well defined: the equations from* (a) *to* (f) *in above definition are invariant for the change* (2.8).

Remark. If we replace $(\phi, \xi, \theta)^*$ by $(\phi', \xi', \theta')^*$, then the 1-form p in the above definition changes to $p' = p - d\mu$.

From this, we can say that a CR Weyl connection D *preserves* the CR structure $(\mathscr{D}, J)$. Let $((\mathscr{D}, J),\ D)$ be a pair of a CR structure $(\mathscr{D}, J)$ and a CR Weyl connection preserving it. The pair $((\mathscr{D}, J),\ D)$ is called a *CR Weyl structure* on M.

Next we closely observe the conditions of a CR Weyl connection. In fact, we don't have to assume the condition (f) if we add a certain condition to the torsion tensor of a linear connection satisfying from (a) to (e) for a 1-form p. To see this, we need the following:

Lemma 3.2. *Let D be a linear connection satisfying from* (a) *to* (e) *for a 1-form p and T the torsion tensor of D. Then T satisfies*

$$\theta(T(\xi_p, V)) = 0, \tag{3.3}$$

$$\phi_p(T(\xi_p, \phi_p V)) + T(\xi_p, V) = -\phi_p(\mathscr{L}_{\xi_p}\phi_p)(V) \tag{3.4}$$

for every $V \in \mathfrak{X}(M)$.

Proof. It is sufficient to show that $T(\xi_p, V)$ belongs to $\Gamma(\mathscr{D})$ for $V = \xi_p$ and $V = X \in \Gamma(\mathscr{D})$. When $V = \xi_p$, $T(\xi_p, \xi_p) = 0$. When $V = X$, we have

$$T(\xi_p, X) = D_{\xi_p}X - D_X\xi_p - [\xi_p, X] = D_{\xi_p}X - 2p(X)\xi_p - [\xi_p, X]$$

because of (b). The condition (a) implies that $D_V\Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D})$. The $\mathscr{D}$-component $[\xi_p, X]_{\mathscr{D}_{\xi_p}}$ with respect to ξ_p is given by

$$\begin{aligned}[\xi_p, X]_{\mathscr{D}_{\xi_p}} &= [\xi_p, X] - \theta([\xi_p, X])\xi_p \\ &= [\xi_p, X] - \theta([\xi - 2Q, X])\xi_p \\ &= [\xi_p, X] + 2\theta([Q, X])\xi_p \\ &= [\xi_p, X] + 2\omega(Q, X)\xi_p \\ &= [\xi_p, X] + 2g(P, X)\xi_p = [\xi_p, X] + 2p(X)\xi_p.\end{aligned}$$

Therefore we have

$$T(\xi_p, X) = D_{\xi_p}X - 2p(X)\xi_p - ([\xi_p, X]_{\mathscr{D}_{\xi_p}} - 2p(X)\xi_p)$$

which proves (3.3). Since

$$\begin{aligned}0 = (D_{\xi_p}\phi_p)V &= D_{\xi_p}(\phi_p V) - \phi_p(D_{\xi_p}V) \\ &= D_{\phi_p V}\xi_p + [\xi_p, \phi_p V] + T(\xi_p, \phi_p V) - \phi_p(D_V\xi_p + [\xi_p, V] + T(\xi_p, V)),\end{aligned}$$

we have

$$T(\xi_p, \phi_p V) - \phi_p(T(\xi_p, V)) = -2p(\phi_p V)\xi_p - (\mathscr{L}_{\xi_p}\phi_p)V$$

because of (b) and the equation $\phi_p\xi_p = 0$. Thus if we apply ϕ_p to the both hand sides of the above equation, we obtain (3.4). □

Now put $F_pV = T(\xi_p, V)$ for $V \in \mathfrak{X}(M)$. Then we have

$$\theta \circ F_p = 0, \tag{3.5}$$

$$\phi_p \circ F_p \circ \phi_p + F_p = -\phi_p(\mathscr{L}_{\xi_p}\phi_p). \tag{3.6}$$

We demand for F_p the condition that F_p anticommutes with ϕ_p. Then F_p must be $-1/2\phi_p(\mathscr{L}_{\xi_p}\phi_p)$. Conversely we see that F_p anticommutes with ϕ_p if $F_p = -1/2\phi_p(\mathscr{L}_{\xi_p}\phi_p)$. Therefore if we add the condition that F_p anticommutes with ϕ_p to the conditions from (a) to (e) for a 1-form p, D becomes a CR Weyl connection. For F_p, we also have

Lemma 3.3. *Let D be a connection satisfying from* (a) *to* (e) *for a 1-form p and T the torsion tensor of D. Then F_p satisfies*

$$g(F_pY, Z) + g(Y, F_pZ) = -g(\phi_p(\mathscr{L}_{\xi_p}\phi_p)Y, Z) - 4dp(JY, Z) \tag{3.7}$$

for every $Y, Z \in \Gamma(\mathscr{D})$.

Proof. Since $F_pY = T(\xi_p, Y)$, we have, from (b),

$$D_{\xi_p}Y = 2p(Y)\xi_p + [\xi_p, Y] + F_pY.$$

We substitute this equation into the right hand side of $(D^\circ_{\xi_p}g)(Y, Z) = \xi_p \cdot \omega(\phi_pY, Z) - \omega(\phi_pD_{\xi_p}Y, Z) - \omega(\phi_pY, D_{\xi_p}Z)$. Since $\phi|_{\mathscr{D}} = \phi_p|_{\mathscr{D}} = J$ on $\mathscr{D}$, we consequently obtain

$$\begin{aligned}(3.8) \quad & g(F_pY, Z) + g(Y, F_pZ) \\ &= \xi_p \cdot \omega(\phi_pY, Z) - \omega(\phi_p[\xi_p, Y], Z) - \omega(\phi_pY, 2p(Z)\xi_p + [\xi_p, Z]) - (D^\circ_{\xi_p}g)(Y, Z).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(3.9) \quad -2(d\,\mathscr{L}_{\xi_p}\theta)(\phi_pY, Z) = {} & (\phi_pY) \cdot \theta([\xi_p, Z]) - Z \cdot \theta([\xi_p, \phi_pY]) + \xi_p \cdot \omega(\phi_pY, Z) \\ & - \theta([\phi_pY, \mathscr{L}_{\xi_p}Z]) - \theta([\mathscr{L}_{\xi_p}(\phi_pY), Z])\end{aligned}$$

by using Jacobi identity. Combining (3.9) with (3.8), we obtain

$$\begin{aligned}g(F_pY, Z) + g(Y, F_pZ) = {} & -2(d\,\mathscr{L}_{\xi_p}\theta)(\phi_pY, Z) - (\phi_pY) \cdot \theta([\xi_p, Z]) + Z \cdot \theta([\xi_p, \phi_pY]) \\ & - \theta([\phi_pY, 2p(Z)\xi_p]) + \theta([(\mathscr{L}_{\xi_p}\phi_p)Y, Z]) - (D^\circ_{\xi_p}g)(Y, Z).\end{aligned}$$

Furthermore by (2.7) and (d), the above equation becomes

$$\begin{aligned}(3.10) \quad & g(F_pY, Z) + g(Y, F_pZ) \\ &= 4(d\,\mathscr{L}_Q\theta)(\phi Y, Z) + 2(\phi Y) \cdot \omega(Q, Z) - 2Z \cdot \omega(Q, \phi Y) \\ &\quad - \theta([\phi Y, 2p(Z)\xi_p]) + \theta([(\mathscr{L}_{\xi_p}\phi_p)(Y), Z]) + 2p(\xi_p)\omega(\phi Y, Z).\end{aligned}$$

Next we shall calculate $4(d\,\mathscr{L}_Q\theta)(\phi Y, Z)$. If we use (c), we have

$$(3.11) \quad 2(D^\circ_Q g)(Y, Z) = 2Q \cdot \omega(\phi Y, Z) - 2\omega(D_Q(\phi Y), Z) - 2\omega(\phi Y, D_QZ).$$

We obtain $p(Q) = 0$ since g is Hermitian, so that the left hand side of (3.11) vanishes by (d). Applying this fact and (e) to (3.11), we have

$$\begin{aligned}(3.12) \quad 0 = {} & 2Q \cdot \omega(\phi Y, Z) - 2\omega(D_{\phi Y}Q, Z) - 2\theta([\mathscr{L}_Q(\phi Y) - \omega(Q, \phi Y)\xi_p, Z]) \\ & - 2\omega(\phi Y, D_ZQ) - 2\theta([\phi Y, \mathscr{L}_QZ - \omega(Q, Z)\xi_p]).\end{aligned}$$

On the other hand, a straightforward computation shows

$$\begin{aligned}4(d\,\mathscr{L}_Q\theta)(\phi Y, Z) = {} & -2(\phi Y) \cdot \omega(Q, Z) + 2Z \cdot \omega(Q, \phi Y) - 2Q \cdot \omega(\phi Y, Z) \\ & + 2\theta([\mathscr{L}_Q(\phi Y), Z]) + 2\theta([\phi Y, \mathscr{L}_QZ]).\end{aligned}$$

Combining this equation with (3.12), we obtain

$$\begin{aligned}(3.13) \quad & 4(d\,\mathscr{L}_Q\theta)(\phi Y, Z) \\ &= -2(\phi Y) \cdot \omega(Q, Z) + 2Z \cdot \omega(Q, \phi Y) - 2\omega(D_{\phi Y}Q, Z) - 2\omega(\phi Y, D_ZQ) \\ &\quad + 2\theta([\omega(Q, \phi Y)\xi_p, Z]) + 2\theta([\phi Y, \omega(Q, Z)\xi_p]).\end{aligned}$$

Moreover, we directly calculate $4dp(\phi Y, Z)$. Then we obtain

$$\begin{aligned}(3.14) \quad 4dp(\phi Y, Z) &= 2(\phi Y) \cdot p(Z) - 2Z \cdot p(\phi Y) - 2p([\phi Y, Z]) \\ &= 2g(D_{\phi Y}Q, \phi Z) + 2g(Q, D_{\phi Y}(\phi Z)) - 4p(\phi Y)g(Q, \phi Z) \\ &\quad - 2g(D_ZQ, \phi^2Y) - 2g(Q, D_Z(\phi^2Y)) + 4p(Z)g(Q, \phi^2Y) \\ &\quad - 2p(D_{\phi Y}Z - D_Z(\phi Y) + \omega(\phi Y, Z)\xi_p) \\ &= 2\omega(D_{\phi Y}Q, Z) + 2\omega(\phi Y, D_ZQ) - 2p(\xi_p)\omega(\phi Y, Z).\end{aligned}$$

Substitute (3.13) into (3.10) and use (3.14). Then we have

$$(3.15)\qquad \begin{aligned} &g(F_pY,\, Z) + g(Y,\, F_pZ) \\ &= -\,4dp(\phi Y,\, Z) + 2\theta([\omega(Q,\, \phi Y)\xi_p,\, Z]) + \theta([(\mathscr{L}_{\xi_p}\phi_p)Y,\, Z]). \end{aligned}$$

Finally since the $\mathscr{D}$-component of $(\mathscr{L}_{\xi_p}\phi_p)Y$ with respect to ξ_p is given by

$$((\mathscr{L}_{\xi_p}\phi_p)Y)_{\mathscr{D}_{\xi_p}} = (\mathscr{L}_{\xi_p}\phi_p)Y + 2\omega(Q,\, \phi Y)\xi_p,$$

substituting which into (3.15), we obtain (3.7). □

By Lemma 3.3, we have

Lemma 3.4. *Let D be a CR Weyl connection and p the corresponding 1-form. Then p satisfies*

$$(3.16)\qquad dp(JX,\, JY) + dp(X,\, Y) = 0$$

for every $X, Y \in \Gamma(\mathscr{D})$.

Proof. Applying the assumption (f) or the condition that F_p anticommutes with J to the equation (3.7), we have

$$(3.17)\qquad g(F_pX,\, Y) - g(X,\, F_pY) = 4dp(JX,\, Y).$$

Thus by anticommutativity of F_p with J, we obtain (3.16). □

Now as we deal with $\mathscr{D}$-preserving almost contact structures $(\phi, \xi, \theta)^*$ belonging to a CR structure $(\mathscr{D}, J)$, we have a unique linear connection called Tanaka connection associated with $(\phi, \xi, \theta)^*$. Therefore we have to compute the difference between a CR Weyl connection D and Tanaka connection with respect to a fixed $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$.

Proposition 3.5. *Let $(\phi, \xi, \theta)^*$ be a $\mathscr{D}$-preserving almost contact structure, D a CR Weyl connection and ∇ Tanaka connection associated with $(\phi, \xi, \theta)^*$. Define the difference H between D and ∇ by*

$$H(V,\, W) = D_VW - \nabla_VW, \qquad V, W \in \mathfrak{X}(M)$$

Then we have

$$(3.18)\quad H(X,\, Y) = p(X)Y + p(Y)X - g(X,\, Y)P + q(X)JY + q(Y)JX - g(JX,\, Y)Q,$$

$$(3.19)\qquad H(\xi,\, X) = \nabla_{JX}P + \nabla_XQ - 2q(X)P + 2p(X)Q + 2g(P,\, P)JX,$$

for every $X, Y \in \Gamma(\mathscr{D})$, where p is the 1-form of D corresponding to $(\phi, \xi, \theta)^*$, P $(\in \Gamma(\mathscr{D}))$ defined by $g(P, X) = p(X)$ for $X \in \Gamma(\mathscr{D})$, $Q = JP$ and q a 1-form defined by $q = -p \circ \phi$.

Proof. First we denote the torsion tensor of Tanaka connection by T^∇ and note that

$$(3.20)\qquad T^\nabla(Y,\, Z) = -\omega(Y,\, Z)\xi$$

for $Y, Z \in \Gamma(\mathscr{D})$ since $T^\nabla_{\mathscr{D}_\xi} = 0$ and $\theta(T^\nabla(Y,\, Z))\xi = -\omega(Y,\, Z)\xi$ by Lemma 2.2.

Computing $H(Y,\, Z) - H(Z,\, Y)$ directly, we have

$$\begin{aligned} H(Y,\, Z) - H(Z,\, Y) &= D_YZ - \nabla_YZ - D_ZY + \nabla_ZY \\ &= T(Y,\, Z) + [Y,\, Z] - (T^\nabla(Y,\, Z) + [Y,\, Z]) \\ &= T(Y,\, Z) - T^\nabla(Y,\, Z) \end{aligned}$$

for $Y, Z \in \Gamma(\mathscr{D})$. Using (e) and (3.20), we obtain

$$H(Y, Z) - H(Z, Y) = 2\omega(Y, Z)Q. \tag{3.21}$$

On the other hand, since $(D^\circ_X g)(Y, Z) = -2p(X)g(Y, Z)$ and $(\nabla^\circ_X g)(Y, Z) = 0$ for $X, Y, Z \in \Gamma(\mathscr{D})$, we have

$$g(H(X, Y), Z) + g(Y, H(X, Z)) = 2p(X)g(Y, Z). \tag{3.22}$$

In the equation (3.22) we permute X, Y and Z cyclically and subtract one from the sum of the other two. Applying (3.21) to the resulting equation, we have the equation (3.18). Next we compute $H(\xi, X)$ for $X \in \Gamma(\mathscr{D})$. Since $F_p X = D_{\xi_p} X - 2p(X)\xi_p - [\xi_p, X]$ and $FX = \nabla_\xi X - [\xi, X]$, we have

$$\begin{aligned} H(\xi_p, X) &= D_{\xi_p} X - \nabla_{\xi_p} Y = F_p X + [\xi_p, X] + 2p(X)\xi_p - \nabla_\xi X + 2\nabla_Q X \\ &= F_p X + [\xi - 2Q, X] + 2p(X)\xi_p - (FX + [\xi, X]) + 2\nabla_Q X \\ &= F_p X - FX - 2[Q, X] + 2\nabla_Q X + 2p(X)\xi_p. \end{aligned}$$

Furthermore, applying (3.20) to this equation and noting that $\omega(Q, X) = p(X)$, we have

$$H(\xi_p, X) = F_p X - FX + 2\nabla_X Q - 4p(X)Q. \tag{3.23}$$

Now computing $F_p X - FX$ directly by the equation $F_p = -1/2\phi_p(\mathscr{L}_{\xi_p}\phi_p)$ and $F = -1/2\phi(\mathscr{L}_\xi \phi)$, we have

$$\begin{aligned} F_p X - FX &= -\frac{1}{2}\{\phi_p([\xi_p, JX] - \phi_p[\xi_p, X]) - J([\xi, JX] - J[\xi, X])\} \\ &= -\frac{1}{2}\{\phi_p([\xi, JX] - 2[Q, JX] - J[\xi, X] + 2\phi_p[Q, X]) \\ &\quad - J([\xi, JX] - J[\xi, X])\} \\ &= (\phi - 2\theta \otimes P)[Q, JX] - (\phi - 2\theta \otimes P)^2[Q, X] \\ &= \phi(\nabla_Q JX - \nabla_{JX} Q + \omega(Q, JX)\xi) \\ &\quad - 2\theta(\nabla_Q JX - \nabla_{JX} Q + \omega(Q, JX)\xi)P \\ &\quad - (\phi - 2\theta \otimes P)\{\phi(\nabla_Q X - \nabla_X Q + \omega(Q, X)\xi) \\ &\quad - 2\theta(\nabla_Q X - \nabla_X Q + \omega(Q, X)\xi)P\} \\ &= \nabla_{JX} P - \nabla_X Q + 2q(X)P + 2p(X)Q. \end{aligned} \tag{3.24}$$

Therefore we have

$$H(\xi_p, X) = \nabla_{JX} P + \nabla_X Q + 2q(X)P - 2p(X)Q. \tag{3.25}$$

In the equation $H(\xi_p, X) = H(\xi, X) - 2H(Q, X)$, we use (3.18) for $H(Q, X)$ and (3.25) for $H(\xi_p, X)$. Then we obtain the equation (3.19). □

Remark. We can compute $H(X, \xi)$ and $H(\xi, \xi)$ by the same way as the equation (3.19). They are given by

$$H(X, \xi) = 2\nabla_X Q - 4p(X)Q - 4q(X)P + 2g(P, P)JX + 2p(X)\xi, \tag{3.26}$$

$$H(\xi, \xi) = 2(\nabla_\xi Q - \nabla_P P + \nabla_Q Q - 4g(P, P)P - 2p(\xi)Q) + 2p(\xi)\xi. \tag{3.27}$$

Conversely, one may ask whether given Tanaka connection ∇ and p define a CR Weyl connection. We have the following answer to this question.

Proposition 3.6. *Let $(\phi,\xi,\theta)^*$ be a $\mathscr{D}$-preserving almost contact structure belonging to CR structure $(\mathscr{D},J)$ and ∇ Tanaka connection associated with $(\phi,\xi,\theta)^*$. If D is defined by $D_V W = \nabla_V W + H(V, W)$ for a given p satisfying* (3.16), *where H is defined by* (3.18), (3.19), (3.26) *and* (3.27), *then it becomes a CR Weyl connection.*

By Propositon 3.6 we see that an arbitrary pair $(p, (\phi,\xi,\theta)^*)$ of a 1-form p satisfying (3.16) and a $\mathscr{D}$-preserving almost contact structure $(\phi,\xi,\theta)^*$ determines a CR Weyl structure.

4. The View from G-structure

Let M be an oriented $(2n+1)$-dimensional manifold and $\pi : F^+(M) \to M$ the principal bundle of positively oriented frames over M. Assume that a pair $(\mathscr{D}, J)$ of a hyperdistribution $\mathscr{D}$ and a complex structure J on $\mathscr{D}$ is given on M. In addition, we assume that $(\mathscr{D}, J)$ is a nondegenerate CR structure. Now we define the subspace $\mathscr{D}_0$ in $\mathbb{R}^{2n+1}$, the matrix $\tilde{J}_0 \in GL(2n+1;\mathbb{R})$ and the matrix $J_0 \in GL(2n;\mathbb{R})$ by

$$(4.1)\quad \mathscr{D}_0 = \left\{ \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} \in \mathbb{R}^{2n+1} \,\middle|\, x^0 = 0 \right\}, \quad \tilde{J}_0 = \begin{pmatrix} 0 & {}^t\mathbf{0} \\ \mathbf{0} & J_0 \end{pmatrix} \text{ and } J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

respectively, where I_n is $n \times n$ unit matrix and the boldface denotes a column vector of degree $2n$. We have a principal subbundle $\overline{\mathfrak{P}}$ of $F^+(M)$:

$$\overline{\mathfrak{P}} = \{\, u \in F(M) \mid u\mathscr{D}_0 \subset \mathscr{D},\ Ju|_{\mathscr{D}_0} = u\tilde{J}_0|_{\mathscr{D}_0} \,\}$$

whose structure group is

$$\overline{G} = \left\{ \begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix} \,\middle|\, a > 0,\ \mathbf{b} \in \mathbb{R}^{2n},\ CJ_0 = J_0 C \right\},$$

where the linear frame u is considered as a linear map from $\mathbb{R}^{2n+1}$ to $T_{\pi(u)}M$(cf. [4]). Futhermore we define $\theta_0 \in (\mathbb{R}^{2n+1})^*$ and $\xi_0 \in \mathbb{R}^{2n+1}$ by

$$(4.2)\qquad \theta_0 = \begin{pmatrix} 1 & {}^t\mathbf{0} \end{pmatrix} \qquad \text{and} \qquad \xi_0 = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$$

respectively. By using a local cross section $\bar{\sigma}$ of $\overline{\mathfrak{P}}$, we define a 1-form $\theta^{\bar{\sigma}}$ and vector field $\xi^{\bar{\sigma}}$ on an open set $U^{\bar{\sigma}}$ by

$$(4.3)\qquad \theta^{\bar{\sigma}} = \theta_0 \bar{\sigma}^{-1} \qquad \text{and} \qquad \xi^{\bar{\sigma}} = \bar{\sigma}\xi_0$$

respectively. Then we obtain

$$(4.4)\qquad \theta^{\bar{\sigma}}|_{\mathscr{D}} = 0\,, \qquad \theta^{\bar{\sigma}}(\xi^{\bar{\sigma}}) = 1$$

because of their definitions. Note that the definitions of $\theta^{\bar{\sigma}}$ and $\xi^{\bar{\sigma}}$ are dependent of the local section $\bar{\sigma}$. Later on, we shall study between $\theta^{\bar{\sigma}}$ (resp. $\xi^{\bar{\sigma}}$) and $\theta^{\bar{\tau}}$ (resp. $\xi^{\bar{\tau}}$) defined by another local section $\bar{\tau}$ whose domain has non empty intersection with $U^{\bar{\sigma}}$.

Next, we define a 2-form $\omega^{\bar{\sigma}}$ by

$$(4.5)\qquad \omega^{\bar{\sigma}} = -2d\theta^{\bar{\sigma}}.$$

Then since we assume that $(\mathscr{D}, J)$ is a nondegenerate CR structure, we see that $\omega^{\bar{\sigma}}$ is a nondegenerate and Hermitian 2-form when it is restricted to $\mathscr{D}$:

$$(4.6)\qquad \omega^{\bar{\sigma}}(JX, JY) = \omega^{\bar{\sigma}}(X, Y)$$

for $X, Y \in \Gamma(U^{\bar\sigma}, \mathscr{D})$, where $\Gamma(U^{\bar\sigma}, \mathscr{D})$ denotes the set of cross sections on $U^{\bar\sigma}$ of the vector bundle $\mathscr{D}$. By using $\omega^{\bar\sigma}$, we define $B^{\bar\sigma} \in \Gamma(U^{\bar\sigma}, \mathscr{D})$ by

$$\omega^{\bar\sigma}(B^{\bar\sigma}, X) = -\omega^{\bar\sigma}(\xi^{\bar\sigma}, X) \tag{4.7}$$

for every $X \in \Gamma(U^{\bar\sigma}, \mathscr{D})$. This is uniquely defined since $\omega^{\bar\sigma}$ is nondegenerate. Moreover, define a local bilinear form $g^{\bar\sigma}$ on $\mathscr{D}$ by

$$g^{\bar\sigma}(X, Y) = \omega^{\bar\sigma}(JX, Y), \tag{4.8}$$

which satisfies the equations

$$g^{\bar\sigma}(X, Y) = g^{\bar\sigma}(Y, X), \qquad g^{\bar\sigma}(JX, JY) = g^{\bar\sigma}(X, Y) \tag{4.9}$$

for $X, Y \in \Gamma(U^{\bar\sigma}, \mathscr{D})$. Thus it becomes a fiber pseudo-metric of $\mathscr{D}$ defined on $U^{\bar\sigma}$. When we take two local cross sections $\bar\sigma$ and $\bar\tau$ of $\overline{\mathfrak{P}}$ defined on $U^{\bar\sigma}$ and $U^{\bar\tau}$ respectively, we suppose that they are related by $\bar\tau = \bar\sigma\bar h$ on $U^{\bar\sigma\bar\tau}$, where $U^{\bar\sigma\bar\tau}$ denotes the intersection of $U^{\bar\sigma}$ and $U^{\bar\tau}$, $\bar h$ is a $\overline{G}$-valued function of the form

$$\bar h = \begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix} = \begin{pmatrix} e^{-2\mu} & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix}, \tag{4.10}$$

and μ a function on $U^{\bar\sigma\bar\tau}$. Then we obtain

$$\theta^{\bar\tau} = e^{2\mu}\theta^{\bar\sigma} \tag{4.11}$$

on $U^{\bar\sigma\bar\tau}$. Thus we have

$$\omega^{\bar\tau} = e^{2\mu}\omega^{\bar\sigma} - 4d\mu \wedge \theta^{\bar\tau} \tag{4.12}$$

because of (4.5). Furthermore we have

$$\omega^{\bar\tau}|_{\mathscr{D}} = e^{2\mu}\omega^{\bar\sigma}|_{\mathscr{D}}, \qquad g^{\bar\tau}|_{\mathscr{D}} = e^{2\mu}g^{\bar\sigma}|_{\mathscr{D}}. \tag{4.13}$$

Therefore we have the conformal structure $[g^{\bar\sigma}]$ over $\mathscr{D}$. Let $a(p, \bar\sigma)$ be the dimension of the maximal subspace in $\mathscr{D}_p$ where $g^{\bar\sigma}_p$ is negative definite for each point $p \in M$ and local section $\bar\sigma$ defined on a neighborhood of p. These numbers are necessary even and we see from the equation (4.13) that $a(p, \bar\sigma)$ depends only on p. So we put $\gamma(p) = a(p, \bar\sigma)$. Since γ is a lower semicontinuous function on M and M is connected, we easily see that it is constant.

Now we define a subbundle $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ by

$$\mathfrak{P} = \left\{ u \in \overline{\mathfrak{P}} \;\middle|\; g^{\bar\sigma}_{\pi(u)}\left(u\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}, u\begin{pmatrix} 0 \\ \mathbf{y} \end{pmatrix}\right) = {}^t\mathbf{x}\widetilde{E_\gamma}\mathbf{y},\ \bar\sigma(\pi(u)) = u \right\}$$

whose structure group is

$$G = \left\{ \begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix} \;:\; a > 0,\ \mathbf{b} \in \mathbb{R}^{2n},\ CJ_0 = J_0C,\ {}^tC\widetilde{E_\gamma}C = a\widetilde{E_\gamma} \right\}, \tag{4.14}$$

where

$$\widetilde{E_\gamma} = \begin{pmatrix} E_\gamma & 0 \\ 0 & E_\gamma \end{pmatrix}, \qquad E_\gamma = \begin{pmatrix} -I_\gamma & 0 \\ 0 & I_{n-\gamma} \end{pmatrix}.$$

We remark that $C \in CU_\gamma = GL(n, \mathbb{C}) \cap CO(2\gamma, 2n - 2\gamma)$. A local cross section σ of $\mathfrak{P}$ is witten as

$$\sigma = \langle \xi^\sigma, X_1, \ldots, X_n, JX_1, \ldots, JX_n \rangle, \tag{4.15}$$

where $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ is a local orthonormal frame field of $\mathscr{D}$ with respect to g^σ. And we can also express (4.15) as follows:

$$X_i = \sigma e_i\ , \qquad JX_i = \sigma \widetilde{J}_0 e_i \qquad (i = 1, \ldots, n) \tag{4.16}$$

where $e_i = {}^t(\ 0 \mid 0\ \cdots\ 1\ \cdots\ 0 \mid 0\ \cdots\ 0\ )$.

Let $\mathfrak{g}$ and $\mathfrak{cu}_\gamma$ denote the Lie algebra of G and CU_γ respectively. Let $\vartheta = \begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix}$ be a connection form of a linear connection D reducible to $\mathfrak{P}$, where ϕ is $\mathbb{R}$-valued 1-form, η $\mathbb{R}^{2n}$-valued 1-form and α $\mathfrak{cu}_\gamma$-valued 1-form on $\mathfrak{P}$. The connection form ϑ satisfies

$$\vartheta(A^*) = A\ , \qquad R_h^*\vartheta = Ad(h^{-1})\vartheta, \tag{4.17}$$

where A^* denotes the fundamental vector field corresponding to $A \in \mathfrak{g}$ and h an element of G. Since

$$\begin{aligned} R_h^*\begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix} &= Ad(h^{-1})\begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix} = \begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix}^{-1}\begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix}\begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix} \\ &= \begin{pmatrix} 1/a & {}^t\mathbf{0} \\ -(1/a)C^{-1}\mathbf{b} & C^{-1} \end{pmatrix}\begin{pmatrix} \phi a & {}^t\mathbf{0} \\ \eta a + \alpha\mathbf{b} & \alpha C \end{pmatrix} = \begin{pmatrix} \phi & {}^t\mathbf{0} \\ * & C^{-1}\alpha C \end{pmatrix} \end{aligned}$$

where $* = C^{-1}(-\phi\mathbf{b} + a\eta + \alpha\mathbf{b})$, we have

$$R_h^*\phi = \phi, \qquad R_h^*\eta = C^{-1}(-\phi\mathbf{b} + a\eta + \alpha\mathbf{b}), \qquad R_h^*\alpha = Ad(C^{-1})\alpha. \tag{4.18}$$

Now let σ and τ be local cross sections of $\mathfrak{P}$ defined on U^σ and U^τ respectively. Suppose that they are related by $\tau = \sigma h$ on $U^{\sigma\tau}$, where h is a G-valued function of the form as (4.10) with $C \in CU_\gamma$. Then, for the differential maps of σ and τ, we have

$$d\tau(V) = dR_h(d\sigma(V)) + (\,h^{-1}(dh)(V)\,)^* \tag{4.19}$$

for $V \in \mathfrak{X}(U^{\sigma\tau})$ (cf. [4]). Applying the connection form ϑ to (4.19), we obtain, from (4.17),

$$\tau^*\vartheta = Ad(h^{-1})\sigma^*\vartheta + h^{-1}dh. \tag{4.20}$$

On the other hand, we have

$$h^{-1}dh = \begin{pmatrix} 1/a & 0 \\ -(1/a)C^{-1}\mathbf{b} & C^{-1} \end{pmatrix}\begin{pmatrix} da & {}^t\mathbf{0} \\ d\mathbf{b} & dC \end{pmatrix} = \begin{pmatrix} a^{-1}da & {}^t\mathbf{0} \\ ** & C^{-1}dC \end{pmatrix} \tag{4.21}$$

where $** = C^{-1}(-a^{-1}\mathbf{b}da + d\mathbf{b})$. In particular, we have

$$a^{-1}da = -2d\mu \tag{4.22}$$

by (4.10), and hence from (4.20) we obtain

$$\tau^*\phi = \sigma^*\phi - 2d\mu. \tag{4.23}$$

We put $2p^\sigma = \sigma^*\phi$ and $2p^\tau = \tau^*\phi$ for local cross section σ and τ respectively. Thus we obtain

$$p^\tau = p^\sigma - d\mu. \tag{4.24}$$

We regard local cross sections of $\mathfrak{P}$ as those of $\overline{\mathfrak{P}}$. Then we also have θ^σ, ξ^σ, ω^σ and g^σ on U^σ. We define P^σ, $Q^\sigma \in \Gamma(U^\sigma, \mathscr{D})$ by

$$g^\sigma(P^\sigma, X) = p^\sigma(X)\,, \qquad Q^\sigma = JP^\sigma \tag{4.25}$$

for every $X \in \Gamma(U^\sigma, \mathscr{D})$. Then we have

Lemma 4.1. *Let σ and τ be two local cross sections of $\mathfrak{P}$ such that $\tau = \sigma h$ on $U^{\sigma\tau}$. We put $\xi_{p^\sigma} = \xi^\sigma + B^\sigma - 2Q^\sigma$, where B^σ is defined by (4.7). Then ξ_{p^σ} and ξ_{p^τ} are related by*

$$\xi_{p^\tau} = e^{-2\mu}\xi_{p^\sigma}. \tag{4.26}$$

It follows that we have a transversal line bundle $\mathscr{L} = \operatorname{span}\{\xi_{p^\sigma}\}$ associated with the connection D.

Proof. Since $0 = \omega^\tau(\xi^\tau + B^\tau, X) = \omega^\sigma(\xi^\sigma + B^\sigma, X)$ because of (4.7), by using (4.12) we have

$$\begin{aligned} 0 = \omega^\tau(\xi^\tau + B^\tau, X) &= e^{2\mu}\omega^\sigma(\xi^\tau + B^\tau, X) - 4(d\mu \wedge \theta^\tau)(\xi^\tau + B^\tau, X) \\ &= e^{2\mu}\omega^\sigma(\xi^\tau + B^\tau, X) - \omega^\sigma(\xi^\sigma + B^\sigma, X) + 2d\mu(X). \end{aligned}$$

Define $(d\mu^\sharp)^\sigma \in \Gamma(U^{\sigma\tau}, \mathscr{D})$ by

$$g^\sigma((d\mu^\sharp)^\sigma, X) = d\mu(X) \tag{4.27}$$

for every $X \in \Gamma(U^{\sigma\tau}, \mathscr{D})$. We have

$$\begin{aligned} e^{2\mu}\omega^\sigma(\xi^\tau + B^\tau, X) &= \omega^\sigma(\xi^\sigma + B^\sigma, X) - 2g^\sigma((d\mu^\sharp)^\sigma, X) \\ &= \omega^\sigma(\xi^\sigma + B^\sigma, X) - 2\omega^\sigma(J(d\mu^\sharp)^\sigma, X). \end{aligned}$$

Therefore, since ω^σ is nondegenerate, we obtain

$$\xi^\tau + B^\tau = e^{-2\mu}(\xi^\sigma + B^\sigma - 2J(d\mu^\sharp)^\sigma). \tag{4.28}$$

We have, from (4.24),

$$\begin{aligned} g^\tau(Q^\tau, X) = g^\tau(JP^\tau, X) &= -g^\tau(P^\tau, JX) \\ &= -p^\tau(JX) \\ &= -(p^\sigma - d\mu)(JX) \\ &= -p^\sigma(JX) + d\mu(JX) \\ &= g^\sigma(Q^\sigma - J(d\mu^\sharp)^\sigma, X). \end{aligned}$$

It follows that

$$Q^\tau = e^{-2\mu}(Q^\sigma - J(d\mu^\sharp)^\sigma). \tag{4.29}$$

Combining (4.28) with (4.29), we obtain (4.26). □

Next we investigate the covariant derivative D of TM determined by ϑ. We take a fixed local frame field (4.15) of TM. Note that, for a fixed $W \in \mathfrak{X}(U^\sigma)$, the local frame field σ induces a map $\sigma^{-1}W : x \in U^\sigma \mapsto \sigma(x)^{-1}W(x) \in \mathbb{R}^{2n+1}$. The covariant derivative of $W \in \mathfrak{X}(U^\sigma)$ in the direction $V \in TU^\sigma$ is given by

$$D_V W = \sigma(d(\sigma^{-1}W)(V) + (\sigma^*\vartheta(V))(\sigma^{-1}W)). \tag{4.30}$$

Futhermore, since $\sigma^{-1}X \in \mathbb{R}^{2n}$ for $X \in \Gamma(U^\sigma, \mathscr{D})$, we obtain

$$D_V X = \sigma(d(\sigma^{-1}X)(V) + (\sigma^*\alpha(V))(\sigma^{-1}X)). \tag{4.31}$$

Note that the product of the second term of the right hand side in the equations above is the matrix multiplication. From (4.31), it is clear that

$$D_V\Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D}). \tag{4.32}$$

It follows that D induces the covariant differentiation of the vector bundle $\mathscr{D}$, which is denoted by D°.

Lemma 4.2. *Let D be the covariant derivative of TM determined by ϑ and D° the covariant derivative on $\mathscr{D}$ determined by α. Then D and D° satisfy*

$$D_V\theta^\sigma = -2p^\sigma(V)\theta^\sigma, \qquad D^\circ_V J = 0 \qquad \text{and} \qquad D^\circ_V g^\sigma = -2p^\sigma(V)g^\sigma \tag{4.33}$$

for $V \in TU^\sigma$.

Proof. Since

$$(D_V\theta^\sigma)(X) = V\cdot\theta^\sigma(X) - \theta^\sigma(D_VX) = -\theta^\sigma(D^\circ_VX) = 0$$

for every $X \in \Gamma(U^\sigma, \mathscr{D})$, we have $(D_V\theta^\sigma)(X) = -2p^\sigma(V)\theta^\sigma(X)$. Futhermore, we have

$$(D_V\theta^\sigma)(\xi^\sigma) = V\cdot(\theta^\sigma\xi^\sigma) - \theta^\sigma(D_V\xi_{p^\sigma}) = -\theta^\sigma(\,\sigma(\,(\sigma^*\vartheta V)(\sigma^{-1}\xi^\sigma)\,)\,) = -\sigma^*\phi(V)\theta^\sigma(\xi^\sigma).$$

Thus we obtain $(D_V\theta^\sigma)(W) = -2p^\sigma(V)\theta^\sigma(W)$ for every $W \in \mathfrak{X}(U^\sigma)$. Next we have

$$\begin{aligned}
J(D_VY_\lambda) &= J\{\,\sigma((\sigma^*\alpha(V))\sigma^{-1}Y_\lambda)\,\}\\
&= \sigma\{\,\widetilde{J}_0(\sigma^*\alpha(V))\sigma^{-1}Y_\lambda\,\}\\
&= \sigma\{\,(\sigma^*\alpha(V))\sigma^{-1}(\sigma\widetilde{J}_0\sigma^{-1}Y_\lambda)\,\}\\
&= \sigma\{\,(\sigma^*\alpha(V))(\sigma^{-1}JY_\lambda)\,\}\\
&= D_V(JY_\lambda)
\end{aligned}$$

for $\lambda = 1,\cdots,2n$, where we have put $Y_i = X_i$, $Y_{n+i} = JX_i$ $(i = 1,\cdots,n)$. Therefore, since $(D^\circ_VJ)(Y_\lambda) = D_V(JY_\lambda) - J(D_VY_\lambda)$, we obtain $D^\circ_VJ = 0$. At last we show that $D^\circ_Vg^\sigma = -2p^\sigma(V)g^{\bar\sigma}$. Since

$${}^t\alpha\widetilde{E_\gamma} + \widetilde{E_\gamma}\alpha = \phi\widetilde{E_\gamma},$$

we have

$$\begin{aligned}
g^\sigma(D^\circ_VY_\lambda,\, Y_\mu) &= {}^t\{(\sigma^*\alpha(V))\sigma^{-1}Y_\lambda\}\widetilde{E_\gamma}(\sigma^{-1}Y_\mu)\\
&= {}^t(\sigma^{-1}Y_\lambda){}^t(\sigma^*\alpha(V))\widetilde{E_\gamma}(\sigma^{-1}Y_\mu)\\
&= {}^t(\sigma^{-1}Y_\lambda)\{\,-\widetilde{E_\gamma}(\sigma^*\alpha(V)) + \sigma^*\phi(V)\widetilde{E_\gamma}\,\}(\sigma^{-1}Y_\mu)\\
&= -\,g^\sigma(Y_\lambda,\, D^\circ_VY_\mu) + 2p^\sigma(V)g^\sigma(Y_\lambda,\, Y_\mu).
\end{aligned}$$

Therefore, for local frame $\{Y_\lambda\}$ of $\mathscr{D}$, we have

$$(D^\circ_Vg^\sigma)(Y_\lambda,\, Y_\mu) = -2p^\sigma(V)g^\sigma(Y_\lambda,\, Y_\mu) \tag{4.34}$$

from which we obtain $D^\circ_Vg^\sigma = -2p^\sigma(V)g^\sigma$. □

Now assume that the torsion tensor T of D satisfies

$$T(X,\, Y) \in \mathscr{L}, \tag{4.35}$$

$$T(L,\, X) \in \mathscr{D}, \qquad T(L,\, JX) = -JT(L,\, X), \tag{4.36}$$

$$(D_UT)(X,\, Y) = 0 \tag{4.37}$$

for $U \in TM$, $X, Y \in \mathscr{D}$ and $L \in \mathscr{L}$. Then we have

$$\theta^\sigma(T(X,\, Y)) = -\omega^\sigma(X,\, Y), \qquad \theta^\sigma(\xi_{p^\sigma}) = 1$$

because of (4.32) and (4.35). Therefore we obtain

$$T(X,\, Y) = -\omega^\sigma(X,\, Y)\xi_{p^\sigma} \tag{4.38}$$

for $X, Y \in \mathscr{D}$. We define $(1,1)$ tensor ϕ^σ by $\phi^\sigma \xi^\sigma = 0$ and $\phi^\sigma X = JX$ for $X \in \mathscr{D}$, and moreover ϕ_{p^σ} by

$$\phi_{p^\sigma} = \phi^\sigma - 2\theta \otimes \left(P^\sigma + \frac{1}{2}JB^\sigma\right). \tag{4.39}$$

It is easy to show that

$$\phi_{p^\sigma} X = JX, \qquad \phi_{p^\sigma} = \phi_{p^\tau}, \qquad \theta^\sigma \circ \phi_{p^\sigma} = 0, \qquad \phi_{p^\sigma}\xi_{p^\sigma} = 0. \tag{4.40}$$

Since $\theta^\sigma(D_V\xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma}) = 0$, we have

$$\begin{aligned} g^\sigma(D_V\xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma},\, Y) &= \omega^\sigma(J(D_V\xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma}),\, Y) \\ &= \omega^\sigma(\phi_{p^\sigma}(D_V\xi_{p^\sigma}) - 2p^\sigma(V)\phi_{p^\sigma}\xi_{p^\sigma},\, Y) \\ &= \omega^\sigma(\phi_{p^\sigma}(D_V\xi_{p^\sigma}),\, Y) \\ &= \omega^\sigma(-(D_V\phi_{p^\sigma})\xi_{p^\sigma},\, Y) \\ &= g^\sigma(J(D_V\phi_{p^\sigma})\xi_{p^\sigma},\, Y). \end{aligned}$$

Therefore we obtain

$$D_V\xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma} = J(D_V\phi_{p^\sigma})\xi_{p^\sigma} \tag{4.41}$$

for $V \in \mathfrak{X}(U^\sigma)$. From (4.37) and (4.33), we have

$$\begin{aligned} 0 =& D_V(T(X,\, Y)) - T(D_VX,\, Y) - T(X,\, D_VY) \\ =& - D_V(\omega^\sigma(X,\, Y)\xi_{p^\sigma}) + \omega^\sigma(D_VX,\, Y)\xi_{p^\sigma} + \omega^\sigma(X,\, D_VY)\xi_{p^\sigma} \\ =& 2p^\sigma(V)\omega^\sigma(X,\, Y)\xi_{p^\sigma} - \omega^\sigma(X,\, Y)D_V\xi_{p^\sigma}. \end{aligned}$$

Combining this equation with (4.41), we obtain

$$D_V\xi_{p^\sigma} = 2p^\sigma(V)\xi_{p^\sigma}, \qquad D_V\phi_{p^\sigma} = 0. \tag{4.42}$$

In particular, $D_{\xi_{p^\sigma}}\phi_{p^\sigma} = 0$ and hence

$$\begin{aligned} 0 &= D_{\xi_{p^\sigma}}(\phi_{p^\sigma}X) - \phi_{p^\sigma}(D_{\xi_{p^\sigma}}X) \\ &= F_{p^\sigma}\phi_{p^\sigma}X + D_{\phi_{p^\sigma}X}\xi_{p^\sigma} + [\xi_{p^\sigma},\, \phi_{p^\sigma}X] \\ &\quad - \phi_{p^\sigma}(F_{p^\sigma}X + D_X\xi_{p^\sigma} + [\xi_{p^\sigma},\, X]) \\ &= [F_{p^\sigma},\, J]X + (\mathscr{L}_{\xi_{p^\sigma}}\phi_{p^\sigma})X + 2p^\sigma(JX)\xi_{p^\sigma}, \end{aligned}$$

where we have put $F_{p^\sigma}X = T(\xi_{p^\sigma},\, X)$. Equation (4.36) implies that

$$T(\xi_{p^\sigma},\, X) = -\frac{1}{2}\phi_{p^\sigma}(\mathscr{L}_{\xi_{p^\sigma}}\phi_{p^\sigma})X. \tag{4.43}$$

Finally, if σ satisfies $\omega^\sigma(X,\, \xi^\sigma) = 0$, then we obtain, from Lemma 3.4,

$$dp^\sigma(JX,\, JY) + dp^\sigma(X,\, Y) = 0 \tag{4.44}$$

for $X, Y \in \mathscr{D}$.

Proposition 4.3. *Let $\mathfrak{P}(M, G)$ be the subbundle determined by the CR structure and D a linear connection reducible to $\mathfrak{P}(M, G)$. Then there is a 1-dimensional distribution $\mathscr{L}$ on M transversal to $\mathscr{D}$. For a local cross section σ of $\mathfrak{P}(M, G)$, D satisfies*

$$D_V\theta^\sigma = -2p^\sigma(V)\theta^\sigma, \qquad D_V^\circ J = 0, \qquad D_V^\circ g^\sigma = -2p^\sigma(V)g^\sigma$$

for $V \in \mathfrak{X}(M)$. Moreover, if the torsion tensor T of D satisfies

$$T(X,\, Y) \in \mathscr{L}, \quad T(L,\, X) \in \mathscr{D}, \quad T(L,\, JX) = -JT(L,\, X), \quad (D_VT)(X,\, Y) = 0$$

for $X, Y \in \mathscr{D}$ and $L \in \mathscr{L}$, then D satisfies

$$T(X, Y) = -\omega^\sigma(X, Y)\xi_{p^\sigma}, \qquad T(\xi_{p^\sigma}, X) = -\frac{1}{2}\phi_{p^\sigma}(\mathscr{L}_{\xi_{p^\sigma}}\phi_{p^\sigma})X,$$
$$D_V\xi_{p^\sigma} = 2p^\sigma(V)\xi_{p^\sigma}, \qquad D_V\phi_{p^\sigma} = 0$$

and if $\omega^\sigma(X, \xi^\sigma) = 0$ holds for $X \in \mathscr{D}$, then p^σ satisfies

$$dp^\sigma(JX, JY) + dp^\sigma(X, Y) = 0.$$

Remark. We assume that M is orientable. Then we have a nonvanishing globally defined vector field ξ transversal to $\mathscr{D}$. Then, for the local cross section σ and τ of the form (4.15), h reduces to a matrix that

$$\begin{pmatrix} 1 & {}^t0 \\ \mathrm{b} & C \end{pmatrix},$$

where ${}^tC\widetilde{E_\gamma}C = \widetilde{E_\gamma}$. It follows from (4.11), (4.13) and (4.24) that θ, ϕ, ω, g and p are globally defined on M, and ξ_p is a global section of $\mathscr{L}$. Moreover, if we take ξ such that $\omega(X, \xi) = 0$ for every $X \in \mathscr{D}$, then p satisfies (3.16).

5. Curvature of CR Weyl connection

In this section, we investigate the property of the curvature of a CR Weyl connection. Let D be a CR Weyl connection of the CR structure $(\mathscr{D}, J)$. Let R be the curvature tensor field of D defined by

$$R(U, V)W = D_UD_VW - D_VD_UW - D_{[U,V]}W$$

for $U, V, W \in \mathfrak{X}(M)$. We fix a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ and let p be the 1-form of D corresponding to $(\phi, \xi, \theta)^*$. Since $D\xi_p = 2p \otimes \xi_p$, we see easily that

(5.1) $$R(U, V)\xi_p = 4dp(U, V)\xi_p, \qquad U, V \in TM.$$

The property $D_U\Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D})$ implies that

(5.2) $$R(U, V)\mathscr{D} \subset \mathscr{D}, \qquad U, V \in TM.$$

Since $D\phi_p = 0$, we have

(5.3) $$R(U, V)\phi_p = \phi_pR(U, V), \qquad U, V \in TM.$$

If we put $R(U, V, X, Y) = g(R(U, V)X, Y)$ for $U, V \in TM$ and $X, Y \in \mathscr{D}$, then we have the equation

(5.4) $$R(U, V, X, Y) = -R(U, V, Y, X) + 4dp(U, V)g(X, Y).$$

The first Bianchi identity is the formula (cf. [4]):

$$\mathfrak{S}\{R(U, V)W\} = \mathfrak{S}\{T(T(U, V), W) + (D_UT)(V, W)\},$$

where $U, V, W \in TM$ and $\mathfrak{S}$ denotes the cyclic sum with respect to U, V and W. Replacing U, V, W with $X, Y, Z \in \mathscr{D}$ respectively in the first Bianchi identity above, we have, from (3.2),

$$\mathfrak{S}\{R(X, Y)Z\} = \mathfrak{S}\{T(T(X, Y), Z)\}.$$

Moreover, applying the condition (e) in the definition of a CR Weyl connection to the above equation, we obtain

(5.5) $$\mathfrak{S}\{R(X, Y)Z\} = -\mathfrak{S}\{\omega(X, Y)F_pZ\}$$

for every $X, Y, Z \in \mathscr{D}$. Putting $U = \xi_p$ and replacing V, W with $Y, Z \in \mathscr{D}$ in the first Bianchi identity, we have

$$
\begin{aligned}
&R(\xi_p, Y)Z + R(Y, Z)\xi_p + R(Z, \xi_p)Y \\
&= T(T(\xi_p, Y), Z) + T(T(Y, Z), \xi_p) + T(T(Z, \xi_p), Y) \\
&\quad + (D_{\xi_p}T)(Y, Z) + (D_Y T)(Z, \xi_p) + (D_Z T)(\xi_p, Y) \\
&= T(F_pY, Z) + T(-\omega(Y, Z)\xi_p, \xi_p) - T(T(\xi_p, Z), Y) - (D_Y T)(\xi_p, Z) + (D_Z T)(\xi_p, Y) \\
&= -\omega(F_pY, Z)\xi_p + \omega(F_pZ, Y)\xi_p - D_Y(T(\xi_p, Z)) + T(D_Y\xi_p, Z) + T(\xi_p, D_Y Z) \\
&\quad + D_Z(T(\xi_p, Y)) - T(D_Z\xi_p, Y) - T(\xi_p, D_Z Y) \\
&= -\omega(F_pY, Z)\xi_p + \omega(F_pZ, Y)\xi_p - (D_Y F_p)Z + 2p(Y)F_pZ + (D_Z F_p)Y - 2p(Z)F_pY,
\end{aligned}
$$

where we have used (3.2) and (b), (e) in the definition of a CR Weyl connection. In addition, when we rewrite (3.17) with ω, we have

$$\omega(F_pX, Y) + \omega(X, F_pY) = -4dp(X, Y). \tag{5.6}$$

Substituting (5.1) and (5.6) into the first Bianchi identity including ξ_p above, we obtain

$$R(\xi_p, Y)Z - R(\xi_p, Z)Y = -\{(D_Y F_p)Z - 2p(Y)F_pZ\} + \{(D_Z F_p)Y - 2p(Z)F_pY\} \tag{5.7}$$

for $Y, Z \in \mathscr{D}$. Since the second Bianchi identity is the formula:

$$\mathfrak{S}\{(D_U R)(V, W)\} = -\mathfrak{S}\{R(T(U, V), W)\}$$

for $U, V, W \in TM$, we have immediately

$$\mathfrak{S}\{(D_X R)(Y, Z)\} = \mathfrak{S}\{\omega(X, Y)R(\xi_p, Z)\} \tag{5.8}$$

for $X, Y, Z \in \mathscr{D}$. Furthermore, if we put $U = \xi_p$ and replace V, W with $Y, Z \in \mathscr{D}$ respectively in the second Bianchi identity, then

$$(D_{\xi_p}R)(Y, Z) - (D_Y R)(\xi_p, Z) + (D_Z R)(\xi_p, Y) = -R(F_pY, Z) + R(F_pZ, Y). \tag{5.9}$$

We shall prove the following formula:

$$
\begin{aligned}
&R(X, Y, Z, W) - 2g(JX, Y)dp(JZ, W) + 2g(X, Y)dp(Z, W) \\
&-R(Z, W, X, Y) + 2g(JZ, W)dp(JX, Y) - 2g(Z, W)dp(X, Y) \\
&= -g(JX, Z)g(F_pY, W) + 2g(JX, Z)dp(JY, W) + 2g(X, Z)dp(Y, W) \\
&\quad + g(JY, Z)g(F_pX, W) - 2g(JY, Z)dp(JX, W) - 2g(Y, Z)dp(X, W) \\
&\quad - g(JY, W)g(F_pX, Z) + 2g(JY, W)dp(JX, Z) + 2g(Y, W)dp(X, Z) \\
&\quad + g(JX, W)g(F_pY, Z) - 2g(JX, W)dp(JY, Z) - 2g(X, W)dp(Y, Z),
\end{aligned}
\tag{5.10}
$$

where $X, Y, Z, W \in \mathscr{D}$. If we put

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W),$$

then we have

$$
\begin{aligned}
&\widetilde{R}(X, Y, Z, W) - \widetilde{R}(Y, Z, W, X) - \widetilde{R}(Z, W, X, Y) + \widetilde{R}(W, X, Y, Z) \\
&=2\{R(Y, Z, X, W) - R(W, X, Z, Y)\} \\
&\quad + 4dp(X, Y)g(Z, W) - 4dp(Y, Z)g(X, W) + 4dp(Z, X)g(Y, W) \\
&\quad - 4dp(Z, W)g(Y, X) + 4dp(Y, W)g(Z, X) + 4dp(W, X)g(Y, Z)
\end{aligned}
$$

because of (5.4). The equation (5.5) shows

$$\widetilde{R}(X, Y, Z, W) = -\{\omega(X, Y)g(F_pZ, W) + \omega(Y, Z)g(F_pX, W) + \omega(Z, X)g(F_pY, W)\}.$$

Combining the two equations above, applying (3.16) and (3.17) to the obtained equation and changing Y for X, Z for Y and X for Z, we have (5.10). From (5.7) we have

$$\begin{aligned} R(\xi_p, Y, Z, W) - R(\xi_p, Z, Y, W) = &- g((D_Y F_p)Z, W) + 2p(Y)g(F_pZ, W) \\ &+ g((D_Z F_p)Y, W) - 2p(Z)g(F_pY, W), \end{aligned}$$

in which we permute the letters Y, Z and W cyclically and subtract one from the sum of the other two. Then we have

$$\begin{aligned} (5.11) \qquad & 2R(\xi_p, Z, W, Y) \\ & \quad + 4dp(\xi_p, Y)g(Z, W) - 4dp(\xi_p, Z)g(W, Y) - 4dp(\xi_p, W)g(Z, Y) \\ &= - g((D_Y F_p)Z, W) + 2p(Y)g(F_pZ, W) \\ &\quad + g((D_Z F_p)Y, W) - 2p(Z)g(F_pY, W) \\ &\quad + g((D_W F_p)Y, Z) - 2p(W)g(F_pY, Z) \\ &\quad - g((D_Y F_p)W, Z) + 2p(Y)g(F_pW, Z) \\ &\quad - g((D_Z F_p)W, Y) + 2p(Z)g(F_pW, Y) \\ &\quad + g((D_W F_p)Z, Y) - 2p(W)g(F_pZ, Y) \end{aligned}$$

because of (5.4). Note that $D_V F_p$ satisfies the following equation

$$(5.12) \qquad g((D_V F_p)X, Y) = g(X, (D_V F_p)Y) + 4(D_V dp)(JX, Y) + 8p(V)dp(JX, Y)$$

for $V \in TM$ and $X, Y \in \mathscr{D}$, which is obtained from (3.17). Moreover note that (3.16) shows that

$$(5.13) \qquad (D_V dp)(JX, JY) = -(D_V dp)(X, Y)$$

for $V \in TM$ and $X, Y \in \mathscr{D}$. Applying (5.12) and (5.13) to (5.11), we obtain

$$\begin{aligned} (5.14) \qquad & R(\xi_p, Y, Z, W) \\ &= g(Y, (D_Z F_p)W - (D_W F_p)Z) + g(Y, 2p(W)F_pZ - 2p(Z)F_pW) \\ &\quad - 2(D_W dp)(JY, Z) + 2(D_Y dp)(JW, Z) + 2(D_Z dp)(JY, W) \\ &\quad - 2dp(\xi_p, W)g(Y, Z) + 2dp(\xi_p, Y)g(Z, W) + 2dp(\xi_p, Z)g(Y, W) \end{aligned}$$

for every $Y, Z, W \in \mathscr{D}$.

Next we get the following formula for the difference of $R(JX, JY)$ and $R(X, Y)$:

$$\begin{aligned} (5.15) \qquad & R(JX, JY)Z - R(X, Y)Z \\ &= g(JX, Z)F_pY - g(JY, Z)F_pX + g(X, Z)F_pJY - g(Y, Z)F_pJX \\ &\quad + f_p(X, Z)JY - f_p(Y, Z)JX + f_p(JX, Z)Y - f_p(JY, Z)X \\ &\quad - 4dp(X, Y)Z + 4dp(JX, Y)JZ, \end{aligned}$$

where $X, Y, Z \in \mathscr{D}$ and we have defined f_p by

$$f_p(X, Y) = g(F_pX, Y), \qquad X, Y \in \mathscr{D}.$$

This formula can be proved by using equations (3.16), (5.3) and (5.10). In fact we see that

$$\begin{aligned}
&R(JX, JY, Z, W) - 2g(J^2X, JY)dp(JZ, W) + 2g(JX, JY)dp(Z, W)\\
=&R(Z, W, JX, JY) - 2g(JZ, W)dp(J^2X, JY) + 2g(Z, W)dp(JX, JY)\\
&- g(J^2X, Z)g(F_pJY, W) + 2g(J^2X, Z)dp(J^2Y, W) + 2g(JX, Z)dp(JY, W)\\
&+ g(J^2Y, Z)g(F_pJX, W) - 2g(J^2Y, Z)dp(J^2X, W) - 2g(JY, Z)dp(JX, W)\\
&- g(J^2Y, W)g(F_pJX, Z) + 2g(J^2Y, W)dp(J^2X, Z) + 2g(JY, W)dp(JX, Z)\\
&+ g(J^2X, W)g(F_pJY, Z) - 2g(J^2X, W)dp(J^2Y, Z) - 2g(JX, W)dp(JY, Z)\\
=&\{R(Z, W, X, Y) - 2g(JZ, W)dp(JX, Y) + 2g(Z, W)dp(X, Y)\}\\
&\quad + 4g(JZ, W)dp(JX, Y) - 4g(Z, W)dp(X, Y)\\
&+ g(X, Z)g(F_pJY, W) + 2g(X, Z)dp(Y, W) + 2g(JX, Z)dp(JY, W)\\
&- g(Y, Z)g(F_pJX, W) - 2g(Y, Z)dp(X, W) - 2g(JY, Z)dp(JX, W)\\
&+ g(Y, W)g(F_pJX, Z) + 2g(Y, W)dp(X, Z) + 2g(JY, W)dp(JX, Z)\\
&- g(X, W)g(F_pJY, Z) - 2g(X, W)dp(Y, Z) - 2g(JX, W)dp(JY, Z)\\
=&\{R(X, Y, Z, W) - 2g(JX, Y)dp(JZ, W) + 2g(X, Y)dp(Z, W)\\
&+ g(JX, Z)g(F_pY, W) - 2g(JX, Z)dp(JY, W) - 2g(X, Z)dp(Y, W)\\
&- g(JY, Z)g(F_pX, W) + 2g(JY, Z)dp(JX, W) + 2g(Y, Z)dp(X, W)\\
&+ g(JY, W)g(F_pX, Z) - 2g(JY, W)dp(JX, Z) - 2g(Y, W)dp(X, Z)\\
&- g(JX, W)g(F_pY, Z) + 2g(JX, W)dp(JY, Z) + 2g(X, W)dp(Y, Z)\}\\
&\quad + 4g(JZ, W)dp(JX, Y) - 4g(Z, W)dp(X, Y)\\
&+ g(X, Z)g(F_pJY, W) + 2g(X, Z)dp(Y, W) + 2g(JX, Z)dp(JY, W)\\
&- g(Y, Z)g(F_pJX, W) - 2g(Y, Z)dp(X, W) - 2g(JY, Z)dp(JX, W)\\
&+ g(Y, W)g(F_pJX, Z) + 2g(Y, W)dp(X, Z) + 2g(JY, W)dp(JX, Z)\\
&- g(X, W)g(F_pJY, Z) - 2g(X, W)dp(Y, Z) - 2g(JX, W)dp(JY, Z).
\end{aligned}$$

We turn to the study of the Ricci tensor field of a CR Weyl connection. We shall define two kinds of Ricci tensors. In general, Ricci tensor field s is defined by

$$s(V, W) = \text{trace of } (U \to R(U, V)W) \tag{5.16}$$

for $V, W \in TM$. We define another Ricci tensor field k by

$$k(V, W) = \frac{1}{2}\text{trace}\,(\phi_p R(V, \phi_p W)) \tag{5.17}$$

for $V, W \in TM$. Restricting s to $\mathscr{D}$, we obtain the following equation

$$s(X, Y) - s(Y, X) = -4(n+1)dp(X, Y) \tag{5.18}$$

for every $X, Y \in \mathscr{D}$. The proof of (5.18) is as follows: Noting that R satisfies (5.2), we may consider the contraction in only $\mathscr{D}$. Since

$$\text{trace}_{\mathscr{D}}\,(R(V, W)) = 4n dp(V, W), \qquad V, W \in TM, \tag{5.19}$$

where we have used (5.4) and $\text{trace}_{\mathscr{D}}$ denotes the trace in only $\mathscr{D}$, we have

$$s(X, Y) - s(Y, X) = \text{trace}_{\mathscr{D}}\,(Z \to \mathfrak{S}\{R(Z, X)Y\}) - 4n dp(X, Y).$$

Therefore, from (5.5), (3.17) and the fact that $\mathrm{trace}_{\mathscr{D}} F_p = 0$,

$$\begin{aligned} s(X, Y) - s(Y, X) &= -\,\mathrm{trace}_{\mathscr{D}}\,(Z \to \mathfrak{S}\{\omega(Z, X)F_pY\}) - 4ndp(X, Y) \\ &= g(F_pX, JY) - g(F_pJY, X) - \omega(X, Y)\mathrm{trace}_{\mathscr{D}}\, F_p - 4ndp(X, Y) \\ &= -\,4(n+1)dp(X, Y). \end{aligned}$$

Next we obtain the relation between s and k:

$$k(X, Y) = s(X, Y) - (n-1)f_p(JX, Y) - 2ndp(X, Y), \quad X, Y \in \mathscr{D}. \tag{5.20}$$

The equation (5.20) can be shown as follows:

$$\begin{aligned} s(X, Y) &= \mathrm{trace}_{\mathscr{D}}\,(Z \to -JR(Z, X)JY) \\ &= \mathrm{trace}_{\mathscr{D}}\,(Z \to JR(X, JY)Z + JR(JY, Z)X \\ &\quad + \omega(X, JY)JF_pZ + \omega(JY, Z)JF_pX + \omega(Z, X)F_pY) \\ &= 2k(X, Y) + \mathrm{trace}_{\mathscr{D}}\,(Z \to JR(JY, Z)X) \\ &\quad - \omega(X, JY)\,\mathrm{trace}_{\mathscr{D}}\,(F_pJ) + g(JF_pX, Y) + g(F_pY, JX) \\ &= 2k(X, Y) + \mathrm{trace}_{\mathscr{D}}\,(Z \to JR(JY, Z)X) + 4dp(X, Y), \end{aligned}$$

where we have used (5.5), (3.17) and the fact that F_p anticommutes with J, and using (5.15) and (3.17) again, we have

$$\begin{aligned} &\mathrm{trace}_{\mathscr{D}}\,(Z \to JR(JY, Z)X) \\ &= \mathrm{trace}_{\mathscr{D}}\,(JZ \to JR(JY, JZ)X) \\ &= \mathrm{trace}_{\mathscr{D}}\,(Z \to R(JY, JZ)X) \\ &= \mathrm{trace}_{\mathscr{D}}\,(Z \to R(Y, Z)X + g(JY, X)F_pZ - g(JZ, X)F_pY + g(Y, X)F_pJZ \\ &\quad - g(Z, X)F_pJY + f_p(Y, X)JZ - f_p(Z, X)JY + f_p(JY, X)Z - f_p(JZ, X)Y \\ &\quad - 4dp(Y, Z)X + 4dp(JY, Z)JX) \\ &= -\,s(Y, X) + g(JY, X)\,\mathrm{trace}_{\mathscr{D}}\, F_p + g(F_pY, JX) + g(Y, X)\,\mathrm{trace}_{\mathscr{D}}\,(F_pJ) \\ &\quad - g(F_pJY, X) + f_p(Y, X)\,\mathrm{trace}_{\mathscr{D}}\, J - g(F_pJY, X) + 2ng(F_pJY, X) - g(F_pJY, X) \\ &\quad - 4dp(Y, X) + 4dp(X, Y) \\ &= -\,s(Y, X) + 2(n-1)f_p(JX, Y) + 8ndp(X, Y), \end{aligned}$$

which shows (5.20). From equations (5.18) and (5.20) we obtain

$$k(X, Y) - k(Y, X) = -4(n+2)dp(X, Y) \tag{5.21}$$

for every $X, Y \in \mathscr{D}$. The defining equation (5.17) of k shows the following property

$$k(JX, JY) - k(X, Y) = 4(n+2)dp(X, Y) \tag{5.22}$$

for every $X, Y \in \mathscr{D}$. It follows that

$$s(JX, JY) - s(X, Y) = -2(n-1)f_p(JX, Y) + 8dp(X, Y) \tag{5.23}$$

for every $X, Y \in \mathscr{D}$. It is easy to show

$$s(X, \xi_p) = -4dp(X, \xi_p), \quad X \in \mathscr{D}. \tag{5.24}$$

Furthermore, by making use of (5.7) and (5.19) we obtain

$$\begin{aligned} s(\xi_p, X) &= \mathrm{trace}_{\mathscr{D}}\,(Z \to (D_ZF_p)X) - 2p(F_pX) - 4ndp(\xi_p, X) \\ &= \mathrm{trace}_{\mathscr{D}}\,(Z \to (D_ZT)(\xi_p, X)) - 4ndp(\xi_p, X), \quad X \in \mathscr{D}. \end{aligned} \tag{5.25}$$

We introduce two notations for later use. Define $S \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$g(SX, Y) = s(X, Y), \quad X, Y \in \mathscr{D} \tag{5.26}$$

and ρ by

$$\rho = \mathrm{trace}_{\mathscr{D}}\, S \tag{5.27}$$

which is a smooth function on M and will be called scalar curvature.

Finally we state the following lemma and conclude this section.

Proposition 5.1. *The Ricci tensor field s satisfies*

$$\sum_{i=1}^{2n} \varepsilon_i (D_{e_i} s)(X, e_i) = \frac{1}{2}(d\rho - 2\rho p)(X) \tag{5.28}$$

for $X \in \mathscr{D}$, where $\{e_i\}$ denotes an orthonormal frame of $\mathscr{D}$ with respect to the pseudo metric g and $\varepsilon_i = g(e_i, e_i) = \pm 1$.

Proof. From the second Bianchi identity (5.8) we have

$$\mathfrak{S}\{(D_X R)(Y, Z)W\} = \mathfrak{S}\{\omega(X, Y)R(\xi_p, Z)W\}$$

for $W \in \mathscr{D}$. Therefore, if, in the above equation, we replace Y with e_i and take the inner product with e_i, we have

$$\begin{aligned}(D_X s)(Z, W) + \sum_{i=1}^{2n} \varepsilon_i g((D_{e_i} R)(Z, X)W, e_i) - (D_Z s)(X, W) \\ = - g(JX, R(\xi_p, Z)W) - g(JR(\xi_p, X)W, Z) + g(JZ, X)s(\xi_p, W),\end{aligned} \tag{5.29}$$

where we have used the following equation

$$D_X e_i = - \sum_{j=1}^{2n} \varepsilon_j g(D_X e_j, e_i) e_j + 2p(X)e_i. \tag{5.30}$$

Moreover, replace both Z and W with e_j and sum with respect to j. Then we have

(5.31)

$$\begin{aligned}\sum_{j=1}^{2n} \varepsilon_j (D_X s)(e_j, e_j) + \sum_{i,j=1}^{2n} \varepsilon_i \varepsilon_j g((D_{e_i} R)(e_j, X)e_j, e_i) - \sum_{j=1}^{2n} \varepsilon_j (D_{e_j} s)(X, e_j) \\ = - \sum_{j=1}^{2n} \varepsilon_j g(JX, R(\xi_p, e_j)e_j) - \sum_{j=1}^{2n} \varepsilon_j g(JR(\xi_p, X)e_j, e_j) + \sum_{j=1}^{2n} \varepsilon_j g(Je_j, X)s(\xi_p, e_j).\end{aligned}$$

We calculate the each term of the equation (5.31). Applying (5.30) to the first term of the left hand side of (5.31), we have

$$\sum_{j=1}^{2n} \varepsilon_j (D_X s)(e_j, e_j) = (d\rho - 2\rho p)(X). \tag{5.32}$$

Applying (5.4) and (5.30) to the second term of the left hand side of (5.31), we have

$$\sum_{i,j=1}^{2n} \varepsilon_i \varepsilon_j g((D_{e_i} R)(e_j, X)e_j, e_i) = - \sum_i \varepsilon_i (D_{e_i} s)(X, e_i) + 4 \sum_i (D_{e_i} dp)(e_i, X). \tag{5.33}$$

For the first term of the right hand side of (5.31), we have, from (5.4),

$$-\sum_{j=1}^{2n} \varepsilon_j g(JX,\, R(\xi_p,\, e_j)e_j) = -s(\xi_p,\, JX) - 4dp(\xi_p,\, JX). \tag{5.34}$$

To compute the second term of the right hand side of (5.31), we prepare the following equation

$$\sum_i \varepsilon_i g((D_X F_p)e_i,\, Je_i) = \sum_i \varepsilon_i g((D_X F_p J)e_i,\, e_i) = \mathrm{trace}_{\mathscr{D}}\, D_X(F_p J) = 0.$$

By using (5.3), (5.7) and the equation $\mathrm{trace}_{\mathscr{D}}(F_p J) = 0$, we have

$$-\sum_{i=1}^{2n} \varepsilon_i g(JR(\xi_p,\, X)e_i,\, e_i) = \mathrm{trace}_{\mathscr{D}}\,(Z \to (D_Z F_p)JX) - 2p(F_p JX) + s(\xi_p,\, JX). \tag{5.35}$$

For the third term of the right hand side of (5.31), we have

$$\sum_{j=1}^{2n} \varepsilon_j g(Je_j,\, X)s(\xi_p,\, e_j) = -s(\xi_p,\, JX). \tag{5.36}$$

We see from (5.34), (5.35), (5.36) and (5.25) that the right hand side of (5.31) becomes $4(n-1)dp(\xi_p,\, JX)$. Substituting (5.32) and (5.33) into (5.31), we have

$$\begin{aligned} &-2\sum_i \varepsilon_i(D_{e_i}s)(X,\, e_i) + 4\sum_i \varepsilon_i(D_{e_i}dp)(e_i,\, X) + (d\rho - 2\rho\, p)(X) \\ &= 4(n-1)dp(\xi_p,\, JX). \end{aligned} \tag{5.37}$$

If we prove

$$2\sum_i \varepsilon_i(D_{e_i}dp)(JX,\, e_i) = 2(n-1)dp(\xi_p,\, X), \tag{5.38}$$

then we conclude (5.28). The proof of (5.38) is as follows. We calculate the exterior derivative of dp.

$$3d(dp)(Y,\, Z,\, W) = \mathfrak{S}\{(D_Y dp)(Z,\, W) - \omega(Y,\, Z)dp(\xi_p,\, W)\}$$

for $Y, Z, W \in \mathscr{D}$, where we have used (e) in the definition of the torsion tensor of CR Weyl connection. Replacing Y with e_i, Z with Je_i and W with X in the above equation, and summing with respect to i, we have

$$\begin{aligned} &\sum_i \varepsilon_i(D_{e_i}dp)(Je_i,\, X) + \sum_i \varepsilon_i(D_{Je_i}dp)(X,\, e_i) \\ &= -(2n-1)dp(\xi_p,\, X). \end{aligned} \tag{5.39}$$

For the first term of the left hand side of (5.39), we have

$$-\sum_i \varepsilon_i(D_{e_i}dp)(Je_i,\, X) = \sum_i \varepsilon_i(D_{e_i}dp)(JX,\, e_i). \tag{5.40}$$

For the second term of the left hand side of (5.39), we also have

$$\begin{aligned} -\sum_i \varepsilon_i(D_{Je_i}dp)(X,\, e_i) &= \sum_i \varepsilon_i(D_{Je_i}dp)(JX,\, Je_i) \\ &= \sum_i \varepsilon_i(D_{e_i}dp)(JX,\, e_i), \end{aligned} \tag{5.41}$$

where we have used (3.16). Substituting (5.40) and (5.41) into (5.39), we obtain (5.38). □

6. CR Einstein-Weyl structures

Let D be a CR Weyl connection on a CR manifold $(M, \mathscr{D}, J)$. Fixing a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ belonging to the CR structure $(\mathscr{D}, J)$, we know that there exists uniquely a Tanaka connection ∇ associated with the almost contact structure $(\phi, \xi, \theta)^*$ (cf. [9], [12]). Then the difference tensor H between D and ∇ is given in Proposition 3.5. Thus we may calculate the difference $R(X, Y)Z - R^{\nabla}(X, Y)Z$ for $X, Y, Z \in \mathscr{D}$, where R^{∇} denotes the curvature tensor of ∇. We introduce suitable 2-forms and rewrite the resulting long equation comfortably. Next we shall calculate $k - k^{\nabla}$ and $\rho - \rho^{\nabla}$. In this way, the curvature tensor R will be expressed as the equation including Bochner curvature tensor. Making use of this equation, we can define a CR Einstein-Weyl structure on a CR manifold.

To begin with, we calculate the difference $R - R^{\nabla}$. Since

$$\begin{aligned} D_X D_Y Z =& D_X(\nabla_Y Z + H(Y, Z)) \\ =& \nabla_X \nabla_Y Z + H(X, \nabla_Y Z) + (\nabla_X H)(Y, Z) + H(\nabla_X Y, Z) \\ &+ H(Y, \nabla_X Z) + H(X, H(Y, Z)), \end{aligned}$$

$$[X, Y] = \nabla_X Y - \nabla_Y X - T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X + \omega(X, Y)\xi$$

for $X, Y, Z \in \Gamma(\mathscr{D})$, where we have used the equation $T^{\nabla}(X, Y) = -\omega(X, Y)\xi$, we have

$$\begin{aligned} &R(X, Y)Z - R^{\nabla}(X, Y)Z \\ &= (\nabla_X H)(Y, Z) - (\nabla_Y H)(X, Z) \\ &+ H(X, H(Y, Z)) - H(Y, H(X, Z)) - \omega(X, Y)H(\xi, Z). \end{aligned} \tag{6.1}$$

We substitute (3.18) and (3.19) into (6.1). The calculation is long but routine and hence we omit the proof. The result is as follows (cf. [9]):

$$\begin{aligned} &R(X, Y)Z - R^{\nabla}(X, Y)Z \\ =& - \{(\nabla_Y p)(Z) - p(Y)p(Z) + q(Y)q(Z) + p(P)g(Y, Z)\}X \\ &+ \{(\nabla_X p)(Z) - p(X)p(Z) + q(X)q(Z) + p(P)g(X, Z)\}Y \\ &- \{(\nabla_Y q)(Z) - q(Y)p(Z) - p(Y)q(Z) + p(P)g(JY, Z)\}JX \\ &+ \{(\nabla_X q)(Z) - q(X)p(Z) - p(X)q(Z) + p(P)g(JX, Z)\}JY \\ &- g(Y, Z)\{\nabla_X P - p(X)P + q(X)Q\} + g(X, Z)\{\nabla_Y P - p(Y)P + q(Y)Q\} \\ &- g(JY, Z)\{\nabla_X Q - q(X)P - p(X)Q\} + g(JX, Z)\{\nabla_Y Q - q(Y)P - p(Y)Q\} \\ &+ \{(\nabla_X p)(Y) - (\nabla_Y p)(X)\}Z + \{(\nabla_X q)(Y) - (\nabla_Y q)(X)\}JZ \\ &+ g(JX, Y)\{\nabla_{JZ} P + \nabla_Z Q + 2p(P)JZ\}. \end{aligned} \tag{6.2}$$

Now we define $\alpha \in \Gamma(\mathscr{D}^* \otimes \mathscr{D}^*)$ by

$$\alpha(Y, Z) = (\nabla_Y p)(Z) - p(Y)p(Z) + q(Y)q(Z) + \frac{1}{2}p(P)g(Y, Z) + \frac{1}{2}p(\xi)g(JY, Z) \tag{6.3}$$

and $\gamma \in \Gamma(\mathscr{D}^* \otimes \mathscr{D}^*)$ by

$$\gamma(Y, Z) = (\nabla_Y q)(Z) - q(Y)p(Z) - p(Y)q(Z) + \frac{1}{2}p(P)g(JY, Z) - \frac{1}{2}p(\xi)g(Y, Z). \tag{6.4}$$

Then they are related as

$$\alpha(Y, Z) = \gamma(Y, JZ). \tag{6.5}$$

Rewriting the exterior differentiation dp and dq of the 1-form p and q in terms of the Tanaka connection respectively, we obtain

$$2dp(Y, Z) = (\nabla_Y p)(Z) - (\nabla_Z p)(Y) - p(\xi)\omega(Y, Z), \tag{6.6}$$

$$2dq(Y, Z) = (\nabla_Y q)(Z) - (\nabla_Z q)(Y) \tag{6.7}$$

for $Y, Z \in \mathscr{D}$, where we have used $q(\xi) = 0$. From (6.3) and (6.6), we have

$$\alpha(Y, Z) - \alpha(Z, Y) = 2dp(Y, Z). \tag{6.8}$$

We also have, from (6.4) and (6.7),

$$\gamma(Y, Z) - \gamma(Z, Y) = 2dq(Y, Z) + p(P)g(JY, Z). \tag{6.9}$$

Furthermore, define $A, C \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$AY = \nabla_Y P - p(Y)P + q(Y)Q + \frac{1}{2}p(P)Y + \frac{1}{2}p(\xi)JY, \tag{6.10}$$

$$CY = \nabla_Y Q - q(Y)P - p(Y)Q + \frac{1}{2}p(P)JY - \frac{1}{2}p(\xi)Y. \tag{6.11}$$

Then we have

$$g(AY, Z) = \alpha(Y, Z), \qquad g(CY, Z) = \gamma(Y, Z), \tag{6.12}$$

and from (6.5)

$$JA = C. \tag{6.13}$$

Substituting (6.3), (6.4), (6.10) and (6.11) into (6.2), we easily obtain the following equation and we omit the proof (cf. [10]).

Lemma 6.1. *$R - R^{\nabla}$ is given by*

$$\begin{aligned} &R(X, Y)Z - R^{\nabla}(X, Y)Z \\ &= -\alpha(Y, Z)X + \alpha(X, Z)Y - \gamma(Y, Z)JX + \gamma(X, Z)JY \\ &\quad - g(Y, Z)AX + g(X, Z)AY - g(JY, Z)CX + g(JX, Z)CY \\ &\quad + \{\alpha(X, Y) - \alpha(Y, X)\}Z + \{\gamma(X, Y) - \gamma(Y, X)\}JZ \\ &\quad + g(JX, Y)(AJZ + CZ). \end{aligned} \tag{6.14}$$

Remark. We can represent the equation (6.14) in the form similar to [10]:

$$\begin{aligned} &R(X, Y)Z - R^{\nabla}(X, Y)Z \\ &= -\alpha(Y, Z)X + \alpha(X, Z)Y - \gamma(Y, Z)JX + \gamma(X, Z)JY \\ &\quad - g(Y, Z)AX + g(X, Z)AY - g(JY, Z)CX + g(JX, Z)CY \\ &\quad + \{\gamma(X, Y) - \gamma(Y, X)\}JZ + g(JX, Y)\{CZ - {}^tCZ\} \\ &\quad + 2dp(X, Y)Z + 2g(JX, Y)dp^{\sharp}(JZ), \end{aligned}$$

where ${}^tC$ denotes the transpose of the linear transformation C of $\mathscr{D}$ with respect to g and $dp^{\sharp}$ is the linear transformation of $\mathscr{D}$ defined by $g(dp^{\sharp}X, Y) = dp(X, Y)$.

Next we shall compute $k(Y, Z) - k^{\nabla}(Y, Z)$ for $Y, Z \in \mathscr{D}$, where k^{∇} is the Ricci tensor of the fixed Tanaka connection ∇. Before contracting the equation (6.14), we consider the symmetric part of γ. For $f_p(Y, Z) - f(Y, Z)$, we obtain

$$\gamma(Y, Z) + \gamma(Z, Y) = -f_p(Y, Z) + f(Y, Z) + 2dp(JY, Z), \tag{6.15}$$

where $f(Y, Z) = g(FY, Z)$. In fact, since

$$f_p(Y, Z) - f(Y, Z) = (\nabla_{JY}p)(Z) - (\nabla_Y q)(Z) + 2p(Y)q(Z) + 2q(Y)p(Z)$$

because of (3.24), the bilinear form α satisfies

$$\alpha(JY, Z) + \alpha(Y, JZ) = f_p(Y, Z) - f(Y, Z), \tag{6.16}$$

which implies (6.15).

Well we compute $s(Y, Z) - s^{\nabla}(Y, Z)$, where s^{∇} is the Ricci tensor of ∇. Contracting (6.14), we see that

$$\begin{aligned} & s(Y, Z) - s^{\nabla}(Y, Z) \\ = & - 2n\alpha(Y, Z) + \alpha(Y, Z) - \gamma(Y, Z)\operatorname{trace}_{\mathscr{D}} J + \gamma(JY, Z) \\ & - g(Y, Z)\operatorname{trace}_{\mathscr{D}} A + \alpha(Y, Z) - g(JY, Z)\operatorname{trace}_{\mathscr{D}} C - \gamma(Y, JZ) \\ & + \{\alpha(Z, Y) - \alpha(Y, Z)\} - \{\gamma(JZ, Y) - \gamma(Y, JZ)\} \\ & - g(AJZ, JY) - g(CZ, JY). \end{aligned}$$

Since $\operatorname{trace}_{\mathscr{D}} F_p = \operatorname{trace}_{\mathscr{D}} F = 0$, we obtain $\operatorname{trace}_{\mathscr{D}} C = 0$ by virtue of the equation (6.15). Making use of (6.5), (6.8), (6.15) and (6.16), we have

$$\begin{aligned} s(Y, Z) - s^{\nabla}(Y, Z) = & - 2(n+2)\alpha(Y, Z) - 3f_p(JY, Z) \\ & + f(JY, Z) - g(Y, Z)\operatorname{trace}_{\mathscr{D}} A - 4dp(Y, Z). \end{aligned} \tag{6.17}$$

Therefore, by the equation (5.20), we get

Lemma 6.2. *The difference $k(Y, Z) - k^{\nabla}(Y, Z)$ is given by*

$$\begin{aligned} k(Y, Z) - k^{\nabla}(Y, Z) = & - (n+2)\{\alpha(Y, Z) + \alpha(JY, JZ)\} \\ & - g(Y, Z)\operatorname{trace}_{\mathscr{D}} A - 2(n+2)dp(Y, Z) \end{aligned} \tag{6.18}$$

for every $Y, Z \in \mathscr{D}$.

Using the equation (6.17), we have

$$S - S^{\nabla} = -2(n+2)A - 3F_pJ + 3FJ - (\operatorname{trace}_{\mathscr{D}} A)I_{\mathscr{D}} - 4dp^{\sharp}, \tag{6.19}$$

where S^{∇} denotes the linear transformation of $\mathscr{D}$ defined by $g(S^{\nabla}Y, Z) = s^{\nabla}(Y, Z)$ and $I_{\mathscr{D}}$ denotes the identity transformation of $\mathscr{D}$. We obtain, from (6.19),

Lemma 6.3. *The difference $\rho - \rho^{\nabla}$ is given by*

$$\rho - \rho^{\nabla} = -4(n+1)\operatorname{trace}_{\mathscr{D}} A, \tag{6.20}$$

where ρ^{∇} denotes the scalar curvature of ∇.

Let us define l and m by

$$l(Y, Z) = -\frac{1}{2(n+2)}k(Y, Z) + \frac{1}{8(n+1)(n+2)}\rho g(Y, Z) \tag{6.21}$$

and

$$(6.22)\qquad m(Y,Z) = -\frac{1}{2(n+2)}k(JY,Z) + \frac{1}{8(n+1)(n+2)}\rho\, g(JY,Z)$$

respectively, where $Y, Z \in \mathscr{D}$. From the equation (5.21) and (5.22) we obtain

$$(6.23)\qquad l(Y,Z) - l(Z,Y) = 2dp(Y,Z),$$

$$(6.24)\qquad l(JY,JZ) - l(Y,Z) = -2dp(Y,Z).$$

Also we similarly obtain

$$(6.25)\qquad m(Y,Z) = -m(Z,Y),$$

$$(6.26)\qquad m(JY,JZ) - m(Y,Z) = -2dp(JY,Z).$$

The forms l and m are related as

$$(6.27)\qquad m(Y,Z) = l(JY,Z).$$

We define $L \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ and $M \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$(6.28)\qquad g(LY,Z) = l(Y,Z),$$

$$(6.29)\qquad g(MY,Z) = m(Y,Z)$$

for every $Y, Z \in \mathscr{D}$ respectively.

We express α, γ, A and C by the above notations:

Lemma 6.4. *The bilinear form α on $\mathscr{D}$ is given by*

$$(6.30)\qquad \alpha(Y,Z) = l(Y,Z) - l^{\nabla}(Y,Z) - \frac{1}{2}\{f_p(JY,Z) - f(JY,Z)\} - dp(Y,Z),$$

so that we have

$$(6.31)\qquad A = L - L^{\nabla} - \frac{1}{2}(F_pJ - FJ) - dp^{\sharp},$$

and the bilinear form γ is given by

$$(6.32)\qquad \gamma(Y,Z) = m(Y,Z) - m^{\nabla}(Y,Z) - \frac{1}{2}\{f_p(Y,Z) - f(Y,Z)\} - dp(JY,Z),$$

so that we have

$$(6.33)\qquad C = M - M^{\nabla} - \frac{1}{2}(F_p - F) - dp^{\sharp}J,$$

where l^{∇}, m^{∇}, L^{∇} and M^{∇} denote the tensors similarly defined by (6.21), (6.22), (6.28) *and* (6.29) *with respect to ∇ respectively.*

Remark. In [10], the following equations are easily verified:

$$l^{\nabla}(Y,Z) = l^{\nabla}(Z,Y)\,, \qquad m^{\nabla}(Y,Z) = -m^{\nabla}(Z,Y)$$

$$l^{\nabla}(JY,JZ) = l(Y,Z)\,, \qquad m^{\nabla}(JY,JZ) = m^{\nabla}(Y,Z)$$

for $Y, Z \in \mathscr{D}$. These are derived from the fact that k^{∇} is symmetric on $\mathscr{D}$ and satisfies $k^{\nabla}(JY,JZ) = k(Y,Z)$ for $Y, Z \in \mathscr{D}$.

Proof. It suffices to prove the equation (6.30) from which the others are trivially derived from the above remark. From the defining equation (6.21), we have

$$l(Y,\,Z) - l^{\nabla}(Y,\,Z) = -\frac{1}{2(n+2)}\{k(Y,\,Z) - k^{\nabla}(Y,\,Z)\} + \frac{1}{8(n+1)(n+2)}(\rho - \rho^{\nabla})g(Y,\,Z).$$

We substitute the equation (6.18) and (6.20) into the above equation. Then we have

$$l(Y,\,Z) - l^{\nabla}(Y,\,Z) = \frac{1}{2}\{\alpha(Y,\,Z) + \alpha(JY,\,JZ)\} + dp(Y,\,Z),$$

and hence, we obtain (6.30) from (6.16). □

Next we shall rewrite the equation (6.14) by making use of Lemma 6.4. Before we do so, we need to state the Bochner curvature tensor which is invariant under the change (2.8).

Sakamoto and Takemura (cf. [10]) state the Bochner curvature tensor in the following form.

Lemma 6.5. *Let B_0, $B_1 \in \Gamma(\mathscr{D}^{*3} \otimes \mathscr{D})$ be defined by*

$$\begin{aligned}(6.34)\qquad & B_0(X,\,Y)Z \\ =& R^{\nabla}(X,\,Y)Z + l^{\nabla}(Y,\,Z)X - l^{\nabla}(X,\,Z)Y + m^{\nabla}(Y,\,Z)JX - m^{\nabla}(X,\,Z)JY \\ &+ g(Y,\,Z)L^{\nabla}X - g(X,\,Z)L^{\nabla}Y + g(JY,\,Z)M^{\nabla}X - g(JX,\,Z)M^{\nabla}Y \\ &- 2\{m^{\nabla}(X,\,Y)JZ + g(JX,\,Y)M^{\nabla}Z\},\end{aligned}$$

$$(6.35)\qquad B_1(X,\,Y)Z = \frac{1}{2}\{R^{\nabla}(JX,\,JY)Z - R^{\nabla}(X,\,Y)Z\}.$$

Then $B = B_0 + B_1$ is invariant under the change (2.8). (*The tensor field B on $\mathscr{D}$ is called Bochner curvature tensor.*)

The right hand side of the definition of B_1 is given by

$$\begin{aligned}(6.36)\qquad & R^{\nabla}(JX,\,JY)Z - R^{\nabla}(X,\,Y)Z \\ =& g(JX,\,Z)FY - g(JY,\,Z)FX + g(X,\,Z)FJY - g(Y,\,Z)FJX \\ &+ f(X,\,Z)JY - f(Y,\,Z)JX + f(JX,\,Z)Y - f(JY,\,Z)X\end{aligned}$$

for $X, Y, Z \in \mathscr{D}$ (cf. [10]).

We introduce the important notations for a CR Einstein-Weyl structure by which we rewrite the equation (6.14). We define ric^D by

$$(6.37)\qquad ric^D(Y,\,Z) = l(Y,\,Z) - dp(Y,\,Z)$$

for $Y, Z \in \mathscr{D}$. From the equation (6.23) we see that the tensor ric^D is symmetric and hence ric^D is the symmetric part of l. We obtain, from (6.27),

$$(6.38)\qquad ric^D(JY,\,Z) = m(Y,\,Z) - dp(JY,\,Z).$$

Furthermore we define $Ric^D \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$(6.39)\qquad g(Ric^D Y,\,Z) = ric^D(Y,\,Z)$$

for $Y, Z \in \mathscr{D}$. It follows that

$$(6.40)\qquad Ric^D = L - dp^{\sharp}\,, \qquad Ric^D J = M - dp^{\sharp} J.$$

We obtain, from Lemma 6.1,

Theorem 6.6. *Let $(\mathscr{D}, J)$ be a nodegenerate CR structure on M^{2n+1} and $(\phi, \xi, \theta)^*$ a $\mathscr{D}$-preserving almost contact structure belonging to $(\mathscr{D}, J)$. Let D be a CR Weyl connection. Then the curvature tensor R of D satisfies*

$$
\begin{aligned}
(6.41)\quad & \frac{1}{2}\{R(JX,\, JY)Z + R(X,\, Y)Z\} \\
&= - ric^D(Y,\, Z)X + ric^D(X,\, Z)Y - ric^D(JY,\, Z)JX + ric^D(JX,\, Z)JY \\
&\quad - g(Y,\, Z)Ric^D X + g(X,\, Z)Ric^D Y - g(JY,\, Z)Ric^D JX + g(JX,\, Z)Ric^D JY \\
&\quad + 2\{ric^D(JX,\, Y)JZ + g(JX,\, Y)Ric^D JZ\} \\
&\quad + B(X,\, Y)Z
\end{aligned}
$$

for every $X, Y, Z \in \mathscr{D}$.

Proof. Substitute the equations from (6.30) to (6.40) into (6.14). Then we obtain (6.41). □

Remark. For $X, Y \in \mathscr{D}$ we define the transformation $X \wedge Y$ on $\mathscr{D}$ by

$$(6.42)\qquad (X \wedge Y)Z = g(Y,\, Z)X - g(X,\, Z)Y$$

for $Z \in \mathscr{D}$. Furthermore, for $Ric^D, I_{\mathscr{D}} \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ we define $Ric^D \wedge I_{\mathscr{D}}$ by

$$(6.43)\qquad (Ric^D \wedge I_{\mathscr{D}})_{X,Y}Z = Ric^D Y \wedge I_{\mathscr{D}}X - Ric^D X \wedge I_{\mathscr{D}}Y$$

for $X, Y, Z \in \mathscr{D}$. Using such notations as (6.42) and (6.43) and rewriting the equation (6.41) and (5.15), we obtain

$$
\begin{aligned}
(6.44)\quad \frac{1}{2}\{R(JX,\, JY)Z + R(X,\, Y)Z\} &= \{Ric^D \wedge I_{\mathscr{D}} + Ric^D J \wedge J\}_{X,Y}Z \\
&\quad + 2\{ric^D(JX,\, Y)JZ + g(JX,\, Y)Ric^D JZ\} \\
&\quad + B(X,\, Y)Z, \\
(6.45)\quad \frac{1}{2}\{R(JX,\, JY)Z - R(X,\, Y)Z\} &= \frac{1}{2}\{F_p \wedge J + F_p J \wedge I_{\mathscr{D}}\}_{X,Y}Z \\
&\quad - 2\{dp(X,\, Y)Z - dp(JX,\, Y)JZ\}.
\end{aligned}
$$

We find that the equations (6.44) and (6.45) are similar to the equation in [2] which describes the relation between the curvature R of a Weyl connection and the Weyl conformal curvatrue tensor W. The definition of an Einstein-Weyl connection is that the symmetric part of h^D in [2] is proportional to g pointwise. Therefore it will be appropriate that we define a CR Einstein-Weyl connection as follows:

Definition. A pair of a nondegenerate CR structure $(\mathscr{D}, J)$ and a CR Weyl connection D is *CR Einstein-Weyl* if the bilinear form ric^D is proportional to g pointwise, where g is the Levi metric of arbitrary $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathscr{D}, J)$. And a CR manifold M furnished with a CR Einstein-Weyl structure is called a *CR Einstein-Weyl manifold.*

Remark. The bilinear form ρg does not depend on the choice of $(\phi, \xi, \theta)^*$ and so does ric^D. Therefore the definition that the CR Weyl connection is CR Einstein-Weyl is independent of the choice of $(\phi, \xi, \theta)^*$.

By the following proposition, we may state that a certain pair of a 1-form p and $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ determines a CR Einstein-Weyl structure as in the case of Einstein-Weyl structure.

Proposition 6.7. *The CR structure $(\mathscr{D}, J)$ admits a CR Einstein-Weyl connection D if and only if D is determined by a pair of a 1-from p satisfying* (3.16) *and a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ which satisfy*

$$k^{\nabla}(Y, Z) - (n+2)\{(\nabla_Y p)(Z) - (\nabla_{JY} q)(Z) + p(\xi)g(JY, Z)\} = \Lambda g(Y, Z) \tag{6.46}$$

for every Y, $Z \in \mathscr{D}$.

Proof. First we assume that $ric^D(Y, Z)$ is proportional to $g(Y, Z)$ pointwise. Then by the definition of ric^D and (6.30), we have

$$ric^D(Y, Z) = l^{\nabla}(Y, Z) + \alpha(Y, Z) + \frac{1}{2}\{f_p(JY, Z) - f(JY, Z)\}. \tag{6.47}$$

Moreover, applying (3.24) to (6.47), we have

$$\begin{aligned} ric^D(Y, Z) = l^{\nabla}(Y, Z) &+ \frac{1}{2}\{(\nabla_Y p)(Z) - (\nabla_{JY} q)(Z) + p(\xi)g(JY, Z)\} \\ &+ \frac{1}{2}p(P)g(Y, Z). \end{aligned} \tag{6.48}$$

Substituting the definition of l^{∇} into (6.48), we have

(6.49)

$$\begin{aligned} ric^D(Y, Z) = &-\frac{1}{2(n+2)}k^{\nabla}(Y, Z) + \frac{1}{2}\{(\nabla_Y p)(Z) - (\nabla_{JY} q)(Z) + p(\xi)g(JY, Z)\} \\ &+ \frac{1}{8(n+1)(n+2)}\{\rho^{\nabla} + 4(n+1)(n+2)p(P)\}g(Y, Z), \end{aligned}$$

which implies (6.46).

Conversely, we assume that there exist p and $(\phi, \xi, \theta)^*$ which satisfy (6.46). By Proposition 3.6, we have a CR Weyl connection D. Then we define the tensor ric^D of the CR Weyl connection D. Substituting (6.46) into (6.49), we see that ric^D is proportional to g pointwise. □

Next we state the main theorem in terms of a holomorphic 1-form. If $(\phi, \xi, \theta)^*$ is a $\mathscr{D}$-preserving almost contact structure such that the Ricci tensor k^{∇} of the Tanaka connection ∇ associated with $(\phi, \xi, \theta)^*$ is proportional to g pointwise, that is,

$$k^{\nabla}(Y, Z) = c\, g(Y, Z) \tag{6.50}$$

for $Y, Z \in \mathscr{D}$, where c is a smooth function on M and g is the Levi metric of $(\phi, \xi, \theta)^*$, then $(\phi, \xi, \theta)^*$ is said to be *pseudo-Einstein* (cf. [6]).

Theorem 6.8. *Let $(\mathscr{D}, J)$ be a nondegenerate CR structure on a $(2n+1)$-dimensional manifold M. Assume that there exists a $\mathscr{D}$-preserving pseudo-Einstein almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathscr{D}, J)$. If there exists a holomorphic 1-form $p + \sqrt{-1}q$, where p is a real 1-form and $q = -p \circ \phi$, then the CR Weyl connection D determined by p and ∇ (Tanaka connection associated with $(\phi, \xi, \theta)^*$) is CR Einstein-Weyl.*

Proof. First we put $u = p + \sqrt{-1}q$. We see from (2.19) that $d''u = 0$ if and only if u satisfies the following equations:

$$(6.51) \qquad (\nabla_{Z+\sqrt{-1}JZ}u)(Y - \sqrt{-1}JY) - u(T^{\nabla}(Y - \sqrt{-1}JY,\, Z + \sqrt{-1}JZ)) = 0$$

$$(6.52) \qquad (\nabla_{Z+\sqrt{-1}JZ}u)(\xi) - u(T^{\nabla}(\xi,\, Z + \sqrt{-1}JZ)) = 0$$

for $Y, Z \in \mathscr{D}$. We have, from (6.51),

$$(6.53) \qquad \begin{aligned} &(\nabla_Z p)(Y) - (\nabla_{JZ} q)(Y) + p(\xi)g(JZ, Y) \\ &+ \sqrt{-1}\{(\nabla_{JZ} p)(Y) + (\nabla_Z q)(Y) - p(\xi)g(Z, Y)\} = 0. \end{aligned}$$

Combining (6.53) with the assumption that $(\phi, \xi, \theta)^*$ is pseudo-Einstein, we see that (6.46) is satisfied. Since

$$\begin{aligned} 2\{dp(X, Y) + dp(JX, JY)\} =& (\nabla_X p)(Y) - (\nabla_Y p)(X) + p(T(X, Y)) \\ &+ (\nabla_{JX} p)(JY) - (\nabla_{JY} p)(JX) + p(T(JX, JY)) \\ =& (\nabla_X p)(Y) - (\nabla_{JX} q)(Y) + p(\xi)g(JX, Y) \\ &- \{(\nabla_Y p)(X) - (\nabla_{JY} q)(X) + p(\xi)g(JY, X)\} \end{aligned}$$

for $X, Y \in \mathscr{D}$, we also obtain (3.16). Therefore, by Theorem 6.7, the CR Weyl connection D determined by p and ∇ is CR Einstein-Weyl. □

7. Example of CR Einstein-Weyl Manifolds

We shall explain an example of a CR Einstein-Weyl manifold. We shall show that the total space of $SO(3)$-principal bundle over a quaternion Kähler manifold has a CR Einstein-Weyl structure.

Let M be a Riemannian manifold of dimension $4m$ ($m \geq 2$). The manifold M is a quaternion Kähler manifold if the holonomy group of the Levi-Civita connection is contained in $Sp(m) \cdot Sp(1)$, where $Sp(m)$ acts on $\mathbb{H}^m$ on the left and $Sp(1)$ acts on $\mathbb{H}^m$ as $\vec{q} \mapsto \vec{q} \cdot \bar{u}$ on the right for $\vec{q} \in \mathbb{H}^m$. Thus $Sp(m) \cdot Sp(1)$ is a subgroup of $SO(4m)$, which is isomorphic to $Sp(m) \times Sp(1)/\{\pm 1\}$ (cf. [1]).

A Riemannian manifold (M, g) is a quaternion Kähler manifold if and only if there are an open covering $\{U\}$ of M and $(1,1)$ tensor fields J_1, J_2, J_3 (defined on U) satisfying

$$J_1^2 = -I, \qquad J_2^2 = -I, \qquad J_3^2 = -I$$
$$J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,$$
$$g(J_iX,\, J_iY) = g(X,\, Y) \qquad (i = 1, 2, 3)$$

and

$$(7.1) \qquad \begin{cases} \nabla^g_X J_1 &= \qquad\qquad\qquad 2\theta_3(X)J_2 \quad -2\theta_2(X)J_3 \\ \nabla^g_X J_2 &= -2\theta_3(X)J_1 \qquad\qquad\qquad +2\theta_1(X)J_3 \\ \nabla^g_X J_3 &= \;\; 2\theta_2(X)J_1 \quad -2\theta_1(X)J_2 \end{cases}$$

for $X, Y \in TU$, where ∇^g is the Levi-Civita connection of g. The tensors J_1, J_2 and J_3 form a local basis of a vector bundle $V(M)$ over M. For another local basis J_1', J_2' and J_3' on U', we have, on $U \cap U'$,

$$(7.2) \qquad (J_1', J_2', J_3') = (J_1, J_2, J_3)\, s_{U\,U'}, \qquad s_{U\,U'} \in SO(3),$$

where the product of the right hand side is the matrix multiplication.

Let $\mathscr{P}$ be the principal $SO(3)$-bundle associated with $V(M)$, that is, $\mathscr{P}$ is the principal bundle consisting of frames of $V(M)$. The dimension of the total space of $\mathscr{P}$ is equal to $4m+3$. We shall show that the total space $\mathscr{P}$ admits a CR Einstein-Weyl structure. We take a basis of the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ as follows:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

Then the basis satisfies

$$[e_1, e_2] = 2e_3, \qquad [e_2, e_3] = 2e_1, \qquad [e_3, e_1] = 2e_2. \tag{7.4}$$

By using 1-forms θ_1, θ_2 and θ_3 appearing in (7.1), we define ω_U by

$$\omega_U = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3 \tag{7.5}$$

on the each U. Hence ω_U is a $\mathfrak{so}(3)$-valued 1-form. We have, from (7.1) and (7.2),

$$\omega_{U'} = {s_{UU'}}^{-1} ds_{UU'} + {s_{UU'}}^{-1}\omega_U s_{UU'} \tag{7.6}$$

on $U \cap U'$. Therefore the family $\mathfrak{so}(3)$-valued 1-form $\{\omega_U\}$ determines a connection ω in the principal bundle $\mathscr{P}$. If we consider $\sigma = \{J_1, J_2, J_3\}$ as a cross section of $\mathscr{P}$ on U, then $\sigma^*\omega = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3$. We put

$$\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3, \tag{7.7}$$

where ω_1, ω_2 and ω_3 are 1-forms on $\mathscr{P}$. The curvature form Ω of ω is given by

$$\Omega_1 = d\omega_1 + \omega_2 \wedge \omega_3, \quad \Omega_2 = d\omega_2 + \omega_3 \wedge \omega_1, \quad \Omega_3 = d\omega_3 + \omega_1 \wedge \omega_2,$$

where $\Omega = \Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3$. Let ζ_i be the fundamental vector field corresponding to e_i $(i = 1, 2, 3)$. Then we have from (7.4)

$$[\zeta_1, \zeta_2] = 2\zeta_3, \qquad [\zeta_2, \zeta_3] = 2\zeta_1, \qquad [\zeta_3, \zeta_1] = 2\zeta_2 \tag{7.8}$$

and

$$\omega_i(\zeta_j) = \delta_{ij}. \tag{7.9}$$

The Ricci identity for J_i and (7.1) imply that

$$\begin{cases} [R^g(X, Y), J_1] = 4\sigma^*\Omega_3(X, Y)J_2 - 4\sigma^*\Omega_2(X, Y)J_3 \\ [R^g(X, Y), J_2] = -4\sigma^*\Omega_3(X, Y)J_1 + 4\sigma^*\Omega_1(X, Y)J_3 \\ [R^g(X, Y), J_3] = 4\sigma^*\Omega_2(X, Y)J_1 - 4\sigma^*\Omega_1(X, Y)J_2 \end{cases} \tag{7.10}$$

for $X, Y \in TU$. If $m \geq 2$, then it can be shown that (M, g) is Einstein. For the proof of this fact, [p. 403, 2] or [3] where (7.10) is used as a key equation. Let X^H denote the horizontal lift of $X \in TM$. Then we have

$$\Omega_i(X^H, Y^H)_\sigma = -\frac{c}{2}g(J_i X, Y) \qquad (i = 1, 2, 3), \tag{7.11}$$

where $c = \rho^g/\{8m(m+2)\}$ and ρ^g is the scalar curvature of (M, g). In the sequel, we assume that the constant ρ^g does not vanish. We put $\theta = -\omega_1/c$ and $\xi = -c\zeta_1$. Let $\mathscr{D}$ be the hyperdistribution spanned by the horizontal distribution $\mathscr{H}$ of ω, ζ_2 and ζ_3 at each point of $\mathscr{P}$. Then we have $\theta(\xi) = 1$ and $\theta(\mathscr{D}) = 0$. Moreover

$$\begin{aligned} -2d\theta(X^H, Y^H)_\sigma &= \frac{2}{c}\{\Omega_1(X^H, Y^H) - 2(\omega_2 \wedge \omega_3)(X^H, Y^H)\}_\sigma \\ &= -g(J_1 X, Y) \end{aligned} \tag{7.12}$$

for $X, Y \in TU$. We define $J_u : \mathscr{D}_u \to \mathscr{D}_u$ at $u = \{J_1, J_2, J_3\} \in \mathscr{P}$ by

$$J_u V = (J_1 X)^H - \omega_3(V_3)\zeta_2 + \omega_2(V_2)\zeta_3 \tag{7.13}$$

for $V = X^H + V_2 + V_3 \in \mathscr{H}_u \oplus \operatorname{span}\{\zeta_2\} \oplus \operatorname{span}\{\zeta_3\}$. It is easily verified that $J\zeta_2 = \zeta_3$, $J\zeta_3 = -\zeta_2$ and hence J is a complex structure on $\mathscr{D}$. We also define ω_L and g_L by $\omega_L = -2d\theta$ and $g_L(\cdot, \cdot) = \omega_L(J\cdot, \cdot)$, respectively. Then

$$\omega_L(X^H, Y^H)_u = -g(J_1 X, Y), \quad \omega_L(X^H, \zeta_2) = \omega_L(X^H, \zeta_3) = 0, \tag{7.14}$$
$$\omega_L(\zeta_2, \zeta_3) = -\frac{2}{c},$$

since $[\zeta_2, \zeta_3] = 2\zeta_1$ and $[X^H, \zeta_i] = 0$. Putting $\xi_j = \sqrt{|c|/2}\,\varepsilon\zeta_j$ $(j = 2, 3)$, we have

$$g_L(X^H, Y^H) = g(X, Y), \quad g_L(X^H, \xi_j) = 0, \quad g_L(\xi_j, \xi_j) = \varepsilon \tag{7.15}$$

for $j = 2, 3$, where ε is the signature of c. It follows that g_L is nondegenerate and positive definite (resp. pseudo-metric with $\gamma = 2$) if the scalar curvature ρ^g is positive (resp. negative). It is easy to show that g_L is Hermitian, that is, $g_L(JV, JW) = g_L(V, W)$ for $V, W \in \mathscr{D}$. Thus we see that the nondegenerate pair $(\mathscr{D}, J)$ satisfies (C.1) in Section 1. To prove (C.2), we first show

$$[JX^H, JY^H] - [X^H, Y^H] - J([X^H, JY^H] + [JX^H, Y^H]) = 0 \tag{7.16}$$

for $X, Y \in \mathfrak{X}(U)$. For an arbitrarily fixed $u \in \mathscr{P}$, we can take a cross section $\sigma = \{J_1, J_2, J_3\}$ on U such that $\sigma(x) = u$ and $d\sigma(T_x M) = \mathscr{H}_u$, where $\pi(u) = x$, π being the projection $\mathscr{P} \to M$. Then the left hand side of the above equation is equal to

$$[(J_1X)^H, (J_1Y)^H] - [X^H, Y^H] - J([X^H, (J_1Y)^H] + [(J_1X)^H, Y^H]) \tag{7.17}$$

at u, since $JX^H = (J_1X)^H$ along σ. The horizontal component of (7.17) is the horizontal lift of

$$[J_1X, J_1Y] - [X, Y] - J_1([X, J_1Y] + [J_1X, Y]). \tag{7.18}$$

Since $\theta_i = 0$ at x $(i = 1, 2, 3)$, we see from (7.1) that (7.18) vanishes at x. The vertical component of (7.17) also vanishes at u since

$$\begin{aligned}
&\omega_j([(J_1X)^H, (J_1Y)^H] - [X^H, Y^H]) \\
&= 2\{-\Omega_j((J_1X)^H, (J_1Y)^H) + \Omega_j(X^H, Y^H)\} \\
&= -2cg(J_jX, Y)
\end{aligned}$$

and

$$\begin{aligned}
&\omega_j(J[X^H, (J_1Y)^H] + J[(J_1X)^H, Y^H]) \\
&= \omega_j((J_1[X, J_1Y])^H - \omega_3([X^H, (J_1Y)^H])\zeta_2 + \omega_2([X^H, (J_1Y)^H])\zeta_3) \\
&\quad + \omega_j((J_1[J_1X, Y])^H - \omega_3([(J_1X)^H, Y^H])\zeta_2 + \omega_2([(J_1X)^H, Y^H])\zeta_3) \\
&= -2cg(J_jX, Y)
\end{aligned}$$

at u for $j = 2, 3$. Secondly we show, for $j = 2, 3$,

$$[JX^H, J\zeta_j] - [X^H, \zeta_j] - J([X^H, J\zeta_j] + [JX^H, \zeta_j]) = 0. \tag{7.19}$$

Note that $[X^H, \zeta_j] = [X^H, J\zeta_j] = 0$. Thus it suffices to show that

$$[JX^H, J\zeta_j] - J[JX^H, \zeta_j] = 0 \tag{7.20}$$

at u. Since

$$\begin{aligned}[JX^H,\, \zeta_j]_u &= \lim_{t\to 0}\frac{1}{t}\{(d\varphi_t(JX^H))_u - (JX^H)_u\} \\ &= \lim_{t\to 0}\frac{1}{t}\{(J_1(-t) - J_1)X\}_u^H \\ &= -((ue_j)_1 X)_u^H \qquad\qquad (\varphi_t = R\exp(te_j))\,,\end{aligned}$$

where $(J_1(t), J_2(t), J_3(t)) = (J_1, J_2, J_3)\exp(te_j)$ and $((ue_j)_1, (ue_j)_2, (ue_j)_3) = (J_1, J_2, J_3)e_j$. If $j = 2$, then $[JX^H,\, \zeta_2]_u = 2(J_3X)_u^H$ and hence $J[JX^H,\, \zeta_2]_u = 2(J_1J_3X)_u^H = -2(J_2X)_u^H$. Since $[JX^H,\, \zeta_3]_u = -2(J_2X)_u^H$, we see that

$$[JX^H,\, J\zeta_2] - J[JX^H,\, \zeta_2] = 0$$

at u. We have (7.20) for $j = 3$ in the similar way. Thirdly it is easy to show

$$[J\zeta_2,\, J\zeta_3] - [\zeta_2,\, \zeta_3] - J([\zeta_2,\, J\zeta_3] + [J\zeta_2,\, \zeta_3]) = 0.$$

We have proved that the condition (C.2) is satisfied. The pair $(\mathscr{D}, J)$ is a nondegenerate CR structure on $\mathscr{P}$.

Let ϕ be defined by $\phi\xi = 0$ and $\phi|_{\mathscr{D}} = J$. Then (ϕ, ξ, θ) is an almost contact structure belonging to $(\mathscr{D}, J)$. For $fX^H + g\zeta_2 + h\zeta_3 \in \Gamma(\mathscr{D})$, we have

$$[fX^H + g\zeta_2 + h\zeta_3,\, \xi] \equiv 2cg\zeta_3 - 2ch\zeta_2 \quad (\bmod\ \mathscr{D}).$$

Therefore (ϕ, ξ, θ) is a $\mathscr{D}$-preserving almost contact structure.

Next we compute the curvature tensor of the Tanaka connection ∇ associated with (ϕ, ξ, θ). Since $F = -1/2\phi(\mathscr{L}_\xi\phi)$ on $\mathscr{D}$, we easily have $F\xi_j = 0$ $(j = 2, 3)$. Moreover, $\phi(\mathscr{L}_\xi\phi)X^H = J[\xi,\, JX^H]$ and hence $FX^H = 0$ by the same method as the proof of (7.20). Thus we have $F = 0$. It follows that

(7.21) $$\nabla_\xi\xi_2 = -2c\xi_3, \qquad \nabla_\xi\xi_3 = 2c\xi_2.$$

Since

$$\begin{aligned}-\omega_L(\xi_2,\, \xi_3)\xi = T(\xi_2,\, \xi_3) &= \nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2 - [\xi_2,\, \xi_3] \\ &= \nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2 + \varepsilon\xi\end{aligned}$$

and $\omega_L(\xi_2,\, \xi_3) = -\varepsilon$, we see that $\nabla_{\xi_2}\xi_3 = \nabla_{\xi_3}\xi_2$. Note that

$$\begin{aligned}\nabla_{\xi_2}\xi_3 &= \nabla_{\xi_2}(\phi\xi_2) = \phi\nabla_{\xi_2}\xi_2\,, \\ \nabla_{\xi_3}\xi_3 &= \phi\nabla_{\xi_3}\xi_2 = \phi\nabla_{\xi_2}\xi_3 = \phi^2\nabla_{\xi_2}\xi_2.\end{aligned}$$

To prove

(7.22) $$\nabla_{\xi_j}\xi_k = 0 \qquad (j, k = 2, 3),$$

we have only to show $\nabla_{\xi_2}\xi_2 = 0$. Since $\nabla^\circ g_L = 0$ and $T_{\mathscr{D}_\xi} = 0$ on $\mathscr{D}$, we have

$$\begin{aligned}2g_L(\nabla_{\xi_2}\xi_2,\, W) =& \xi_2\cdot g_L(\xi_2,\, W) + \xi_2\cdot g_L(\xi_2,\, W) - W\cdot g_L(\xi_2,\, \xi_2) \\ &- g_L(\xi_2,\, [\xi_2,\, W]_{\mathscr{D}_\xi}) - g_L(\xi_2,\, [\xi_2,\, W]_{\mathscr{D}_\xi}) + g_L(W,\, [\xi_2,\, \xi_2]_{\mathscr{D}_\xi})\end{aligned}$$

for $W \in \Gamma(\mathscr{D})$. If $W = \xi_j$ $(j = 2, 3)$, then the right hand side vanishes. If $W = X^H$, then the right hand side also vanishes because of the equation $[\xi_2,\, X^H] = 0$. By the equation $\nabla_\xi X^H = FX^H + [\xi,\, X^H]$, we obtain

(7.23) $$\nabla_\xi X^H = 0$$

for every $X \in \mathfrak{X}(U)$. Note that

$$2g_L(\nabla_{\xi_j} X^H, W) = \xi_j \cdot g_L(X^H, W) + X^H \cdot g_L(\xi_j, W) - W \cdot g_L(\xi_j, X^H) \\ - g_L(\xi_j, [X^H, W]_{\mathscr{D}_\xi}) - g_L(X^H, [\xi_j, W]_{\mathscr{D}_\xi}) + g_L(W, [\xi_j, X^H]_{\mathscr{D}_\xi}).$$

If $W = \xi_k$ $(k = 2, 3)$, then the right hand side vanishes and if $W = Y^H$, then

$$\begin{aligned} 2g_L(\nabla_{\xi_j} X^H, Y^H) &= - g_L(\xi_j, [X^H, Y^H]_{\mathscr{D}_\xi}) \\ &= - g_L(\xi_j, \omega_2([X^H, Y^H])\zeta_2 + \omega_3([X^H, Y^H])\zeta_3) \\ &= - \frac{1}{a}\omega_j([X^H, Y^H]) \\ &= \frac{2}{a}\Omega_j(X^H, Y^H), \end{aligned}$$

where $a = \sqrt{|c|/2}$. We define K_j $(j = 2, 3)$ by $(K_j)_u X^H = (J_j X)^H_u$ and $K_j \xi_2 = K_j \xi_3 = 0$ at $u = \{J_1, J_2, J_3\} \in \mathscr{P}$. Then K_j is a linear transformation of $\mathscr{D}$ such that

$$\Omega_j(V, W) = -\frac{c}{2} g_L(K_j V, W) \tag{7.24}$$

for every $V, W \in \mathscr{D}$. With this notation, we have

$$\nabla_{\xi_j} X^H = -\varepsilon a K_j X^H. \tag{7.25}$$

Since $[X^H, \xi_j] = 0$ and $\omega_L(X^H, \xi_j) = 0$, we obtain

$$\nabla_{X^H} \xi_j = -\varepsilon a K_j X^H. \tag{7.26}$$

We also use

$$\begin{aligned} 2g_L(\nabla_{X^H} Y^H, Z^H) &= X^H \cdot g_L(Y^H, Z^H) + Y^H \cdot g_L(X^H, Z^H) - Z^H \cdot g_L(X^H, Y^H) \\ &\quad - g_L(X^H, [Y^H, Z^H]_{\mathscr{H}}) - g_L(Y^H, [X^H, Z^H]_{\mathscr{H}}) + g_L(Z^H, [X^H, Y^H]_{\mathscr{H}}) \\ &= X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \\ &= 2g(\nabla^g_X Y, Z) = 2g_L((\nabla^g_X Y)^H, Z^H), \end{aligned}$$

from which

$$\nabla_{X^H} Y^H = a\{g_L(K_2 X^H, Y^H)\xi_2 + g_L(K_3 X^H, Y^H)\xi_3\} + (\nabla^g_X Y)^H. \tag{7.27}$$

To calculate the curvature tensor easily, we assume that $\nabla^g X = \nabla^g Y = \nabla^g Z = 0$ at x and $\sigma = \{J_1, J_2, J_3\}$ is a cross section of $\mathscr{P}|_U$ such that $\sigma(x) = u$ and $d\sigma(T_x M) = \mathscr{H}_u$ for an arbitrarily fixed $u \in \mathscr{P}|_U$. So the calculation is always evaluated at u. Using (7.27), we have

$$\begin{aligned} R^\nabla(X^H, Y^H)Z^H &= a(X^H \cdot g_L(K_2 Y^H, Z^H))\xi_2 + a g_L(K_2 Y^H, Z^H)\nabla_{X^H}\xi_2 \\ &\quad + a(X^H \cdot g_L(K_3 Y^H, Z^H))\xi_3 + a g_L(K_3 Y^H, Z^H)\nabla_{X^H}\xi_3 \\ &\quad - a(Y^H \cdot g_L(K_2 X^H, Z^H))\xi_2 - a g_L(K_2 X^H, Z^H)\nabla_{Y^H}\xi_2 \\ &\quad - a(Y^H \cdot g_L(K_3 X^H, Z^H))\xi_3 - a g_L(K_3 X^H, Z^H)\nabla_{Y^H}\xi_3 \\ &\quad + (R^g(X, Y)Z)^H \\ &\quad - 2\Omega_2(X^H, Y^H)K_2 Z^H - 2\Omega_3(X^H, Y^H)K_3 Z^H, \end{aligned} \tag{7.28}$$

from which and (7.26),

(7.29)
$$g_L(R^\nabla(X^H, Y^H)Z^H, X^H) = -\frac{3}{2}c\{g(X, J_2Z)g(J_2Y, X) + g(X, J_3Z)g(J_3Y, X)\} + g(R^g(X, Y)Z, X).$$

Similarly we have

$$\varepsilon g_L(R^\nabla(\xi_j, Y^H)Z^H, \xi_j) = a^2\varepsilon\zeta_j \cdot g_L(K_jY^H, Z^H) + \frac{c}{2}g(Y, Z)$$

for $j = 2, 3$. Since

$$\begin{aligned}\zeta_j \cdot g_L(K_jY^H, Z^H) =& \frac{d}{dt}g_L(K_jY^H, Z^H)_{u\exp te_j}|_{t=0}\\ =& \frac{d}{dt}g(J_j(t)Y, Z)|_{t=0}\\ =& g((ue_j)_jY, Z)\\ =& 0,\end{aligned}$$

where the notation $J_j(t)$ and $(ue_j)_j$ are defined in the proof of (7.20), we have

$$\varepsilon g_L(R^\nabla(\xi_j, Y^H)Z^H, \xi_j) = \frac{c}{2}g(Y, Z). \tag{7.30}$$

It follows from (7.29) and (7.30) that

$$s^\nabla(Y^H, Z^H) = \frac{m+1}{4m(m+2)}\rho^g g_L(Y^H, Z^H). \tag{7.31}$$

The calculation of $g_L(R^\nabla(X^H, \xi_j)\xi_j, X^H)$ and $\varepsilon g_L(R^\nabla(\xi_k, \xi_j)\xi_j, \xi_k)$ $(j, k = 2, 3,\ j \neq k)$ is easy. The results are

$$g_L(R^\nabla(X^H, \xi_j)\xi_j, X^H) = \frac{1}{2}c\varepsilon g(X, X) \tag{7.32}$$

and

$$\varepsilon g_L(R^\nabla(\xi_k, \xi_j)\xi_j, \xi_k) = 2c\varepsilon. \tag{7.33}$$

It follows from (7.32) and (7.33) that

$$s^\nabla(\xi_j, \xi_j) = \frac{m+1}{4m(m+2)}\rho^g g_L(\xi_j, \xi_j) \tag{7.34}$$

for $j = 2, 3$. The equation (7.28) implies that $g_L(R^\nabla(X^H, Y^H)\xi_j, X^H) = 0$. We have, from $R^\nabla(\xi_k, \xi_j)\xi_j = 2c\varepsilon\xi_j$,

$$\begin{aligned}g_L(R^\nabla(\xi_k, Y^H)\xi_j, \xi_k) =& g_L(R^\nabla(\xi_j, \xi_k)\xi_k, Y^H)\\ =& 0,\end{aligned} \tag{7.35}$$

where we note that the first equality is derived from $F = 0$. Therefore we obtain

$$s^\nabla(Y^H, \xi_j) = 0. \tag{7.36}$$

Similarly we have

$$g_L(R^\nabla(X^H, \xi_2)\xi_3, X^H) = 0, \qquad g_L(R^\nabla(\xi_j, \xi_2)\xi_3, \xi_j) = 0$$

and hence

$$s^\nabla(\xi_2, \xi_3) = 0. \tag{7.37}$$

The two Ricci tensors s^∇ and k^∇ coincide when $F = 0$ (cf. [10]). Therefore we conclude that (ϕ, ξ, θ) is pseudo-Einstein.

Finally we show that $p = \alpha\omega_2 + \beta\omega_3$ (α, β : constant) satisfies (3.16) and $(\nabla_V p)(W) - (\nabla_{JV} q)(W) + p(\xi) g_L(JV, W) = 0$ for $V, W \in \mathscr{D}$. It is easy to show that p satisfies (3.16) by virtue of the structure equation of the connection ω. Since

$$\begin{aligned}&(\nabla_V p)(W) - (\nabla_{JV} q)(W)\\&= V \cdot p(W) - p(\nabla_V W) + JV \cdot p(JW) - p(J\nabla_{JV} W),\end{aligned}$$

we easily see that $(\nabla_V p)(W) - (\nabla_{JV} q)(W) = 0$ in the cases where $(V = X^H, W = Y^H)$, $(V = X^H, W = \xi_j)$, $(V = \xi_j, W = X^H)$ and $(V = \xi_j, W = \xi_k)$. Noting that $p(\xi) = 0$, we obtain the assertion.

In this way, we have shown that the total space of the $SO(3)$-bundle associated with a quaternion Kähler manifold of dimension $4m$ ($m \geq 2$) with non vanishing scalar curvature admits a CR Einstein-Weyl structure.

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Twistor spaces and the general adiabatic expansions

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Abstract. We investigate the behavior of the derivatives of the fundamental solution of the parabolic equation for the square of the Dirac operator on a twistor space when the metric is blown up in the base space direction. Such a blowing up operation is expected to be an effective method for extracting some intrinsic values from various geometric invariants, most of whose cores consist of the derivatives of the fundamental solution.

§0. INTRODUCTION

Let $M = (M, g^M)$ be an n-dimensional compact oriented Riemannian manifold equipped with a Spin^q structure introduced in [17]

$$\Xi^q : P_{\mathrm{Spin}^q(n)}(M) = P_{\mathrm{Spin}(n)}(M) \times_{\mathbf{Z}_2} P_{Sp(1)} \to P_{SO(n)}(M) \times P_{SO(3)} \tag{0.1}$$

where $P_{SO(n)}(M)$ is the $SO(n)$-bundle consisting of $SO(n)$-frames of TM, $P_{SO(3)}$ is a given $SO(3)$-bundle and the bundle map Ξ^q is equivariant to the Lie group homomorphism Ξ^q : $\mathrm{Spin}^q(n) \equiv \mathrm{Spin}(n) \times_{\mathbf{Z}_2} Sp(1) \to SO(n) \times SO(3)$, $\Xi^q([\phi, h]) = (\Xi(\phi), \mathrm{Ad}(h))$ with $\mathrm{Ad}(h) = \left(\mathrm{Im}\,\mathbb{H} \ni \mu \mapsto \mathrm{Ad}(h)\mu = h\mu h^{-1} \in \mathrm{Im}\,\mathbb{H}\right) \in SO(3)$. Given a set of local trivializations $\{f_{U^b} = [f_{0U^b}, f_{1U^b}]\}$ of the (globally defined) principal $\mathrm{Spin}^q(n)$-bundle $P_{\mathrm{Spin}^q(n)}(\mathrm{M})$, the sets $\{f_{0U^b}\}$, $\{f_{1U^b}\}$ define locally defined principal bundles $P_{\mathrm{Spin}(n)}(M)$, $P_{Sp(1)}$ respectively, whose transition functions f_{0,U^bV^b} etc. may not satisfy the cocycle condition in the sense $f_{0,U^bV^b}(p)f_{0,V^bW^b}(p) = f_{0,U^bW^b}(p)$ or $-f_{0,U^bW^b}(p)$ on $U^b \cap V^b \cap W^b\ (\ni p)$, etc. Now, using the canonical action of $\mathrm{Spin}^q(n)$ or $Sp(1)$ on $\mathrm{Spin}^q(n)/\mathrm{Spin}^c(n) = Sp(1)/U(1)$ and the identification $Sp(1)/U(1) = \mathbb{C}P^1$ through the representation $r_H : Sp(1) \to GL_{\mathbf{C}}(H) = GL_{\mathbf{C}}(\mathbb{C}^2)$ with $r_H(\alpha + j\beta) = \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix}$, we get a $\mathbb{C}P^1$-fibration

$$\pi : Z = P_{\mathrm{Spin}^q(n)}(M) \times_{\mathrm{can}} \mathbb{C}P^1 = P_{Sp(1)} \times_{\mathrm{can}} \mathbb{C}P^1 \to M. \tag{0.2}$$

Let us then take an $Sp(1)$-connection A of $P_{Sp(1)}$, so that the **twistor space** Z possesses a canonical Spin structure ([18], [19]). Namely, the connection induces a splitting of TZ into horizontal and vertical components, $TZ = \mathcal{H} \oplus \mathcal{V}$, with natural orientation and the metric $g^Z = \pi^* g^M + g^{\mathcal{V}}$ ($\pi^* g^M = g^Z|\mathcal{H}$) where $g^{\mathcal{V}}$ is the Riemannian metric on $\mathcal{V}$ induced from the Fubini-Study one $ds^{\mathbb{C}P}$ of $\mathbb{C}P^1$. Further we have the locally defined spinor bundle $\$_{g^M}$ associated to $P_{\mathrm{Spin}(n)}(M)$ and a locally defined hermitian vector bundle $\not{H} = P_{Sp(1)} \times_{\tau_H} H$, which together produce the globally defined vector bundle $\pi^* \$_{g^M} \otimes \pi^* \not{H} \equiv \pi^* \$_{g^M} \otimes \$_{g^{\mathcal{V}}} \equiv \$_{g^Z}$ on Z, whose rank is certainly equal to $2^{[\dim Z/2]}$. Then, the locally defined Clifford action ρ_{g^M} of $\mathbb{C}l(T^*M, g^M)$ on $\$_{g^M}$, together with the action $\rho_{g^{\mathcal{V}}}$ of $\mathbb{C}l(\mathcal{V}^*, g^{\mathcal{V}})$ on $\$_{g^{\mathcal{V}}}$ induced from the fiberwise global canonical Spin structure, gives the globally defined action ρ_{g^Z} of $\mathbb{C}l(T^*Z, g^Z)$ on $\$_{g^Z}$, i.e., $\rho_{g^Z}(\pi^*\xi_b) = \pi^*\rho_{g^M}(\xi_b) \otimes 1$ ($\xi_b \in T^*M$) and $\rho_{g^Z}(\xi_f) = \pi^*\rho_{g^M}(\tau_{g^M}) \otimes \rho_{g^{\mathcal{V}}}(\xi_f)$ ($\xi_f \in \mathcal{V}^*$) where τ_{g^M} is the complex volume element of (M, g^M). Thus (Z, g^Z) has a canonical Spin structure, which gives the Dirac operator $\partial\!\!\!/_{g^Z} : \Gamma(\$_{g^Z}) \to \Gamma(\$_{g^Z})$. Now, consider the semi-group with C^∞-kernel $e^{-t\partial\!\!\!/^2_{g^Z}}$ associated to the parabolic equation with the initial condition

$$\text{(0.3)} \qquad \left(\frac{\partial}{\partial t} + \partial\!\!\!/^2_{g^Z}\right)\psi = 0, \quad \psi\Big|_{t=0} = \psi_0 \in L^2\Gamma(\$_{g^Z}).$$

The purpose of the paper is to study, replacing g^Z by $g^Z_\varepsilon = \varepsilon^{-1}\pi^* g^M + g^{\mathcal{V}} = \pi^* g^M_\varepsilon + g^{\mathcal{V}}$ with $\varepsilon > 0$, the behavior of $e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}$ and its derivatives when $\varepsilon \to 0$.

Such an operation of blowing up the metric in the base space direction (or shrinking each fiber into one point), called the adiabatic operation, is expected to be an effective method for extracting some intrinsic values from various geometric invariants of Z. For example, Witten ([21]) found that the adiabatic limit of a certain η-invariant is closely related to the so-called global gravitational anomaly which may impose some restriction on our universe, and his result was further extended mathematically by Cheeger([8], [9]), Bismut-Freed ([6]), Bismut-Cheeger ([5]), Dai([12]), etc. Getzler ([13]) also essentially used the operation to give a new and amazingly short proof of the Atiyah-Singer index theorem for Dirac operators. It seems, however, that the arguments used in [6], [5], etc. are too specialized to be applicable for the study of other various invariants (refer to Remark on Proposition 2.2). In the paper, to settle their studies upon a sound basis, we intend to show the fundamental properties of the behavior of $e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}$ itself which is an essential component of various ones. That is, we will show for example that $e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0, P^0)$ has a series expansion at $\varepsilon^{1/2} = 0$ using $\varepsilon^{m/2}$ ($0 \le m < \infty$), which we will call an **adiabatic expansion**, and study the basic properties of the coefficients and describe the constant term explicitly. Its derivatives are also studied and consequently, with no ad-hoc argument, the formula for the index of Dirac operator is canonically derived, and, moreover, the study of several invariants can be reduced commonly to further investigations of some derivatives of the above coefficients. Last, the author would like to mention that only the situation (0.2) in which he has been interested is treated here, but it will be obviously easy to extend our results into general fibrations.

§1. The Main Theorems

Let us take a coordinate neighborhood ($U = U_{P^0} = U^b \times U^f, x = (x^b, x^f)$) around $P^0 \in Z$ in the following way. First, at $p^0 = \pi(P^0)$, take a g^M-normal coordinate neighborhood ($U^b = U^b_{p^0}, x^{b,p^0}$) and set $x^b(P) = x^{b,p^0}(\pi(P))$. Then we fix a trivialization f_{1U^b} of $P_{Sp(1)}|U^b$ given by a cross-section

$$h \text{ with } \nabla^A_{\partial/\partial r_b} h = 0 \ \ (r_b \text{ is the distance function from } p^0 \text{ in } (M, g^M)), \tag{1.1}$$

which induces an identification

$$\iota_A^{p^0,p} = \iota_A^{p^0} : Z_p \equiv \pi^{-1}(p) \cong Z_{p^0} = \mathbb{C}P^1. \tag{1.2}$$

In other word, this is the A-parallel displacement along the g^M-geodesic from p to p^0 and, together with a $g^{\mathcal{V}}$-normal coordinate neighborhood ($U^f = U^f_{P^0}, x^{f,P^0}$) at $P^0 \in Z_{p^0}$, gives the coordinates x^f with $x^f(P) = x^{f,P^0}(\iota_A^{p^0}(P))$. Further, let us take a local $SO(n)$-frame $e^b = (e^b_1, \cdots, e^b_n)$ of (TM, g^M) which is parallel along the geodesics from p^0 and is equal to $(\partial/\partial x^b)_{p^0} \equiv (\partial/\partial x^b_1, \cdots, \partial/\partial x^b_n)_{p^0}$ at p^0, and similarly a local $SO(2)$-frame $e^f = (e^f_1, e^f_2)$ of $(\mathcal{V}, g^{\mathcal{V}})|Z_{p^0}$ with $e^f(P^0) = (\partial/\partial x^f)_{P^0}$. Note that, referring to [1, Appendix II], we have, for example,

$$e^b_i(x^b) = \sum (\partial/\partial x^b_j)_{x^b} \cdot v^b_{ji}(x^b)\,, \ v^b_{ji}(x^b) = \delta_{ji} + \frac{1}{6} \sum_{j_1, j_2} x^b_{j_1} x^b_{j_2} R^{g^M}_{ij_1jj_2}(0) + O(|x^b|^3) \tag{1.3}$$

where we put $R^{g^M}_{i_2i_1ij}(p) = g^M(F(\nabla^{g^M})(\partial/\partial x^b_i, \partial/\partial x^b_j)\partial/\partial x^b_{i_1}, \partial/\partial x^b_{i_2})(p)$. Moreover, let $e^b(A)$ be the A-horizontal lift of e^b and let us spread e^f out on U by using (1.2) and denote it by the same symbol. They give a local $SO(n+2)$-frame $e_*(A) = (e^b(A), e^f)$ of (TZ, g^Z), a local $SO(n+2)$-frame $e^\varepsilon_*(A) = (e^{b\varepsilon}(A), e^f) = (\varepsilon^{1/2} e^b(A), e^f)$ of (TZ, g^Z_ε) ($e^{b\varepsilon}(A)$ is the A-horizontal lift of $e^{b\varepsilon} = \varepsilon^{1/2} e^b$), and the dual frames $e^*(A) = (e_b, e_f(A))$, $e^*_\varepsilon(A) = (e_{b\varepsilon}, e_f(A)) = (\varepsilon^{-1/2} e_b, e_f(A))$. The frame $e^{\varepsilon}_*(A)$ then gives local $SU(2^{[(n+2)/2]})$-frames $s(e^{\varepsilon}_*(A)) = (s(e^{\varepsilon}_*(A))_1, \cdots) = \pi^* s(\varepsilon^{1/2}e^b) \otimes s(e^f)$, $s(e^{\varepsilon}_*(A))^* = (s(e^{\varepsilon}_*(A))^1, \cdots)$ of $\$_{g^Z_\varepsilon}$, $\$^*_{g^Z_\varepsilon}$. Now, on the neighborhood let us express the Dirac operator as $\partial\!\!\!/_{g^Z_\varepsilon} = \sum \rho_{g^Z_\varepsilon}(e^i_\varepsilon(A))\nabla^{\$_{g^Z_\varepsilon}}_{e^\varepsilon_i(A)}$ where $\nabla^{\$_{g^Z_\varepsilon}}$ is a spinor connection associated to the Levi-Civita one $\nabla^{g^Z_\varepsilon}$, and express the kernel as

$$(1.4)\qquad e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,P') \equiv \sum s(e^{\varepsilon}_*(A))_i(P) \otimes s(e^{\varepsilon}_*(A))^j(P') \cdot \left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)^{i,j}(P,P').$$

Differentiate it by $\partial^\alpha = (\partial/\partial x)^\alpha = (\partial/\partial x_{\alpha_1})(\partial/\partial x_{\alpha_2})\cdots(\partial/\partial x_{\alpha_{|\alpha|}}) = (\partial/\partial x^b_{\alpha^b_1})(\partial/\partial x^b_{\alpha^b_2})\cdots (\partial/\partial x^b_{\alpha^b_{|\alpha^b|}})(\partial/\partial x^f_{\alpha^f_1})(\partial/\partial x^f_{\alpha^f_2})\cdots(\partial/\partial x^f_{\alpha^f_{|\alpha^f|}}) = \partial^{\alpha^b}\partial^{\alpha^f}$ with respect to P and by $\partial^{\alpha'} = (\partial/\partial x)^{\alpha'} = \partial^{\alpha'^b}\partial^{\alpha'^f}$ with respect to P', namely, set

$$(1.5)\qquad \partial^\alpha\partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,P') \equiv \sum s(e^{\varepsilon}_*(A))_i(P) \otimes s(e^{\varepsilon}_*(A))^j(P') \cdot \partial^\alpha\partial^{\alpha'}\left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)^{i,j}(P,P').$$

Precisely the purpose of the paper is to investigate mainly its behavior at $(P,P') = (P^0,P^0)$ when $\varepsilon \to 0$. To clarify what should be studied, let us consider the canonical inclusion

$$(1.6)\qquad \begin{aligned} &\$_{g^Z_\varepsilon,P^0} \otimes \$^*_{g^Z_\varepsilon,P^0} \hookrightarrow \wedge T^*_{p^0}M \otimes \$_{g^V,P^0} \otimes \$^*_{g^V,P^0} \\ &\rho_{g^Z_\varepsilon}(e^I_{b\varepsilon}) \mapsto \begin{cases} e^I_{b\varepsilon}(p^0) & : n \text{ is even (in the case, this is isomorphic)} \\ \frac{1}{2}\{e^I_{b\varepsilon} + (\sqrt{-1})^{(n+1)/2}(-1)^{|I|(|I|+1)} *_{g^M_\varepsilon} e^I_{b\varepsilon}\}(p^0) & : n \text{ is odd} \end{cases}\end{aligned}$$

and regard (1.5) with $(P,P') = (P^0,P^0)$ as an element of its right hand side, that is,

$$(1.7)\quad \partial^\alpha\partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0,P^0) \equiv \sum e^I_b(p^0)\otimes s(e^f)_k(P^0)\otimes s(e^f)^\ell(P^0) \cdot \partial^\alpha\partial^{\alpha'}\left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P^0,P^0)$$

where the multi-index I is always lined up in increasing order, i.e., $I = (i_1 < i_2 < \cdots < i_{|I|})$. In contrast to the expression (1.5), the terms $e^I_b(p^0)\otimes s(e^f)_k(P^0)\otimes s(e^f)^\ell(P^0)$ above do not depend on ε and we have only to investigate the coefficients $\partial^\alpha\partial^{\alpha'}\left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P^0,P^0)$ for our purpose. Because of the advantage, hereafter we will use not (1.5) but the expression (1.7). Further (1.7) can be interpreted canonically as follow: Let $\wedge T^*M \otimes_\pi \$_{g^V} \boxtimes \$^*_{g^V}$ be the pull-back of $\wedge T^*M \boxtimes \$_{g^V} \boxtimes \$^*_{g^V}$ by the map $\pi^{-1}(U^b) \times \pi^{-1}(U^b) \to M \times Z \times Z$, $(P,P') \mapsto (\pi(P),P,P')$ and let us extend the inclusion map (1.6) to

$$(1.8)\qquad \$_{g^Z_\varepsilon}|\pi^{-1}(U^b) \boxtimes \$^*_{g^Z_\varepsilon}|\pi^{-1}(U^b) \hookrightarrow \wedge T^*U^b \otimes_\pi \$_{g^V}|\pi^{-1}(U^b) \boxtimes \$^*_{g^V}|\pi^{-1}(U^b)$$

by the chain of maps $\$_{g^Z_\varepsilon,P}\otimes\$^*_{g^Z_\varepsilon,P'} \cong \$_{g^Z_\varepsilon,\iota^{p^0}_A(P)}\otimes\$^*_{g^Z_\varepsilon,\iota^{p^0}_A(P')} \hookrightarrow \wedge T^*_{p^0}M\otimes\$_{g^V,\iota^{p^0}_A(P)}\otimes\$^*_{g^V,\iota^{p^0}_A(P')} \cong \wedge T^*_{\pi(P)}M\otimes\$_{g^V,P}\otimes\$^*_{g^V,P'}$ where the first isomorphism is given by $s(e^{\varepsilon}_*(A))_i(P)\otimes s(e^{\varepsilon}_*(A))^j(P') \leftrightarrow s(e^{\varepsilon}_*(A))_i(\iota^{p^0}_A(P)) \otimes s(e^{\varepsilon}_*(A))^j(\iota^{p^0}_A(P'))$, the second is by (1.6) and the third is by $e^I_b(p^0) \otimes$

$s(e^f)_k(\iota_A^{p^0}(P)) \otimes s(e^f)^\ell(\iota_A^{p^0}(P')) \leftrightarrow e_b^I(\pi(P)) \otimes s(e^f)_k(P) \otimes s(e^f)^\ell(P')$. Now, regard (1.4) which is a cross-section of the left hand side of (1.8) as one of its right hand side, i.e.,

$$(1.9) \qquad e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P') \equiv \sum e_b^I(\pi(P)) \otimes s(e^f)_k(P) \otimes s(e^f)^\ell(P') \cdot \left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P,P'),$$

and differentiate it into

$$(1.10) \quad \partial^\alpha\partial^{\alpha'} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P') \equiv \sum e_b^I(\pi(P)) \otimes s(e^f)_k(P) \otimes s(e^f)^\ell(P') \cdot \partial^\alpha\partial^{\alpha'}\left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P,P').$$

Then its value at $(P,P') = (P^0,P^0)$ is obviously equal to (1.7). Next, let us define the pointwise norm of (1.9), etc. with respect to the metric g^Z by

$$(1.11) \qquad \left|e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right|_{g^Z} = \{\sum \left|\left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P,P')\right|^2\}^{1/2},$$

etc., and, using various metrics, various pointwise norms of cross-sections of the right hand side of (1.8) are similarly defined. Note that the so-called pointwise operator norm of (1.4) is equivalent to the norm $\left|e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right|_{g^Z_\varepsilon}$ with respect to g^Z_ε uniformly for all ε with $0 < \varepsilon \le \varepsilon_0$:

$$e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P') = \sum (\varepsilon^{-1/2}e_b)^I(\pi(P)) \otimes s(e^f)_k(P) \otimes s(e^f)^\ell(P') \cdot \varepsilon^{|I|/2}\left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P,P'),$$

$$(1.12) \quad \left|e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right|_{g^Z_\varepsilon} = \{\sum \left|\varepsilon^{|I|/2}\left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right)^{(k,\ell)}_{g^Z,I}(P,P')\right|^2\}^{1/2}$$

In the paper we will argue mainly on the right hand side of (1.8) using the norm of (1.11) type. Its merit lies in the fact that (the inclusion map changes, but) the bundle of the side does not depend on the parameter ε (refer also to the comment following (1.7)), so that it makes sense to ask whether or not its elements which depend on ε, such as $e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)$ etc., can be expanded into series with respect to the variable ε. Anyway, to expand them, they need to inhabit some bundle not depending on ε. In [6] etc., a certain bundle isometry $\$_{g^Z_\varepsilon} \cong \$_{g^Z}$ (see Remark to Lemma 5.1, and see [7] for its further generalization) was adopted, by which they inhabit $\$_{g^Z} \boxtimes \$^*_{g^Z}$ not depending on ε. This scheme is of course equivalent to ours. But, comparing these, ours will be much simply introduced and the results in our scheme will describe more clearly how (1.5) etc. depend on the parameter ε.

Now the first result is

Theorem 1.1. *Given α, α' and $\bar{r} > 0$, there exist constants $C_1 > 0$, $C_2 > 0$ and an integer $N > 0$ satisfying*

$$(1.13) \qquad \left|\partial^\alpha\partial^{\alpha'} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P^0)\right|_{g^Z_\varepsilon}$$

$$\leq \frac{C_1}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big)\begin{cases}1 & : (\textit{with no condition}),\\ e^{-(r_{g_\varepsilon^Z}(P,P^0)-\bar{r})^2/C_2 t} & : r_{g_\varepsilon^Z}(P,P^0)>\bar{r},\end{cases}$$

(1.14)
$$\Big|\partial^\alpha\partial^{\alpha'}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P,P^0)\Big|_{g^Z}$$
$$\leq \frac{C_1}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\Big)\begin{cases}1 & : (\textit{with no condition})\\ e^{-(r_{g_\varepsilon^Z}(P,P^0)-\bar{r})^2/C_2 t} & : r_{g_\varepsilon^Z}(P,P^0)>\bar{r}\end{cases}$$
$$(\,0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall P^0\in Z,\ \forall P\in\pi^{-1}(U^b_{\pi(P^0)})\,)$$

where $r_{g_\varepsilon^Z}(P,P^0)$ is the distance from P to P^0 with respect to the metric g_ε^Z. Further, given α, α', $\bar{r}>0$ and $T_0>0$, there exists a constant $C>0$ satisfying

(1.15)
$$\Big|\partial^\alpha\partial^{\alpha'}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P,P^0)\Big|_{g^Z}\leq\frac{C\,e^{-r_{g_\varepsilon^Z}(P,P^0)^2/5t}}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}\,t^{(n+2+|\alpha|+|\alpha'|)/2}}$$
$$(\,0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ 0<\forall t\leq T_0,\ \forall P^0\in Z,\ \forall P\in U_{P^0}\,).$$

Next, in the case $P=P^0$, we have

Theorem 1.2. *For any integer $m_0\geq 0$, there exist C^∞-sections $K_{(m/2)}(t,P^0,P,P')$ ($m=0,1,\cdots,m_0$), $K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0,P,P')$ of the right hand side of (1.8), which are also C^∞ with respect to the variable P^0 (and $\varepsilon^{1/2}$), satisfying the following two conditions.*

(1) *Define the differentiations of $K_{(m/2)}(t,P^0,P,P')$ etc. at $(P,P')=(P^0,P^0)$ in the same way as at (1.10). Then, at $\varepsilon^{1/2}=0$, (1.7) has a series expansion called an* **adiabatic expansion**

(1.16)
$$\partial^\alpha\partial^{\alpha'}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P^0,P^0)=\sum_{m=0}^{m_0}\varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+m/2}\,\partial^\alpha\partial^{\alpha'}K_{(m/2)}(t,P^0,P^0,P^0)$$
$$+\varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+(m_0+1)/2}\,\partial^\alpha\partial^{\alpha'}K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0,P^0,P^0).$$

(2) *Given α, α', there exist constants $\lambda>0$, $C>0$ and an integer $N>0$ satisfying*

(1.17)
$$|\,\partial^\alpha\partial^{\alpha'}K_{(m/2)}(t,P^0,P^0,P^0)|_{g^Z}\leq C\,e^{-t\lambda}\,t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big),$$
$$|\,\partial^\alpha\partial^{\alpha'}K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0,P^0,P^0)|_{g^Z}\leq C\,t^{1/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\Big)$$
$$(\,0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall P^0\in Z\,)$$

where $\delta_{0m}=1$ if $m=0$ and $\delta_{0m}=0$ if $m\neq 0$.

Moreover, the first term $K_{(0)}(t,P^0,P,P')$ can be described concretely. After some preparations, we will introduce it. First, let us define the functions $\nu^k(A(e_i^b))$ on U by

(1.18)
$$e_i^b(A)=e_i^b-2\sum\nu^k(A(e_i^b))\,e_k^f,\quad\text{hence,}\quad e_f^k(A)=e_f^k+2\sum\nu^k(A(e_i^b))\,e_b^i.$$

Considering various connections A, the map $A(e_i^b)(\pi(P)) \mapsto \nu(A(e_i^b))^\natural(P) \equiv \sum \nu^k(A(e_i^b))(P)$ $e_k^f(P)$ gives a linear map $sp(1) \ni a \mapsto \nu(a)^\natural(P) \in \mathcal{V}_P$. Hence we obtain cross-sections $\nu(i)^\natural$, $\nu(j)^\natural$, $\nu(k)^\natural$, which are independent of the choice of A, and using which we have the expression $\nu(A(e_i^b))^\natural = \nu(i)^\natural A(e_i^b)^{(i)} + \nu(j)^\natural A(e_i^b)^{(j)} + \nu(k)^\natural A(e_i^b)^{(k)}$ for $A(e_i^b) = iA(e_i^b)^{(i)} + jA(e_i^b)^{(j)} + kA(e_i^b)^{(k)}$. Further, for its curvature 2-form $F_A = iF_A^{(i)} + jF_A^{(j)} + kF_A^{(k)} \in \Omega^2(sp(1)_{P_{Sp(1)}})$ etc., we set

$$(1.19) \qquad \nu(F_A)^\natural = \nu(i)^\natural \otimes F_A^{(i)} + \nu(j)^\natural \otimes F_A^{(j)} + \nu(k)^\natural \otimes F_A^{(k)} = \sum e_k^f \otimes \nu^k(F_A),$$

etc. Then, let us consider the elliptic operator acting on $\Gamma(\wedge T_p^* M \otimes \$_{g^\mathcal{V}} | Z_p)$

$$(1.20) \quad \mathcal{A}_{p^0}^2 = \mathcal{A}_{p^0,p}^2 = 1 \otimes \partial\!\!\!/_{g^\mathcal{V}}^2 - \sum \nu^k(F_A)_{p^0} \wedge \cdot 1 \otimes \nabla_{e_k^f}^{\$_{g^\mathcal{V}}} + \frac{1}{4}\Big(\sum \nu^k(F_A)_{p^0} \wedge \cdot \rho_{g^Z}(e_f^k(A))\Big)^2$$
$$\text{with } \nu^k(F_A)_{p^0}(P) = \frac{1}{2}\sum \nu^k(F_A(\partial/\partial x_i^b, \partial/\partial x_j^b))(\iota_A^{p^0}(P))\,(e_b^{i_1} \wedge e_b^{i_2})(p)$$

where $\nabla^{\$_{g^\mathcal{V}}}$ is the spinor connection for $\$_{g^\mathcal{V}}|Z_p$ associated to the Levi-Civita one $\nabla^{g^\mathcal{V}}$ on Z_p, which coincides with ∇^{g^Z} restricted to Z_p because each fiber is totally geodesic, and $\partial\!\!\!/_{g^\mathcal{V}}$ is the associated fiberwise Dirac operator. (Note that $\nabla_{e_i^b}^{\$_{g^\mathcal{V}}}$ has no meaning now.) Clearly (1.20) gives a (C^0) semi-group with C^∞-kernel $\exp\big(-t\mathcal{A}_{p^0}^2\big)$, which is a cross-section of $\wedge T_p^* M \otimes \$_{g^\mathcal{V}}|Z_p \boxtimes \$_{g^\mathcal{V}}^*|Z_p \subset \mathrm{End}(\wedge T_p^* M) \otimes \$_{g^\mathcal{V}}|Z_p \boxtimes \$_{g^\mathcal{V}}^*|Z_p$ expressed as

$$(1.21) \quad \exp\big(-t\mathcal{A}_{p^0}^2\big)(P,P') = \sum e_b^I(p) \otimes s(e^f)_k(P) \otimes s(e^f)^\ell(P') \cdot \exp\big(-t\mathcal{A}_{p^0}^2\big)_{g^Z,I}^{(k,\ell)}(P,P').$$

We take then a cross-section of the right hand side of (1.8) defined by

$$(1.22) \quad \exp\big(-t\mathcal{A}_{p^0}^2\big)(P,P')$$
$$= \sum e_b^I(\pi(P)) \otimes s(e^f)_k(P) \otimes s(e^f)^\ell(P') \cdot \exp\big(-t\mathcal{A}_{p^0}^2\big)_{g^Z,I}^{(k,\ell)}(P, \iota_A^{\pi(P),\pi(P')}(P')).$$

Next, on the coordinate neighborhood (U^b, x^b) at $p^0 = \pi(P^0)$, let us set $R_{ij}^{g^M}(p^0) = R_{ij}^{g^M}(p^0,p)$ $= \frac{1}{2}\sum R_{i_2 i_1 ij}^{g^M}(p^0)\,(e_b^{i_1} \wedge e_b^{i_2})(p)$, which belongs to $\Gamma(\wedge T^*M|U^b)$, and denote by $R^{g^M}(p^0) = R^{g^M}(p^0,p)$ the anti-symmetric matrix whose (i,j)-entries are equal to $R_{ij}^{g^M}(p^0)$. And, putting $x^b = x^b(P)$, $x'^b = x^b(P')$, we set

$$(1.23) \quad K_M(t,P^0,P,P') = \frac{1}{(4\pi t)^{n/2}} \det{}^{1/2}\left(\frac{tR^{g^M}(p^0)/2}{\sinh\big(tR^{g^M}(p^0)/2\big)}\right)$$
$$\times \exp\Big(-\frac{1}{4t}\Big\langle (x^b - x'^b)\,\Big|\, \frac{tR^{g^M}(p^0)}{2} \coth \frac{tR^{g^M}(p^0)}{2} \,\Big|\, (x^b - x'^b)\Big\rangle + \frac{1}{4}\Big\langle x^b \,\Big|\, R^{g^M}(p^0) \,\Big|\, x'^b\Big\rangle\Big),$$

which is a cross-section of the right hand side of (1.8). In particular, $K_M(t,P^0,P,P^0)$ (that is, $x'^b = 0$) was originally introduced by Getzler([13], [3, Theorem 4.20]) as a formal solution of a certain parabolic equation (see (5.25)). Note that $\jmath(tR^{g^M}(p^0)) \equiv \det\Big(\sinh(tR^{g^M}(p^0)/2)/(tR^{g^M}(p^0)/2)\Big)$ is a polynomial with respect to t and $\jmath(0R^{g^M}(p^0)) = 1$, so that $\jmath^{-1/2}(tR^{g^M}(p^0)) = \det^{1/2}\Big((tR^{g^M}(p^0)/2)/(\sinh(tR^{g^M}(p^0)/2))\Big)$ is well-defined as an analytic function of t. Hence, by considering the degrees of differential forms $R_{ij}^{g^M}(p^0)$, we find it also a polynomial with respect to t and so is (1.23) devided by the Gaussian kernel $(4\pi t)^{-n/2}e^{-|x^b-x'^b|^2/4t}$.

Now, under these preparations and referring to (1.3), we have

Theorem 1.3. *We may set*

$$K_{(0)}(t, P^0, P, P') = K_M(t, P^0, P, P') \exp\Big(-t\mathcal{A}_{p^0}^2\Big)(P, P') \cdot \det v^b(P'). \tag{1.24}$$

Note that $\det v^b(P') = \det(g^M(\partial/\partial x_i^b, \partial/\partial x_j^b)(x'^b))^{-1/2} = 1 + \mathcal{O}(|x'^b|^2)$.

§2. Two Propositions and the Proof of Theorem 1.2

Let us introduce two propositions, using which we will prove Theorem 1.2. The proofs of the propositions will be postponed to the following two sections.

As in [13], [5], etc., let us start our discussion with showing that the proof of Theorem 1.2 can be reduced to a study of parabolic equation (0.3) for $\not{\partial}_{g_\varepsilon^Z}^2$ localized at each point $p^0 \in M$ in the following way. First, since the injectivity radius $i(g_\varepsilon^M)$ does not decrease when $\varepsilon \to 0$, there exists a constant $r_0 > 0$ with $i(g_\varepsilon^M) \geq 3r_0$ $(0 < \varepsilon \leq \varepsilon_0)$. Fix $p^0 \in M$ and let us identify its normal coordinate neighborhood $(U^b, x^b = x^{b,p^0})$ with an open ball B_{2r_0} in $M(p^0) \equiv (\mathbb{R}^n, x^b)$ centered at the origin and with radius $2r_0$. We take a metric $g^{M(p^0)}$ on $M(p^0)$ so that its restriction to B_{2r_0} is equal to g^M through the identification, outside B_{3r_0} it is trivial, and, moreover, x^b are its normal coordinates at the origin all over $M(p^0)$. Further, let us spread the frames e^b, e_b on $U^b = B_{2r_0}$ all over $M(p^0)$ by the parallel displacement along the geodesics from the origin. Consequently we have a trivial Spin^q structure

$$\begin{aligned}\Xi^q(p^0) : P_{\mathrm{Spin}^q(n)}(M(p^0)) &= P_{\mathrm{Spin}(n)}(M(p^0)) \times_{\mathbf{Z}_2} P_{Sp(1)}(M(p^0)) \\ &\to P_{SO(n)}(M(p^0)) \times P_{SO(3)}(M(p^0))\end{aligned} \tag{2.1}$$

which coincides with (0.1) on $U^b = B_{2r_0}$. Note that the bundles above are all globally defined and canonically trivial, so that $M(p^0)$ has a Spin structure $P_{\mathrm{Spin}(n)}(M(p^0)) \to P_{SO(n)}(M(p^0))$, from which the Spin^q structure is induced. Accordingly, (2.1) gives a trivial $\mathbb{C}P^1$-fibration

$$\pi(p^0) : Z(p^0) \equiv P_{\mathrm{Spin}^q(n)}(M(p^0)) \times_{\mathrm{can}} \mathbb{C}P^1 = M(p^0) \times Z_{p^0} \to M(p^0) \tag{2.2}$$

with a fiberwise metric $g^{\mathcal{V}(p^0)}$ which coincides with $g^{\mathcal{V}}$ on $B_{2r_0} = U^b$ and is independent of x^b outside B_{3r_0}. Moreover, we will take a connection $A(p^0)$ of $P_{Sp(1)}(M(p^0))$ which coincides with the original A on $B_{2r_0} = U^b$ and satisfies $A(p^0)(e_i^b) = 0$ for all i outside B_{3r_0}.

Now, under the setting such localized at p^0, certainly there exists a canonical Spin structure on $Z(p^0)$, a spinor bundle $\$_{g_\varepsilon^{Z(p^0)}}$ and the Dirac operator $\partial\!\!\!/_{g_\varepsilon^{Z(p^0)}}$, etc. as in §1, all of which coincide with the original ones on $B_{2r_0} = U^b$. And, on the coordinate neighborhood ($U = M(p^0) \times U^f, x = (x^b, x^f)$) (taken as in §1) and using the frame $e_b \otimes s(e^f)$, if we write $\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}$ as $-\sum \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum a_i(x) \frac{\partial}{\partial x_i} + c(x)$ $(a_{ij} = a_{ji})$, then the coefficients $a_{ij}(x)$, $a_i(x)$, $c(x)$ can be expressed canonically using the derivatives of the coefficients of $g_\varepsilon^{Z(p^0)}$ and $A(p^0)$. Since $g_\varepsilon^{Z(p^0)}$ is independent of x^b outside B_{3r_0} and also $A(p^0)$ is flat outside B_{3r_0}, and, moreover, the principal symbol of $\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}$ at $(x, \xi) \in T_x Z(p^0)$ is equal to $g_\varepsilon^{Z(p^0)}(\xi, \xi) \cdot \mathrm{id}_{\$_{g_\varepsilon^{Z(p^0)}, x}}$, we know that

(2.3) the finite-times derivatives of $a_{ij}(x)$, $a_i(x)$ and $c(x)$ are all bounded on U, and the principal symbol is hermitian and uniformly elliptic.

Hence, the Yosida's theorem ([22, Chapter IX]) says that the parabolic equation (0.3) for $\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}$ with the canonical domain has a (C^0) semi-group with C^∞-kernel $e^{-t\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}}$.

The first proposition is then as follows. Refer to [12, Lemma 3.3] which gives the similar estimates.

Proposition 2.1 *(the Duhamel's principle). Given α and α', there exist constants $C_1 > 0$ and $C_2 > 0$ such that, for each $p^0 \in M$, we have*

$$\left| \partial^\alpha \partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g_\varepsilon^Z}}(P, P') - \partial^\alpha \partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}}(P, P') \right|_{g^Z} \le \frac{C_1 t}{\varepsilon^{n+(|\alpha^b|+|\alpha'^b|-1)/2}} e^{-C_2/\varepsilon t} \tag{2.4}$$

$$\left(\begin{array}{l} 0 < \forall \varepsilon^{1/2} \le \varepsilon_0^{1/2},\ 0 < \forall t < \infty,\ \forall (P, P') \in \pi^{-1}(U^b_{p^0}) \times \pi^{-1}(U^b_{p^0}) \text{ with (putting} \\ \pi(P) = p, \text{ etc.) } r_{g^M}(p, p^0) \le r_0/2,\ r_{g^M}(p', p^0) \le r_0/2 \text{ and } r_{g^M}(p, p') \le r_0/3 \end{array} \right)$$

where we regard $\partial^\alpha \partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}}(P, P')$ also as a cross-section of the right hand side of (1.8) with Z replaced by $Z(p^0)$.

Hence, the difference $e^{-t\partial\!\!\!/^2_{g_\varepsilon^Z}}(P, P') - e^{-t\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}}(P, P')$ $(P, P' \in U \subset Z)$ may be counted in the remainder term $K_{((m_0+1)/2, \varepsilon^{1/2})}(t, P^0, P, P')$ at Theorem 1.2 and the study of (1.7) is now reduced to that of $e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}$. Abbreviating $Z(p^0)$, $M(p^0)$, etc. to Z, M, etc. to simplify the description, we will investigate it in the following.

To do so, let us consider, not $\not{\partial}_{g_\varepsilon^Z}(=\not{\partial}_{g_\varepsilon^{Z(p^0)}})$, but $\not{\partial}_{g_\varepsilon^Z}\otimes \mathrm{id}_{\not{S}^*_{g_\varepsilon^M,p^0}}$ acting on the cross-sections of the right hand side of

$$\not{S}_{g_\varepsilon^Z}\otimes \not{S}^*_{g_\varepsilon^M,p^0}\hookrightarrow \wedge T^*M\otimes_\pi \not{S}_{g^V}. \tag{2.5}$$

In the same way as in (1.8), the inclusion map is given as $\not{S}_{g_\varepsilon^Z,P}\otimes\not{S}^*_{g_\varepsilon^M,p^0}\cong \not{S}_{g_\varepsilon^Z,\iota_A^{p^0}(P)}\otimes\not{S}^*_{g_\varepsilon^M,p^0}\cong (\not{S}_{g_\varepsilon^M,p^0}\otimes\not{S}^*_{g_\varepsilon^M,p^0})\otimes\not{S}_{g^V,\iota_A^{p^0}(P)}\hookrightarrow \wedge T^*_{p^0}M\otimes\not{S}_{g^V,\iota_A^{p^0}(P)}\cong\wedge T^*_{\pi(P)}M\otimes\not{S}_{g^V,P}$. Its action, originally on the left hand side, can be obviously extended to the action on the right hand side by regarding $\rho_{g_\varepsilon^Z}(\varepsilon^{-1/2}e_b^i)$ as $(\varepsilon^{-1/2}e_b^i)\wedge-(\varepsilon^{-1/2}e_b^i)\vee$ where $\vee$ is the inner product. Thus we obtain an elliptic operator

$$\not{\partial}_{g_\varepsilon^Z}\,(=\not{\partial}_{g_\varepsilon^Z}\otimes\mathrm{id}_{\not{S}^*_{g_\varepsilon^M,p^0}}):\Gamma(\wedge T^*M\otimes_\pi\not{S}_{g^V})\to\Gamma(\wedge T^*M\otimes_\pi\not{S}_{g^V}) \tag{2.6}$$

and, observing (2.3) around, its square $\not{\partial}^2_{g_\varepsilon^Z}\,(=(\not{\partial}_{g_\varepsilon^Z})^2\otimes\mathrm{id}_{\not{S}^*_{g_\varepsilon^M,p^0}})$ with the canonical domain certainly gives a (C^0) semi-group with C^∞-kernel $e^{-t\not{\partial}^2_{g_\varepsilon^Z}}$, which is a cross-section of the bundle

$$(\wedge T^*M\boxtimes(\wedge T^*M)^*)\otimes_\pi(\not{S}_{g^V}\boxtimes\not{S}^*_{g^V}). \tag{2.7}$$

Note that, if we express (1.9) for $g_\varepsilon^Z=g_\varepsilon^{Z(p^0)}$ as

$$\begin{aligned} &e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}(P,P')\\ &=\sum e_b^I(P)\otimes\left(e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}\right)_{g^Z,I}(P,P')=\sum(\varepsilon^{-1/2}e_b)^I(P)\otimes\left(e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}\right)_I(P,P')\end{aligned} \tag{2.8}$$

with abbreviating $e_b^I(\pi(P))$ to $e_b^I(P)$, then the above kernel can be written as

$$\begin{aligned} &e^{-t\not{\partial}^2_{g_\varepsilon^Z}}\\ &=\sum((\varepsilon^{-1/2}e_b)\wedge-(\varepsilon^{-1/2}e_b)\vee)^I(\varepsilon^{-1/2}e_b)^J(P)\otimes((\varepsilon^{-1/2}e_b)^J)^*(P')\otimes\left(e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}\right)_I(P,P')\\ &\equiv\sum(\varepsilon^{-1/2}e_b)^I(P)\otimes((\varepsilon^{-1/2}e_b)^J)^*(P')\otimes\left(e^{-t\not{\partial}^2_{g_\varepsilon^Z}}\right)_{(I,J)}(P,P'),\end{aligned} \tag{2.9}$$

that is,

$$\partial^\alpha\partial^{\alpha'}\left(e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}\right)_{g^Z,I}=\varepsilon^{-|I|/2}\,\partial^\alpha\partial^{\alpha'}\left(e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}\right)_I=\varepsilon^{-|I|/2}\,\partial^\alpha\partial^{\alpha'}\left(e^{-t\not{\partial}^2_{g_\varepsilon^Z}}\right)_{(I,\emptyset)}. \tag{2.10}$$

Thus the study of (1.7) or (2.8) was reduced to that of (2.9). Next, let us replace the metric $g_\varepsilon^Z=\pi_A^*\,g_\varepsilon^M+g^V$ which diverges when $\varepsilon\to0$ by a non-divergent metric. That is, consider a (global) diffeomorphism of $Z=Z(p^0)$ given by

$$\iota_\varepsilon: Z\cong Z,\ x=(x^b,x^f)\mapsto(\varepsilon^{1/2}x^b,x^f) \tag{2.11}$$

and set

$$(2.12)\qquad g^Z_{(\varepsilon)} = \iota^*_\varepsilon g^Z_\varepsilon = \pi^*_{\iota^*_\varepsilon A}\, g^M_{(\varepsilon)} + g^V \ \text{ with } g^M_{(\varepsilon)} = \iota^*_\varepsilon g^M_\varepsilon = \sum e^i_b(\varepsilon)\otimes e^i_b(\varepsilon).$$

This change of metric is the generalization of the change in [9] (from (1.15) to (1.16) in it). Since $\lim_{\varepsilon\to 0} g^M_{(\varepsilon)} = \sum dx^b_i \otimes dx^b_i \equiv g^M_{(0)}$ and $\lim_{\varepsilon\to 0}(\iota^*_\varepsilon A)((\partial/\partial x^b_i)(x)) = 0$, certainly (2.12) converges to the product metric $g^M_{(0)} + \sum e^k_f \otimes e^k_f = g^M_{(0)} + g^V \equiv g^Z_{(0)}$ on $Z = M \times Z_{p^0}$. Let us then consider (2.6) with g^Z_ε replaced by $g^Z_{(\varepsilon)}$. Its square $\not{\partial}^2_{g^Z_{(\varepsilon)}}$ with the canonical domain (see (2.3)) also gives a (C^0) semi-group with C^∞-kernel $e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}$ and obviously we have

$$(2.13)\quad e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}(x,x') = \iota^*_\varepsilon \circ e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(\iota_\varepsilon(x),\iota_\varepsilon(x'))\circ(\iota^*_\varepsilon)^{-1} \ \text{ with}$$
$$\iota^*_\varepsilon : \wedge T^*M\otimes_\pi \$_{g^V} \to \wedge T^*M\otimes_\pi \$_{g^V},\ ((\varepsilon^{-1/2}e_b)^I\otimes s(e^f)_k)(\iota_\varepsilon(x)) \mapsto (e^I_b(\varepsilon)\otimes s(e^f)_k)(x).$$

Hence, if we express it similarly to (2.9) as

$$(2.14)\qquad e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}(x,x') = \sum e^I_b(\varepsilon)(x)\otimes e^J_b(\varepsilon)^*(x')\cdot\left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(x,x'),$$

then we have

$$(2.15)\qquad \partial^\alpha_x\partial^{\alpha'}_{x'}\left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right)_{(I,J)}(x,x') = \partial^\alpha_x\partial^{\alpha'}_{x'}\left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(\iota^{-1}_\varepsilon(x),\iota^{-1}_\varepsilon(x'))$$
$$= \varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2}\,\partial^\alpha_{\iota^{-1}_\varepsilon(x)}\partial^{\alpha'}_{\iota^{-1}_\varepsilon(x')}\left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(\iota^{-1}_\varepsilon(x),\iota^{-1}_\varepsilon(x')).$$

That is, referring to (2.10), finally we have

$$(2.16)\qquad \partial^\alpha\partial^{\alpha'}\left(e^{-t\not{\partial}^2_{g^{Z(p^0)}_\varepsilon}}\right)_{g^Z,I}(P^0,P^0) = \varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2-|I|/2}\,\partial^\alpha\partial^{\alpha'}\left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)_{(I,\emptyset)}(0,0).$$

Thus, the study of (1.7) or (2.8) was further reduced to that of (2.14). The limit metric $g^Z_{(0)}$ is of product type as was explained and (2.6) etc. associated to it are certainly as follows:

$$(2.17)\qquad \not{\partial}_{g^Z_{(0)}} = (d+\delta_{g^M_{(0)}}) + \not{\partial}_{g^V},\ \ \not{\partial}^2_{g^Z_{(0)}} = (d+\delta_{g^M_{(0)}})^2 + \not{\partial}^2_{g^V}$$

where d is the exterior derivative and $\delta_{g^M_{(0)}}$ is its formal adjoint. Hence, obviously we have

$$(2.18)\qquad e^{-t\not{\partial}^2_{g^Z_{(0)}}}(x,x') \equiv K(t,0,x,x') \equiv K_{g^M_{(0)}}(t,0,x^b,x'^b)\cdot K_{g^V}(t,0,x^f,x'^f)$$
$$\equiv \sum (dx^b)^I(x)\otimes((dx^b)^I)^*(x')\,\frac{e^{-|x^b-x'^b|^2/4t}}{(4\pi t)^{n/2}}\cdot e^{-t\not{\partial}^2_{g^V}}(x^f,x'^f).$$

Referring to (2.14) and (1.3), now put

$$(2.19)\qquad e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}(x,x') \equiv K(t,\varepsilon,x,x') = \sum (dx^b)^I(x) \otimes ((dx^b)^J)^*(x') \cdot K(t,\varepsilon,x,x')_{(I,J)}$$
$$= \sum (dx^b)^I(x) \otimes ((dx^b)^J)^*(x') \otimes s(e^f)_k \otimes s(e^f)^\ell \cdot K(t,\varepsilon,x,x')^{(k,\ell)}_{(I,J)},$$

$$(2.20)\qquad \text{hence, } \left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(x,x') = \sum K(t,\varepsilon,x,x')_{(I',J')}\, v^b_{I'I}(\varepsilon^{1/2}x^b)\, v^{J'J}_b(\varepsilon^{1/2}x'^b)$$
$$\text{with } (dx^b)^{I'}(x^b) = \sum v^b_{I'I}(x^b)\cdot e^I_b(x^b),\ ((dx^b)^{J'})^*(x'^b) = \sum v^{J'J}_b(x'^b)\cdot (e^J_b)^*(x'^b),$$

then we can state the second proposition.

Proposition 2.2. *The kernel $K(t,\varepsilon,x,x')$, i.e., each $K(t,\varepsilon,x,x')_{(I,J)}$, is C^∞ with respect to $(\varepsilon^{1/2},t,x,x') \in [0,\varepsilon_0^{1/2}] \times (0,\infty) \times Z \times Z$. Consider then the Taylor expansion*

$$(2.21)\qquad K(t,\varepsilon,x,x') = \sum_{m=0}^{m_0} \varepsilon^{m/2} K(t,m/2:x,x') + \varepsilon^{(m_0+1)/2} K(t,(m_0+1)/2:\varepsilon^{1/2},x,x').$$

Let us here define the differential $\partial^\alpha\partial^{\alpha'}K(t,\varepsilon,x,x')$ etc. by the differentials of the coefficients $K(t,\varepsilon,x,x')^{(k,\ell)}_{(I,J)}$ etc. (not of the coefficients $\left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)^{(k,\ell)}_{(I,J)}$ etc.), set $|\partial^\alpha\partial^{\alpha'}K(t,\varepsilon,x,x')|_{g^Z_{(0)}} = \{\sum |\partial^\alpha\partial^{\alpha'}K(t,\varepsilon,x,x')^{(k,\ell)}_{(I,J)}|^2\}^{1/2}$ (compare with (1.11)), and put $r_{g^Z_{(0)}}(x) = r_{g^Z_{(0)}}(x,0)$. Then we have:

(1) *Given α, α' and $\bar r > 0$, there exist constants $\lambda_0 > 0$, $C_1 > 0$, $C_2 > 0$ and an integer $N > 0$ satisfying*

$$(2.22)\qquad |\partial^\alpha\partial^{\alpha'}K(t,m/2:x,x')|_{g^Z_{(0)}} \le C_1\,(1+r_{g^Z_{(0)}}(x'))^m$$
$$\cdot e^{-t\lambda_0}\left(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\right) t^{(1-\delta_{0m})/2} \begin{cases} 1 : (\text{with no condition}), \\ e^{-(r_{g^Z_{(0)}}(x,x')-\bar r)^2/C_2 t} & : r_{g^Z_{(0)}}(x,x') > \bar r, \end{cases}$$

$$(2.23)\qquad |\partial^\alpha\partial^{\alpha'}K(t,(m_0+1)/2:\varepsilon^{1/2},x,x')|_{g^Z_{(0)}} \le C_1\,(1+r_{g^Z_{(0)}}(x'))^{m_0+1}$$
$$\cdot\left(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\right) t^{1/2} \begin{cases} 1 : (\text{with no condition}) \\ e^{-(r_{g^Z_{(0)}}(x,x')-\bar r)^2/C_2 t} & : r_{g^Z_{(0)}}(x,x') > \bar r \end{cases}$$
$$(0 < \forall\varepsilon^{1/2} \le \varepsilon_0^{1/2},\ 0 < \forall t < \infty,\ \forall(x,x') \in Z \times Z).$$

(2) *(the detailed estimate for $t > 0$ small) Given α, α', $\bar r > 0$ and $T_0 > 0$, there exists a constant $C > 0$ such that, for every m with $0 \le m \le m_0+1$, we have*

$$(2.24)\quad |\partial^\alpha\partial^{\alpha'}K(t,m/2:\cdots)|_{g^Z_{(0)}} \le C\,(1+r_{g^Z_{(0)}}(x'))^m\, t^{-(n+2+|\alpha|+|\alpha'|)/2+(1-\delta_{0m})/2}\, e^{-r_{g^Z_{(0)}}(x,x')^2/5t}$$
$$(0 < \forall\varepsilon^{1/2} \le \varepsilon_0^{1/2},\ 0 < \forall t \le T_0,\ \forall(x,x') \in Z \times Z).$$

(3) *For every m with $0 \le m \le m_0 + 1$, let us set*

$$K(t, m/2 : \cdots) = \sum (dx^b)^I(x) \otimes ((dx^b)^J)^*(x') \cdot K(t, m/2 : \cdots)_{(I,J)} \tag{2.25}$$

as at (2.19). Then we have $K(t, m/2 : \cdots)_{(I,J)} = 0$ if $|(I,J)| \equiv |I| - |J| > m$. Further, for (I,J) with $|(I,J)| = m$, if $K(t, m/2 : \cdots)_{(I,J)} \neq 0$, then m is even.

Remark. In the study of the adiabatic limit of the η-invariant $\eta(\partial\!\!\!/_{g^Z_\varepsilon})$ in [6], [5], it was an important point to show that that it has no term which diverges when $\varepsilon \to 0$. To show it they introduced an auxiliary Grassmann variable and used a certain transformation by Bismut-Freed (see [5, (4.58)]). It will be, however, unfortunately difficult to apply their method to study the adiabatic hehavior of other various invariants, or, to investigate their divergent terms as a first step. In contrast to it, Proposition 2.2 (**3**) certifies not only that $\eta(\partial\!\!\!/_{g^Z_\varepsilon})$ has no such a divergent term, but, more strongly, that $\partial^\alpha \partial^{\alpha'} \left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)$ which forms the core of various invariants has no such one. That is, it is (**3**) that plays an important role in proving the fact that the expansion at (1.16) starts from the term with $m = 0$.

Now, using the fact in Proposition 2.2 (**3**), let us rewrite Proposition 2.2 (**1**)(**2**) into an assertion which naturally implies Theorem 1.2. First, it is easily found out by referring to (**3**), (2.20) and (1.3) that the term $\varepsilon^{-|I|/2} \partial^\alpha \partial^{\alpha'} \left(e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}\right)_{(I,\emptyset)}(0,0)$ appearing in (2.16) is C^∞ with respect to the variable $\varepsilon^{1/2}$ up to $\varepsilon^{1/2} = 0$. Since the theorem is concerned with the estimates of the coefficients of its Taylor expansion, we want to rewrite (**1**), (**2**) into the assertion for $\varepsilon^{-|(I,J)|/2}\left(e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(x,x')$. Thereupon, let us consider the bundle isomorphism over the identity map $1 : Z \to Z$

$$1_\varepsilon : \wedge T^*M \otimes_\pi \$_g\nu \cong \wedge T^*M \otimes_\pi \$_g\nu, \quad e^I_b(x) \otimes h(x) \mapsto (\varepsilon^{1/2} e_b(\varepsilon))^I(x) \otimes h(x). \tag{2.26}$$

Observing (2.14) and (2.19), we have then

$$\begin{aligned} 1_\varepsilon^{-1} \circ e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}(x,x') \circ 1_\varepsilon &= \sum e^I_b(x) \otimes (e^J_b)^*(x') \cdot \varepsilon^{-|(I,J)|/2} \left(e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(x,x') \\ &\equiv \sum e^I_b(x) \otimes (e^J_b)^*(x') \cdot K^{(\varepsilon)}(t,x,x')_{(I,J)} \equiv K^{(\varepsilon)}(t,x,x') \\ &= \sum e^I_b(x) \otimes (e^J_b)^*(x') \cdot \sum \varepsilon^{-|(I',J')|/2} K(t,\varepsilon,x,x')_{(I',J')}\, v^b_{I'I}(\varepsilon^{1/2}x^b)\, v^{J'J}_b(\varepsilon^{1/2}x'^b). \end{aligned} \tag{2.27}$$

Hence, we will rewrite them into the assertion for this. Proposition 2.2(**3**) and the expansion at (1.3), etc. say that there exists a Taylor expansion

$$\begin{aligned} K^{(\varepsilon)}(t,x,x')_{(I,J)} &= \sum_{m=m_1+m_2+m_3 \ge 0} \varepsilon^{(m-|(I',J')|)/2} \sum K(t, m_1/2 : x, x')_{(I',J')}\, \mathcal{O}(|x^b|^{m_2} |x'^b|^{m_3}) \\ &= \sum_{m=m_1+m_2+m_3 \ge 0} \varepsilon^{m/2} \sum K(t, (m_1 + |(I',J')|)/2 : x, x')_{(I',J')}\, \mathcal{O}(|x^b|^{m_2} |x'^b|^{m_3}) \end{aligned}$$

and now, using Proposition 2.2 (1)(2), certainly we have

Corollary 2.3. *The kernel $K^{(\varepsilon)}(t,x,x')$, i.e., each $K^{(\varepsilon)}(t,x,x')_{(I,J)}$, is C^∞ with respect to $(\varepsilon^{1/2},t,x,x') \in [0,\varepsilon_0^{1/2}] \times (0,\infty) \times Z \times Z$ and has the Taylor expansion*

$$K^{(\varepsilon)}(t,x,x') = \sum_{m=0}^{m_0} \varepsilon^{m/2} K^{(m/2:)}(t,x,x') + \varepsilon^{(m_0+1)/2} K^{((m_0+1)/2:\varepsilon^{1/2})}(t,x,x') \tag{2.28}$$

$$\text{with } K^{(0:)}(t,x,x') = \sum e_b^I(x) \otimes (e_b^J)^*(x') \cdot K(t,|(I,J)|/2 : x,x')_{(I,J)}. \tag{2.29}$$

Further we have:

(1) *Given α, α' and $\bar{r}>0$, there exist constants $\lambda_0>0$, $C_1>0$, $C_2>0$ and an integer $N>0$ satisfying*

$$\begin{aligned} |\partial^\alpha \partial^{\alpha'} K^{(m/2:)}(t,x,x')|_{gZ} &\le C_1\,(1+r_{gZ}(x))^m\,(1+r_{gZ}(x'))^{n+m} \\ &\cdot e^{-t\lambda_0}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big)\,t^{(1-\delta_{0m})/2} \begin{cases} 1 : (\text{with no condition}), \\ e^{-(r_{gZ}(x,x')-\bar{r})^2/C_2 t} : r_{gZ}(x,x')>\bar{r}, \end{cases} \end{aligned} \tag{2.30}$$

$$\begin{aligned} |\partial^\alpha \partial^{\alpha'} K^{((m_0+1)/2:\varepsilon^{1/2})}(t,x,x')|_{gZ} &\le C_1\,(1+r_{gZ}(x))^{m_0+1}\,(1+r_{gZ}(x'))^{n+m_0+1} \\ &\cdot \Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\Big)\,t^{1/2} \begin{cases} 1 : (\text{with no condition}) \\ e^{-(r_{gZ}(x,x')-\bar{r})^2/C_2 t} : r_{gZ}(x,x')>\bar{r} \end{cases} \\ &(0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall(x,x')\in Z\times Z). \end{aligned} \tag{2.31}$$

(2) *(the detailed estimate for $t>0$ small) Given α, α', $\bar{r}>0$ and $T_0>0$, there exists a constant $C>0$ such that, for every m with $0\le m\le m_0+1$, we have*

$$\begin{aligned} &|\partial^\alpha \partial^{\alpha'} K^{(m/2:\cdot)}(t,x,x')|_{gZ} \\ &\le C\,(1+r_{gZ}(x))^m(1+r_{gZ}(x'))^{n+m}\,t^{-(n+2+|\alpha|+|\alpha'|)/2+(1-\delta_{0m})/2}\,e^{-r_{gZ}(x,x')^2/5t} \\ &\qquad (0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t\le T_0,\ \forall(x,x')\in Z\times Z). \end{aligned} \tag{2.32}$$

Assume that Propositions 2.1, 2.2, and, hence, also Corollary 2.3 hold. Then, Theorem 1.2 will be already obvious. That is, we have

Proof of Theorem 1.2. Observing (2.16), (2.27) and Corollary 2.3, it is clear that we have only to set

$$\begin{aligned} K^{(m/2:\cdot)}(t,x,x') &= \sum e_b^I(x) \otimes (e_b^J)^*(x') \cdot K^{(m/2:\cdot)}(t,x,x')_{(I,J)}, \\ K_{(m/2)}(t,P^0,P,P') &= \sum e_b^I(P) \otimes K^{(m/2:)}(t,x(P),x(P'))_{(I,\emptyset)}. \end{aligned} \tag{2.33}$$

By using $K^{((m_0+1)/2:\varepsilon^{1/2})}(t,x,x') + \Big(e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(x,x') - e^{-t\not{\partial}^2_{g_\varepsilon^{Z(P^0)}}}(x,x')\Big)$ (see Proposition 2.1), the remainder term $K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0,P,P')$ is given similarly.

§3. Proof of Proposition 2.1

Let us start with investigating the connection $\nabla^{g^Z_\epsilon}$ and the distance function $r_{g^Z_\epsilon}$.

We will first collect what will be needed for their investigations. For the point $P^0 \in Z_{p^0} = \mathbb{C}P^1$, select wisely its representative $\tilde{P}^0 \in H = \mathbb{C}^2 (= \mathbb{H})$ with $|\tilde{P}^0| = 1$ (hence, $\tilde{P}^0 \in Sp(1)$) so that the local coordinate $w = w_1 + \sqrt{-1}\, w_2$ around P^0 defined by $U^f \ni P = r_H(\tilde{P}^0)\begin{bmatrix} 1 \\ w(P) \end{bmatrix} \in \mathbb{C}P^1$ satisfies

$$e^f(P^0) = (\frac{1}{2}\,(\partial/\partial w_1)_0, \frac{1}{2}\,(\partial/\partial w_2)_0). \tag{3.1}$$

Then, on $U^f\,(\subset Z_{p^0})$, clearly we have

$$e^f_k = \frac{1+|w|^2}{2}\,\frac{\partial}{\partial w_k},\; e^k_f = \frac{2}{1+|w|^2}\,dw_k \tag{3.2}$$

and the connection form of the hermitian covariant derivative $\nabla^{ds^{\mathbb{C}P}}$ for $T^{(1,0)}\mathbb{C}P^1$ may be expressed, with respect to the $U(1)$-frame $(1/2)(1+|w|^2)\,\partial/\partial w = (1/2)\big(e^f_1 - \sqrt{-1}\, e^f_2\big)$, as $\sqrt{-1}\big(w_2 e^1_f - w_1 e^2_f\big) \equiv \omega^{\mathbb{C}P} \equiv \sqrt{-1}\,\omega^{\mathbb{C}P}_{\mathcal{I}}$, so that we have $\nabla^{g^{\mathcal{V}}_{p^0}}_{e^f_k} e^f_1 = -\omega^{\mathbb{C}P}_{\mathcal{I}}(e^f_k)\, e^f_2$, $\nabla^{g^{\mathcal{V}}_{p^0}}_{e^f_k} e^f_2 = \omega^{\mathbb{C}P}_{\mathcal{I}}(e^f_k)\, e^f_1$. Further, the form $\nu^k(F)$ (for general F) given at (1.19) may be interpreted now as follows: The (locally defined) bundle $\$_{g^{\mathcal{V}}} = \pi^*\mathcal{H}$ can be splitted into the locally defined universal bundle (or the tautological bundle) $\$^+_{g^{\mathcal{V}}} = \{([v], cv) \in \pi^*\mathcal{H}\}$ and its orthogonal complement $\$^-_{g^{\mathcal{V}}}\,(\cong (\$^+_{g^{\mathcal{V}}})^*)$. Accordingly, let us take an $SU(2)$-frame (μ^+, μ^-) of $\$_{g^{\mathcal{V}}}|U = \pi^*\mathcal{H}|U = U \times H$ given by

$$\begin{aligned} &\mu^\pm(P) = (P, \mu^\pm(w(P))) \in U \times H \quad \text{with} \\ &(\mu^+(w), \mu^-(w)) = r_H(\tilde{P}^0)\,\frac{1}{(1+|w|^2)^{1/2}}\begin{pmatrix} 1 & -\bar{w} \\ w & 1 \end{pmatrix} \in H \oplus H, \end{aligned} \tag{3.3}$$

using which the pull-back π^*F may be expressed as $\begin{pmatrix} \nu_{\mathrm{u}}(F) & -\overline{\nu_{\mathrm{m}}(F)} \\ \nu_{\mathrm{m}}(F) & \overline{\nu_{\mathrm{u}}(F)} \end{pmatrix}$. Then we have $\nu_{\mathrm{m}}(F) = \nu^1(F) + \sqrt{-1}\,\nu^2(F)$. In addition, $\nu_{\mathrm{u}}(F) = \sqrt{-1}\,\nu_{\mathrm{u}\mathcal{I}}(F)$ is purely imaginary and, moreover, obviously we have the formulas

$$\begin{aligned} &e^f_1(\nu^1(F)) = -\omega^{\mathbb{C}P}_{\mathcal{I}}(e^f_1)\,\nu^2(F),\quad e^f_2(\nu^1(F)) = \nu_{\mathrm{u}\mathcal{I}}(F) - \omega^{\mathbb{C}P}_{\mathcal{I}}(e^f_2)\,\nu^2(F), \\ &e^f_1(\nu^2(F)) = -\nu_{\mathrm{u}\mathcal{I}}(F) + \omega^{\mathbb{C}P}_{\mathcal{I}}(e^f_1)\,\nu^1(F),\quad e^f_2(\nu^2(F)) = \omega^{\mathbb{C}P}_{\mathcal{I}}(e^f_2)\,\nu^1(F), \\ &\nu_{\mathrm{u}\mathcal{I}}(F_2) \wedge \nu^2(F_1) - \nu^2(F_2) \wedge \nu_{\mathrm{u}\mathcal{I}}(F_1) = \frac{1}{2}\,\nu^1([F_2 \wedge F_1]), \end{aligned}$$

(3.4)
$$\nu_{u\mathcal{I}}(F_2)\wedge\nu^1(F_1)-\nu^1(F_2)\wedge\nu_{u\mathcal{I}}(F_1)=-\frac{1}{2}\nu^2([F_2\wedge F_1]),$$
$$e_1^f(\nu_{u\mathcal{I}}(F))=\nu^2(F),\quad e_2^f(\nu_{u\mathcal{I}}(F))=-\nu^1(F),$$
$$\nu^2(F_2)\wedge\nu^1(F_1)-\nu^1(F_2)\wedge\nu^2(F_1)=\frac{1}{2}\nu_{u\mathcal{I}}([F_2\wedge F_1]).$$

Next, let us take a connection $\nabla^{g^{\mathcal{V}}}\equiv P^{\mathcal{V}}\circ\nabla^{g^Z}$ of $\mathcal{V}$ where $P^{\mathcal{V}}:TZ=\mathcal{H}\oplus\mathcal{V}\to\mathcal{V}$ is the projection. Hence, for $v,V\in\Gamma(\mathcal{V})$ and $u\in\Gamma(\mathcal{H})$, we have $\nabla_u^{g^{\mathcal{V}}}V=P^{\mathcal{V}}\big([u,V]\big)$ and $\nabla_{v(p)}^{g^{\mathcal{V}}}V=\nabla_{v(p)}^{g^Z}V$. Note that the latter is the fiberwise one and has already appeared (see (1.20) around). The Levi-Civita one ∇^{g^M} on (M,g^M), together with the above $\nabla^{g^{\mathcal{V}}}$, defines a new connection $\nabla^{g^Z\oplus}=\pi^*\nabla^{g^M}\oplus\nabla^{g^{\mathcal{V}}}$ of $TZ=\mathcal{H}\oplus\mathcal{V}$, which certainly induces the concept of $\nabla^{g^Z\oplus}$-geodesic, etc. as usual.

Lemma 3.1. *The connection $\nabla^{g^Z\oplus}$ is compatible with the metric g^Z and its torsion is equal to $2\nu(F_A)^\natural$. Further, the coordinates $x=(x^b,x^f)$ at P^0 are the $\nabla^{g^Z\oplus}$-normal coordinates with $(\partial/\partial x)=e_*(A)$ at P^0 and the $SO(n+2)$-frame $e_*(A)$ is $\nabla^{g^Z\oplus}$-parallel along the $\nabla^{g^Z\oplus}$-geodesics from P^0.*

Proof. The compatibility will be obvious. Let us compute its torsion $T(X,Y)\equiv\nabla_X^{g^Z\oplus}Y-\nabla_Y^{g^Z\oplus}X-[X,Y]$. First clearly we have $T(e_i^b(A),e_j^b(A))=-P^{\mathcal{V}}([e_i^b(A),e_j^b(A)])$, $T(e_k^f,e_{k'}^f)=0$, $T(e_k^f,e_i^b(A))=0$. And since the formulas (3.4) yield

$$[e_i^b(A),e_j^b(A)]=[e_i^b,e_j^b](A)-2\,\nu(F_A(e_i^b,e_j^b))^\natural \tag{3.5}$$

where $[e_i^b,e_j^b](A)$ is the A-horizontal lift of $[e_i^b,e_j^b]$, we have $T=\frac{1}{2}\sum T(e_i^b(A),e_j^b(A))\,e_b^i\wedge e_b^j=\sum\nu(F_A(e_i^b,e_j^b))^\natural e_b^i\wedge e_b^j=2\,\nu(F_A)^\natural$. Next, we will show that the curve $c(s)=(c^b(s),c^f(s))=sx$ $(0\le s\le 1)$ is a $\nabla^{g^Z\oplus}$-geodesic, i.e., $(\nabla_{\dot c}^{g^Z\oplus}\dot c)(c(s))\equiv 0$, which implies that the coordinates x are $\nabla^{g^Z\oplus}$-normal coordinates. Set $r_b(x^b)=|x^b|$, $r_f(x^f)=|x^f|$. (1.1) implies

$$(\partial/\partial r_b)(A)(x)=(\partial/\partial r_b)(x),\ \text{hence,}\ \nu(A(\partial/\partial r_b))^\natural(x)=0 \tag{3.6}$$

where $(\partial/\partial r_b)(A)(x)$ is the A-horizontal lift of $(\partial/\partial r_b)(x)$. Hence, we have

$$\dot c=\sum x_i^b\cdot(\partial/\partial x_i^b)_{c(s)}+\sum x_i^f\cdot(\partial/\partial x_i^f)_{c(s)}=r_b(x^b)\cdot(\partial/\partial r_b)_{c(s)}+r_f(x^f)\cdot(\partial/\partial r_f)_{c(s)}$$
$$=r_b(x^b)\cdot(\partial/\partial r_b)(A)_{c(s)}+r_f(x^f)\cdot(\partial/\partial r_f)_{c(s)}\equiv\dot c^{\mathcal{H}}+\dot c^{\mathcal{V}}\ (=P^{\mathcal{H}}(\dot c)+P^{\mathcal{V}}(\dot c)),$$
$$(\nabla_{\dot c}^{g^Z\oplus}\dot c)(c(s))=(\pi^*\nabla^{g^M})_{\dot c^{\mathcal{H}}}\dot c^{\mathcal{H}}+(\pi^*\nabla^{g^M})_{\dot c^{\mathcal{V}}}\dot c^{\mathcal{H}}+\nabla_{\dot c^{\mathcal{H}}}^{g^{\mathcal{V}}}\dot c^{\mathcal{V}}+\nabla_{\dot c^{\mathcal{V}}}^{g^{\mathcal{V}}}\dot c^{\mathcal{V}}$$
$$=\pi^*(\nabla_{\dot c^b}^{g^M}\dot c^b)+\nabla_{\dot c^{\mathcal{H}}}^{g^{\mathcal{V}}}\dot c^{\mathcal{V}}+\nabla_{\dot c^{\mathcal{V}}}^{g^{\mathcal{V}}}\dot c^{\mathcal{V}}=\pi^*(\nabla_{\dot c^b}^{g^M}\dot c^b)+P^{\mathcal{V}}[\dot c^{\mathcal{H}},\dot c^{\mathcal{V}}]+\nabla_{\dot c^{\mathcal{V}}}^{g^{\mathcal{V}}}\dot c^{\mathcal{V}}$$
$$=\pi^*(\nabla_{\dot c^b}^{g^M}\dot c^b)+P^{\mathcal{V}}[\dot c^b,\dot c^f]+\nabla_{\dot c^f}^{g^{\mathcal{V}}}\dot c^f=0\quad(\nabla_{\dot c^b}^{g^M}\dot c^b=0,\ \nabla_{\dot c^f}^{g^{\mathcal{V}}}\dot c^f=0,\ [\dot c^b,\dot c^f]=0).$$

Also, we have $\nabla^{g^Z\oplus}_{\dot c} e^b_i(A) = \pi^*(\nabla^{g^M}_{\dot c^b} e^b_i) = 0$, $\nabla^{g^Z\oplus}_{\dot c} e^f_k = P^{\mathcal V}[\dot c^b, e^f_k] + \nabla^{g^{\mathcal V}}_{\dot c^f} e^f_k = 0$, which mean that $e_*(A)$ is A-parallel.

Now, as for $\nabla^{g^Z_\varepsilon}$ and $r_{g^Z_\varepsilon}$, we have

Lemma 3.2. *We have*

$$e^{b\varepsilon}_i(A) = e^{b\varepsilon}_i - \varepsilon^{1/2}\, 2\sum \nu^k(A(e^b_i))\, e^f_k,\ \ e^k_{f\varepsilon}(A) = e^k_f + \varepsilon^{1/2}\, 2\sum \nu^k(A(e^b_i))\, e^i_{b\varepsilon} \tag{3.7}$$

and, putting $\nabla^{g^M}_{e^b_i} e^b_j = \sum C(\nabla^{g^M})(e^b_i)_{i_1 j}\, e^b_{i_1}$ *and* $\omega^{\mathbb{C}P}_{\mathcal I}(\nu(A(e^b_i))^\natural) = \sum \omega^{\mathbb{C}P}_{\mathcal I}(e^f_k)\, \nu^k(A(e^b_i))$, *we have*

$$\begin{aligned}
&\nabla^{g^Z_\varepsilon}_{e^{b\varepsilon}_i(A)} e^{b\varepsilon}_j(A) = \varepsilon^{1/2}\sum C(\nabla^{g^M})(e^b_i)_{i_1 j}\, e^{b\varepsilon}_{i_1}(A) - \varepsilon\sum \nu^k(F_A(e^b_i, e^b_j))\, e^f_k,\\
&\nabla^{g^Z_\varepsilon}_{e^{b\varepsilon}_i(A)} e^f_1 = \varepsilon^{1/2}\Big\{-2\,\nu_{u\mathcal I}(A(e^b_i)) + 2\omega^{\mathbb{C}P}_{\mathcal I}(\nu(A(e^b_i))^\natural)\Big\}\, e^f_2 + \varepsilon\sum \nu^1(F_A(e^b_i, e^b_j))\, e^{b\varepsilon}_j(A),\\
&\nabla^{g^Z_\varepsilon}_{e^{b\varepsilon}_i(A)} e^f_2 = \varepsilon^{1/2}\Big\{2\,\nu_{u\mathcal I}(A(e^b_i)) - 2\omega^{\mathbb{C}P}_{\mathcal I}(\nu(A(e^b_i))^\natural)\Big\}\, e^f_1 + \varepsilon\sum \nu^2(F_A(e^b_i, e^b_j))\, e^{b\varepsilon}_j(A),\\
&\nabla^{g^Z_\varepsilon}_{e^f_k} e^{b\varepsilon}_i(A) = \varepsilon\sum \nu^k(F_A(e^b_i, e^b_j))\, e^{b\varepsilon}_j(A),\ \nabla^{g^Z_\varepsilon}_{e^f_k} e^f_1 = -\omega^{\mathbb{C}P}_{\mathcal I}(e^f_k)\, e^f_2,\ \nabla^{g^Z_\varepsilon}_{e^f_k} e^f_2 = \omega^{\mathbb{C}P}_{\mathcal I}(e^f_k)\, e^f_1.
\end{aligned} \tag{3.8}$$

Further, the square $r_{g^Z_\varepsilon}(x, x')^2$ *can be expanded as*

$$\begin{aligned}
r_{g^Z_\varepsilon}(x, x')^2 &= \varepsilon^{-1} r_{g^M}(x^b, x'^b)^2 + \Big(r_{g^{\mathcal V}}(x^f, x'^f)^2 + O(|x^b - x'^b|^2 |x^f - x'^f|)\Big)\\
&\quad + \sum_{m=1}^{m_0} \varepsilon^m\, r_{(2), g^Z}(m/2 : x, x') + \varepsilon^{m_0+1}\, r_{(2), g^Z}((m_0+1)/2 : \varepsilon, x, x'),\\
r_{(2), g^Z}&(m/2 : \cdot, x, x') = O(|x^b - x'^b||x^f - x'^f||x - x'|) \quad (1 \le m \le m_0 + 1).
\end{aligned} \tag{3.9}$$

This is termwise differentiable with respect to the variables x, x', *and the difference* $r_{g^Z_\varepsilon}(x, x')^2 - \varepsilon^{-1} r_{g^M}(x^b, x'^b)^2$ *is analytic with respect to* ε *near* $\varepsilon = 0$.

Proof. (1.18) implies (3.7). As for (3.8): We want to prove them with $\varepsilon = 1$ and clearly it suffices to show

$$\begin{aligned}
&\nabla^{g^Z\oplus}_{e^b_i(A)} e^b_j(A) = \sum C(\nabla^{g^M})(e^b_i)_{i_1 j}\, e^b_{i_1}(A),\\
&\nabla^{g^Z\oplus}_{e^b_i(A)} e^f_1 = \Big\{-2\,\nu_{u\mathcal I}(A(e^b_i)) + 2\omega^{\mathbb{C}P}_{\mathcal I}(\nu(A(e^b_i))^\natural)\Big\}\, e^f_2,\\
&\nabla^{g^Z\oplus}_{e^b_i(A)} e^f_2 = \Big\{2\,\nu_{u\mathcal I}(A(e^b_i)) - 2\omega^{\mathbb{C}P}_{\mathcal I}(\nu(A(e^b_i))^\natural)\Big\}\, e^f_1,\\
&\nabla^{g^Z\oplus}_{e^f_k} e^b_i(A) = 0,\ \nabla^{g^Z\oplus}_{e^f_k} e^f_1 = -\omega^{\mathbb{C}P}_{\mathcal I}(e^f_k)\, e^f_2,\ \nabla^{g^Z\oplus}_{e^f_k} e^f_2 = \omega^{\mathbb{C}P}_{\mathcal I}(e^f_k)\, e^f_1,\\
&\nabla^{g^Z}_{e^b_i(A)} = \nabla^{g^Z\oplus}_{e^b_i(A)} + \sum \nu^k(F_A(e^b_i, e^b_j))\Big\{e^b_j(A) \otimes e^k_f(A) - e^f_k \otimes e^j_b\Big\},\\
&\nabla^{g^Z}_{e^f_k} = \nabla^{g^Z\oplus}_{e^f_k} + \sum \nu^k(F_A(e^b_i, e^b_j))\, e^b_j(A) \otimes e^i_b.
\end{aligned} \tag{3.10}$$

The formula for $\nabla^{g^Z\oplus}_{e^b_i(A)} e^f_k = P^{\mathcal{V}}([e^b_i(A), e^f_k])$ comes from (3.4) and the others for $\nabla^{g^Z\oplus}$ will be all obvious. And, if we set $S \equiv \nabla^{g^Z} - \nabla^{g^Z\oplus}$, then we have

$$
\begin{aligned}
&g^Z(S(e^b_i(A))e^f_k, e^b_j(A)) = -g^Z(S(e^b_i(A))e^b_j(A), e^f_k) = g^Z(S(e^f_k)e^b_i(A), e^b_j(A)) \\
(3.11)\qquad &= \frac{1}{2}\, g^Z(T(e^b_i(A), e^b_j(A)), e^f_k) = \nu^k(F_A(e^b_i, e^b_j)), \quad g^Z(S(\cdot)\cdot,\cdot) = 0 \text{ (otherwise)}
\end{aligned}
$$

where T is the torsion of $\nabla^{g^Z\oplus}$ as before (see [4, Chapter 9]). Therefore, we have

$$
\begin{aligned}
S(e^b_i(A)) &= \sum \nu^k(F_A(e^b_i, e^b_j))\,(e^b_j(A) \otimes e^k_f(A) - e^f_k \otimes e^j_b), \\
S(e^f_k) &= \sum \nu^k(F_A(e^b_i, e^b_j))\, e^b_j(A) \otimes e^i_b,
\end{aligned}
$$

which imply the remained formulas. As for (3.9): It suffices to prove the case $x' = 0$, i.e.,

$$
\begin{aligned}
(3.12)\qquad & r_{g^Z_\varepsilon}(x, P^0)^2 = \varepsilon^{-1}|x^b|^2 + \left(|x^f|^2 + \mathcal{O}(|x^b|^2|x^f|)\right) \\
&+ \sum_{m=1}^{m_0} \varepsilon^m\, r_{(2),g^Z_\varepsilon}(m/2 : x, P^0) + \varepsilon^{m_0+1}\, r_{(2),g^Z_\varepsilon}((m_0+1)/2 : \varepsilon, x, P^0), \\
& r_{g^Z_\varepsilon,(2)}(m/2 : \cdot, x, P^0) = \mathcal{O}(|x^b||x^f||x|) \quad (1 \le m \le m_0 + 1).
\end{aligned}
$$

To prove it, let us show that the g^Z_ε-geodesic $c^\varepsilon(s,x)$ $(0 \le s \le 1)$ from 0 to x is analytic with respect to ε at $\varepsilon = 0$ and its expansion $c^\varepsilon(s,x) = \sum_{\ell \ge 0} \varepsilon^m\, c^{(m)}(s,x)$ satisfies

$$
\begin{aligned}
& c^{(0)}(s,x) = sx, \;\; c^{(m)}(s,x) = \mathcal{O}(s|x^b||x^f|) \;\; (m > 0), \\
(3.13)\qquad & c^\varepsilon(s,(x^b,0)) = s(x^b,0), \;\; c^\varepsilon(s,(0,x^f)) = s(0,x^f).
\end{aligned}
$$

If these are true, then the g^Z_ε-normal coordinates $x^\varepsilon = (x^{b\varepsilon}, x^{f\varepsilon})$ with $\partial/\partial x^\varepsilon_i = e^\varepsilon_i(A)$ at 0 can be written as

$$
(3.14)\qquad x^\varepsilon = (\varepsilon^{-1/2}x^b + \sum_{m=1}^{\infty} \varepsilon^{m-1/2}\,(\partial c^{b(m)}/\partial s)(0,x),\; x^f + \sum_{m=1}^{\infty} \varepsilon^m\,(\partial c^{f(m)}/\partial s)(0,x))
$$

where we put $c^{(m)} = (c^{b(m)}, c^{f(m)})$, which obviously implies (3.12). Now let us investigate the g^Z_ε-geodesic. For $\xi = \sum \xi^{\mathcal{H}}_i\, e^b_i(A) + \sum \xi^{\mathcal{V}}_k\, e^f_k \equiv \xi^{\mathcal{H}} + \xi^{\mathcal{V}}$ and $\eta = \sum \eta^{\mathcal{H}}_i\, e^b_i(A) + \sum \eta^{\mathcal{V}}_k\, e^f_k \equiv \eta^{\mathcal{H}} + \eta^{\mathcal{V}}$, (3.10) and (3.8) imply

$$
\begin{aligned}
(3.15)\quad \nabla^{g^Z_\varepsilon}_\xi \eta &= \nabla^{g^Z_\varepsilon}_{\xi^{\mathcal{H}}}\eta^{\mathcal{H}} + \nabla^{g^Z_\varepsilon}_{\xi^{\mathcal{H}}}\eta^{\mathcal{V}} + \nabla^{g^Z_\varepsilon}_{\xi^{\mathcal{V}}}\eta^{\mathcal{H}} + \nabla^{g^Z_\varepsilon}_{\xi^{\mathcal{V}}}\eta^{\mathcal{V}} \\
&= \left\{\nabla^{g^Z_\varepsilon\oplus}_{\xi^{\mathcal{H}}}\eta^{\mathcal{H}} - \sum \nu^k(F_A(\xi^{\mathcal{H}}, \eta^{\mathcal{H}}))\, e^f_k\right\} + \left\{\nabla^{g^Z_\varepsilon\oplus}_{\xi^{\mathcal{H}}}\eta^{\mathcal{V}} + \varepsilon \sum \nu^k(F_A(\xi^{\mathcal{H}}, e^b_j))\, \eta^{\mathcal{V}}_k e^b_j(A)\right\} \\
&+ \left\{\nabla^{g^Z_\varepsilon\oplus}_{\xi^{\mathcal{V}}}\eta^{\mathcal{H}} + \varepsilon \sum \nu^k(F_A(\eta^{\mathcal{H}}, e^b_j))\, \xi^{\mathcal{V}}_k\, e^b_j(A)\right\} + \nabla^{g^Z_\varepsilon\oplus}_{\xi^{\mathcal{V}}}\eta^{\mathcal{V}} \\
&= \left\{\nabla^{g^Z_\varepsilon\oplus}_\xi \eta - \sum \nu^k(F_A(e^b_i, e^b_j))\, \xi^{\mathcal{H}}_i \eta^{\mathcal{H}}_j e^f_k\right\} + \varepsilon \sum \nu^k(F_A(e^b_i, e^b_j)) \left\{\xi^{\mathcal{V}}_k \eta^{\mathcal{H}}_i + \xi^{\mathcal{H}}_i \eta^{\mathcal{V}}_k\right\} e^b_j(A).
\end{aligned}
$$

And, setting $\partial/\partial x_i^b = \sum v_b^{ji} e_i^b$ and $\partial/\partial x_{k'}^f = \sum v_f^{k'k} e_k^f$ (see (1.3) and (2.20)), we have, for a curve $c(s)$,

$$\begin{aligned}
\dot{c}(s) &= \sum \dot{c}_i(s)\, \partial/\partial x_i = \sum \dot{c}_j(s)\, v_b^{ji}\, e_i^b + \sum \dot{c}_{n+k'}(s)\, v_f^{k'k}\, e_k^f \\
&= \sum v_b^{ji}\, \dot{c}_j(s)\, e_i^b(A) + \sum_k \Big\{ \sum_{k'} v_f^{k'k}\, \dot{c}_{n+k'}(s) + 2 \sum_{i,j} v_b^{ji} \nu^k(A(e_i^b))\, \dot{c}_j(s) \Big\} e_k^f \\
&\equiv \sum \dot{c}_i^{\mathcal{H}}(s)\, e_i^b(A) + \sum \dot{c}_k^{\mathcal{V}}(s)\, e_k^f \equiv \dot{c}^{\mathcal{H}}(s) + \dot{c}^{\mathcal{V}}(s)\ (= P^{\mathcal{H}}(\dot{c}(s)) + P^{\mathcal{V}}(\dot{c}(s))), \\
\dot{c}_i(s) &= \sum v_{ij}^b\, \dot{c}_j^{\mathcal{H}}(s)\ \ (i \le n), \quad \dot{c}_{n+k}(s) = \sum v_{kk'}^f \Big\{ \dot{c}_{k'}^{\mathcal{V}}(s) - 2 \sum_i \nu^{k'}(A(e_i^b))\, \dot{c}_i^{\mathcal{H}}(s) \Big\}.
\end{aligned}$$

Hence, the curve $c(s)$ is g_ε^Z-geodesic, i.e., $\nabla_{\dot{c}}^{g_\varepsilon^Z} \dot{c} = 0$, if and only if

$$\nabla_{\dot{c}}^{g^Z \oplus} \dot{c} + 2\varepsilon \sum \nu^k(F_A(e_i^b, e_j^b))\, \dot{c}_i^{\mathcal{H}} \dot{c}_k^{\mathcal{V}} e_j^b(A) = 0 \ \ (\text{and } c(0) = 0,\ x(1) = x), \tag{3.16}$$

which implies that $c^\varepsilon(s, x)$ is certainly analytic with respect to ε and satisfies (3.13). For example, as for $c^\varepsilon(s, (x^b, 0)) = s(x^b, 0)$: $c(s) \equiv s(x^b, 0)$ satisfies $\nabla_{\dot{c}}^{g^Z \oplus} \dot{c} = 0$ and $\dot{c}_k^{\mathcal{V}}(s) = 0$ because (3.6) yields $\sum_{i,j} v_b^{ji} \nu^k(A(e_i^b))\, \dot{c}_j(s) = \nu^k(A(\sum x_j^b \partial/\partial x_j^b)) = 0$. That is, $c(s) \equiv s(x^b, 0)$ satisfies (3.16), which means $c^\varepsilon(s, (x^b, 0)) = s(x^b, 0)$. Moreover, as for $c^{(m)}(s, x) = \mathcal{O}(s|x^b||x^f|)$ $(m > 0)$: Certainly we have $c^{(m)}(s, x) = \mathcal{O}(s|x^b|^{m_b}|x^f|^{m_f})$. And, observing the second line of (3.13), we have $m_b \ge 1$, $m_f \ge 1$.

Next, let us introduce an important formula for the proof of Proposition 2.1 (and also for the proof of Proposition 2.2). Consider a complete Spin manifold (N, g^N). The square $\partial\!\!\!/_{g^N}^2$ of its Dirac operator acting on compactly supported smooth cross-sections of the spinor bundle $\$_{g^N}$ is non-positive and essentially self-adjoint. The square root of its closure $\partial\!\!\!/_{g^N}^2 : L^2\Gamma(\$_{g^N}) \to L^2\Gamma(\$_{g^N})$ can be defined by the spectral theorem, $\sqrt{\partial\!\!\!/_{g^N}^2} = \int_0^\infty \lambda\, dE_\lambda$. Thus we obtain a bounded and self-adjoint operator acting on $L^2\Gamma(\$_{g^N})$

$$e^{-t\partial\!\!\!/_{g^N}^2} = \int_0^\infty e^{-t\lambda^2} dE_\lambda \quad (t > 0), \tag{3.17}$$

which has a C^∞-kernel. Note that this is just a definition and whether this defines a (C^0) semi-group or not is an another story. Now, applying to it the same argument as in [10, Example 2.1 and Theorem 4.1] which deals with the Laplacian acting on functions, we obtain

Lemma 3.3. *Given constants $R > -\infty$, $\bar{r} > 0$ and integers $m > 0$, $k, k' \ge 0$, there exists a constant $C > 0$ such that, for every m-dimensional complete Spin manifold (N, g^N) with* $\mathrm{Ric}(g^N) \ge R$ *at every point, we have*

$$\left| \partial\!\!\!/_{g^N}^k \partial\!\!\!/_{g^N}^{k'}\, e^{-t\partial\!\!\!/_{g^N}^2}(P, P') \right|_{g^N} \tag{3.18}$$

$$\leq C\Big(\frac{1}{t^{(m+k+k')/2}}+\frac{1}{t^{[(k+k')/2]}}\Big)\begin{cases}1 & : (\textit{with no condition})\\ e^{-(r_{g^N}(P,P')-\bar{r})^2/8t} & : r_{g^N}(P,P')>\bar{r}\end{cases}$$
$$(0<\forall t<\infty,\ \forall(P,P')\in N\times N)$$

where $\partial\!\!\!/^k_{g^N}\partial\!\!\!/^{k'}_{g^N}\, e^{-t\partial\!\!\!/^2_{g^N}}(P,P')$ *is the derivative of the kernel by* $\partial\!\!\!/^k_{g^N}$, $\partial\!\!\!/^{k'}_{g^N}$ *with respect to* P, P', *and the left hand side is the pointwise operator norm (see (1.12)).*

Remark. Refer to [10, §1] and set $f_t(\lambda)\equiv e^{-t\lambda^2}$. Then we have $\hat{f}_t(u)\equiv\int_{-\infty}^{\infty}f_t(\lambda)\,e^{-\sqrt{-1}u\lambda}\,d\lambda$ $=\sqrt{\frac{\pi}{t}}\,e^{-u^2/4t}$ and $f_t(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}_t(u)\,e^{\sqrt{-1}u\lambda}\,du=\int_{-\infty}^{\infty}\frac{e^{-u^2/4t}}{(4\pi t)^{1/2}}\cos u\lambda\,du$, which implies

$$e^{-t\partial\!\!\!/^2_{g^N}}=f_t\big(\sqrt{\partial\!\!\!/^2_{g^N}}\big)=\int_{-\infty}^{\infty}\frac{e^{-u^2/4t}}{(4\pi t)^{1/2}}\cos u\sqrt{\partial\!\!\!/^2_{g^N}}\,du. \tag{3.19}$$

Here, $\cos u\sqrt{\partial\!\!\!/^2_{g^N}}$ is the wave kernel and the wave equation $\big(\frac{\partial^2}{\partial u^2}+\partial\!\!\!/^2_{g^N}\big)\psi=0$ with $\psi|_{u=0}=\psi_0\in\Gamma(\$_{g^N})$ and $\frac{\partial\psi}{\partial u}|_{u=0}=0$ has a unique solution, denoted $\cos u\sqrt{\partial\!\!\!/^2_{g^N}}\,\psi_0$. Importantly the kernel has the finite propagation speed property, i.e.,

$$\operatorname{supp}\cos u\sqrt{\partial\!\!\!/^2_{g^N}}\subset\{(P,P')\mid r_{g^N}(P,P')\leq|u|\}. \tag{3.20}$$

Note that the property and the formula (3.19) yield the existence of the term $e^{-(r_{g^N}(P,P')-\bar{r})^2/8t}$ at (3.18).

For example, for our compact (Z,g^Z_ε), there exists a constant $R>-\infty$ satisfying

$$\mathrm{Ric}(g^Z_\varepsilon)(P)\geq R\quad(0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ \forall P\in Z). \tag{3.21}$$

Actually, (3.8) implies

$$\begin{aligned}&g^Z_\varepsilon(F(\nabla^{g^Z_\varepsilon})(e^{b\varepsilon}_i(A),e^{b\varepsilon}_j(A))e^{b\varepsilon}_i(A),e^{b\varepsilon}_j(A))\\&=\varepsilon g^M(F(\nabla^{g^M})(e^b_i,e^b_j)e^b_i,e^b_j)+3\varepsilon^2\sum\nu^k(F_A(e^b_i,e^b_j))^2,\\&g^Z_\varepsilon(F(\nabla^{g^Z_\varepsilon})(e^f_k,e^f_l)e^f_k,e^f_l)=g^V(F(\nabla^{g^V})(e^f_k,e^f_l)e^f_k,e^f_l),\\&g^Z_\varepsilon(F(\nabla^{g^Z_\varepsilon})(e^{b\varepsilon}_i(A),e^f_k)e^{b\varepsilon}_i(A),e^f_k)=-\varepsilon^2\sum\nu^k(F_A(e^b_i,e^b_j))^2.\end{aligned} \tag{3.22}$$

(More strongly, the curvature coefficients are certainly all bounded.) Hence, applying Lemma 3.3 to it, for given $\bar{r}>0$, k, k', there exists a constant $C>0$ satisfying

$$\begin{aligned}&\Big|\partial\!\!\!/^k_{g^Z_\varepsilon}\partial\!\!\!/^{k'}_{g^Z_\varepsilon}\,e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,P')\Big|_{g^Z_\varepsilon}\\&\leq C\Big(\frac{1}{t^{(n+2+k+k')/2}}+\frac{1}{t^{[(k+k')/2]}}\Big)\begin{cases}1 & : (\text{with no condition})\\ e^{-(r_{g^Z_\varepsilon}(P,P')-\bar{r})^2/8t} & : r_{g^Z_\varepsilon}(P,P')>\bar{r}\end{cases}\\&\qquad(0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall(P,P')\in Z\times Z).\end{aligned} \tag{3.23}$$

(3.9) says that $r_{g_\varepsilon^Z}(x, P^0) = r_0$ is almost equivalent to $\dfrac{|x^b|^2}{(\varepsilon^{1/2} r_0)^2} + \dfrac{|x^f|^2}{r_0^2} = 1$ and we may apparently know by observing the term $e^{-(r_{g_\varepsilon^Z}(P,P')-\bar r)^2/8t}$ how the norm is deformed when ε, t, P, P' move, and how powerful the property (3.20) which produces the term is. It will be clear that, in our case where Z is compact, (3.17) with $\not{\partial}^2_{g^N}$ replaced by $\not{\partial}^2_{g_\varepsilon^Z}$ coincides with the (C^0) semi-group generated by $\not{\partial}^2_{g_\varepsilon^Z}$, and, further the estimate (3.23) implies that, for given α, α' and $\bar r > 0$, there exists a constant $C > 0$ satisfying

$$
\begin{aligned}
(3.24)\qquad & \left|\partial^\alpha \partial^{\alpha'} e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P, P')\right|_{g_\varepsilon^Z} \\
& \le \frac{C}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}} \left(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}} + 1\right) \begin{cases} 1 & : \text{(with no condition)} \\ e^{-(r_{g_\varepsilon^Z}(P,P')-\bar r)^2/8t} & : r_{g_\varepsilon^Z}(P,P') > \bar r \end{cases} \\
& \quad (\, 0 < \forall \varepsilon^{1/2} \le \varepsilon_0^{1/2},\ 0 < \forall t < \infty,\ \forall (P, P') \in \pi^{-1}(U^b) \times \pi^{-1}(U^b)\,).
\end{aligned}
$$

Note that, for cross-sections ψ of $\$_{g_\varepsilon^Z}$, the two kinds of pointwise norms $\sum_{k\le \ell} |\not{\partial}^k_{g_\varepsilon^Z} \psi(P)|_{g_\varepsilon^Z}$, $\sum_{|\alpha|\le \ell} \varepsilon^{|\alpha^b|/2} |\partial^\alpha \psi(P)|_{g_\varepsilon^Z}$ over $\pi^{-1}(U^b)$ are equivalent independently of the choices of $P \in U$ and $0 < \varepsilon^{1/2} \le \varepsilon_0^{1/2}$. The equivalence will be clear from the facts that $\not{\partial}_{g_\varepsilon^Z} = \sum \rho_{g_\varepsilon^Z}(e_\varepsilon^i(A)) \nabla^{\$_{g_\varepsilon^Z}}_{e_i^\varepsilon(A)} = \sum \rho_{g_\varepsilon^Z}((\varepsilon^{-1/2} dx^b, dx^f)^i) \nabla^{\$_{g_\varepsilon^Z}}_{(\varepsilon^{1/2}\partial/\partial x^b, \partial/\partial x^f)_i}$ is elliptic, the norm $|\cdot|_{g_\varepsilon^Z}$ is invariant under the Clifford actions $\rho_{g_\varepsilon^Z}(e_\varepsilon^i(A))$, and the expressions of $\nabla^{\$_{g_\varepsilon^Z}}_{\varepsilon^{1/2}\partial/\partial x_i^b}$, $\nabla^{\$_{g_\varepsilon^Z}}_{\partial/\partial x_k^f}$ with respect to the frame $s(e_*^\varepsilon(A))$ are $\varepsilon^{1/2}\{\partial/\partial x_i^b + \mathcal{O}(\varepsilon, x)\}$, $\partial/\partial x_k^f + \mathcal{O}(\varepsilon, x)$, where $\mathcal{O}(\varepsilon, x)$ is a square matrix any differentials of whose entries are all bounded on $[0, \varepsilon_0^{1/2}] \times U$.

Let us now prove Proposition 2.1.

Proof of Proposition 2.1. First note that, though $Z(p^0)$ is non-compact, the kernel $e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}$ coincides with the one abstractly given using the spectral theorem (see (3.17)). Assume that P, P' belong to $\pi^{-1}(U^b)$ from now on. Take a cut-off function φ on $[0, \infty)$ ($\ni \delta$) with $\varphi(\delta) = 1$ if $\delta \le r_0^2/4$ and with $\varphi(\delta) = 0$ if $\delta > r_0^2$. Then, if $y \equiv x^b(P)$, $y' \equiv x^b(P') \in B_{r_0/2}$, we have

$$
\begin{aligned}
& e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P, P')\, \varphi(|y - y'|^2) - e^{-t\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}(P, P')\, \varphi(|y - y'|^2) \\
&= -\int_0^t d\theta \, \frac{\partial}{\partial \theta} \int dg_\varepsilon^Z(P'')\, e^{-(t-\theta)\not{\partial}^2_{g_\varepsilon^Z}}(P, P'')\, e^{-\theta\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}(P'', P')\, \varphi(|y - y''|^2) \varphi(|y'' - y'|^2) \\
&= -\int_0^t d\theta \int dg_\varepsilon^Z(P'') \Big\{ \Big(\not{\partial}^2_{g_\varepsilon^Z, P''}\, e^{-(t-\theta)\not{\partial}^2_{g_\varepsilon^Z}}(P, P'')\Big)\, e^{-\theta\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}(P'', P')\, \varphi(|y - y''|^2) \varphi(|y'' - y'|^2) \\
&\quad - e^{-(t-\theta)\not{\partial}^2_{g_\varepsilon^Z}}(P, P'') \Big(\not{\partial}^2_{g_\varepsilon^{Z(p^0)}, P''}\, e^{-\theta\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}(P'', P')\Big) \varphi(|y - y''|^2) \varphi(|y'' - y'|^2) \Big\} \\
&= -\int_0^t d\theta \int dg_\varepsilon^Z(P'') \Big\{ \Big(\not{\partial}^2_{g_\varepsilon^Z, P''}\, e^{-(t-\theta)\not{\partial}^2_{g_\varepsilon^Z}}(P, P'')\Big)\, e^{-\theta\not{\partial}^2_{g_\varepsilon^{Z(p^0)}}}(P'', P')\, \varphi(|y - y''|^2) \varphi(|y'' - y'|^2)
\end{aligned}
$$

$$
\begin{aligned}
&- e^{-(t-\theta)\not{D}^2_{g^Z_\varepsilon}}(P,P'')\,\not{D}^2_{g_\varepsilon^{Z(p^0)},P''}\Big(e^{-\theta\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P'',P')\,\varphi(|y-y''|^2)\varphi(|y''-y'|^2)\Big)\\
&+ e^{-(t-\theta)\not{D}^2_{g^Z_\varepsilon}}(P,P'')\sum \varepsilon^{1/2}e^b_{i,y''}\Big(\varphi(|y-y''|^2)\,\varphi(|y''-y'|^2)\Big)\Big(\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\,\not{D}_{g_\varepsilon^{Z(p^0)},P''}\\
&\qquad\qquad + \not{D}_{g_\varepsilon^{Z(p^0)},P''}\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\Big)\,e^{-\theta\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P'',P')\\
&+ e^{-(t-\theta)\not{D}^2_{g^Z_\varepsilon}}(P,P'')\sum \varepsilon e^b_{i,y''}\,e^b_{j,y''}\Big(\varphi(|y-y''|^2)\,\varphi(|y''-y'|^2)\Big)\\
&\qquad\qquad \times \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\,\rho_{g^Z_\varepsilon}(e^j_{b\varepsilon}(A))\,e^{-\theta\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P'',P')\Big\}\\
&= -\int_0^t d\theta\int dg^Z_\varepsilon(P'')\cdot e^{-(t-\theta)\not{D}^2_{g^Z_\varepsilon}}(P,P'')\\
&\quad\cdot\Big\{\varepsilon^{1/2}\sum e^b_i\Big(\varphi(|y-y''|^2)\,\varphi(|y''-y'|^2)\Big)\Big(\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\,\not{D}_{g_\varepsilon^{Z(p^0)}} + \not{D}_{g_\varepsilon^{Z(p^0)}}\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\Big)\\
&\quad+\varepsilon\sum e^b_i\,e^b_j\Big(\varphi(|y-y''|^2)\,\varphi(|y''-y'|^2)\Big)\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\,\rho_{g^Z_\varepsilon}(e^j_{b\varepsilon}(A))\Big\}\,e^{-\theta\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P'',P').
\end{aligned}
$$

Hence, further if $|y-y'|\le r_0/3$, then we have

$$
\begin{aligned}
(3.25)\quad &\partial^\alpha\partial^{\alpha'}e^{-t\not{D}^2_{g^Z_\varepsilon}}(P,P') - \partial^\alpha\partial^{\alpha'}e^{-t\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P,P')\\
&= -\int_0^t d\theta\int_{r_0/2\le|y''-y|\le r_0}^{r_0/6\le|y''-y'|\le r_0} dg^Z_\varepsilon(P'')\cdot\partial^\alpha_{(P)}e^{-(t-\theta)\not{D}^2_{g^Z_\varepsilon}}(P,P'')\\
&\quad\cdot\partial^{\alpha'}_{(P')}\Big\{\varepsilon^{1/2}\sum e^b_i\Big(\varphi(|y-y''|^2)\Big)\varphi(|y''-y'|^2)\Big(\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\not{D}_{g_\varepsilon^{Z(p^0)}} + \not{D}_{g_\varepsilon^{Z(p^0)}}\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\Big)\\
&\quad+\varepsilon\sum e^b_i\Big(e^b_j\Big(\varphi(|y-y''|^2)\Big)\varphi(|y''-y'|^2)\Big)\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\,\rho_{g^Z_\varepsilon}(e^j_{b\varepsilon}(A))\Big\}\,e^{-\theta\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P'',P')\\
&\quad-\int_0^t d\theta\int_{r_0/6\le|y''-y|\le r_0}^{r_0/2\le|y''-y'|\le r_0} dg^Z_\varepsilon(P'')\cdot\partial^\alpha_{(P)}e^{-(t-\theta)\not{D}^2_{g^Z_\varepsilon}}(P,P'')\\
&\quad\cdot\partial^{\alpha'}_{(P')}\Big\{\varepsilon^{1/2}\sum \varphi(|y-y''|^2)e^b_i\Big(\varphi(|y''-y'|^2)\Big)\Big(\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\not{D}_{g_\varepsilon^{Z(p^0)}} + \not{D}_{g_\varepsilon^{Z(p^0)}}\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\Big)\\
&\quad+\varepsilon\sum e^b_i\Big(\varphi(|y-y''|^2)e^b_j\Big(\varphi(|y''-y'|^2)\Big)\Big)\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}(A))\,\rho_{g^Z_\varepsilon}(e^j_{b\varepsilon}(A))\Big\}\,e^{-\theta\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P'',P').
\end{aligned}
$$

In the above we used the estimate (3.24) (with $\bar r = r_0$) for g^Z_ε and $g_\varepsilon^{Z(p^0)}$. Remark that the latter metric also has the property (3.21) so that $(Z(p^0), g_\varepsilon^{Z(p^0)})$ has the same estimation as (3.24). That is, we have

$$
\begin{aligned}
&\varepsilon^{n+(|\alpha^b|+|\alpha'^b|)/2}\left|\partial^\alpha\partial^{\alpha'}e^{-t\not{D}^2_{g^Z_\varepsilon}}(P,P') - \partial^\alpha\partial^{\alpha'}e^{-t\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P,P')\right|_{g^Z}\\
&\le \varepsilon^{(n+|\alpha^b|+|\alpha'^b|)/2}\left|\partial^\alpha\partial^{\alpha'}e^{-t\not{D}^2_{g^Z_\varepsilon}}(P,P') - \partial^\alpha\partial^{\alpha'}e^{-t\not{D}^2_{g_\varepsilon^{Z(p^0)}}}(P,P')\right|_{g^Z_\varepsilon}\ \text{(see (1.11), (1.12))}\\
&\le C\varepsilon^{1/2}\int_0^t d\theta\Big(\frac{1}{(t-\theta)^{(n+2+|\alpha|)/2}}+1\Big)e^{-r_0^2/(39)^2(t-\theta)\varepsilon}\Big(\frac{1}{\theta^{(n+3+|\alpha'|)/2}}+1\Big)e^{-r_0^2/(39)^2\theta\varepsilon}\\
&\le C'\varepsilon^{1/2}\int_0^t d\theta\Big(\varepsilon^{(n+2+|\alpha|)/2}+1\Big)e^{-r_0^2/(40)^2(t-\theta)\varepsilon}\Big(\varepsilon^{(n+3+|\alpha'|)/2}+1\Big)e^{-r_0^2/(40)^2\theta\varepsilon}\\
&\le C''\varepsilon^{1/2}\int_0^t d\theta\,e^{-r_0^2/(40)^2(t-\theta)\varepsilon-r_0^2/(40)^2\theta\varepsilon}\le C''\varepsilon^{1/2}\int_0^t d\theta\,e^{-r_0^2/(40)^2t\varepsilon} = C''t\varepsilon^{1/2}e^{-r_0^2/(40)^2t\varepsilon}.
\end{aligned}
$$

Thus we obtained the estimate (2.4).

§4. Proof of Proposition 2.2

Let us start with investigating $\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}$ acting on $\Gamma(\wedge T^*M \otimes_\pi \$_g\nu)$. Referring to (1.18) and (3.7), take the local $SO(n+2)$-frames for $(Z, g^Z_{(\varepsilon)})$

$$
(4.1)\qquad
\begin{aligned}
&e_*(\varepsilon, \iota_\varepsilon^* A) \equiv \iota_\varepsilon^* e_*^\varepsilon(A) = (e^b(\varepsilon, \iota_\varepsilon^* A), e^f),\\
&e^*(\varepsilon, \iota_\varepsilon^* A) \equiv \iota_\varepsilon^* e_\varepsilon^*(A) = (e_b(\varepsilon), e_f(\varepsilon, \iota_\varepsilon^* A)),\\
&e_i^b(\varepsilon, \iota_\varepsilon^* A) = e_i^b(\varepsilon) - \varepsilon^{1/2}\, 2 \sum \nu^k(A(e_i^b))(\iota_\varepsilon(x))\, e_k^f,\\
&e_f^k(\varepsilon, \iota_\varepsilon^* A) = e_f^k + \varepsilon^{1/2}\, 2 \sum \nu^k(A(e_i^b))(\iota_\varepsilon(x))\, e_b^i(\varepsilon),
\end{aligned}
$$

then we have the Lichnerowicz formula (see (3.8))

$$
(4.2)\quad \partial\!\!\!/^2_{g^Z_{(\varepsilon)}} = -\sum \Big(\nabla^{\$_{g^Z_{(\varepsilon)}}}_{e_i(\varepsilon,\iota_\varepsilon^* A)} \nabla^{\$_{g^Z_{(\varepsilon)}}}_{e_i(\varepsilon,\iota_\varepsilon^* A)} - \nabla^{\$_{g^Z_{(\varepsilon)}}}_{\nabla^{g^Z_{(\varepsilon)}}_{e_i(\varepsilon,\iota_\varepsilon^* A)} e_i(\varepsilon,\iota_\varepsilon^* A)} \Big) + \frac{\kappa_{g^Z_{(\varepsilon)}}}{4} \quad \text{with}
$$

$$
\begin{aligned}
&\nabla^{\$_{g^Z_{(\varepsilon)}}}_{e_i(\varepsilon,\iota_\varepsilon^* A)} = e_i(\varepsilon, \iota_\varepsilon^* A) + \frac{1}{4} \sum C(\nabla^{g^Z_{(\varepsilon)}})(e_i(\varepsilon, \iota_\varepsilon^* A))_{i_2 i_1} \rho_{g^Z_{(\varepsilon)}}(e^{i_1}(\varepsilon, \iota_\varepsilon^* A))\, \rho_{g^Z_{(\varepsilon)}}(e^{i_2}(\varepsilon, \iota_\varepsilon^* A)),\\
&\rho_{g^Z_{(\varepsilon)}}(e_b^i(\varepsilon, \iota_\varepsilon^* A)) = e_b^i(\varepsilon) \wedge - e_b^i(\varepsilon) \vee,\ \rho_{g^Z_{(\varepsilon)}}(e_f^k(\varepsilon, \iota_\varepsilon^* A)) = \rho_{g^Z}(e_f^k(A)),
\end{aligned}
$$

$$
(4.3)\quad
\begin{aligned}
&e_b(\varepsilon)(x) = (dx^b)_x \cdot v_b(\iota_\varepsilon(x)),\ C(\nabla^{g^Z_{(\varepsilon)}})(e_i(\varepsilon, \iota_\varepsilon^* A))_{i_1 i_2}(x) = C(\nabla^{g^Z_\varepsilon})(e_i^\varepsilon(A))_{i_1 i_2}(\iota_\varepsilon(x)),\\
&\kappa_{g^Z_{(\varepsilon)}}(x) = \kappa_{g^M_{(\varepsilon)}}(x) + 2 - \sum \nu^k(F_{\iota_\varepsilon^* A}(e_i^b(\varepsilon), e_j^b(\varepsilon)))^2(x)\\
&= \varepsilon \kappa_{g^M}(\iota_\varepsilon(x)) + 2 - \varepsilon^2 \sum \nu^k(F_A(e_i^b, e_j^b))^2(\iota_\varepsilon(x)).
\end{aligned}
$$

The last formula for the scalar curvature $\kappa_{g^Z_{(\varepsilon)}}$ will be obvious from (3.22). And, corresponding to $\lim_{\varepsilon\to 0} g^Z_{(\varepsilon)} = g^Z_{(0)}$ (see (2.12)), clearly we have $\lim_{\varepsilon\to 0} \partial\!\!\!/^2_{g^Z_{(\varepsilon)}} = \partial\!\!\!/^2_{g^Z_{(0)}}$ (see (2.17)).

Next, let us regard (2.14)=(2.19) which was originally a cross-section of (2.7) as a cross-section of

$$
(4.4)\qquad (\wedge T^*M \boxtimes \wedge T^*M) \otimes_\pi (\$_g\nu \boxtimes \$_g^*\nu),
$$

that is, we set

$$
(4.5)\qquad
\begin{aligned}
e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}(x, x') &= \sum e_b^I(\varepsilon)(x) \otimes e_b^J(\varepsilon)(x') \cdot \Big(e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}\Big)_{(I,J)}(x, x')\\
&\equiv E(t, \varepsilon, x, x') \equiv \sum (dx^b)^I(x) \otimes (dx^b)^J(x') \cdot E(t, \varepsilon, x, x')_{(I,J)}.
\end{aligned}
$$

Then the coefficients appearing in (2.19) and (2.27) can be expressed as

$$
(4.6)\qquad K(t, \varepsilon, x, x')_{(I,J)} = \sum E(t, \varepsilon, x, x')_{(I,J')}\, v^b_{J'J''}(\varepsilon^{1/2} x'^b)\, v^b_{JJ''}(\varepsilon^{1/2} x'^b),
$$

$$(4.7)\qquad K^{(\varepsilon)}(t,x,x')_{(I,J)}=\sum\varepsilon^{-|(I',J')|/2}E(t,\varepsilon,x,x')_{(I',J')}\,v^b_{I'I}(\varepsilon^{1/2}x^b)\,v^b_{J'J}(\varepsilon^{1/2}x'^b),$$

$$(4.8)\qquad \text{with }\ v^b_{IJ}(\varepsilon^{1/2}x^b),\ v_b^{IJ}(\varepsilon^{1/2}x^b)=\delta_{IJ}+\sum_{m\geq 2}\varepsilon^{m/2}\mathcal{O}(|x^b|^m)$$

and the action of (2.14)=(2.19) on $\psi(x')=\sum e_b(\varepsilon)^J(x')\cdot\psi_J(e_b(\varepsilon):x')\in\Gamma(\wedge T^*M\otimes_\pi \$_{g^\mathcal{V}})$ may be expressed as

$$(4.9)\qquad \begin{aligned}(E(t,\varepsilon)\psi)(x)&=\int_Z\sum e^I_b(\varepsilon)(x)\cdot\left(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\right)_{(I,J)}(x,x')\left(\psi_J(e_b(\varepsilon):x')\right)dg^Z_{(\varepsilon)}(x')\\ &\equiv\int_Z\langle E(t,\varepsilon,x,x'),\psi(x')\rangle_{g^Z_{(\varepsilon)}}\,dg^Z_{(\varepsilon)}(x')\equiv\int_Z\langle E(t,\varepsilon,x,x')\star_{g^M_{(\varepsilon)}}\psi(x')\rangle_{\$_{g^\mathcal{V}}}dg^\mathcal{V}(x')\\ &=\int_Z\sum(dx^b)^I(x)\cdot\langle E(t,\varepsilon,x,x')_{(I,J)}(dx^b)^J(x')\star_{g^M_{(\varepsilon)}}\psi(x')\rangle_{\$_{g^\mathcal{V}}}dg^\mathcal{V}(x')\end{aligned}$$

where $\star_{g^M_{(\varepsilon)}}$ denotes the star operator associated to $g^M_{(\varepsilon)}$. Note that we have hence

$$(4.10)\qquad \begin{aligned}(E(t,0)\psi)(x)&=\int_Z\langle E(t,0,x,x'),\psi(x')\rangle_{g^Z_{(0)}}\,dg^Z_{(0)}(x')\\ &=\int_Z\langle\star_{g^M_{(\varepsilon)},x'}\star^{-1}_{g^M_{(0)},x'}E(t,0,x,x'),\psi(x')\rangle_{g^Z_{(\varepsilon)}}\,dg^Z_{(\varepsilon)}(x').\end{aligned}$$

The purpose of the section is to prove Proposition 2.2 and, observing (4.6) and (4.8), obviously it suffices to prove it with $K(t,\varepsilon,x,x')$ replaced by $E(t,\varepsilon,x,x')$. Hereupon, paying attention to the formula

$$(4.11)\qquad \begin{aligned}E(t,0,x,x')&\equiv E_{g^M_{(0)}}(t,0,x^b,x'^b)\cdot K_{g^\mathcal{V}}(t,0,x^f,x'^f)\\ &\equiv\sum(dx^b)^I(x)\otimes(dx^b)^I(x')\,\frac{e^{-|x^b-x'^b|^2/4t}}{(4\pi t)^{n/2}}\cdot e^{-t\not{\partial}^2_{g^\mathcal{V}}}(x^f,x'^f)\quad\text{(compare with (2.18))},\end{aligned}$$

let us prove **(1)** and **(2)** separated into two cases, the case where $t>0$ is large and the case where $t>0$ is small. **(3)** will be shown in the former case. The smoothness of $E(t,\varepsilon,x,x')$ will be obvious from either discussion in both cases. We put $\star_{(\varepsilon)}=\star_{g^M_{(\varepsilon)}}$, $\langle\,,\rangle_{(\varepsilon)}=\langle\,,\rangle_{g^Z_{(\varepsilon)}}$, $|\cdot|_{(\varepsilon)}=|\cdot|_{g^Z_{(\varepsilon)}}$, $r_{(\varepsilon)}=r_{g^Z_{(\varepsilon)}}$, $r=r_{(0)}$ to simplify the description in the following.

Proof of Proposition 2.2 for $E(t,\varepsilon)$ with t large. On the model of the argument by Cheeger [9, §3, §4], let us prove first (2.22), (2.23) with $t>T_0$. Remark that (4.27), (4.28) which we intend to show as a first step will hold with no restriction on t. The restriction becomes necessary on and after (4.42) at which the proofs of (2.22), (2.23) with $t>T_0$ will start substantially. Now, we start our argument with three preparations.

First, the metric $g^Z_{(\varepsilon)}$ has the property (3.21) (see (4.3) and (3.22)) so that we have the estimate similar to (3.23) for $g^Z_{(\varepsilon)}$. Further the two pointwise norms $\sum_{k\leq\ell}|\not{\partial}^k_{g^Z_{(\varepsilon)}}\psi(x)|_{(\varepsilon)}$ and

$\sum_{|\alpha|\le\ell}|\partial^\alpha\psi(x)|_{(\varepsilon)}$ are equivalent to each other and also $|\cdot|_{(\varepsilon)}$, $r_{(\varepsilon)}$ are equivalent to $|\cdot|_{(0)}$, $r_{(0)}=r$, respectively. Thus, for given α, α' and $\bar r>0$, there exists a constant $C>0$ satisfying

$$(4.12)\qquad |\partial^\alpha\partial^{\alpha'}E(t,\varepsilon,x,x')|_{(0)}\le C\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big)\begin{cases}1 & :(\text{with no condition})\\ e^{-(r(x,x')-\bar r)^2/8t} & : r(x,x')>\bar r\end{cases}$$
$$(0\le\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall(x,x')\in Z\times Z).$$

Let us take then a cut-off function ϕ_a on $\mathbb{R}$ depending on the parameter $a\ge\bar r(>0)$ and satisfying: $0\le\phi_a\le 1$, $\operatorname{supp}\phi_a\subset\{u\mid |u|\le\bar r+a\}$, $\operatorname{supp}(1-\phi_a)\subset\{u\mid |u|\ge a\}$ and $|(d/du)^i\phi_a(u)|\le C_i$. (The constants $C_i>0$ are independent of $a\ge\bar r$.) And, set

$$(4.13)\qquad e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}=\int_{-\infty}^{\infty}\frac{e^{-u^2/4t}}{(4\pi t)^{1/2}}\Big(\phi_a(u)\cos u\sqrt{\not{\partial}^2_{g^Z_{(\varepsilon)}}}+(1-\phi_a(u))\cos u\sqrt{\not{\partial}^2_{g^Z_{(\varepsilon)}}}\Big)\,du$$
$$\equiv\Big(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\Big)_a+\Big(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\Big)_{a,\infty}$$

and denote the kernels of $\Big(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\Big)_a$, $\Big(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\Big)_{a,\infty}$ by $E_a(t,\varepsilon)$, $\bar E_a(t,\varepsilon)$, respectively. Then the estimate (4.12) and the finite propagation property (see (3.20)) which $\cos t\sqrt{\not{\partial}^2_{g^Z_{(\varepsilon)}}}$ has imply the following: For given α, α' and $\bar r>0$, there exists a constant $C>0$ such that, for $\forall a\ge\bar r$, $0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2}$, $0<\forall t<\infty$ and $\forall(x,x')\in Z\times Z$, we have

$$(4.14)\qquad \|E(t,\varepsilon)\|_{op}\le 1,\ \|E_a(t,\varepsilon)\|_{op}\le 1,\ \|\bar E_a(t,\varepsilon)\|_{op}\le C\,e^{-a^2/8t},$$

$$(4.15)\qquad |\partial^\alpha\partial^{\alpha'}E_a(t,\varepsilon,x,x')|_{(0)}\le C\begin{cases}1+t^{-(\dim Z+|\alpha|+|\alpha'|)/2} & : r(x,x')\le\bar r,\\ e^{-r(x,x')^2/8t} & : \bar r\le r(x,x')\le\bar r+a,\\ 0 & : \bar r+a\le r(x,x'),\end{cases}$$

$$(4.16)\qquad |\partial^\alpha\partial^{\alpha'}\bar E_a(t,\varepsilon,x,x')|_{(0)}\le C\begin{cases}e^{-a^2/8t} & : r(x,x')\le\bar r+a,\\ e^{-r(x,x')^2/8t} & : \bar r+a\le r(x,x')\end{cases}$$

where $\|\cdot\|_{op}$ is the global operator norm of an operator acting on $L^2\Gamma(\wedge T^*M\otimes_\pi \$_{g^V},g^Z_{(0)})$, and, setting $L_a(t,\varepsilon)\equiv(\partial/\partial t+\not{\partial}^2_{g^Z_{(\varepsilon)}})E_a(t,\varepsilon)=-(\partial/\partial t+\not{\partial}^2_{g^Z_{(\varepsilon)}})\bar E_a(t,\varepsilon)$, we have

$$(4.17)\qquad \|L_a(t,\varepsilon)\|_{op}\le C\,e^{-a^2/8t},$$

$$(4.18)\qquad |\partial^\alpha\partial^{\alpha'}L_a(t,\varepsilon,x,x')|_{(0)}\le C\begin{cases}e^{-a^2/8t} & : r(x,x')\le\bar r+a,\\ 0 & : \bar r+a\le r(x,x').\end{cases}$$

Second, the Duhamel's principle says

$$(4.19)\qquad \partial^\alpha\partial^{\alpha'}E(t,\varepsilon,x,x')=\partial^\alpha\partial^{\alpha'}E_a(t,\varepsilon,x,x')+\partial^\alpha\partial^{\alpha'}\bar E_a(t,\varepsilon,x,x')$$

$$(4.20)\qquad =\sum_{m=0}^{m_0+1}\partial^\alpha\partial^{\alpha'}\star_{(\varepsilon),x'}\star^{-1}_{(0),x'}E_a(,0)\,\sharp\,(\widetilde{-\not{\partial}^2_{g^Z_{(\varepsilon)}}E_a(,0)})^m\quad(\text{see }(4.10))$$

(4.21) $$+\partial^{\alpha}\partial^{\alpha'}E_a(,0)\,\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,E_a(,0))^{m_0+1}\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,E_a(,\varepsilon))$$

(4.22) $$+\sum_{m=0}^{m_0+1}\partial^{\alpha}\partial^{\alpha'}E_a(,0)\,\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,E_a(,0))^{m}\sharp\,L_a(,\varepsilon)$$

(4.23) $$-\sum_{m=0}^{m_0+1}\partial^{\alpha}\partial^{\alpha'}E_a(,0)\,\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,E_a(,0))^{m-1}\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,L_a(,0))\,\sharp\,E_a(,\varepsilon)$$

(4.24) $$+\partial^{\alpha}\partial^{\alpha'}\bar{E}_a(t,\varepsilon,x,x')$$

where we set $(E^0\,\sharp\,E^1)(t,x,x')=\int_0^t dt_1\int_Z dg^Z_{(0)}(Q^1)\,\langle E^0(t-t_1,x,Q^1),E^1(t_1,Q^1,x')\rangle_{(0)}$, etc., and $\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\equiv\not\partial^2_{g^Z_{(\varepsilon)}}-\not\partial^2_{g^Z_{(0)}}$, $E_a(,0))\,\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}E_a(,0))^0\equiv E_a(,0)=E_a(t,0,x,x')$, $(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}E_a(,0))^i(t,x,x')$ $\equiv((-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}E_a(,0))\,\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}E_a(,0))^{i-1})(t,x,x')$, $E_a(,0)\,\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}E_a(,0))^{-1}\sharp\,(-\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}L_a(,0))\equiv L_a(,0)$, etc. Since we perform the convolution operation $\sharp$ repeatedly, strict description will be rather hard to read, so that, for example, we denote $(E^0\,\sharp\,E^1)(t,x,x')$ above by $E^0\,\sharp\,E^1$, $E^0(t-t_1)\,\sharp\,E^1(t_1)$, $E^0(t-t_1,x,Q^1)\,\sharp\,E^1(t_1,Q^1,x')$ or $\int_0^t dt_1E^0(t-t_1,x,Q^1)\,\sharp\,E^1(t_1,\,Q^1,x')$, etc. according to the situations. Judging from the circumstances it will be easy to understand such expressions correctly. To prove the formula (4.19), first note that

$$\begin{aligned}
&E_a(t,\varepsilon,x,x')-\star_{(\varepsilon),x'}\star^{-1}_{(0),x'}E_a(t,0,x,x')=\int\langle E(0,0,x,Q^1),E_a(t,\varepsilon,Q^1,x')\rangle_{(0)}\,dg^Z_{(0)}\\
&\qquad-\int\langle\star_{(\varepsilon),Q^1}\star^{-1}_{(0),Q^1}E_a(t,0,x,Q^1),E(0,\varepsilon,Q^1,x')\rangle_{(\varepsilon)}\,dg^Z_{(\varepsilon)}\\
&=\int\langle E(0,0,x,Q^1),E_a(t,\varepsilon,Q^1,x')\rangle_{(0)}\,dg^Z_{(0)}-\int\langle E_a(t,0,x,Q^1),E(0,\varepsilon,Q^1,x')\rangle_{(0)}\,dg^Z_{(0)}\\
&=\int\langle E_a(0,0,x,Q^1),E_a(t,\varepsilon,Q^1,x')\rangle_{(0)}\,dg^Z_{(0)}-\int\langle E_a(t,0,x,Q^1),E_a(0,\varepsilon,Q^1,x')\rangle_{(0)}\,dg^Z_{(0)}\\
&=\int_0^t dt_1\,\frac{\partial}{\partial t_1}\int\langle E_a(t-t_1,0,x,Q^1),E_a(t_1,\varepsilon,Q^1,x')\rangle_{(0)}\,dg^Z_{(0)}\\
&=E_a(t-t_1,0,x,Q^1)\,\sharp\,\frac{\partial E_a}{\partial t_1}(t_1,\varepsilon,Q^1,x')-\frac{\partial E_a}{\partial t_1}(t-t_1,0,x,Q^1)\,\sharp\,E_a(t_1,\varepsilon,Q^1,x')\\
&=-E_a(t-t_1,0,x,Q^1)\,\sharp\,\not\partial^2_{g^Z_{(\varepsilon)}}E_a(t_1,\varepsilon,Q^1,x')+\not\partial^2_{g^Z_{(0)}}E_a(t-t_1,0,x,Q^1)\,\sharp\,E_a(t_1,\varepsilon,Q^1,x')\\
&\quad+E_a(t-t_1,0,x,Q^1)\,\sharp\,L_a(t_1,\varepsilon,Q^1,x')-L_a(t-t_1,0,x,Q^1)\,\sharp\,E_a(t_1,\varepsilon,Q^1,x')\\
&=-E_a(t-t_1,0,x,Q^1)\,\sharp\,\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,E_a(t_1,\varepsilon,Q^1,x')\\
&\quad+E_a(t-t_1,0,x,Q^1)\,\sharp\,L_a(t_1,\varepsilon,Q^1,x')-L_a(t-t_1,0,x,Q^1)\,\sharp\,E_a(t_1,\varepsilon,Q^1,x').
\end{aligned}$$

Then, replace $E_a(t_1,\varepsilon,Q^1,x')$ in the second line from below by $\star_{(\varepsilon),x'}\star^{-1}_{(0),x'}E_a(t_1,0,Q^1,x')-E_a(t_1-t_2,0,Q^1,Q^2)\,\sharp\,\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\,E_a(t_2,\varepsilon,Q^2,x')+\cdots$ which was found out to be equal to it. Repeating such a replacement again and again, we obtain (4.19).

Third, observing (4.2), (4.3) and (3.8), obviously we have the series expansion

(4.25) $$\widetilde{\not\partial^2_{g^Z_{(\varepsilon)}}}\equiv\not\partial^2_{g^Z_{(\varepsilon)}}-\not\partial^2_{g^Z_{(0)}}\equiv\sum_{2\le m\le m_0}\varepsilon^{m/2}(\not\partial^2_{g^Z_{(0)}})^{(m/2)}(x^b)+\varepsilon^{(m_0+1)/2}(\not\partial^2_{g^Z_{(0)}})^{((m_0+1)/2)}(\varepsilon^{1/2},x^b),$$

$$(\not{\partial}^2_{g^Z_{(0)}})^{(m/2)}(\cdot\cdot) = \sum (dx^b\wedge)^I \circ (dx^b\vee)^J \cdot \sum_{|\alpha|+|\beta|\le 2} \mathcal{O}(|x^b|^m)(\partial/\partial x^b)^\alpha (e^f)^\beta \quad (|x^b|\to\infty),$$

$$\max\{|I| \mid I \text{ appears in the above expression of } (\not{\partial}^2_{g^Z_{(0)}})^{(m/2)}(\cdot\cdot)\} \le \begin{cases} 2 & (m=2,3), \\ 4 & (m \ge 4). \end{cases}$$

Now, for each $x' \in Z$ in (4.19), we assume that the parameter a we use belongs to the interval $[r(x'),\infty)$. Then, since $E_a(t,\varepsilon,x,x') = 0$ if $\bar{r}+a \le r(x,x')$ (see (4.15)) for any $\varepsilon \le \varepsilon_0$, all of the domains of integral appearing in (4.19) can be restricted to the bounded domain $\mathcal{N}_{P^0} = \mathcal{N}_{P^0}^{m_0+3} = \mathcal{N}_{P^0,\bar{r},a}^{m_0+3} \equiv \{Q \mid r(Q) \le (m_0+3)(\bar{r}+a)\}$. And, (4.25) says that, for given α, m_0 and $\bar{r} > 0$, there exists a constant $C > 0$ satisfying

$$(4.26) \qquad |\partial^\alpha(\text{each coefficient of } (\not{\partial}^2_{g^Z_{(0)}})^{(m/2)}(\cdot\cdot)) \text{ at } Q| \le C\, a^m \quad (2 \le m \le m_0+1)$$
$$(\forall Q \in \mathcal{N}_{P^0,\bar{r},a}^{m_0+3},\ \forall a \ge \bar{r})$$

where "each coefficient of $\cdots$" means each term denoted by $\mathcal{O}(|x^b|^m)$ at (4.25). Now, by investigating (4.20)-(4.24) using the above estimates, we intend to show that there exist constants $\lambda_0 > 0$, $C_1 > 0$ and (sufficiently large) integers N, N' satisfying

$$\begin{aligned}(4.27)\quad & \partial^\alpha\partial^{\alpha'}E(t,\varepsilon,x,x') \\ &= \sum_{m=0}^{m_0+1} \varepsilon^{m/2}\Big\{\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0}(t^{-(n+2+N')/2}+1)) + \mathcal{O}(e^{-t\lambda_0}e^{-a^2/9t})\Big\} \\ &\quad + \varepsilon^{m_0+2}\,\mathcal{O}(a^{n+2+m_0+2}\,(t^{-(n+2+N')/2}+t^N)) + \mathcal{O}(e^{-a^2/9t}\,t^N) \ : \text{(with no condition)},\end{aligned}$$

$$\begin{aligned}(4.28)\quad & \partial^\alpha\partial^{\alpha'}E(t,\varepsilon,x,x') \\ &= \sum_{m=0}^{m_0+1} \varepsilon^{m/2}\Big\{\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0}e^{-(r(x,x')-\bar{r})^2/C_1 t}) + \mathcal{O}(e^{-t\lambda_0}e^{-a^2/9t})\Big\} \\ &\quad + \varepsilon^{m_0+2}\,\mathcal{O}(a^{n+2+m_0+2}\,t^N\,e^{-(r(x,x')-\bar{r})^2/C_1 t}) + \mathcal{O}(e^{-a^2/9t}\,t^N) \ : \text{if } r(x,x') > \bar{r} \\ &\quad (0 < \forall\varepsilon^{1/2} \le \varepsilon_0^{1/2},\ 0 < \forall t < \infty,\ \bar{r} \le \forall a < \infty,\ \forall P^0 \in (\text{compact } Z),\ \forall (x,x') \in Z\times Z)\end{aligned}$$

where the terms $\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0}(t^{-(n+2+N')/2}+1))$, etc., that is, the first terms of the coefficients of $\varepsilon^{m/2}$ $(0 \le m \le m_0)$, do not depend on $\varepsilon^{1/2}$. In the following we will show this by investigating (4.24), (4.23), (4.22), (4.21), (4.20) in the order named.

First, let us show an estimate commonly available to the cases "with no condition" and "$r(x,x') > \bar{r}$", that is,

$$(4.29) \qquad (4.24) + (4.23) + (4.22) = \mathcal{O}(e^{-a^2/9t}\,t^N).$$

Clearly (4.16) implies that the term (4.24) has such an estimate. (4.23) and (4.22) vanish if $x \notin \mathcal{N}_{P^0}^{m_0+3}$, so that we assume $x \in \mathcal{N}_{P^0}^{m_0+3} = \mathcal{N}_{P^0}$. The estimate of (4.23): Consider (4.25), (4.26) with $m_0 = -1$ and set $\langle \not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},x''}\rangle \equiv 1 + \not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},x''}$, $\check{\not\partial}_{g^Z_{(\epsilon)}} \equiv \star_{(0)}\star^{-1}_{(\epsilon)}\not\partial_{g^Z_{(\epsilon)}}\star_{(\epsilon)}\star^{-1}_{(0)}$. Further, let $|f(x,x')|_{(0),L^2(x')}$ denote the pointwise norm with respect to the variable x and the L^2-norm with respect to x'. Interpreting the other notations of norms similarly and using the elliptic estimates appropriately, we have

$$
\begin{aligned}
&|-\partial^\alpha_x \partial^{\alpha'}_{x'} E_a(,0) \sharp (-\widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(,0))^{-1} \sharp (-\widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(,0)) \sharp E_a(,\epsilon)|_{(0)} \\
&\le C \int dt_1 |\partial^\alpha_x L_a(t-t_1,0,x,Q^1) \sharp \langle \not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},x''}\rangle E_a(t_1,\epsilon,Q^1,x'')|_{(0),L^2(x'')} \\
&= C \int dt_1 |\partial^\alpha_x L_a(t-t_1,0,x,Q^1) \sharp \langle \not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^1}\rangle E_a(t_1,\epsilon,Q^1,x'')|_{(0),L^2(x'')} \\
&= C \int dt_1 |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^1}\rangle \partial^\alpha_x L_a(t-t_1,0,x,Q^1) \sharp E_a(t_1,\epsilon,Q^1,x'')|_{(0),L^2(x'')} \\
&\le C \int dt_1 |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^1}\rangle \partial^\alpha_x L_a(t-t_1,0,x,Q^1)|_{(0),L^2(Q^1)} \quad \text{(by (4.14))} \\
&= C \int dt_1 |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^1}\rangle \partial^\alpha_x L_a(t-t_1,0,x,Q^1)|_{(0),L^2(Q^1\in\mathcal{N}^{m_0+4}_{p0})} \\
&\le C_1 e^{-a^2/8t} a^{(n+2)/2}\, t \le C_2\, e^{-a^2/9t}\, t^{(n+2)/4+1},
\end{aligned}
$$

$$
\begin{aligned}
&|-\partial^\alpha_x \partial^{\alpha'}_{x'} E_a(,0) \sharp (-\widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(,0))^0 \sharp (-\widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(,0)) \sharp E_a(,\epsilon)|_{(0)} \\
&\le C \int dt_1 dt_2 |\langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},x''}\rangle E_a(t-t_1,0,x'',Q^1) \\
&\qquad \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2) \sharp \partial^{\alpha'}_{x'} E_a(t_2,\epsilon,Q^2,x')|_{L^2(x''),(0)} \\
&= C \int dt_1 dt_2 |E_a(t-t_1,0,x'',Q^1) \sharp \langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2) \\
&\qquad \sharp \partial^{\alpha'}_{x'} E_a(t_2,\epsilon,Q^2,x')|_{L^2(x''),(0)} \\
&\le C \int dt_1 dt_2 |\langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2) \sharp \partial^{\alpha'}_{x'} E_a(t_2,\epsilon,Q^2,x')|_{L^2(Q^1),(0)} \\
&= C \int dt_1 dt_2 |\langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2) \sharp \partial^{\alpha'}_{x'} E_a(t_2,\epsilon,Q^2,x')|_{L^2(Q^1\in\mathcal{N}_{p0}),(0)} \\
&\le C_1 \int dt_1 dt_2 |\langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2) \\
&\qquad\qquad \sharp \langle \not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},x''}\rangle E_a(t_2,\epsilon,Q^2,x'')|_{L^2(Q^1\in\mathcal{N}_{p0}),L^2(x'')} \\
&= C_1 \int dt_1 dt_2 |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^2}\rangle \langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2) \\
&\qquad\qquad \sharp E_a(t_2,\epsilon,Q^2,x'')|_{L^2(Q^1\in\mathcal{N}_{p0}),L^2(x'')} \\
&\le C_1 \int dt_1 dt_2 |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^2}\rangle \langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2)|_{L^2(Q^1\in\mathcal{N}_{p0}),L^2(Q^2)} \\
&= C_1 \int dt_1 dt_2 \\
&\qquad |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^2}\rangle \langle \not\partial^{[(n+2)/2]+|\alpha|+1}_{g^Z_{(0)},Q^1}\rangle \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} L_a(t_1-t_2,0,Q^1,Q^2)|_{L^2(Q^1\in\mathcal{N}_{p0}),L^2(Q^2\in\mathcal{N}_{p0})} \\
&\le C_2\, e^{-a^2/8t}\, a^{n+2}\, t^2 \le C_3\, e^{-a^2/9t}\, t^{(n+2)/2+2}
\end{aligned}
$$

and, in the case $m \geq 2$, similarly we have

$$
\begin{aligned}
&|-\partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} E_a(,0))^{m-1} \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} L_a(,0)) \sharp E_a(,\varepsilon)|_{(0)} \\
&\leq C \int dt_1 \cdot\cdot dt_{m+1} |\langle \partial^{[(n+2)/2]+|\alpha|+2m+1}_{g^Z_{(0)},Q^m} \rangle \langle \tilde{\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\varepsilon)},Q^{m+1}} \rangle \\
&\qquad\qquad L_a(t_m - t_{m+1}, 0, Q^m, Q^{m+1})|_{L^2(Q^m \in \mathcal{N}_{p0}), L^2(Q^{m+1} \in \mathcal{N}_{p0})} \\
&\leq C_1 e^{-a^2/8t} a^{n+2} t^{m+1} \leq C_2 e^{-a^2/9t} t^{(n+2)/2+m+1}.
\end{aligned}
$$

Thus we have such an estimate of (4.23) as in (4.29). The estimate of (4.22): Using (4.25), (4.26) with $m_0 = -1$, we have

$$
\begin{aligned}
&|\partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} E_a(,0))^m \sharp L_a(,\varepsilon)|_{(0)} \\
&\leq C \int dt_1 \cdot\cdot dt_{m+1} |\langle \partial^{[(n+2)/2]+|\alpha|+2m+1}_{g^Z_{(0)},Q^{m+1}} \rangle \partial_{x'}^{\alpha'} L_a(t_{m+1}, \varepsilon, Q^{m+1}, x')|_{L^2(Q^{m+1} \in \mathcal{N}_{p0}),(0)} \\
&\leq C_1 e^{-a^2/8t} a^{(n+2)/2} t^{m+1} \leq C_2 e^{-a^2/9t} t^{(n+2)/4+m+1}.
\end{aligned}
$$

Thus we have obtained the estimate (4.29).

Next, let us show

$$
\begin{aligned}
(4.30)\quad (4.21) + (4.20) = \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \Big\{ &\mathcal{O}((1 + r(x', P^0))^m e^{-t\lambda_0} (t^{-(n+2+N')/2} + 1)) \\
+ \mathcal{O}(e^{-t\lambda_0} e^{-a^2/9t}) \Big\} &+ \varepsilon^{m_0+2} \mathcal{O}(a^{n+2+2(m_0+2)} (t^{-(n+2+N')/2} + t^N)) \ : \text{(with no condition)}
\end{aligned}
$$

with $x \in \mathcal{N}_{p0}^{m_0+3} = \mathcal{N}_{p0}$. The estimate of (4.21) with "no condition" : Here we use (4.25), (4.26) with $m_0 = 1$. Set

$$
\begin{aligned}
&\partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} E_a(,0))^{m_0+1} \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} E_a(,\varepsilon)) \\
&= \int_0^t dt_1 \int_0^{t_1} dt_2 \cdot\cdot \int_0^{t_{m_0+1}} dt_{m_0+2}\, \partial_x^\alpha E_a(t - t_1, 0) \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} E_a(t_1 - t_2, 0)) \\
&\qquad \sharp \cdots \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} E_a(t_{m_0+1} - t_{m_0+2}, 0)) \sharp (-\widetilde{\partial^2_{g^Z_{(\varepsilon)}}} \partial_{x'}^{\alpha'} E_a(t_{m_0+2}, \varepsilon)) \\
&\qquad\qquad (\, 0 \equiv t_{m_0+3} < t_{m_0+2} < t_{m_0+1} < \cdots < t_2 < t_1 < t_0 \equiv t \,) \\
&= \sum_{i=1}^{m_0+2} \int_{\frac{m_0+2}{m_0+3}t}^{t} dt_1 \cdot\cdot \int_{\frac{m_0+4-i}{m_0+5-i}t_{i-2}}^{t_{i-2}} dt_{i-1} \int_0^{\frac{m_0+3-i}{m_0+4-i}t_{i-1}} dt_i \int_0^{t_i} dt_{i+1} \cdot\cdot \int_0^{t_{m_0+1}} dt_{m_0+2} (\cdots) \\
&\qquad\qquad (t_{i-1} - t_i \geq t/(m_0 + 3)\,) \\
&+ \int_{\frac{m_0+2}{m_0+3}t}^{t} dt_1 \cdot\cdot \int_{\frac{2}{3}t_{m_0}}^{t_{m_0}} dt_{m_0+1} \int_{\frac{1}{2}t_{m_0+1}}^{t_{m_0+1}} dt_{m_0+2} (\cdots) \equiv \sum_{i=1}^{m_0+3} \int^{(i)} dt_1 \cdot\cdot dt_{m_0+2} (\cdots). \\
&\qquad (t_{m_0+2} \geq t/(m_0 + 3)\,)
\end{aligned}
$$

Then we have

$$
|\int^{(1)} dt_1 \cdot\cdot dt_{m_0+2} (\cdots)|_{(0)}
$$

$$\le \int^{(1)} dt_1 \cdot\cdot dt_{m_0+2}\, |\partial_x^\alpha E_a(t-t_1,0,x,Q^1) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_1-t_2,0) \sharp \cdots$$
$$\sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_{m_0+1}-t_{m_0+2},0) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} \partial_{x'}^{\alpha'} E_a(t_{m_0+2},\epsilon,Q^{m_0+2},x')|_{(0)}$$
$$\le C_0 \int^{(1)} dt_1 \cdot\cdot dt_{m_0+2}\, |\partial_x^\alpha E_a(t-t_1,0,x,Q^1) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_1-t_2,0) \sharp \cdots$$
$$\sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_{m_0+1}-t_{m_0+2},0) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} \langle \not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},x''} \rangle E_a(t_{m_0+2},\epsilon,Q^{m_0+2},x'')|_{(0),L^2(x'')}$$
$$\le C_0 \int^{(1)} dt_1 \cdot\cdot dt_{m_0+2}\, |\partial_x^\alpha E_a(t-t_1,0,x,Q^1) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_1-t_2,0) \sharp \cdots$$
$$\sharp \langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(\epsilon)},Q^{m_0+2}} \rangle \widetilde{\check{\not\partial}^2_{g^Z_{(\epsilon)},Q^{m_0+2}} \not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_{m_0+1}-t_{m_0+2},0,Q^{m_0+1},Q^{m_0+2})|_{(0),L^2(Q^{m_0+2}\in\mathcal{N}_{p0})}$$
$$\le C_1(\epsilon a^2) \int^{(1)} dt_1 \cdot\cdot dt_{m_0+2}\, |\partial_x^\alpha E_a(t-t_1,0,x,Q^1) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_1-t_2,0) \sharp \cdots$$
$$\sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} \langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+3}_{g^Z_{(\epsilon)},Q^{m_0+2}} \rangle E_a(t_{m_0+1}-t_{m_0+2},0,Q^{m_0+1},Q^{m_0+2})|_{(0),L^2(Q^{m_0+2})}$$
$$\le \cdots$$
$$\le C_{m_0+2}(\epsilon a^2)^{m_0+2} \int^{(1)} dt_1 \cdot\cdot dt_{m_0+2}\, |\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1+2(m_0+2)}_{g^Z_{(\epsilon)},Q^1} \rangle \partial_x^\alpha E_a(t-t_1,0,x,Q^1)|_{(0),L^2(Q^1)}$$
$$= C_{m_0+2}(\epsilon a^2)^{m_0+2} \int_c^{(1)} dt_1 \cdot\cdot dt_{m_0+2}$$
$$|\langle \check{\not\partial}^{[(n+2)/2]+|\alpha'|+1+2(m_0+2)}_{g^Z_{(\epsilon)},Q^1} \rangle \partial_x^\alpha E_a(t-t_1,0,x,Q^1)|_{(0),L^2(Q^1\in\mathcal{N}^{m_0+4}_{p0})}$$
$$\le C\,\epsilon^{m_0+2}\, a^{(n+2)/2+2(m_0+2)}(1+t^{-(n+2+[(n+2)/2]+|\alpha|+|\alpha'|+1+2(m_0+2))/2})\, t^{m_0+2},$$

$$|\int^{(2)} dt_1 \cdot\cdot dt_{m_0+2}\, (\cdots)|_{(0)}$$
$$\le \int^{(2)} dt_1 \cdot\cdot dt_{m_0+2}\, |\partial_x^\alpha E_a(t-t_1,0,x,Q^1) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_1-t_2,0) \sharp \cdots$$
$$\sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} E_a(t_{m_0+1}-t_{m_0+2},0) \sharp \widetilde{\not\partial^2_{g^Z_{(\epsilon)}}} \partial_{x'}^{\alpha'} E_a(t_{m_0+2},\epsilon,Q^{m_0+2},x')|_{(0)}$$
$$\le C_{m_0+2}(\epsilon a^2)^{m_0+2} \int^{(2)} dt_1 \cdot\cdot dt_{m_0+2}$$
$$|\langle \check{\not\partial}^{2[(n+2)/2]+|\alpha|+|\alpha'|+1+2(m_0+2)}_{g^Z_{(\epsilon)},Q^1} \rangle E_a(t_1-t_2,0,Q^1,Q^2)|_{L^2(Q^1\in\mathcal{N}_{p0}),L^2(Q^2\in\mathcal{N}_{p0})}$$
$$\le C\,\epsilon^{m_0+2}\, a^{n+2+2(m_0+2)}(1+t^{-(n+2+2[(n+2)/2]+|\alpha|+|\alpha'|+2(m_0+3))/2})\, t^{m_0+2},$$

etc. Thus, the term (4.21) with "no condition" appears as the term $\epsilon^{m_0+2}\,\mathcal{O}(a^{n+2+2(m_0+2)}(t^{-(n+2+N')/2}+t^N))$ in (4.30). The estimate of (4.20) with "no condition" : We want to show that it produces the first term in the right hand side of (4.30). Note that $\star_{(\epsilon),x'}\star^{-1}_{(0),x'}$ appearing at (4.20) has a series expansion

$$\star_{(\epsilon),x'}\star^{-1}_{(0),x'} = \sum_{m\ge 0} \epsilon^{m/2} \star_{(m/2:),x'} \star^{-1}_{(0),x'} = \mathrm{id} + \sum_{m\ge 2} \epsilon^{m/2}\mathcal{O}(|x'^b|^m), \tag{4.31}$$

which produces $(1+r(x',P^0))^m$ at (4.30). Hence, it suffices to show the following estimates (4.33), (4.34) of (4.20) with $\star_{(\epsilon),x'}\star^{-1}_{(0),x'}$ deleted. (If $r(x',P^0)\le a$, then we have $(1+r(x',P^0))^m$

$e^{-\lambda_0 t}e^{-a^2/9t} \le (1+a)^m e^{-\lambda_0 t}e^{-a^2/9t} \le C\,e^{-\lambda_0 t/2}e^{-a^2/10t}$. Hence, at (4.30), we have no need to add $(1+r(x',P^0))^m$ to the part $\mathcal{O}(e^{-\lambda_0 t}e^{-a^2/9t})$.) That is, we have

$$
\begin{aligned}
(4.32)\quad & \sum_{m=0}^{m_0+1} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0)\,\sharp\,(\widetilde{-\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}E_a(,0)})^m = \sum_{m=0}^{m_0+1} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \\
& \qquad \sharp\Big(\sum_{2\le i<i_0} \varepsilon^{i/2}(-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i/2)})E_a(,0) + \varepsilon^{i_0/2}(-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i_0/2)})(\theta\varepsilon^{1/2})E_a(,0)\Big)^m \\
&= \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \\
& \qquad \sharp\,(-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i_1/2)})E_a(,0)\,\sharp\cdots\sharp\,(-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i_p/2)})E_a(,0) \\
& + \varepsilon^{m_0+2}\,\mathcal{O}(a^{n+2+2(m_0+2)}(1+t^{-(n+2+2[(n+2)/2]+|\alpha|+|\alpha'|+2(m_0+3))/2})\,t^{m_0+2}) \\
&\equiv \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0)\,\sharp\,\Big((-\partial\!\!\!/^2_{g^Z_{(0)}})E_a(,0)\Big)^{(i_1/2),\cdots,(i_p/2)} \\
& + \varepsilon^{m_0+2}\,\mathcal{O}(a^{n+2+2(m_0+2)}(1+t^{-(n+2+2[(n+2)/2]+|\alpha|+|\alpha'|+2(m_0+3))/2})\,t^{m_0+2}) \\
&= \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_1(,0)\,\sharp\,\Big((-\partial\!\!\!/^2_{g^Z_{(0)}})E_1(,0)\Big)^{(i_1/2),\cdots,(i_p/2)} \\
& + \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \int_1^a da\,(\partial/\partial a) \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0)\,\sharp\,\Big((-\partial\!\!\!/^2_{g^Z_{(0)}})E_a(,0)\Big)^{(i_1/2),\cdots,(i_p/2)} \\
& + \varepsilon^{m_0+2}\,\mathcal{O}(a^{n+2+2(m_0+2)}(1+t^{-(n+2+2[(n+2)/2]+|\alpha|+|\alpha'|+2(m_0+3))/2})\,t^{m_0+2}).
\end{aligned}
$$

(The above estimate of the remainder term can be obtained in the same way as that of (4.21) with "no condition".) Then we intend to show that there exists a constant $\lambda_0 > 0$ satisfying

$$
\begin{aligned}
(4.33)\quad & \partial_x^\alpha \partial_{x'}^{\alpha'} E_1(,0)\,\sharp\,\Big((-\partial\!\!\!/^2_{g^Z_{(0)}})E_1(,0)\Big)^{(i_1/2),\cdots,(i_p/2)} \\
&= \mathcal{O}(e^{-t\lambda_0}(t^{-(n+2+2[(n+2)/2]+|\alpha|+|\alpha'|+2)/2}+1)),
\end{aligned}
$$

$$
\begin{aligned}
(4.34)\quad & \int_1^a da\,(\partial/\partial a)\partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0)\,\sharp\,\Big((-\partial\!\!\!/^2_{g^Z_{(0)}})E_a(,0)\Big)^{(i_1/2),\cdots,(i_p/2)} \\
&= \mathcal{O}(e^{-t\lambda_0}\,e^{-1/9t}) + \mathcal{O}(e^{-t\lambda_0}\,e^{-a^2/9t}).
\end{aligned}
$$

As for (4.33): Since the Dirac operator $\partial\!\!\!/_{g^V}$ acting on $\Gamma(\$_{g^V}|Z_{p^0})$ (see $E(t,0)$ given at (4.11)) is invertible ([19, (5.15)]), there exists a constant $\mu_0 > 0$ with $\mathrm{Spec}(\partial\!\!\!/^2_{g^V}) \ge \mu_0 > 0$. Hence, more strongly than the estimate (4.12) with $\varepsilon = 0$, there exist constants $\lambda_0 > 0$, $C > 0$ satisfying

$$
\begin{aligned}
(4.35)\quad & |\partial^\alpha \partial^{\alpha'} E(t,0,x,x')|_{(0)} \\
& \le C\,e^{-t\lambda_0}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big)\begin{cases} 1 & : \text{(with no condition)} \\ e^{-(r(x,x')-\bar r)^2/8t} & : r(x,x') > \bar r \end{cases} \\
& \qquad (0 < \forall t < \infty,\ \forall (x,x') \in Z \times Z),
\end{aligned}
$$

which implies further the estimate stronger than (4.15) with $\varepsilon = 0$

$$|\partial^{\alpha}\partial^{\alpha'}E_a(t,0,x,x')|_{(0)} \le C\,e^{-t\lambda_0}\begin{cases} 1+t^{-(\dim Z+|\alpha|+|\alpha'|)/2} : r(x,x')\le \bar r, \\ e^{-r(x,x')^2/8t} : \bar r \le r(x,x')\le \bar r + a, \\ 0 \qquad\qquad : \bar r + a \le r(x,x'). \end{cases} \tag{4.36}$$

Using it, now let us show (4.33). Set

$$\begin{aligned}
&\partial_x^{\alpha}\partial_{x'}^{\alpha'}E_1(,0)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_1/2)}E_1(,0)\,\sharp\cdots\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_p/2)}E_1(,0)\\
&=\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{p-1}}dt_p\,\partial_x^{\alpha}E_1(t-t_1,0)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_1/2)}E_1(t_1-t_2,0)\\
&\qquad\sharp\cdots\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_{p-1}/2)}E_1(t_{p-1}-t_p,0)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_p/2)}\partial_{x'}^{\alpha'}E_1(t_p,0)\\
&\qquad\qquad(\,0\equiv t_{p+1}<t_p<t_{p-1}<\cdots<t_2<t_1<t_0\equiv t\,)\\
&=\sum_{q=1}^{p}\int_{\frac{p}{p+1}t}^{t}dt_1\cdots\int_{\frac{p+2-q}{p+3-q}t_{q-2}}^{t_{q-2}}dt_{q-1}\int_0^{\frac{p+1-q}{p+2-q}t_{q-1}}dt_q\int_0^{t_q}dt_{q+1}\cdots\int_0^{t_{p-1}}dt_p\,(\cdots)\\
&\qquad\qquad(t_{q-1}-t_q\ge t/(p+1)\,)\\
&\quad+\int_{\frac{p}{p+1}t}^{t}dt_1\cdots\int_{\frac{2}{3}t_{p-2}}^{t_{p-2}}dt_{p-1}\int_{\frac{1}{2}t_{p-1}}^{t_{p-1}}dt_p\,(\cdots)\equiv\sum_{q=1}^{p+1}\int^{(q)}dt_1\cdots dt_p\,(\cdots).\\
&\qquad(t_p\ge t/(p+1)\,)
\end{aligned}$$

Then we have

$$\begin{aligned}
&|\int^{(1)}dt_1\cdots dt_p\,(\cdots)|_{(0)}\\
&\le\int^{(1)}dt_1\cdots dt_p\,|\partial_x^{\alpha}E_1(t-t_1,0,x,Q^1)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_1/2)}E_1(t_1-t_2,0)\,\sharp\cdots\\
&\qquad\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_{p-1}/2)}E_1(t_{p-1}-t_p,0)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_p/2)}\partial_{x'}^{\alpha'}E_1(t_p,0,Q^p,x')|_{(0)}\\
&\le C_0\int^{(1)}dt_1\cdots dt_p\,|\partial_x^{\alpha}E_1(t-t_1,0,x,Q^1)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_1/2)}E_1(t_1-t_2,0)\,\sharp\cdots\\
&\quad\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_{p-1}/2)}E_1(t_{p-1}-t_p,0)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_p/2)}\langle\not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(0)},x''}\rangle E_1(t_p,0,Q^p,x'')|_{(0),L^2(x'')}\\
&\le C_0\int^{(1)}dt_1\cdots dt_p\,|\partial_x^{\alpha}E_1(t-t_1,0,x,Q^1)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_1/2)}E_1(t_1-t_2,0)\,\sharp\cdots\\
&\quad\sharp\,\langle\not\partial^{[(n+2)/2]+|\alpha'|+1}_{g^Z_{(0)},Q^p}\rangle(\not\partial^2_{g^Z_{(0)},Q^p})^{(i_p/2)}(\not\partial^2_{g^Z_{(0)}})^{(i_{p-1}/2)}E_1(t_{p-1}-t_p,0,Q^{p-1},Q^p)|_{(0),L^2(Q^p)}\\
&\le C_1\int^{(1)}dt_1\cdots dt_p\,|\partial_x^{\alpha}E_1(t-t_1,0,x,Q^1)\,\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_1/2)}E_1(t_1-t_2,0)\,\sharp\cdots\\
&\quad\sharp\,(\not\partial^2_{g^Z_{(0)}})^{(i_{p-1}/2)}\langle\not\partial^{[(n+2)/2]+|\alpha'|+3}_{g^Z_{(0)},Q^p}\rangle E_1(t_{p-1}-t_p,0,Q^{p-1},Q^p)|_{(0),L^2(Q^p)}\\
&\le\cdots\\
&\le C_p\int^{(1)}dt_1\cdots dt_p\,|\partial_x^{\alpha}\langle\not\partial^{[(n+2)/2]+|\alpha'|+1+2p}_{g^Z_{(0)},Q^1}\rangle E_1(t-t_1,0,x,Q^1)|_{(0),L^2(Q^1)}\\
&=C_p\int^{(1)}dt_1\cdots dt_p\,|\partial_x^{\alpha}\langle\not\partial^{[(n+2)/2]+|\alpha'|+1+2p}_{g^Z_{(0)},Q^1}\rangle E_1(t-t_1,0,x,Q^1)|_{(0),L^2(Q^1\in\mathcal{N}_{p0})}\\
&\le Ce^{-t\lambda_0}(1+t^{-(n+2+[(n+2)/2]+|\alpha|+|\alpha'|+2p+1)/2})\,t^p\\
&\le C'e^{-t\lambda_0/2}(t^{-(n+2+2[(n+2)/2]+|\alpha|+|\alpha'|+1)/2}+1),
\end{aligned}$$

etc. That is, we have obtained (4.33). As for (4.34): In the same way as we have shown (4.16) using (4.12), first we can show, using (4.35),

$$
(4.37)\quad
\begin{aligned}
&\frac{\partial E_a(t,0)}{\partial a} = \int_{-\infty}^{\infty} \frac{e^{-u^2/4t}}{(4\pi t)^{1/2}} \frac{\partial \phi_a}{\partial a}(u) \cos u\sqrt{\not{\partial}^2_{g^Z_{(0)}}}\, du, \quad \mathrm{supp}\, \frac{\partial \phi_a}{\partial a} \subset [a, \bar{r}+a], \\
&|\partial^\alpha \partial^{\alpha'} \frac{\partial E_a(t,0,x,x')}{\partial a}|_{(0)} \le C\, e^{-t\lambda_0} \begin{cases} e^{-a^2/8t} & : r(x,x') \le \bar{r}+a, \\ 0 & : \bar{r}+a \le r(x,x'). \end{cases}
\end{aligned}
$$

Let us set now

$$
\begin{aligned}
&\frac{\partial}{\partial a}\, \partial_x^\alpha \partial_{x'}^{\alpha'}\, E_a(,0) \sharp \left((-\not{\partial}^2_{g^Z_{(0)}})E_a(,0)\right)^{(i_1/2),\cdots,(i_p/2)} \\
&= \partial_x^\alpha \partial_{x'}^{\alpha'}\, \frac{\partial E_a(,0)}{\partial a} \sharp \left((-\not{\partial}^2_{g^Z_{(0)}})E_a(,0)\right)^{(i_1/2),\cdots,(i_p/2)} \\
&+ \sum_{q=1}^{p} \partial_x^\alpha \partial_{x'}^{\alpha'}\, E_a(,0) \sharp (-(\not{\partial}^2_{g^Z_{(0)}})^{(i_1/2)})E_a(,0) \sharp \cdots \\
&\qquad \sharp (-(\not{\partial}^2_{g^Z_{(0)}})^{(i_q/2)}) \frac{\partial E_a(,0)}{\partial a} \sharp \cdots \sharp (-(\not{\partial}^2_{g^Z_{(0)}})^{(i_p/2)})E_a(,0).
\end{aligned}
$$

Then, in the same way as the estimation of (4.23), using (4.37) we have, if $1 \le q \le p-1$,

$$
\begin{aligned}
&|\partial_x^\alpha \partial_{x'}^{\alpha'}\, E_a(,0) \sharp (-(\not{\partial}^2_{g^Z_{(0)}})^{(i_1/2)})E_a(,0) \sharp \cdots \\
&\qquad \sharp (-(\not{\partial}^2_{g^Z_{(0)}})^{(i_q/2)}) \frac{\partial E_a(,0)}{\partial a} \sharp \cdots \sharp (-(\not{\partial}^2_{g^Z_{(0)}})^{(i_p/2)})E_a(,0)|_{(0)} \\
&\le C\, a^{i_1} \cdots a^{i_p} \int dt_1 \cdots dt_p\, |\langle \not{\partial}^{[(n+2)/2]+|\alpha|+2q+1}_{g^Z_{(0)},Q^q} \rangle \langle \not{\partial}^{[(n+2)/2]+|\alpha'|+2(p-q)+1}_{g^Z_{(0)},Q^{q+1}} \rangle \\
&\qquad \frac{\partial E_a(t_q - t_{q+1}, 0, Q^q, Q^{q+1})}{\partial a}|_{L^2(Q^q \in \mathcal{N}_{p0}), L^2(Q^{q+1} \in \mathcal{N}_{p0})} \\
&\le C_1\, a^{i_1} \cdots a^{i_p} e^{-t\lambda_0}\, a^{n+2} t^p e^{-a^2/8t} \le C_2\, e^{-t\lambda_0}\, t^{(n+2+\sum i_j -1)/2+p+1}\, e^{-a^2/9t} (a/t) \\
&\le C_3\, e^{-t\lambda_0/2}\, e^{-a^2/9t}(a/t).
\end{aligned}
$$

The remained cases are similarly shown. Gathering those estimates, finally we obtain

$$
(\partial/\partial a) \partial_x^\alpha \partial_{x'}^{\alpha'}\, E_a(,0) \sharp \left((-\not{\partial}^2_{g^Z_{(0)}})E_a(,0)\right)^{(i_1/2),\cdots,(i_p/2)} = \mathcal{O}(e^{-t\lambda_0/2}\, e^{-a^2/9t}\, (a/t)),
$$

which implies the estimate (4.34). And, we have thus finished the proof of (4.30). Remark the comment preceding (4.32).

Let us then show

$$
(4.38)\quad
\begin{aligned}
&(4.21) + (4.20) = \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \Big\{ \mathcal{O}((1 + r(x', P^0)))^m e^{-t\lambda_0} e^{-(r(x,x')-\bar{r})^2/C_1 t}) \\
&+ \mathcal{O}(e^{-t\lambda_0} e^{-a^2/9t}) \Big\} + \varepsilon^{m_0+2}\, \mathcal{O}(a^{n+2+2(m_0+2)}\, t^N\, e^{-(r(x,x')-\bar{r})^2/C_1 t}) \;\; : r(x,x') > \bar{r}
\end{aligned}
$$

with $x \in \mathcal{N}_{P0}^{m_0+3} = \mathcal{N}_{P0}$, hence, with $\bar{r} < r(x,x') < (m_0+2)(\bar{r}+a)$. The estimate of (4.21) with "$r(x,x') > \bar{r}$": Here we use (4.25), (4.26) with $m_0 = 1$. We set

$$\begin{aligned}
&\partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp (-\widetilde{\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} E_a(,0))^{m_0+1} \sharp (-\widetilde{\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} E_a(,\varepsilon)) \\
&= \int dt_1 \cdot\cdot dt_{m_0+2}\, \partial_x^\alpha E_a(,0,x,Q^1) \sharp (-\widetilde{\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} E_a(,0,Q^1,Q^2)) \sharp \cdots \\
&\qquad \sharp (-\widetilde{\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} E_a(,0,Q^{m_0+1},Q^{m_0+2})) \sharp \partial_{x'}^{\alpha'} (-\widetilde{\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} E_a(,\varepsilon,Q^{m_0+2},x')) \\
&= \sum_{i=1}^{m_0+3} \int dt_1 \cdot\cdot dt_{m_0+2} \int_{r(Q^{i-1},Q^i) \geq r(x,x')/(m_0+3)} dg^Z_{(0)}(Q^1) \cdots dg^Z_{(0)}(Q^{m_0+2}) \equiv \sum_{i=1}^{m_0+3} \int dt_1 \cdot\cdot dt_{m_0+2} \int_{(i)}
\end{aligned}$$

Then, similarly to the estimation of (4.21) with "no condition", we have

$$\begin{aligned}
&| \int dt_1 \cdot\cdot dt_{m_0+2} \int_{(1)} |_{(0)} \\
&\leq C\,(\varepsilon a^2)^{m_0+2} \int dt_1 \cdot\cdot dt_{m_0+2} \int_{(1)} |\langle \partial\!\!\!/^{[(n+2)/2]+|\alpha'|+1+2(m_0+2)}_{g^Z_{(0)},Q^1} \rangle \partial_x^\alpha E_a(,0,x,Q^1)|_{(0),L^2(Q^1)} \\
&= C\,(\varepsilon a^2)^{m_0+2} \int dt_1 \cdot\cdot dt_{m_0+2} \\
&\quad |\langle \partial\!\!\!/^{[(n+2)/2]+|\alpha|+|\alpha'|+1+2(m_0+2)}_{g^Z_{(0)},Q^1} \rangle E_a(,0,x,Q^1)|_{(0),L^2(r(x,x')/(m_0+3)+a \geq r(x,Q^1) \geq r(x,x')/(m_0+3))} \\
&\leq C_1\,(\varepsilon a^2)^{m_0+2}\, t^{m_0+2} a^{(n+2)/2} e^{-(r(x,x')/(m_0+3))^2/8t} \\
&\leq C_1\,\varepsilon^{m_0+2} a^{(n+2)/2+2(m_0+2)}\, t^{m_0+2} e^{-(r(x,x')-\bar{r})^2/(m_0+3)^2 8t},
\end{aligned}$$

$$\begin{aligned}
&| \int dt_1 \cdot\cdot dt_{m_0+2} \int_{(2)} |_{(0)} \leq C_{m_0+2} (\varepsilon a^2)^{m_0+2} \int dt_1 \cdot\cdot dt_{m_0+2} \int_{(2)} |\langle \partial\!\!\!/^{[(n+2)/2]+|\alpha|+|\alpha'|+1+2(m_0+2)}_{g^Z_{(0)},Q^1} \rangle \\
&\qquad \cdot E_a(,0,Q^1,Q^2)|_{L^2(Q^1 \in \mathcal{N}_{P0}),L^2(r(x,x')/(m_0+3)+a \geq r(Q^1,Q^2) \geq r(x,x')/(m_0+3))} \\
&\leq C\,(\varepsilon a^2)^{m_0+2} t^{m_0+2} a^{n+2} e^{-(r(x,x')/(m_0+3))^2/8t} \\
&= C\,\varepsilon^{m_0+2} a^{n+2+2(m_0+2)}\, t^{m_0+2} e^{-(r(x,x')-\bar{r})^2/(m_0+3)^2 8t},
\end{aligned}$$

etc. Thus, the term (4.21) with "$r(x,x') > \bar{r}$" appears as the term $\varepsilon^{m_0+2}\, \mathcal{O}(a^{n+2+2(m_0+2)}\, t^N e^{-(r(x,x')-\bar{r})^2/C_1 t})$ in (4.38). The estimate of (4.20) with "$r(x,x') > \bar{r}$": We want to show that it produces the first term in the right hand side of (4.38). To do so, according to the comment preceding (4.32), it suffices also to show the following estimates (4.40), (4.41). That is, we have

$$\begin{aligned}
(4.39)\quad &\sum_{m=0}^{m_0+1} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp (-\widetilde{\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} E_a(,0))^m = \sum_{m=0}^{m_0+1} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \\
&\qquad \sharp \Big(\sum_{2 \leq i < i_0} \varepsilon^{i/2} (-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i/2)}) E_a(,0) + \varepsilon^{i_0/2} (-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i_0/2)})(\theta \varepsilon^{1/2}) E_a(,0) \Big)^m \\
&= \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp (-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i_1/2)}) E_a(,0) \sharp \cdots \sharp (-(\partial\!\!\!/^2_{g^Z_{(0)}})^{(i_p/2)}) E_a(,0) \\
&\quad + \varepsilon^{m_0+2}\, \mathcal{O}(a^{n+2+2(m_0+2)}\, t^{m_0+2} e^{-(r(x,x')-\bar{r})^2/(m_0+3)^2 8t})
\end{aligned}$$

$$\equiv \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp \left((-\not\partial^2_{g^Z_{(0)}})E_a(,0)\right)^{(i_1/2),\cdots,(i_p/2)}$$
$$+\varepsilon^{m_0+2}\, \mathcal{O}(a^{n+2+2(m_0+2)}\, t^{m_0+2} e^{-(r(x,x')-\bar r)^2/(m_0+3)^2 8t})$$
$$= \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_d(,0) \sharp \left((-\not\partial^2_{g^Z_{(0)}})E_d(,0)\right)^{(i_1/2),\cdots,(i_p/2)} \quad (d = r(x,x') - \bar r)$$
$$+ \sum_{m=0}^{m_0+1} \varepsilon^{m/2} \int_d^a da\,(\partial/\partial a) \sum_{m=i_1+\cdots+i_p} \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp \left((-\not\partial^2_{g^Z_{(0)}})E_a(,0)\right)^{(i_1/2),\cdots,(i_p/2)}$$
$$+\varepsilon^{m_0+2}\, \mathcal{O}(a^{n2+2(m_0+2)}\, t^{m_0+2} e^{-(r(x,x')-\bar r)^2/(m_0+3)^2 8t}).$$

Then we want to show that there exists a constant $\lambda_0 > 0$ satisfying

$$\begin{aligned}(4.40)\qquad &\partial_x^\alpha \partial_{x'}^{\alpha'} E_d(,0) \sharp \left((-\not\partial^2_{g^Z_{(0)}})E_d(,0)\right)^{(i_1/2),\cdots,(i_p/2)} \\ &= \mathcal{O}(e^{-t\lambda_0}\, e^{-(r(x,x')-\bar r)^2/(p+1)^2 8t}) \quad (d = r(x,x') - \bar r),\end{aligned}$$

$$\begin{aligned}(4.41)\qquad &\int_d^a da\,(\partial/\partial a) \partial_x^\alpha \partial_{x'}^{\alpha'} E_a(,0) \sharp \left((-\not\partial^2_{g^Z_{(0)}})E_a(,0)\right)^{(i_1/2),\cdots,(i_p/2)} \\ &= \mathcal{O}(e^{-t\lambda_0}\, e^{-(r(x,x')-\bar r)^2/9t}) + \mathcal{O}(e^{-t\lambda_0}\, e^{-a^2/9t}).\end{aligned}$$

Using (4.36), let us show (4.40) first. Set

$$\begin{aligned}&\partial_x^\alpha \partial_{x'}^{\alpha'} E_d(,0) \sharp (\not\partial^2_{g^Z_{(0)}})^{(i_1/2)} E_d(,0) \sharp \cdots \sharp (\not\partial^2_{g^Z_{(0)}})^{(i_p/2)} E_d(,0) \\ &= \int dt_1 \cdots dt_p\, \partial_x^\alpha E_d(,0,x,Q^1) \sharp (\not\partial^2_{g^Z_{(0)}})^{(i_1/2)} E_d(,0,Q^1,Q^2) \sharp \cdots \\ &\quad \sharp (\not\partial^2_{g^Z_{(0)}})^{(i_{p-1}/2)} E_d(,0,Q^{p-1},Q^p) \sharp (\not\partial^2_{g^Z_{(0)}})^{(i_p/2)} \partial_{x'}^{\alpha'} E_d(,0,Q^p,x') \\ &= \sum_{q=1}^{p+1} \int dt_1 \cdots dt_p \int_{r(Q^{q-1},Q^q)\ge r(x,x')/(p+1)}^{r(Q^{q'-1},Q^{q'})\le r(x,x')/(p+1),\, q'<q} dg^Z_{(0)}(Q^1)\cdots dg^Z_{(0)}(Q^p) \equiv \sum_{q=1}^{p+1} \int dt_1 \cdots dt_p \int_{(q)}.\end{aligned}$$

Then, similarly to the estimation of (4.33), we have

$$\begin{aligned}&\left|\int dt_1 \cdots dt_p \int_{(1)}\right|_{(0)} \le C\, d^{i_1+\cdots+i_p} \int dt_1 \cdots dt_p \int_{(1)} |\partial_x^\alpha \langle \not\partial_{g^Z_{(0)},Q^1}\rangle^{[(n+2)/2]+|\alpha'|+1+2p} E_d(,0,x,Q^1)|_{(0),L^2(Q^1)} \\ &\le C_1 d^{(n+2)/2+m} e^{-\lambda_0 t} t^{m_0+2} e^{-(r(x,x')/(p+1))^2/8t} \le C_2 e^{-\lambda_0 t/2} e^{-(r(x,x')-\bar r)^2/(p+1)^2 9t},\end{aligned}$$

etc. That is, we obtained the estimate (4.40). Also the estimate (4.41) can be shown similarly to that of (4.34). And thus we obtained the estimate (4.38).

We have thus obtained the expansions (4.27), (4.28). That is, (4.29), (4.30) imply (4.27), and (4.29), (4.38) imply (4.28). Using the expansions, we can now show (2.22), (2.23) for E with $t \ge T_0$ in the following way. As for the case "with no condition": We will take the parameter a with

$$(4.42)\qquad a^2 = (m_0+2)(9/2)\, t|\log\varepsilon|\ (> r(x,x')^2), \text{ hence, } e^{-a^2/9t} = \varepsilon^{(m_0+2)/2}.$$

Then (4.27) gives the expansion

$$\begin{aligned}(4.43)\ \ &\partial^\alpha\partial^{\alpha'}E(t,\varepsilon,x,x')\\ &=\sum_{m=0}^{m_0+1}\varepsilon^{m/2}\Big\{\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0}(t^{-(n+2+N')/2}+1))+\varepsilon^{(m_0+2)/2}\mathcal{O}(e^{-t\lambda_0})\Big\}\\ &\quad+\varepsilon^{m_0+2}|\log\varepsilon|^{(n+2+m_0+2)/2}\mathcal{O}(t^{N+(n+2+m_0+2)/2})+\varepsilon^{(m_0+2)/2}\mathcal{O}(1)\\ &=\sum_{m=0}^{m_0}\varepsilon^{m/2}\,\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0}(t^{-(n+2+N')/2}+1))+\varepsilon^{(m_0+1)/2}\,\mathcal{O}(t^{N+(n+2+m_0+2)/2})\\ &\qquad\qquad(T_0\le t<\infty,\ \text{and "with no condition"}).\end{aligned}$$

Referring to the comment following (4.28), note that the coefficients of $\varepsilon^{m/2}$ $(0\le m\le m_0)$ in the last line do not depend on $\varepsilon^{1/2}$. Thus we obtained the estimates (2.22), (2.23) for E with $t\ge T_0$ and "with no condition". As for the case "$r(x,x')>\bar r$": We will take the parameter a with

$$\begin{aligned}(4.44)\qquad &a^2=(r(x,x')-\bar r)^2(9/C_1)+(m_0+2)(9/2)\,t|\log\varepsilon|\ \ (\le 2(m_0+2)(9/2)\,t|\log\varepsilon|),\\ &\text{hence, } e^{-a^2/9t}=\varepsilon^{(m_0+2)/2}\,e^{-(r(x,x')^2-\bar r)^2/C_1t}\end{aligned}$$

where $C_1>0$ is the constant appearing at (4.28). (Here we take $\varepsilon_0>0$ and $T_0>0$ such that, if $(\varepsilon^{1/2},t)\in(0,\varepsilon_0^{1/2}]\times[T_0,\infty)$, then we have $(r(x,x')-\bar r)^2(9/C_1)\le(m_0+2)(9/2)\,t|\log\varepsilon|$.) Then (4.28) gives the expansion

$$\begin{aligned}(4.45)\qquad &\partial^\alpha\partial^{\alpha'}E(t,\varepsilon,x,x')=\sum_{m=0}^{m_0+1}\varepsilon^{m/2}\Big\{\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0}e^{-(r(x,x')-\bar r)^2/C_1t})\\ &\quad+\varepsilon^{(m_0+2)/2}\,\mathcal{O}(e^{-t\lambda_0}e^{-(r(x,x')^2-\bar r)^2/C_1t})\Big\}\\ &\quad+\varepsilon^{m_0+2}\,|\log\varepsilon|^{(n+2+m_0+2)/2}\,\mathcal{O}(t^{N+(n+2+m_0+2)/2}\,e^{-(r(x,x')-\bar r)^2/C_1t})\\ &\quad+\varepsilon^{(m_0+2)/2}\,\mathcal{O}(t^N\,e^{-(r(x,x')^2-\bar r)^2/C_1t})\\ &=\sum_{m=0}^{m_0+1}\varepsilon^{m/2}\,e^{-(r(x,x')-\bar r)^2/C_1t}\,\mathcal{O}((1+r(x',P^0))^m e^{-t\lambda_0})\\ &\quad+\varepsilon^{(m_0+1)/2}\,e^{-(r(x,x')^2-\bar r)^2/C_1t}\,\mathcal{O}(t^{N+(n+2+m_0+2)/2}\,e^{-t\lambda_0})\\ &\qquad\qquad(T_0\le t<\infty,\ \text{and "}r(x,x')>\bar r\text{"}).\end{aligned}$$

Note that the coefficients of $\varepsilon^{m/2}$ $(0\le m\le m_0)$ in the last line also do not depend on $\varepsilon^{1/2}$. Thus we obtained the estimates (2.22), (2.23) for E with $t\ge T_0$ and with "$r(x,x')>\bar r$".

We have thus finished the proofs of the estimates (2.22), (2.23) for E with $t\ge T_0$.

As announced let us here prove (3) by referring to (2.21) and (4.27). Assume $m_0\ge n$. (3) for $E(t,(m_0+1)/2:\cdots)$ holds because there is no (I,J) satisfying $|(I,J)|\ge m_0+1\,(>n)$.

Then consider $E(t, m/2 : \cdots)$ with $m \leq m_0$. Observing (4.32) and its preceding comment, its top degree as a differential form in the M-direction, i.e., $\max\{|(I,J)| \mid E(t,m/2:\cdots)_{(I,J)} \neq 0\}$, is certainly equal to such a top degree of

$$(4.46) \qquad \sum_{m=m_1+m_2} \sum_{m_1=i_1+\cdots+i_p} \star_{(m_2/2:),x'} \star^{-1}_{(0),x'} E_a(,0)\, E_a(,0) \sharp \left((-\not{\partial}^2_{g^Z_{(0)}}) E_a(,0) \right)^{(i_1/2),\cdots,(i_p/2)},$$

so that let us inquire into the latter top degree. Referring to (4.11) and (4.13), the degree of $E_a(,0)$ is clearly zero. This fact and (4.25) imply that, denoting by m_j the number of what are equal to j (≥ 2) among $i_1, i_2, \cdots, i_p$ (≥ 2 inevitably), and by k the top degree of (4.46), we have

$$k \leq 2(m_2 + m_3) + 4(m_4 + m_5 + \cdots) \leq 2m_2 + 3m_3 + 4m_4 + 5m_5 \cdots = m.$$

Moreover, if $k = m$, then we have $k = 2(m_2 + m_3) + 4(m_4 + m_5 + \cdots) = m$, which is certainly even. Thus, when $m_0 \geq n$, (3) holds. And clearly this implies that (3) holds even if $m_0 < n$.

Proof of Proposition 2.2 for $E(t, \varepsilon)$ with t small. We have only to show the estimate (2.24), which implies (2.22), (2.23) with t small. Hereafter we assume $0 < t \leq T_0$ and intend to show the estimate (2.24) by constructing the kernel concretely using the well-known Levi method. In constructing it, keep in mind the facts that the manifold $Z = Z(p^0)$ is non-compact and the operator $\not{\partial}^2_{g^Z_{(\varepsilon)}}$ has the extra parameter ε.

Now, consider the parabolic equation (0.3) for $\not{\partial}^2_{g^Z_{(\varepsilon)}}$ with $\psi_0 \in L^2\Gamma(\wedge T^*M \otimes_\pi \$_{g^V}$ with $g^Z_{(\varepsilon)})$ and its formal solution at each point $x' \in Z$, $E(t, \varepsilon, x, x') = q_{g^Z_{(\varepsilon)}}(t, x, x') \sum_{i=0}^\infty t^i E_i(\varepsilon, x, x')$ with $q_{g^Z_{(\varepsilon)}}(t, x, x') = (4\pi t)^{-(n+2)/2} e^{-r_{(\varepsilon)}(x,x')^2/4t}$ and

$$(4.47) \qquad E_i(\varepsilon, x, x') = \begin{cases} G_{g^Z_{(\varepsilon)}}(x,x')^{-1/4} \iota_{g^Z_{(\varepsilon)}}(x,x') \quad (i=0) \\ -G_{g^Z_{(\varepsilon)}}(x,x')^{-1/4} \iota_{g^Z_{(\varepsilon)}}(x,x') \displaystyle\int_0^1 ds\, s^{i-1}\, \iota_{g^Z_{(\varepsilon)}}(s\, x(\varepsilon, x, x'), x')^{-1} \\ \qquad \cdot \left(G^{1/4}_{g^Z_{(\varepsilon)}} \not{\partial}^2_{g^Z_{(\varepsilon)}} E_{i-1}(\varepsilon) \right)(s\, x(\varepsilon, x, x'), x') \quad (i > 0) \end{cases}$$

$$\equiv \sum e^I_b(\varepsilon)(x) \otimes e^J_b(\varepsilon)(x') \cdot E_i(e_b(\varepsilon) : \varepsilon, x, x')_{(I,J)}$$

$$\equiv \sum (dx^b)^I(x) \otimes (dx^b)^J(x') \cdot E_i(\varepsilon, x, x')_{(I,J)}$$

$$= \sum_{m=0}^{m_0} \varepsilon^{m/2} E_i(m/2 : x, x') + \varepsilon^{(m_0+1)/2} E_i(m_0+1)/2 : \varepsilon^{1/2}, x, x'),$$

$$(4.48) \qquad E_i(m/2 : \cdot, x, x') = \sum (dx^b)^I(x) \otimes (dx^b)^J(x') \cdot E_i(m/2 : \cdot, x, x')_{(I,J)}$$

where $x(\varepsilon) = x(\varepsilon, \cdot, x')$ is the $g^Z_{(\varepsilon)}$-normal coordinates at x' with $(\partial/\partial x(\varepsilon)_i)_{x'} = e^b_i(\varepsilon, \iota^*_\varepsilon A)_{x'}$, $\iota_{g^Z_{(\varepsilon)}}(x, x')$ is the $\nabla^{\$_{g^Z_{(\varepsilon)}}}$-parallel transport from x' to x and we set $G_{g^Z_{(\varepsilon)}}(x, x') \equiv \det(g^Z_{(\varepsilon)}(\partial/\partial x(\varepsilon)_i,$

$\partial/\partial x(\varepsilon)_j))(x)$. As above (4.47) has certainly a series expansion. Further, for given α, α' and integers m_0, N, there exists a constant $C > 0$ satisfying

$$|\partial^\alpha \partial^{\alpha'} E_i(m/2 : \cdot, x, x')|_{(0)} \leq C\,(1+r(x,x'))^m(1+r(x'))^m \qquad (4.49)$$
$$(\,0 \leq \forall m \leq m_0+1\,,\ \forall i \leq N\,,\ \forall (x,x') \in Z \times Z\,).$$

Actually, consider the parabolic equation (0.3) for $\not{\partial}^2_{g^Z_\varepsilon}$ with $\psi_0 \in L^2\Gamma(\wedge T^*M \otimes_\pi \$_{g^V}$ with $g^Z_\varepsilon)$, its formal solution at each point $x' \in Z$, $\mathcal{E}(t,\varepsilon,x,x') = q_{g^Z_\varepsilon}(t,x,x')\sum_{i=0}^\infty t^i \mathcal{E}_i(\varepsilon,x,x')$ with $\mathcal{E}_i(\varepsilon,x,x') \equiv \sum e^I_{b\varepsilon}(x) \otimes e^J_{b\varepsilon}(x') \cdot \mathcal{E}_i(e_{b\varepsilon} : \varepsilon, x, x')_{(I,J)}$ and the expression similar to (4.47). Observing (3.8), the derivatives of $\mathcal{E}_i(e_{b\varepsilon} : \varepsilon, x, x')_{(I,J)}$ with respect to $\varepsilon^{1/2}$, x, x' are all bounded on $[0, \varepsilon_0^{1/2}] \times Z \times Z\,(\ni (\varepsilon^{1/2}, x, x'))$, and, similarly to (4.3), we have

$$E_i(e_b(\varepsilon) : \varepsilon, x, x')_{(I,J)} = \mathcal{E}_i(e_{b\varepsilon} : \varepsilon, \iota_\varepsilon(x), \iota_\varepsilon(x'))_{(I,J)}. \qquad (4.50)$$

These facts, together with expansion of $e_b(\varepsilon)$ (see (4.3) and (2.3)), certainly imply (4.49). Meanwhile, the Gaussian kernel part of the formal solution $E(t,\varepsilon,x,x')$ can be expanded as

$$q_{g^Z_{(\varepsilon)}}(t,x,x') = \sum_{m=0}^{m_0} \varepsilon^{m/2} q_{g^Z_{(m/2:)}}(t,x,x') + \varepsilon^{(m_0+1)/2} q_{g^Z_{((m_0+1)/2:\varepsilon^{1/2})}}(t,x,x'), \qquad (4.51)$$

$$q_{g^Z_{(m/2:\cdot)}}(t,x,x') = q_{g^Z_{(0)}}(t,x,x') \sum_{0 \leq \sum|\alpha^b_i| + |\beta^b| \leq m+2}^{3 \leq |\alpha_i|} \frac{(x-x')^{\alpha_1}}{t} \cdots \frac{(x-x')^{\alpha_a}}{t} (x'^b)^{\beta^b} \cdot q^{\alpha,\beta^b}_{g^Z_{(m/2:\cdot)}}(x^f, x'^f), \qquad (4.52)$$
$$(a \geq 1 \text{ if } m \geq 1)$$

because we have the series expansion

$$r_{(\varepsilon)}(x,x')^2 = r_{g^Z_\varepsilon}(\iota_\varepsilon(x), \iota_\varepsilon(x'))^2 = r_{(0)}(x,x')^2 + \sum_{m=2}^\infty \varepsilon^{m/2} r_{(2),g^Z_{(0)}}(m/2 : x, x') \qquad (4.53)$$
$$\text{with } r_{(2),g^Z_{(0)}}(m/2 : x, x') = \sum_{0 < |\alpha^b| + |\beta^b| \leq m+2}^{3 \leq |\alpha|} (x-x')^\alpha (x'^b)^{\beta^b} \cdot r^{\alpha,\beta^b}_{(2),g^Z_{(0)}}(m/2 : x^f, x'^f)$$

which comes from (3.9) by the same argument as in (4.3).

Now, let us take a parametrix of $\partial/\partial t + \not{\partial}^2_{g^Z_{(\varepsilon)}}$ as usual. That is, let φ be the cut-off function taken in the proof of Proposition 2.1 and set

$$\tilde{E}_{(N)}(t,\varepsilon,x,x') \equiv \varphi(r_{(\varepsilon)}(x,x')^2)\, q_{g^Z_{(\varepsilon)}}(t,x,x') \sum_{i=0}^N t^i\, E_i(\varepsilon,x,x') \qquad (4.54)$$
$$= \sum_{m \leq m_0} \varepsilon^{m/2} \tilde{E}_{(N)}(t, m/2 : x, x') + \varepsilon^{(m_0+1)/2} \tilde{E}_{(N)}(t, (m_0+1)/2 : \varepsilon^{1/2}, x, x')$$

$$\text{with } \tilde{E}_{(N)}(t, m/2 : \cdot, x, x') \equiv \sum (dx^b)^I(x) \otimes (dx^b)^J(x') \cdot \tilde{E}_{(N)}(t, m/2 : \cdot, x, x')_{(I,J)} \qquad (4.55)$$

where $N > 0$ is sufficiently large. Then, for given α, α', there exists a constant $C > 0$ satisfying

$$(4.56) \qquad |\partial^\alpha \partial^{\alpha'} \tilde{E}_{(N)}(t, m/2 : \cdot)|_{(0)} \le C\,(1+r(x'))^m t^{-(n+2+|\alpha|+|\alpha'|)/2+(1-\delta_{0m})/2} e^{-r(x,x')^2/5t}$$
$$(\, 0 \le \forall\varepsilon^{1/2} \le \varepsilon_0^{1/2} \,,\, 0 < \forall t \le T_0 \,,\, 0 \le \forall m \le m_0 + 1 \,,\, \forall (x, x') \in Z \times Z \,).$$

Moreover, if $N > (n+2)/2$, then $(\partial/\partial t + \not{\partial}^2_{g^Z_{(\varepsilon)}})\tilde{E}_{(N)}(t, \varepsilon, x, x')$ can be extended continuously to $[0, T_0] \times Z \times Z\,(\ni (t, x, x'))$ and there exists a constant $C > 0$ satisfying

$$(4.57) \qquad \Big| \int_Z \langle \tilde{E}_{(N)}(t, \varepsilon, x, x'), \psi(x') \rangle_{(\varepsilon)} dg^Z_{(\varepsilon)}(x') - \psi(x) \Big|_{(0)} \le C\, t^{1/2} \sup_{x' \in Z} |\varphi(r_{(\varepsilon)}(x, x')^2)\psi(x')|_{(0)}$$
$$(\, 0 \le \forall\varepsilon^{1/2} \le \varepsilon_0^{1/2} \,,\, 0 < \forall t \le T_0 \,,\, \forall (x, x') \in Z \times Z \,).$$

(4.56) comes from (4.49), (4.52) ($t^{(1-\delta_{0m})/2}$ appears in (4.56) because of the conditions "$a \ge 1$ if $m \ge 1$" and $3 \le |\alpha_i|$), and the expansion $\varphi(r_{(\varepsilon)}(x, x')^2) = \varphi(r_{(0)}(x, x')^2) + \sum_{m \ge 2}^{m_0} \varepsilon^{m/2} r_{(2), g^Z_{(0)}}$ $(m/2 : x, x')\, \varphi^{(m)}(r_{(0)}(x, x')^2) + \varepsilon^{(m_0+1)/2} \varphi((m_0 + 1)/2 : \varepsilon^{1/2}, x, x')$ (see (4.53)). (4.57) comes from the usual argument by referring to (4.49) with $m_0 = -1$ (that is, (4.47) is bounded on $Z \times Z$).

Here, taking care of the management of $(1+r(x'))^m$ appearing in (4.56), let us repeat the operation of convolution again and again as usual. We obtain then a fundamental solution. That is, we set

$$(L_{(N)}(,\cdot))^1(t, \varepsilon, x, x') = \big(\partial/\partial t + \not{\partial}^2_{g^Z_{(\varepsilon)}}\big)\tilde{E}_{(N)}(t, \varepsilon, x, x'),$$
$$(L_{(N)}(,\cdot))^q(t, \varepsilon, x, x') \equiv (L_{(N)}(,\cdot) \sharp_{(\cdot)} (L_{(N)}(,\cdot))^{q-1})(t, \varepsilon, x, x')$$
$$\equiv \int_0^t dt_1 \int_Z dg^Z_{(\varepsilon)}(Q^1)\, \langle L_{(N)}(t - t_1, \varepsilon, x, Q^1), (L_{(N)}(,\cdot))^{q-1}(t_1, \varepsilon, Q^1, x') \rangle_{(\varepsilon)},$$
$$\mathcal{L}_{(N)}(t, \varepsilon, x, x') \equiv \sum_{q=1}^{\infty} (-1)^q (L_{(N)}(,\cdot))^q(t, \varepsilon, x, x'),$$

$$(4.58) \qquad \Big(e^{-t\not{\partial}^2_{g^Z_{(\varepsilon)}}}\Big)_{(N)}(x, x') \equiv \tilde{E}_{(N)}(t, \varepsilon, x, x') + (\tilde{E}_{(N)}(,\cdot) \sharp_{(\cdot)} \mathcal{L}_{(N)}(,\cdot))(t, \varepsilon, x, x')$$
$$= \tilde{E}_{(N)}(t, \varepsilon, x, x') + \sum_{m \le m_0} \varepsilon^{m/2} (\tilde{E}_{(N)}(,\cdot) \sharp_{(\cdot)} \mathcal{L}_{(N)}(,\cdot))(t, m/2 : x, x')$$
$$+ \varepsilon^{(m_0+1)/2} (\tilde{E}_{(N)}(,\cdot) \sharp_{(\cdot)} \mathcal{L}_{(N)}(,\cdot))(t, (m_0 + 1)/2 : \varepsilon^{1/2}, x, x')$$
$$= \sum_{m=0}^{m_0} \varepsilon^{m/2} \Big(e^{-t\not{\partial}^2_{g^Z_{(m/2:)}}}\Big)_{(N)} + \varepsilon^{(m_0+1)/2} \Big(e^{-t\not{\partial}^2_{g^Z_{((m_0+1)/2:\varepsilon^{1/2})}}}\Big)_{(N)}.$$

We find out that, if $N > (n+2)/2 + N_0$, then $(\tilde{E}_{(N)}(,\cdot) \sharp_{(\cdot)} \mathcal{L}_{(N)}(,\cdot))(t, m/2 :)$ can be extended smoothly of class $(C^a, C^{|\alpha|}, C^{|\alpha'|})$ $((|\alpha| + |\alpha'|)/2 + a \le N_0)$ to $[0, T_0] \times Z \times Z\,(\ni (t, x, x'))$ and there exists a constant $C > 0$ satisfying

$$(4.59) \qquad |(\partial/\partial t)^a \partial^\alpha \partial^{\alpha'} (\tilde{E}_{(N)}(,\cdot) \sharp_{(\cdot)} \mathcal{L}_{(N)}(,\cdot))(t, m/2 : \cdot)|_{(0)}$$

$$\le C\,(1+r(x'))^m\, t^{N-(n+2+|\alpha|+|\alpha'|)/2-a}\, e^{-r(x,x')^2/5t}$$
$$(\,0 \le \forall\varepsilon^{1/2} \le \varepsilon_0^{1/2}\,,\ 0 < \forall t \le T_0\,,\ 0 \le \forall m \le m_0+1\,,\ \forall(x,x') \in Z\times Z\,).$$

Using the standard argument, (4.57) and (4.59) yield that, if $N > (n+2)/2+2$, then we have $e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}} = \left(e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}\right)_{(N)}$. That is, (4.59) is certainly equal to the C^∞-kernel of $e^{-t\partial\!\!\!/^2_{g^Z_{(\varepsilon)}}}$ and, now, (4.56) and (4.59) imply (2.24).

§5. Proofs of Theorems 1.1, 1.3

First let us prove Theorem 1.1.

Proof of Theorem 1.1. Clearly (3.24) implies (1.13). As for (1.14): (2.15), (2.33) and (2.28), (2.31) with $m_0 = -1$ imply

$$\left|\partial^\alpha\partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}}(x,0)\right|_{g^{Z(p^0)}} \le \frac{C}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}}\,|\partial^\alpha\partial^{\alpha'} K_{(0/2:\varepsilon^{1/2})}(t,\iota_\varepsilon^{-1}(x),0)|_{g^{Z(p^0)}}$$
$$\le \frac{C_1}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}}\left(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\right)\begin{cases} 1\ :\ (\text{with no condition}) \\ e^{-(r_{g^{Z(p^0)}}(\iota_\varepsilon^{-1}(x),0)-\bar r)^2/C_2 t}\ :\ r_{g^{Z(p^0)}}(\iota_\varepsilon^{-1}(x),0) > \bar r\end{cases}$$

and we have $r_{g^{Z(p^0)}}(\iota_\varepsilon^{-1}(x),0) = r_{g_\varepsilon^{Z(p^0)}}(P,P^0)$. Hence, observing (2.4), we obtain (1.14). As for (1.15): Here, using (2.32) with $m_0=-1$, similarly we have

$$\left|\partial^\alpha\partial^{\alpha'} e^{-t\partial\!\!\!/^2_{g_\varepsilon^{Z(p^0)}}}(x,0)\right|_{g^{Z(p^0)}} \le \frac{C\, e^{-r_{g^{Z(p^0)}}(\iota_\varepsilon^{-1}(x),0)^2/5t}}{\varepsilon^{(|\alpha^b|+|\alpha'^b|)/2}\, t^{(n+2+|\alpha|+|\alpha'|)/2}}.$$

Hence, again considering (2.4), we obtain (1.15).

Hereafter the purpose is to prove Theorem 1.3. Let us start with investigating $\partial\!\!\!/^{(\varepsilon)} \equiv 1_\varepsilon^{-1}\circ\partial\!\!\!/_{g^Z_{(\varepsilon)}}\circ 1_\varepsilon$, etc. where 1_ε is the bundle isomorphism given at (2.26).

Lemma 5.1.

(1) *Referring to (4.1)-(4.3), we have*

$$\text{(5.1)}\quad \begin{aligned} &1_\varepsilon^{-1}\circ\rho_{g^Z_{(\varepsilon)}}(e_b^i(\varepsilon,\iota_\varepsilon^*A))\circ 1_\varepsilon = \varepsilon^{-1/2}(e_b^i\wedge -\varepsilon e_b^i\vee) \equiv \varepsilon^{-1/2}\rho^{(\varepsilon)}(e_b^i),\\ &1_\varepsilon^{-1}\circ\rho_{g^Z_{(\varepsilon)}}(e_f^k(\varepsilon,\iota_\varepsilon^*A))\circ 1_\varepsilon = \rho_{g^Z}(e_f^k(A)) \equiv \rho^{(\varepsilon)}(e_f^k(A)),\end{aligned}$$

$$\text{(5.2)}\quad \nabla^{(\varepsilon)}_{e_i^b(\varepsilon,\iota_\varepsilon^*A)(x)} \equiv 1_\varepsilon^{-1}\circ\nabla^{\$_{g^Z_{(\varepsilon)}}}_{e_i^b(\varepsilon,\iota_\varepsilon^*A)(x)}\circ 1_\varepsilon$$
$$= e_i^b(\varepsilon,\iota_\varepsilon^*A)(x) + \frac{\varepsilon^{-1/2}}{4}\sum C(\nabla^{g^M})(e_i^b)_{i_2 i_1}(\iota_\varepsilon(x))\,\rho^{(\varepsilon)}(e_b^{i_1})\rho^{(\varepsilon)}(e_b^{i_2})$$

$$+\varepsilon^{1/2}\Big\{\nu_{\mathrm{u}}(A(e_i^b)) - \omega^{\mathbb{C}P}(\nu(A(e_i^b))^{\natural})\Big\}(\iota_\varepsilon(x))\,\rho^{(\varepsilon)}(\tau_g\nu(A)) + \frac{\varepsilon^{1/2}}{2}\rho^{(\varepsilon)}(\nu(e_b^i\vee F_A))(\iota_\varepsilon(x)),$$

$$(5.3)\quad \nabla^{(\varepsilon)}_{e_k^f(x)} \equiv 1_\varepsilon^{-1}\circ\nabla^{\$_{g^Z_{(\varepsilon)}}}_{e_k^f(x)}\circ 1_\varepsilon = e_k^f(x) + \frac{1}{2}\,\omega^{\mathbb{C}P}(e_k^f)(x^f)\,\rho^{(\varepsilon)}(\tau_g\nu(A)) + \frac{1}{2}\rho^{(\varepsilon)}(\nu^k(F_A))(\iota_\varepsilon(x)),$$

$$(5.4)\quad (\partial\!\!\!/^{(\varepsilon)})^2 = -\sum\Big(\nabla^{(\varepsilon)}_{e_i(\varepsilon,\iota_\varepsilon^* A)}\nabla^{(\varepsilon)}_{e_i(\varepsilon,\iota_\varepsilon^* A)} - \nabla^{(\varepsilon)}_{\nabla^{g^Z_{(\varepsilon)}}_{e_i(\varepsilon,\iota_\varepsilon^* A)}e_i(\varepsilon,\iota_\varepsilon^* A)}\Big) + \frac{\kappa_{g^Z_{(\varepsilon)}}}{4}$$

where we set $\tau_g\nu(A) = \sqrt{-1}\,e_f^1(A)\wedge e_f^2(A)$ *(the complex volume element),* $\rho^{(\varepsilon)}(\nu(F_A))(\iota_\varepsilon(x)) = \frac{1}{2}\sum\nu^k(F_A(e_i^b, e_j^b))(\iota_\varepsilon(x))\,\rho^{(\varepsilon)}(e_f^k(A))\,\rho^{(\varepsilon)}(e_b^i)\,\rho^{(\varepsilon)}(e_b^j)$*, etc.*

(2) $\nabla^{(\varepsilon)}_{e_i(\varepsilon,\iota_\varepsilon^* A)(x)}$, $(\partial\!\!\!/^{(\varepsilon)})^2$, *etc. are all* C^∞ *with respect to* $\varepsilon^{1/2}$ *near* $\varepsilon^{1/2}=0$ *and, referring to (1.20) and (1.23), we have*

$$(5.5)\qquad \lim_{\varepsilon\to 0}\nabla^{(\varepsilon)}_{e_i(\varepsilon,\iota_\varepsilon^* A)(x)} = \begin{cases} \dfrac{\partial}{\partial x_i^b} + \dfrac{1}{4}\sum x_j^b\,R^{g^M}_{ji}(p^0) \equiv \nabla^{(0)}_{\partial/\partial x_i^b}, \\ e_k^f(x) + \dfrac{1}{2}\,\omega^{\mathbb{C}P}(e_k^f)(x^f)\,\rho_{g^Z}(\tau_g\nu(A)) + \dfrac{1}{2}\nu^k(F_A)_{p^0} \equiv \nabla^{(0)}_{e_k^f(x)}, \\ \qquad (i = n+k) \end{cases}$$

$$(5.6)\qquad (\partial\!\!\!/^{(0)})^2 \equiv \lim_{\varepsilon\to 0}(\partial\!\!\!/^{(\varepsilon)})^2 = -\sum\Big(\nabla^{(0)}_{\partial/\partial x_i^b}\Big)^2 - \sum\Big(\nabla^{(0)}_{e_k^f}\nabla^{(0)}_{e_k^f} - \nabla^{(0)}_{\nabla^{g^V}_{e_k^f}e_k^f}\Big) + \frac{1}{2}$$
$$= -\sum\Big(\nabla^{(0)}_{\partial/\partial x_i^b}\Big)^2 + \mathcal{A}^2_{p^0} \equiv (\partial\!\!\!/^{(0)}_{g^M})^2 + \mathcal{A}^2_{p^0}.$$

Remark. Importantly $\partial\!\!\!/^{(\varepsilon)} = 1_\varepsilon^{-1}\circ\partial\!\!\!/_{g^Z_{(\varepsilon)}}\circ 1_\varepsilon = 1_\varepsilon^{-1}\circ\iota_\varepsilon^*\circ\partial\!\!\!/_{g^Z_\varepsilon}\circ(\iota_\varepsilon^*)^{-1}\circ 1_\varepsilon$ coincides with the Getzler transformation ([13], [5]) of $\partial\!\!\!/_{g^Z_\varepsilon}$. Our study up to now will assert that, to study the kernel $e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}$, our interpretation of the transformation is more appropriate than the original one which we review below: We have the global rescaling $P_{SO(n+2)}(Z, g^Z) \cong P_{SO(n+2)}(Z, g^Z_\varepsilon)$, $e_*(A)\mapsto e_*^\varepsilon(A)$, and it gives a pointwise isometry $b_\varepsilon : \$_{g^Z} \cong \$_{g^Z_\varepsilon}$ to which we referred at the comment following (1.12). Then $\partial\!\!\!/_{g^Z_\varepsilon}$ is transformed into

$$b_\varepsilon^{-1}\circ\partial\!\!\!/_{g^Z_\varepsilon}\circ b_\varepsilon = \varepsilon^{1/2}\sum\rho_{g^Z}(e_b^i)\nabla^{\$_{g^Z}\oplus}_{e_i^b(A)} + \sum\rho_{g^Z}(e_f^k(A))\nabla^{\$_{g^Z}\oplus}_{e_k^f} - \frac{\varepsilon}{2}\,\rho_{g^Z}(\nu(F_A))$$

acting on $\Gamma(\wedge T_0^* M\otimes_\pi \$_g\nu)\,(=\Gamma(\wedge T^*M\otimes_\pi \$_g\nu),\ (dx^b)^I\otimes h(x^b,x^f)\leftrightarrow e_b^I(x^b)\otimes h(x^b,x^f))$, (see [5, (4.26)], [12, (1.3)]) where $\nabla^{\$_{g^Z}\oplus}$ is the spinor connection associated to $\nabla^{g^Z\oplus}$ and we regard $\rho_{g^Z}(e_b^i)$ as $dx_i^b\wedge - dx_i^b\vee$. Further let $\mathcal{T}_\varepsilon$ be the transformation of $\wedge T_0^*M\otimes_\pi\$_g\nu$ given by $(dx^b)^I\otimes h(x^b,x^f)\mapsto\varepsilon^{-|I|/2}(dx^b)^I\otimes h(\varepsilon^{1/2}x^b, x^f)$. Then it is $\mathcal{T}_\varepsilon\circ(b_\varepsilon^{-1}\circ\partial\!\!\!/_{g^Z_\varepsilon}\circ b_\varepsilon)\circ\mathcal{T}_\varepsilon^{-1}$ that is the original definition of the Getzler transformation of $\partial\!\!\!/_{g^Z_\varepsilon}$ or $b_\varepsilon^{-1}\circ\partial\!\!\!/_{g^Z_\varepsilon}\circ b_\varepsilon$. The difference between the original one and ours thus lies in which we choose, the isometry b_ε or the inclusion (1.8).

Proof. As for (1): For $\psi = e_b^I\otimes h$ which belongs to the right hand side of (2.5), we have

$$1_\varepsilon^{-1}\circ e_b^i(\varepsilon)\wedge\circ 1_\varepsilon\,\psi(x) = 1_\varepsilon^{-1}\circ e_b^i(\varepsilon)\wedge(\varepsilon^{1/2}e_b(\varepsilon))^I\otimes h(x) = \varepsilon^{-1/2}e_b^i\wedge\psi(x),$$

(5.7)
$$1_\varepsilon^{-1}\circ e_b^i(\varepsilon)\vee\circ 1_\varepsilon\,\psi(x) = 1_\varepsilon^{-1}\circ e_b^i(\varepsilon)\vee(\varepsilon^{1/2}e_b(\varepsilon))^I\otimes h(x) = \varepsilon^{1/2}e_b^i\vee\psi(x).$$

Thus we get the formula in the first line of (5.1). Its second line is obvious. (4.2), (4.3) and (3.8) imply (5.2) and (5.3), which, together with (4.2), yield (5.4). As for (2): By an easy computation (see the expansion at (1.3)), we have

$$C(\nabla^{g^M})(e_i^b)_{i_2 i_1}(x) = g^M(e_{i_2}^b, \nabla_{e_i^b}^{g^M} e_{i_1}^b)(x) = \frac{1}{2}\sum x_j^b\, R_{i_2 i_1 j i}^{g^M}(0) + O(|x|^2), \tag{5.8}$$

which, with referring to (4.3), implies (2).

Now, let us regard (2.29) and the right hand side of (1.24) denoted $K_{(0)}(t,x,x') = \sum e_b^I(x)\cdot K_{(0)}(t,x,x')_I = \sum e_b^I(x)\cdot \det v^b(x')\cdot K_{MV}(t,x,x')_I$ as cross-sections of (4.4) canonically:

(5.9) $E^{(0)}(t,x,x') \equiv \sum e_b^I(x)\otimes e_b^J(x')\cdot K(t,|(I,J)|/2 : x,x')_{(I,J)}$ with
$$(E^{(0)}(t)\psi)(x) = \int_Z \langle E^{(0)}(t,x,x'),\psi(x')\rangle_{g^Z}\, dg^Z(x'),$$

(5.10) $E_{(0)}(t,x,x') \equiv \det v^b(x')\cdot E_{MV}(t,x,x') \equiv \sum e_b^I(x)\wedge e_b^J(x)\otimes e_b^J(x')\cdot K_{(0)}(t,x,x')_I$
$$\equiv \sum e_b^I(x)\otimes e_b^J(x')\cdot K_{(0)}(t,x,x')_{(I,J)}$$
$$\equiv \det v^b(x')\sum e_b^I(x)\otimes e_b^J(x')\cdot K_{MV}(t,x,x')_{(I,J)} \quad\text{with}$$
$$(E_{(0)}(t)\psi)(x) = \int_Z \langle E_{(0)}(t,x,x'),\psi(x')\rangle_{g^Z} dg^Z(x') = \int_Z \langle E_{MV}(t,x,x'),\psi(x')\rangle_{g^Z} dg^Z_{(0)}(x').$$

Then what Theorem 1.3 asserts is their coincidence. We wish to show it by the standard method, that is, by proving that each of them defines a (C^0) semi-group of the parabolic equation (0.3) for $(\not{\partial}^{(0)})^2$ and such a (C^0) semi-group exists uniquely. But, there is a serious obstruction to such a method. If $\varepsilon > 0$, then, with no obstruction, certainly the parabolic equation (0.3) for $(\not{\partial}^{(\varepsilon)})^2$ with $\psi_0 \in L^2\Gamma(\wedge T^*M\otimes_\pi \$_{g^V}$ with $g^Z)$ has a unique (C^0) semi-group with C^∞-kernel $e^{-t(\not{\partial}^{(\varepsilon)})^2} = E^{(\varepsilon)}(t,x,x') \equiv \sum e_b^I(x)\otimes e_b^J(x')\cdot E^{(\varepsilon)}(t,x,x')_{(I,J)}$ which is (2.27) regarded as a cross-section of (4.4), i.e.,

$$\Big(\frac{\partial}{\partial t} + (\not{\partial}^{(\varepsilon)})^2\Big)E^{(\varepsilon)}(t,x,x') = 0,\quad E^{(\varepsilon)}(t)\psi \to \psi \text{ in } L^2\ (t\to 0). \tag{5.11}$$

This comes from the property of $\not{\partial}^2_{g^Z_{(\varepsilon)}}$ through the transformation 1_ε (see (2.27) around), or more straightforwardly, from the fact that the coefficients of the expression of $(\not{\partial}^{(\varepsilon)})^2$ with $\varepsilon > 0$ (using the frame $e_b\otimes s(e^f)$) satisfy the condition (2.3). Now, unfortunately the coefficients of such a expression of $(\not{\partial}^{(0)})^2$ do not satisfy it (observe the existence of x_j^b at (5.5)). Or, more clearly, by observing (1.23) etc., (5.10) which is expected to be a (C^0) semi-group of the parabolic equation (0.3) for $(\not{\partial}^{(0)})^2$ may not transform $L^2\Gamma(\wedge T^*M\otimes_\pi \$_{g^V}$ with $g^Z)$ to itself, or what is worse, for some element ψ of it, the integral in the definition of $E_{(0)}(t)\psi$ may diverge. To overcome such a serious obstruction, it is the idea of using (not L^2-space but) the space of rapidly decreasing cross-sections that attracted the notice of the author.

That is, let us consider the Fréchet space

$$\mathcal{S} \equiv \{\psi \in \Gamma(\wedge T^*M \otimes_\pi \$_g \nu) \mid \lim_{r(P)\to\infty} |(1+r(P))^\ell \partial^\alpha \psi(P)|_{(0)} = 0 \ (\forall \ell, \forall \alpha)\} \tag{5.12}$$
$$\text{with semi-norms } p_{\ell,\alpha}(\psi) \equiv \sup_{P\in Z} |(1+r(P))^\ell \partial^\alpha \psi(P)|_{(0)}, \ \mathfrak{S} \equiv \{p_{\ell,\alpha}\}$$

where we put $r(P) \equiv r_{(0)}(P, P^0)$, and the parabolic equation with the initial condition

$$\Big(\frac{\partial}{\partial t} + (\partial\!\!\!/^{(0)})^2\Big)\psi = 0, \quad \psi\Big|_{t=0} = \psi_0 \in \mathcal{S}. \tag{5.13}$$

Then, first, we want to show that (5.9) defines its (C^0) semi-group with C^∞-kernel.

As a preliminary we have

Lemma 5.2. *Let $\mathfrak{S}^{(0)} = \{p^{(0)}_{\ell,k}\}$ be another family of semi-norms of $\mathcal{S}$ consisting of $p^{(0)}_{\ell,k}(\psi) \equiv \sup_{P\in Z}|(1 + r(P))^\ell (\partial\!\!\!/^{(0)})^{2k}\psi(P)|_{(0)}$. Then the two kinds of families $\mathfrak{S}$ and $\mathfrak{S}^{(0)}$ are equivalent to each other.*

Proof. It suffices to show that, for given ℓ, k and α, there exist ℓ_i, k_i, α_i $(1 \le i \le N)$ and a constant $C > 0$ satisfying

$$p^{(0)}_{\ell,k}(\psi) \le C\sum_{i=1}^{N} p_{\ell_i,\alpha_i}(\psi), \quad p_{\ell,\alpha}(\psi) \le C\sum_{j=1}^{N} p^{(0)}_{\ell_j,k_j}(\psi) \quad (\forall \psi \in \mathcal{S}). \tag{5.14}$$

The first inequality will be obvious. As for the second inequality: Set $R_{ji} = R^{g^M}_{ji}(p^0)$, $\psi = \sum e^I_b \otimes \psi_I$ and note that we have

$$\begin{aligned}(\partial\!\!\!/^{(0)})^2\psi &= \sum e^I_b\otimes \Big(-\sum_i(\partial/\partial x^b_i)^2 + \partial\!\!\!/^2_{g^\nu}\Big)\psi_I + \Big(-\frac{1}{2}\sum_{i,j,I} R_{ji} e^I_b \otimes x^b_j(\partial/\partial x^b_i)\psi_I \\ &\quad - \sum_{k,I}\nu^k(F_A) e^I_b\otimes \nabla^{\$_g \nu}_{e^I_k}\psi_I\Big) + \sum_I\Big(-\frac{1}{16}\sum_i\big(\sum_j R_{ji}x^b_j\big)^2 + \frac{1}{4}\rho^{(0)}(\nu(F_A))^2\Big)e^I_b\otimes\psi_I \\ &= \Big(-\sum_i(\partial/\partial x^b_i)^2 + \partial\!\!\!/^2_{g^\nu}\Big)\psi_\emptyset + \sum_{\|I\|=1} e^I_b\otimes\Big(-\sum_i(\partial/\partial x^b_i)^2 + \partial\!\!\!/^2_{g^\nu}\Big)\psi_I \\ &\quad + \Big\{-\frac{1}{2}\sum_{i,j} R_{ji}\otimes x^b_j(\partial/\partial x^b_i)\psi_\emptyset - \sum_k \nu^k(F_A)\otimes\nabla^{\$_g\nu}_{e^I_k}\psi_\emptyset \\ &\quad + \sum_{\|I\|=2} e^I_b\otimes\Big(-\sum_i(\partial/\partial x^b_i)^2 + \partial\!\!\!/^2_{g^\nu}\Big)\psi_I\Big\} + \Big\{-\frac{1}{2}\sum_{i,j,\|I\|=1} R_{ji}e^I_b\otimes x^b_j(\partial/\partial x^b_i)\psi_I \\ &\quad - \sum_{k,\|I\|=1}\nu^k(F_A)e^I_b\otimes\nabla^{\$_g\nu}_{e^I_k}\psi_I + \sum_{\|I\|=3} e^I_b\otimes\Big(-\sum_i(\partial/\partial x^b_i)^2 + \partial\!\!\!/^2_{g^\nu}\Big)\psi_I\Big\}\end{aligned} \tag{5.15}$$

$$+\Big\{-\frac{1}{16}\sum_i(\sum_j R_{ji}x_j^b)^2\otimes\psi_\emptyset+\frac{1}{4}\rho^{(0)}(\nu(F_A))^2\otimes\psi_\emptyset-\frac{1}{2}\sum_{i,j,\|I\|=2}R_{ji}e_b^I\otimes x_j^b(\partial/\partial x_i^b)\psi_I$$
$$-\sum_{k,\|I\|=2}\nu^k(F_A)e_b^I\otimes\nabla^{\$_g\nu}_{e_k^f}\psi_I+\sum_{\|I\|=4}e_b^I\otimes(-\sum_i(\partial/\partial x_i^b)^2+\partial\!\!\!/_{g\nu}^2)\psi_I\Big\}+\cdots$$
$$\equiv\sum_{\|I\|=0,1}e_b^I\otimes D_0\psi_I+\sum_{p\geq2}\sum_{\|I\|=p}e_b^I\otimes\Big\{\sum_{\|J\|=p-2}D(I,J)_p\psi_J+D_0\psi_I\Big\}$$

where we denote ψ_I with $|I|=0$ by $\psi_\emptyset$. Then it suffices to prove that, for given α and β, there exists a constant $C>0$ satisfying

$$|x^\alpha(\partial/\partial x)^\beta\psi_I(x)|_{(0)}\leq C\sum_{k=0}^{|\beta|}p^{(0)}_{N+|\alpha|,k}(\psi)\quad(\forall\psi\in\mathcal{S}).\tag{5.16}$$

(Note that the first inequality at (5.14) implies $p^{(0)}_{N+|\alpha|,k}(\psi)<\infty$.) Assuming that all of ψ_I are compactly supported on U^f ($\ni x^f$), let us prove it by the induction with respect to $|I|$. First, as for (5.16) with $|I|=0$ or 1, the standard elliptic estimates for the elliptic operator D_0 imply it, i.e.,

$$|x^\alpha(\partial/\partial x)^\beta\psi_I|_{(0)}\leq C\big\{p^{(0)}_{N+|\alpha|,0}(\psi_I)+p^{(0)}_{N+|\alpha|,0}(D_0^{|\beta|}\psi_I)\big\}$$
$$\leq C\big\{p^{(0)}_{N+|\alpha|,0}(\psi)+p^{(0)}_{N+|\alpha|,|\beta|}(\psi)\big\}.$$

As for (5.16) with $|I|=2$: Expand $(\partial\!\!\!/^{(0)})^k\psi$ as in (5.15). Then the coefficient of e_b^I ($|I|=2$) is equal to $\sum_{k_1+k_2=k-1}D_0^{k_1}D(I,\emptyset)_0D_0^{k_2}\psi_\emptyset+D_0^k\psi_I$ and we can apply (5.16) with $|I|=0$ to the part $\sum_{k_1+k_2=k-1}D_0^{k_1}D(I,\emptyset)_0D_0^{k_2}\psi_\emptyset$. Hence, we have

$$|x^\alpha(\partial/\partial x)^\beta\psi_I|_{(0)}\leq C_1\big\{p^{(0)}_{N+|\alpha|,0}(\psi_I)+p^{(0)}_{N+|\alpha|,0}(D_0^{|\beta|}\psi_I)\big\}$$
$$\leq C_1\big\{p^{(0)}_{N+|\alpha|,0}(\psi_I)+p^{(0)}_{N+|\alpha|,0}(\sum_{k_1+k_2=|\beta|-1}D_0^{k_1}D(I,\emptyset)_0D_0^{k_2}\psi_\emptyset+D_0^{|\beta|}\psi_I)$$
$$+p^{(0)}_{N+|\alpha|,0}(\sum_{k_1+k_2=|\beta|-1}D_0^{k_1}D(I,\emptyset)_0D_0^{k_2}\psi_\emptyset)\big\}$$
$$\leq C_1\big\{p^{(0)}_{N+|\alpha|,0}(\psi)+p^{(0)}_{N+|\alpha|,|\beta|}(\psi)+\sum_{k=0}^{|\beta|}p^{(0)}_{N_1+|\alpha|,k}(\psi)\big\}.$$

Thus we have proved (5.16) with $|I|=2$. In this way (5.16) is shown inductively.

Then, as is desired, we have

Lemma 5.3. *$\{E^{(0)}(t)\}_{0<t<\infty}$ defines a (C^0) semi-group with C^∞-kernel associated to the parabolic equation (5.13). That is, we have*

$$\Big(\frac{\partial}{\partial t}+(\partial\!\!\!/^{(0)})^2\Big)E^{(0)}(t,x,x')=0,\tag{5.17}$$

and $E^{(0)}(t)$ *gives a continuous linear map from* $\mathcal{S}$ *to itself, and, for given* $T_0 > 0$, $\ell \geq 0$ *and* α, *there exist* $C > 0$, $\ell_0 > 0$, $\beta_1, \cdots, \beta_N$ *satisfying*

$$p_{\ell,\alpha}(E^{(0)}(t)\psi - \psi) \leq C\, t^{1/2} \sum p_{\ell+\ell_0,\beta_i}(\psi) \quad (0 < \forall t \leq T_0,\ \forall \psi \in \mathcal{S}), \tag{5.18}$$

and, moreover, we have

$$E^{(0)}(t_1)E^{(0)}(t_2) = E^{(0)}(t_1 + t_2) \quad (0 \leq \forall t_1, \forall t_2). \tag{5.19}$$

Further, the semi-group is equicontinuous, that is, for given $\ell \geq 0$ *and* α, *there exist* $C > 0$, $\ell_0 > 0$, $\beta_1, \cdots, \beta_N$ *satisfying*

$$p_{\ell,\alpha}(E^{(0)}(t)\psi) \leq C \sum p_{\ell+\ell_0,\beta_i}(\psi) \quad (0 < \forall t < \infty,\ \forall \psi \in \mathcal{S}). \tag{5.20}$$

Proof. Take $\psi \in \mathcal{S}$. (2.30) implies

$$\begin{aligned}
&|\int_Z \langle (1+r(x))^\ell \partial_x^\alpha E^{(0)}(t,x,x'), \psi(x')\rangle_{g^Z}\, dg^Z(x')|_{(0)} \\
&\leq C_1 \int_Z \langle (1+r(x))^\ell |\partial_x^\alpha E^{(0)}(t,x,x')|_{(0)} |\psi(x')|_{(0)}\, dg^Z(x') \\
&\leq C_2 (t^{-(n+2+|\alpha|)/2} + 1)\, e^{-t\lambda_0} \Big\{ \int_{r(x,x') \leq 2\bar{r}} (1+r(x))^\ell (1+r(x'))^n |\psi(x')|_{(0)} dg^Z_{(0)}(x') \\
&+ \int_{r(x,x') \geq 2\bar{r}} (1+r(x))^\ell (1+r(x'))^n e^{-(r(x,x')-\bar{r})^2/Ct} |\psi(x')|_{(0)}\, dg^Z_{(0)}(x') \Big\} \\
&\leq C_3 (t^{-(n+2+|\alpha|)/2} + 1)\, e^{-t\lambda_0} \Big\{ \int_{r(x,x') \leq 2\bar{r}} r(x,x')^\ell (1+r(x'))^{\ell+n} |\psi(x')|_{(0)} dg^Z_{(0)}(x') \\
&+ \int_{r(x,x') \geq 2\bar{r}} (\bar{r}+1+r(x'))^{\ell+n} (r(x,x') - \bar{r})^\ell e^{-(r(x,x')-\bar{r})^2/Ct} |\psi(x')|_{(0)}\, dg^Z_{(0)}(x') \Big\} \\
&\leq C_4 (t^{-(n+2+|\alpha|)/2} + 1)\, e^{-t\lambda_0} \Big\{ (2\bar{r})^{\ell+n+2} p_{\ell+n,0}(\psi) + t^{\ell/2} p_{\ell+n,0}(\psi) \Big\}.
\end{aligned} \tag{5.21}$$

That is, certainly $E^{(0)}(t)\psi$ belongs to $\mathcal{S}$ and the map $E^{(0)}(t)$ is continuous. Consider then the Taylor expansions of the first equality at (5.11) and of $E^{(\epsilon)}(t_1)E^{(\epsilon)}(t_2) = E^{(\epsilon)}(t_1 + t_2)$. Their constant terms yield (5.17) and (5.19). Let us show the remained assertions below. First, we have

$$(\partial\!\!\!/^{(0)})^2 (E^{(0)}(t)\psi)(x) = (E^{(0)}(t)\, (\partial\!\!\!/^{(0)})^2 \psi)(x). \tag{5.22}$$

Actually, if $\epsilon^{1/2} > 0$, then we have

$$\begin{aligned}
&(\partial\!\!\!/^{(\epsilon)})^2 (E^{(\epsilon)}(t)\psi)(x) = \int_Z \langle (\partial\!\!\!/^{(\epsilon)})^2 E^{(\epsilon)}(t,x,x'), \psi(x')\rangle_{g^Z}\, dg^Z(x') \\
&= 1_\epsilon^{-1} \int_Z \langle (\partial\!\!\!/_{g^Z_{(\epsilon)}})^2 E(t,\epsilon,x,x'), 1_\epsilon \psi(x')\rangle_{(\epsilon)} dg^Z_{(\epsilon)}(x') \\
&= 1_\epsilon^{-1} \int_Z \langle E(t,\epsilon,x,x'), (\partial\!\!\!/_{g^Z_{(\epsilon)}})^2 1_\epsilon \psi(x')\rangle_{(\epsilon)} dg^Z_{(\epsilon)}(x') \\
&= 1_\epsilon^{-1} \int_Z \langle E(t,\epsilon,x,x'), 1_\epsilon (\partial\!\!\!/^{(\epsilon)})^2 \psi(x')\rangle_{(\epsilon)} dg^Z_{(\epsilon)}(x') = (E^{(\epsilon)}(t)\, (\partial\!\!\!/^{(\epsilon)})^2 \psi)(x).
\end{aligned}$$

Hence, by taking the limit, we have (5.22). Next, we want to show that, for given $\ell \geq 0$, there exists an integer $N > 0$ satisfying

$$(5.23) \qquad |(1+r(x))^{\ell}\Big((E^{(0)}(t)\psi)(x)-\psi(x)\Big)|_{(0)} \leq C\, t^{1/2}\, p^{(0)}_{\ell+N,0}(\psi) \quad (\forall \psi \in \mathcal{S}).$$

As for the case $\ell = 0$: Referring to (2.22), (2.18), (4.57) and (4.59), we have

$$\begin{aligned}
&|(E^{(\varepsilon)}(t)\psi)(x)-\psi(x)|_{(0)} = \sum_I |\sum_J \int_Z \langle E^{(\varepsilon)}(t,\varepsilon,x,x')_{(I,J)}, \psi_J(x')\rangle_{\$_g\mathcal{V}} dg^Z(x') - \psi_I(x)|_{(0)} \\
&\leq \sum |\Big(\sum_J \int_Z \langle K(t,|(I,J)|/2 : x,x')_{(I,J)}, \psi_J(x')\rangle_{\$_g\mathcal{V}} dg^Z(x') - \psi_I(x)\Big)|_{(0)} + C_1 \varepsilon^{1/2} t^{1/2} p_{N,0}(\psi) \\
&\leq \sum |\Big(\int_Z \langle K(t,0/2 : x,x')_{(I,I)}, \psi_J(x')\rangle_{\$_g\mathcal{V}} dg^Z(x') - \psi_I(x)\Big)|_{(0)} \\
&\quad + C_2\, t^{1/2} p_{m_0+1,0}(\psi) + C_1\, \varepsilon^{1/2}\, t^{1/2} p_{m_0+1,0}(\psi) \\
&\leq C_3\, t^{1/2} |\psi|_{(0)} + C_2\, t^{1/2} p_{N,0}(\psi) + C_1\, \varepsilon^{1/2}\, t^{1/2} p_{N,0}(\psi) \leq C\, t^{1/2} p_{N,0}(\psi).
\end{aligned}$$

(This is an estimate stronger than the second part of (5.11).) Hence, again taking the limit, we obtain (5.23) with $\ell = 0$. As for the case $\ell > 0$: we have

$$\begin{aligned}
&|(1+r(x))^{\ell}\Big((E^{(0)}(t)\psi)(x)-\psi(x)\Big)|_{(0)} \leq |(E^{(0)}(t)(1+r(\cdot))^{\ell}\psi)(x) - (1+r(x))^{\ell}\psi(x)|_{(0)} \\
&\quad + |(1+r(x))^{\ell}(E^{(0)}(t)\psi)(x) - (E^{(0)}(t)(1+r(\cdot))^{\ell}\psi)(x)|_{(0)} \\
&\leq C\, t^{1/2}\, p^{(0)}_{\ell+N,0}(\psi) + |(1+r(x))^{\ell}(E^{(0)}(t)\psi)(x) - (E^{(0)}(t)(1+r(\cdot))^{\ell}\psi)(x)|_{(0)}
\end{aligned}$$

and, referring to (2.32), we have

$$\begin{aligned}
&|(1+r(x))^{\ell}(E^{(0)}(t)\psi)(x) - (E^{(0)}(t)(1+r(\cdot))^{\ell}\psi)(x)|_{(0)} \\
&= |\int_Z \langle((1+r(x))^{\ell} - (1+r(x'))^{\ell})E^{(0)}(t,x,x'), \psi(x')\rangle_{(0)}|_{(0)} \\
&\leq C_1 \sum_{\ell_1+\ell_2=\ell}^{\ell_1>0} \int_Z |(r(x)-r(x'))^{\ell_1}(1+r(x'))^{\ell_2} E^{(0)}(t,x,x')|_{(0)} |\psi(x')|_{(0)} dg^Z_{(0)}(x') \\
&\leq C_1 \sum \int_Z |r(x,x')^{\ell_1}(1+r(x'))^{\ell_2} E^{(0)}(t,x,x')|_{(0)} |\psi(x')|_{(0)} dg^Z_{(0)}(x') \\
&\leq C_2 \sum \int_Z r(x,x')^{\ell_1}(1+r(x'))^{\ell_2+n} t^{-(n+2)/2}\, e^{-r(x,x')^2/5t} |\psi(x')|_{(0)} dg^Z_{(0)}(x') \\
&\leq C_3 \sum p^{(0)}_{\ell_2+n,0}(\psi) \int_Z t^{-(n+2)/2+\ell_1/2}\, e^{-r(x,x')^2/6t} dg^Z_{(0)}(x') \leq C_4\, t^{1/2}\, p^{(0)}_{\ell+n,0}(\psi).
\end{aligned}$$

Thus we obtained (5.23). Then (5.22), (5.23) imply the inequality $p^{(0)}_{\ell,k}((E^{(0)}(t)\psi) - \psi) \leq C\, t^{1/2} p^{(0)}_{\ell+n,k}(\psi)$, and, using Lemma 5.2, we obtain (5.18). Last, as for the equicontinuity, in the case t small (5.18) implies it, and in the case t large (5.21) implies it.

Now let us prove Theorem 1.3.

Proof of Theorem 1.3. We intend to prove that $E^{(0)}(t)$ coincides with $E_{(0)}(t)$. First let us show that $E_{(0)}(t,x,x')$ also satisfies the conditions from (5.17) to (5.18). We regard $K_M(t,x^b,x'^b)$ as a cross-section of $\wedge T^*M \boxtimes \wedge T^*M$ (see (4.4)) canonically

$$E_M(t,x^b,x'^b) = \sum e_b^I(x^b) \wedge e_b^J(x^b) \otimes e_b^J(x'^b) \cdot K_M(t,x^b,x'^b)_I \tag{5.24}$$
$$= \sum e_b^I(x^b) \otimes e_b^J(x'^b) \cdot K_M(t,x^b,x'^b)_{(I,J)}.$$

And, set $\mathcal{S}_M = \{\varphi \in \Gamma(\wedge T^*M) \mid \lim_{|x^b|\to\infty} |(1+|x^b|)^\ell \partial_{x^b}^\alpha \varphi(x^b)|_{(0)} = 0 \ (\forall \ell, \forall \alpha)\}$ with semi-norms $q_{\ell,\alpha}(\varphi) \equiv \sup_{x^b\in M} |(1+|x^b|)^\ell \partial_{x^b}^\alpha \varphi(x^b)|_{(0)}$. Since the operators $(\not\partial_{g^M}^{(0)})^2$ and $\mathcal{A}_{p^0}^2$ are commutative, it suffices to prove the following: First we have

$$\Big(\frac{\partial}{\partial t} + (\not\partial_{g^M}^{(0)})^2\Big) E_M(t,x^b,x'^b) = 0, \tag{5.25}$$

and, if we set, for $\varphi = \sum e_b^J(x'^b) \cdot \varphi_J(x'^b) \in \mathcal{S}_M$,

$$(E_M(t)\varphi)(x^b) = \int_Z \langle E_M(t,x^b,x'^b), \varphi(x'^b)\rangle_{g^M}\, dg_{(0)}^M(x'^b) \tag{5.26}$$
$$= \sum e_b^I(x^b) \cdot \int_Z K_M(t,x^b,x'^b)_{(I,J)}\, \varphi_J(x'^b)\, dg_{(0)}^M(x'^b) \quad \text{(see (5.10))},$$

then $E_M(t)$ defines a continuous linear map from $\mathcal{S}_M$ to itself, and, last, for given $\ell \geq 0$, α and $T_0 > 0$, there exist $C > 0$, $\ell_0 > 0$, $\beta_1, \cdots, \beta_N$ satisfying

$$q_{\ell,\alpha}(E_M(t)\varphi - \varphi) \leq C\, t^{1/2} \sum q_{\ell+\ell_0,\beta_i}(\varphi) \quad (0 < \forall t \leq T_0,\ \forall \varphi \in \mathcal{S}_M). \tag{5.27}$$

Let us show these with setting $y = x^b$, etc. As for (5.25): As was mentioned in the comment following (1.23), it is Getzler ([13], [3, §4.2]) who showed (5.24) with $y' = x'^b = 0$ satisfies (5.25). Set $R = R^{g^M}(p^0)$ and $Y = y - y'$, then, using his result, we have

$$(\not\partial_{g^M}^{(0)})^2 = -\sum \Big(\frac{\partial}{\partial y_i} + \frac14 \sum y_j R_{ji}\Big)^2 = -\sum \Big\{\Big(\frac{\partial}{\partial Y_i} + \frac14 \sum y'_j R_{ji}\Big) + \frac14 \sum Y_j R_{ji}\Big\}^2,$$
$$e^{-\langle y|R|y'\rangle/4} \circ (\not\partial_{g^M}^{(0)})^2 \circ e^{\langle y|R|y'\rangle/4} = -\sum \Big(\frac{\partial}{\partial Y_i} + \frac14 \sum Y_j R_{ji}\Big)^2,$$
$$\Big(\frac{\partial}{\partial t} + (\not\partial_{g^M}^{(0)})^2\Big) E_M(t,y,y')$$
$$= e^{\langle y|R|y'\rangle/4} \Big(\frac{\partial}{\partial t} - \sum \Big(\frac{\partial}{\partial Y_i} + \frac14 \sum Y_j R_{ji}\Big)^2\Big) e^{-\langle y|R|y'\rangle/4} E_M(t,y,y')$$
$$= e^{\langle y|R|y'\rangle/4} \Big(\frac{\partial}{\partial t} - \sum \Big(\frac{\partial}{\partial Y_i} + \frac14 \sum Y_j R_{ji}\Big)^2\Big) E_M(t,Y,0) = 0.$$

That is, certainly (5.25) for general $y' = x^b$ holds. The continuity of the operator $E_M(t)$ can be shown similarly to that of $E^{(0)}(t)$. As for (5.27): First we have

$$(\partial/\partial y)^\alpha (E_M(t)\varphi)(y) - (E_M(t)(\partial/\partial y')^\alpha \varphi)(y) \tag{5.28}$$
$$= \sum_{\alpha=\beta+\gamma}^{\beta>0} \int_Z \sum_{\beta'(>0)} \langle (y-y')^{\beta'} (\tfrac14 R)_{\beta'\beta}\, E_M(t,y,y'), (\partial/\partial y')^\gamma \varphi(y')\rangle_{g^M}\, dg_{(0)}^M(y')$$

Actually, since we have $(\partial/\partial y_i)E_M(t,y,y') = (-(\partial/\partial y_i') + \frac{1}{4}\sum(y_j - y_j')R_{ji})E_M(t,y,y')$, we obtain (5.28) referring to the definition (5.26) of $E_M(t)\varphi$. Then, since there is a term $(y-y')^{\beta'}$ ($|\beta'| > 0$) in its right hand side, the above can be estimated as

$$|(\partial/\partial y)^\alpha (E_M(t)\varphi)(y) - (E_M(t)(\partial/\partial y')^\alpha \varphi)(y)|_{(0)} \le C\, t^{1/2} \sum_{|\gamma|<|\alpha|} q_{n,\gamma}(\varphi). \tag{5.29}$$

On the other hand, similarly to (5.23), we have

$$|(1+|y|)^\ell (E_M(t)\varphi)(y) - \varphi(y))|_{(0)} \le C\, t^{1/2}\, q_{\ell+N,0}(\varphi). \tag{5.30}$$

Hence we have obtained (5.27).

We have thus showed that $E_{(0)}(t,x,x')$ also satisfies the conditions from (5.17) to (5.18). We do not ask here whether (5.19) for $E_{(0)}(t,x,x')$ holds or not. (Note that once the proof is finished, consequently we know it holds.) However it is obviously continuous with respect to t, that is, there exists a constant $C > 0$ and an integer $\ell_0 > 0$ satisfying

$$\begin{aligned} p_{0,\emptyset}(E_{(0)}(t)\psi - E_{(0)}(t+s)\psi) &\le s(t^{-2} + t^n)\, C\, p_{\ell_0,\emptyset}(\psi) \\ &(0 < \forall t < \infty,\ 0 \le \forall s < t/2,\ \forall \psi \in \mathcal{S}). \end{aligned} \tag{5.31}$$

This weak estimate is enough for our purpose. Now let us show $E^{(0)}(t,x,x') = E_{(0)}(t,x,x')$. First, for given $0 < t_1 < t_2 < t$ and $\psi \in \mathcal{S}$, we have

$$\begin{aligned} &\int_Z\int_Z \langle E^{(0)}(t_2,x,x'), \langle E_{(0)}(t-t_2,x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \\ &= \int_Z\int_Z \langle E^{(0)}(t_1,x,x'), \langle E_{(0)}(t-t_1,x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x''). \end{aligned} \tag{5.32}$$

Actually, by referring to (5.17) for $E^{(0)}$ and $E_{(0)}$ and moreover (5.22), we find out that the difference between the right and left hand sides is equal to

$$\begin{aligned} &\int_{t_1}^{t_2} dt' \frac{\partial}{\partial t'} \int_Z \langle E^{(0)}(t',x,x'), \langle E_{(0)}(t-t',x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \\ &= \int_{t_1}^{t_2} dt' \Big\{ -\int_Z \langle (\not{\partial}^{(0)})^2 E^{(0)}(t',x,x'), \langle E_{(0)}(t-t',x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \\ &+ \int_Z \langle E^{(0)}(t',x,x'), \langle (\not{\partial}^{(0)})^2 E_{(0)}(t-t',x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \Big\} \\ &= \int_{t_1}^{t_2} dt' \Big\{ -\int_Z \langle E^{(0)}(t',x,x'), \langle (\not{\partial}^{(0)})^2 E_{(0)}(t-t',x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \\ &+ \int_Z \langle E^{(0)}(t',x,x'), \langle (\not{\partial}^{(0)})^2 E_{(0)}(t-t',x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \Big\} = 0. \end{aligned}$$

And we have

$$
\begin{aligned}
&|\int_Z\int_Z \langle E^{(0)}(t_2,x,x'), \langle E_{(0)}(t-t_2,x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \\
&\qquad\qquad - \int_Z \langle E^{(0)}(t,x,x'), \psi(x')\rangle_{g^Z,x'}\, dg^Z(x')|_{(0)} \\
&\leq |(E^{(0)}(t_2)(E_{(0)}(t-t_2)\psi-\psi))(x)|_{(0)} + |(E^{(0)}(t_2)(E^{(0)}(t-t_2)\psi-\psi))(x)|_{(0)} \text{ (by (5.19))} \\
&\leq C\sum\{p_{\ell_0,\beta_i}(E_{(0)}(t-t_2)\psi-\psi) + p_{\ell_0,\beta_i}(E^{(0)}(t-t_2)\psi-\psi)\} \text{ (by (5.18) for } E^{(0)}) \\
&\leq C(t-t_2)^{1/2}\sum p_{\ell_0',\gamma_j}(\psi) \to 0 \ \ (t_2\to t) \ \ \text{(by (5.18) for } E^{(0)} \text{ and } E_{(0)}), \\
&|\int_Z\int_Z \langle E^{(0)}(t_1,x,x'), \langle E_{(0)}(t-t_1,x',x''), \psi(x'')\rangle_{g^Z,x''}\rangle_{g^Z,x'}\, dg^Z(x')\, dg^Z(x'') \\
&\qquad\qquad - \int_Z \langle E_{(0)}(t,x,x'), \psi(x')\rangle_{g^Z,x'}\, dg^Z(x')|_{(0)} \\
&\leq |(E^{(0)}(t_1)E_{(0)}(t-t_1)\psi - E_{(0)}(t-t_1)\psi)(x)|_{(0)} + |(E^{(0)}(t-t_1)\psi - E_{(0)}(t)\psi)(x)|_{(0)} \\
&\leq C\sum t_1^{1/2} p_{\ell_0,\beta_i}(E_{(0)}(t-t_1)\psi) + t_1(t^{-2}+t^n)\,C\,p_{\ell_0,\emptyset}(\psi) \ \ \text{(by (5.18), (5.31))} \\
&\leq C\sum t_1^{1/2} p_{\ell_0',\gamma_j}(\psi) + t_1(t^{-2}+t^n)\,C\,p_{\ell_0,\emptyset}(\psi) \to 0 \ \ (t_1\to 0)
\end{aligned}
$$

Hence, for any $\psi\in\mathcal{S}$, $0<t<\infty$ and $x\in Z$, we have

$$
\int_Z \langle E^{(0)}(t,x,x'), \psi(x')\rangle_{g^Z}\, dg^Z(x') = \int_Z \langle E_{(0)}(t,x,x'), \psi(x')\rangle_{g^Z}\, dg^Z(x'). \tag{5.33}
$$

Thus we have proved $E^{(0)}(t,x,x') = E_{(0)}(t,x,x')$.

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On the trace and the infinitesimally deformed chiral anomaly of Dirac operators on twistor spaces and the change of metrics on the base spaces

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Abstract. We show that the trace of a quotient of two Dirac operators and the infinitesimally deformed chiral anomaly of a Dirac operator on a twistor space have adiabatic series expansions. Further their top terms can be explicitly described.

0 Introduction

Let $M = (M, g^M)$ be an even dimensional compact oriented Riemannian manifold equipped with a Spin^q structure introduced in [8]

$$\Xi^q : P_{\mathrm{Spin}^q(n)}(M) = P_{\mathrm{Spin}(n)}(M) \times_{\mathbf{Z}_2} P_{Sp(1)} \to P_{SO(n)}(M) \times P_{SO(3)} \tag{0.1}$$

where $P_{SO(n)}(M)$ ($n = \dim M$) is the reduced structure bundle consisting of $SO(n)$-frames of TM and $P_{SO(3)}$, $P_{\mathrm{Spin}^q(n)}(M)$ are some principal bundles with structure groups $SO(3)$, $\mathrm{Spin}^q(n) \equiv \mathrm{Spin}(n) \times_{\mathbf{Z}_2} Sp(1)$, respectively. Remark that $P_{\mathrm{Spin}(n)}(M)$, $P_{Sp(1)}$ are locally defined bundles and the bundle map Ξ^q is assumed to be equivariant to the canonical Lie group homomorphism $\Xi^q = (\Xi, \mathrm{Ad}) : \mathrm{Spin}^q(n) \to SO(n) \times SO(3)$. Then, using the canonical action of $\mathrm{Spin}^q(n)$ or $Sp(1)$ on $\mathrm{Spin}^q(n)/\mathrm{Spin}^c(n) = Sp(1)/U(1)$ and the identification $Sp(1)/U(1) = \mathbb{C}P^1$ through the representation $r_H : Sp(1) \to GL_{\mathbb{C}}(H) = GL_{\mathbb{C}}(\mathbb{C}^2)$ with $r_H(\alpha + j\beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, we get a $\mathbb{C}P^1$-fibration

$$\pi : Z = P_{\mathrm{Spin}^q(n)}(M) \times_{\mathrm{can}} \mathbb{C}P^1 = P_{Sp(1)} \times_{\mathrm{can}} \mathbb{C}P^1 \to M. \tag{0.2}$$

Let us now take an $Sp(1)$-connection A of $P_{Sp(1)}$, so that the **twistor space** Z possesses a canonical Spin structure ([9], [10]). Namely, the connection induces a splitting of TZ into horizontal and vertical components, $TZ = \mathcal{H} \oplus \mathcal{V}$, with natural orientation and with the metric $g^Z = \pi^* g^M + g^{\mathcal{V}}$ ($\pi^* g^M = g^Z|\mathcal{H}$) where $g^{\mathcal{V}}$ is the Riemannian metric on $\mathcal{V}$ induced from the Fubini-Study one of $\mathbb{C}P^1$. Further we have the locally defined spinor bundle $\not{S}_{g^M}$ associated to $P_{\mathrm{Spin}(n)}(M)$ and a locally defined hermitian vector bundle $\not{H} = P_{Sp(1)} \times_{r_H} H$, which together produce the globally defined vector bundle $\pi^*\not{S}_{g^M} \otimes \pi^*\not{H} \equiv \pi^*\not{S}_{g^M} \otimes \not{S}_{g^{\mathcal{V}}} \equiv \not{S}_{g^Z}$ on Z, whose rank is certainly equal to $2^{n/2+1}$. Then, the locally defined Clifford action ρ_{g^M} of $\mathbb{C}l(T^*M, g^M)$ on $\not{S}_{g^M}$, together with the action $\rho_{g^{\mathcal{V}}}$ of $\mathbb{C}l(\mathcal{V}^*, g^{\mathcal{V}})$ on $\not{S}_{g^{\mathcal{V}}}$ induced from the fiberwise globally defined canonical Spin structure, gives the globally defined action ρ_{g^Z} of $\mathbb{C}l(T^*Z, g^Z)$ on $\not{S}_{g^Z}$, i.e., $\rho_{g^Z}(\pi^*\xi_b) = \pi^*\rho_{g^M}(\xi_b) \otimes 1$ ($\xi_b \in T^*M$) and $\rho_{g^Z}(\xi_f) = \pi^*\rho_{g^M}(\tau_{g^M}) \otimes \rho_{g^{\mathcal{V}}}(\xi_f)$ ($\xi_f \in \mathcal{V}^*$) where τ_{g^M} is the complex volume element of (M, g^M). Thus (Z, g^Z) has a canonical Spin structure, which gives the Dirac operator $\not{\partial}^{(\pm)}_{g^Z} : \Gamma(\not{S}^{(\pm)}_{g^Z}) \to \Gamma(\not{S}^{(\mp)}_{g^Z})$. Note that the canonical splittings $\not{S}_{g^M} = \not{S}^+_{g^M} \oplus \not{S}^-_{g^M}$, $\not{S}_{g^{\mathcal{V}}} = \pi^*\not{H} = \not{S}^+_{g^{\mathcal{V}}} \oplus \not{S}^-_{g^{\mathcal{V}}} = \{([v], cv) \in \pi^*\not{H}\} \oplus (\not{S}^+_{g^{\mathcal{V}}})^\perp$ where $(\not{S}^+_{g^{\mathcal{V}}})^\perp$ is the orthogonal complement induce the splitting $\not{S}_{g^Z} = \not{S}^+_{g^Z} \oplus \not{S}^-_{g^Z}$ canonically.

Now, let us take another metric h^M on M and an associated Spin^q structure with the same $P_{SO(3)}$ as in (0.1), whose twistor space is hence equal to the one given at (0.2). We have thus another Spin structure for Z with metric $h^Z = \pi^* h^M + g^{\mathcal{V}}$, which induces another Dirac operator $\not{\partial}^{(\pm)}_{h^Z} : \Gamma(\not{S}^{(\pm)}_{h^Z}) \to \Gamma(\not{S}^{(\mp)}_{h^Z})$. Let us consider then the invariants called **the traces of the quotient** $\not{\partial}_{h^Z}/\not{\partial}_{g^Z}$

$$\text{(0.3)} \qquad \mathrm{Tr}_\pm(\not{\partial}_{h^Z}/\not{\partial}_{g^Z}) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s \, \mathrm{Tr}_\pm\Big(\not{\partial}_{g^Z}\, \not{\partial}_{h^Z}\, e^{-t\not{\partial}^2_{g^Z}}\Big)\, dt,$$

$$\text{with the equalities } \mathrm{Tr}_\pm\Big(\not{\partial}_{g^Z}\, \not{\partial}_{h^Z}\, e^{-t\not{\partial}^2_{g^Z}}\Big) = \mathrm{Tr}_\mp\Big(\not{\partial}_{h^Z}\, \not{\partial}_{g^Z}\, e^{-t\not{\partial}^2_{g^Z}}\Big).$$

(The equalities at the second line will be shown at (2.1).) Remark that $e^{-t\not{\partial}^2_{g^Z}}$ is a cross-section of the vector bundle $\not{S}^{(\pm)}_{g^Z} \boxtimes \not{S}^{(\pm)*}_{g^Z}$ over $Z \times Z$, on which the operator $\not{\partial}_{h^Z}$ cannot act in a naive sense. In the paper we will let $\not{\partial}_{h^Z}$ act on it (see (1.6)) by using the method introduced by Bourguignon and Gauduchon ([4], [5]), the explanation for which will be offered at the beginning of the next section. The first purpose is then to study the adiabatic series expansions of (0.3) and the difference $\mathrm{STr}(\not{\partial}_{h^Z}/\not{\partial}_{g^Z}) = \mathrm{Tr}_+(\not{\partial}_{h^Z}/\not{\partial}_{g^Z}) - \mathrm{Tr}_-(\not{\partial}_{h^Z}/\not{\partial}_{g^Z})$. Namely, by replacing the metrics g^Z etc. by $g^Z_\varepsilon = \varepsilon^{-1}\pi^* g^M + g^{\mathcal{V}} = \pi^* g^M_\varepsilon + g^{\mathcal{V}}$ ($\varepsilon > 0$) etc., we obtain $\mathrm{Tr}_\pm(\not{\partial}_{h^Z_\varepsilon}/\not{\partial}_{g^Z_\varepsilon})$ etc., and we want to investigate their asymptotic expansions when $\varepsilon \to 0$. Incidentally to express the right hand side of (0.3) by $\mathrm{Tr}_\pm(\not{\partial}_{h^Z}/\not{\partial}_{g^Z})$ will be appropriate in the following sense:

Using the series of eigenvalues $(0 <) \lambda_1^\pm \leq \lambda_2^\pm \leq \cdots \to \infty$ (see Lemma 2.1) and the corresponding series of orthonormal eigen-cross-sections of the operator $\partial\!\!\!/_{gz}^2$ acting on $\Gamma(\$_{gz}^\pm)$, let us set $e^{-t\partial\!\!\!/_{gz}^2} = \sum e^{-t\lambda_j^\pm} \phi_j^\pm \boxtimes \phi_j^{\pm *}$ and put $\mu_j^\pm = \langle \partial\!\!\!/_{gz} \partial\!\!\!/_{hz} \phi_j^\pm, \phi_j^\pm \rangle_{L^2}$ where $\langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{L^2\Gamma(\$_{gz}^\pm)}$ is the global inner product which $\Gamma(\$_{gz}^\pm)$ has. Then, formally the right hand side of (0.3) is equal to

$$\sum \mu_j^\pm \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t\lambda_j^\pm} dt = \sum \mu_j^\pm \int_0^\infty e^{-t\lambda_j^\pm} dt = \sum \frac{\mu_j^\pm}{\lambda_j^\pm}.$$

Second, let us consider some infinitesimal deformation of the so-called chiral anomaly. That is, let us take a symmetric bilinear form X on TM and set $g_{(u)}^M = g^M + uX$ $(0 \leq u \leq u_0)$. The metric induces the Dirac operator $\partial\!\!\!/_{g_{(u)}^Z}$ acting on $\Gamma(\$_{gz})$ as above and we have the infinitesimal deformation of $\partial\!\!\!/_{gz}$

$$\delta_X \partial\!\!\!/_{gz} \equiv \frac{d}{du}\Big|_{u=0} \partial\!\!\!/_{g_{(u)}^Z}. \tag{0.4}$$

We are then interested in the associated invariants called **the infinitesimally deformed chiral anomalies of** $\partial\!\!\!/_{gz}$

$$\log\det(\delta_X \partial\!\!\!/_{gz})^\pm = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{Tr}_\pm\Big(\partial\!\!\!/_{gz}\, \delta_X \partial\!\!\!/_{gz}\, e^{-t\partial\!\!\!/_{gz}^2}\Big)\, dt, \tag{0.5}$$

$$\text{with the equalities } \operatorname{Tr}_\pm\Big(\partial\!\!\!/_{gz}\, \delta_X \partial\!\!\!/_{gz}\, e^{-t\partial\!\!\!/_{gz}^2}\Big) = \operatorname{Tr}_\mp\Big(\delta_X \partial\!\!\!/_{gz}\, \partial\!\!\!/_{gz}\, e^{-t\partial\!\!\!/_{gz}^2}\Big)$$

and we want to investigate the asymptotic expansions of $\log\det(\delta_{X_\varepsilon} \partial\!\!\!/_{g_\varepsilon^Z})^\pm$ and also their difference when $\varepsilon \to 0$. If the operators $\partial\!\!\!/_{gz} \partial\!\!\!/_{g_{(u)}^Z}$ acting on $\Gamma(\$_{gz}^\pm)$ happen to have the spectra consisting of eigenvalues $\{\lambda_j(u) = \lambda_j^\pm(u)\}$ all of which lie in a positive cone about the positive real axis in $\mathbb{C}$ and have the corresponding orthonormal eigen-cross-sections $\{\phi_j(u) = \phi_j^\pm(u)\}$ which are all smooth with respect to the parameter u at $u = 0$, then we have

$$\begin{aligned}
\lambda_j'(0) &\equiv \frac{d}{du}\Big|_{u=0} \langle \partial\!\!\!/_{gz} \partial\!\!\!/_{g_{(u)}^Z} \phi_j(u), \phi_j(u) \rangle_{L^2} \\
&= \langle \partial\!\!\!/_{gz} \delta_X \partial\!\!\!/_{gz} \phi_j(0), \phi_j(0) \rangle_{L^2} + \langle \partial\!\!\!/_{gz}^2 \phi_j'(0), \phi_j(0) \rangle_{L^2} + \langle \partial\!\!\!/_{gz}^2 \phi_j(0), \phi_j'(0) \rangle_{L^2} \\
&= \langle \partial\!\!\!/_{gz} \delta_X \partial\!\!\!/_{gz} \phi_j(0), \phi_j(0) \rangle_{L^2} + \lambda_j(0) \frac{\partial}{\partial u}\Big|_{u=0} \langle \phi_j(u), \phi_j(u) \rangle_{L^2} \\
&= \langle \partial\!\!\!/_{gz} \delta_X \partial\!\!\!/_{gz} \phi_j(0), \phi_j(0) \rangle_{L^2} \quad (\text{hence, } \lambda_j(0) > 0 \text{ if } \lambda_j'(0) \neq 0)
\end{aligned}$$

and the right hand side of (0.5) is formally equal to

$$\begin{aligned}
&\sum_{\lambda_j'(0) \neq 0} \lambda_j'(0) \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t\lambda_j(0)} dt = \sum_{\lambda_j(0) > 0} \frac{\lambda_j'(0)}{\lambda_j(0)} = \frac{d}{du}\Big|_{u=0} \log \prod_{\lambda_j(0) > 0} \lambda_j(u) \\
&= \frac{d}{du}\Big|_{u=0} \Big(-\frac{\partial}{\partial s}\Big|_{s=0} \sum_{(\lambda_j(0) > 0)} e^{-s \log \lambda_j(u)} \Big) = \frac{d}{du}\Big|_{u=0} \log\det(\partial\!\!\!/_{gz} \partial\!\!\!/_{g_{(u)}^Z})^\pm.
\end{aligned}$$

Thus, formally (0.5) are the infinitesimal deformations (into the direction X) of **the chiral anomalies** $\log\det(\not{\partial}_{g^Z}\not{\partial}_{g^Z_{(u)}})^{\pm}$ which were mathematically introduced by I.M. Singer [12, Appendix]. (Note that in general all but a finite number of eigenvalues $\lambda_1(u), \cdots, \lambda_k(u)$ lie in a positive cone about the positive real axis and, moreover, the eigen-cross-sections $\phi_j(u)$ are generalized ones. And strictly Singer said to define the anomalies as $\log\det(\not{\partial}_{g^Z}\not{\partial}_{g^Z_{(u)}})^{\pm} = -\lambda_1(u)\cdots\lambda_k(u)(\partial/\partial s)|_{s=0}\sum_{j>k} e^{-s\log\lambda_j(u)}$.)

Our investigation in the paper is an attempt to embody the idea ([3]) that such an operation as replacing g^Z etc. by g^Z_ε etc. and taking the parameter ε up to 0, that is, blowing up the metric g^Z in the base space direction, will extract some intrinsic values from various geometric invariants of Z. And we want to emphasize here that it is mainly the general adiabatic expansion theory concerning the kernel $e^{-t\not{\partial}^2_{g^Z}}$ ([11] and Lemma 2.3) that induces our main assertions, i.e., Theorem 1.2 and Corollary 1.3.

1 The operator $\not{\partial}_{h^Z}$ acting on $\Gamma(\not{S}_{g^Z})$ and the Main Assertions

According to the Bourguignon and Gauduchon's method ([4], [5]), first we will make $\not{\partial}_{h^Z}$ act on $\Gamma(\not{S}_{g^Z})$. The projection from the set $F^+(T_pM)$ of positively oriented frames on T_pM to the set $I(T_pM)$ of inner products on T_pM, given by $e \mapsto$ " the inner product $\langle\cdot,\cdot\rangle_e$ which has e as an orthonormal frame", has a structure of principal $SO(n)$-bundle, which is trivial since the base space $I(T_pM)$ is contractible. And the tangent space $T_eF^+(T_pM) \cong \mathfrak{gl}(n)$, $(d/da)|_{a=0}(e\cdot B_a) \leftrightarrow (d/da)|_{a=0}B_a$, has a subspace $\mathcal{H}_e(F^+(T_pM)) \cong \{B \in \mathfrak{gl}(n) \mid B = {}^tB\}$ which is projected onto $T_{\langle\cdot,\cdot\rangle_e}I(T_pM)$ isomorphically. Clearly the distribution $e \mapsto \mathcal{H}_e(F^+(T_pM))$ gives then a connection for the bundle, which induces the parallel displacement $\eta^M : P_{SO(n)}(M)_p \cong P_{SO(n)}(M,h^M)_p$ along the segment from g^M_p to h^M_p. Gathering such displacements now we get the bundle isomorphism

$$\eta^M : P_{SO(n)}(M) \cong P_{SO(n)}(M,h^M) \quad (1.1)$$
$$\text{with } \eta^M : T^{(*)}M \cong (T^{(*)}M, h^M),\ \eta^M([e^b,v]) = [\eta^M(e^b),v]$$

where we use the canonical expression $TM = P_{SO(n)}(M)\times_{can}\mathbb{R}^n$ $(\ni [e^b,v])$, etc. More explicitly, for a g^M-$SO(n)$-frame $e^b = (e^b_1,\cdots,e^b_n)$, set $\eta^b = (\eta^b_{ij}) \equiv (h^M(e^b_i,e^b_j))^{-1/2}$, which is positive and symmetric. Then we have

$$\eta^M(e^b) = e^b\cdot\eta^b,\ \eta^M(e^b_i) = \eta^M(e^b)_i = \sum e^b_j\cdot\eta^b_{ji}. \quad (1.2)$$

These come from the fact that, if we take the segment $t \mapsto g_p^M(t) = (1-t)g_p^M + th_p^M$ and for each $g_p^M(t)$ we put $\eta_t^M(e^b) = e^b \cdot (g_p^M(t)(e_i^b, e_j^b))^{-1/2}$, then $\eta_t^M(e^b)$ is a $g_p^M(t)$-$SO(n)$-frame and $(\partial/\partial t)\eta_t^M(e^b)$ is horizontal. Next let us assume to use the common $P_{Sp(1)}$ for the two metrics (see (0.1)), which consequently determines (locally defined) Spin structures $\Xi : P_{\mathrm{Spin}(n)}(M) \to P_{SO(n)}(M)$, $\Xi_{h^M} : P_{\mathrm{Spin}(n)}(M, h^M) \to P_{SO(n)}(M, h^M)$. Since the above connection for the (trivial) bundle $F^+(T_pM) \to I(T_pM)$ induces a connection for the associated (trivial) Spin(n)-bundle $\tilde{F}^+(T_pM) \to I(T_pM)$, similarly to the above we obtain a bundle isomorphism $\eta^M : P_{\mathrm{Spin}(n)}(M) \cong P_{\mathrm{Spin}(n)}(M, h^M)$ and, further, we have the bundle isometry

$$(1.3)\qquad \begin{aligned} &\eta^M : \$_{g^M} \cong \$_{h^M},\ \eta^M([\psi, s]) = [\eta^M(\psi), s] \\ &\text{with } \eta^M \circ \rho_{g^M}(\xi) = \rho_{h^M}(\eta^M(\xi)) \circ \eta^M \quad (\xi \in T^*M). \end{aligned}$$

Thus we get the identifications

$$(1.4)\qquad \begin{aligned} &\eta = \eta^M \oplus \mathrm{id} : TZ = \mathcal{H} \oplus \mathcal{V} \cong (TZ, h^Z) = (\mathcal{H}, \pi^* h^M) \oplus \mathcal{V} \\ &\text{given by } e_i^b(A) \equiv \pi^* e_i^b,\ e_k^f \mapsto \pi^* \eta^M(e_i^b),\ e_k^f, \end{aligned}$$

$$(1.5)\qquad \begin{aligned} &\eta = \eta^M \otimes \mathrm{id} : \$_{g^Z} = \pi^* \$_{g^M} \otimes \$_{g^{\mathcal{V}}} \cong \$_{h^Z} = \pi^* \$_{h^M} \otimes \$_{g^{\mathcal{V}}} \\ &\text{with } \eta \circ \rho_{g^Z}(\xi) = \rho_{h^Z}(\eta(\xi)) \circ \eta \quad (\xi \in T^*Z) \end{aligned}$$

where $e^f = (e_1^f, e_2^f)$ is a $g^{\mathcal{V}}$-$SO(2)$-frame of $\mathcal{V}$. Set $e_*(A) = (e_1(A), \cdots) = (e^b(A), e^f)$, which is a g^Z-$SO(n+2)$-frame, and denote its dual by $e^*(A) = (e^1(A), \cdots) = (e_b, e_f(A))$. Then we have the expressions $\partial\!\!\!/_{g^Z} = \sum \rho_{g^Z}(e^i(A)) \nabla^{\$_{g^Z}}_{e_i(A)} = \sum \rho_{g^Z}(e^i(A))\{e_i(A) + \frac{1}{4} \sum g^Z(\nabla^{g^Z}_{e_i(A)} e_{i_1}(A), e_{i_2}(A)) \rho_{g^Z}(e^{i_1}(A)) \rho_{g^Z}(e^{i_2}(A))\}$ etc. where ∇^{g^Z} is the Levi-Civita connection associated to the metric g^Z, and now

$$(1.6)\qquad \begin{aligned} &\partial\!\!\!/_{h^Z} \equiv \eta^{-1} \circ \partial\!\!\!/_{h^Z} \circ \eta = \sum \rho_{g^Z}(e^i(A)) \nabla^{\$_{g^Z}, h^Z}_{\eta(e_i(A))} : \Gamma(\$_{g^Z}) \to \Gamma(\$_{g^Z}) \ \text{ with} \\ &\nabla^{\$_{g^Z}, h^Z}_v = v + \frac{1}{4} \sum g^Z((\eta^{-1} \circ \nabla^{h^Z}_v \circ \eta)\, e_{i_1}(A), e_{i_2}(A))\, \rho_{g^Z}(e^{i_1}(A)) \rho_{g^Z}(e^{i_2}(A)) \\ &= v + \frac{1}{4} \sum h^Z(\nabla^{h^Z}_v \eta(e_{i_1}(A)), \eta(e_{i_2}(A)))\, \rho_{g^Z}(e^{i_1}(A)) \rho_{g^Z}(e^{i_2}(A)) \end{aligned}$$

is the desired one at (0.3). By putting $e_*^\varepsilon(A) = (e^{b\varepsilon}(A), e^f) = (\varepsilon^{1/2} e^b(A), e^f)$ and $e_\varepsilon^*(A) = (e_{b\varepsilon}, e_f(A)) = (\varepsilon^{-1/2} e_b, e_f(A))$, their adiabatic versions are then expressed as

$$(1.7)\qquad \partial\!\!\!/_{g_\varepsilon^Z} = \sum \rho_{g_\varepsilon^Z}(e_\varepsilon^i(A)) \nabla^{\$_{g_\varepsilon^Z}}_{e_i^\varepsilon(A)}, \quad \partial\!\!\!/_{h_\varepsilon^Z} = \sum \rho_{g_\varepsilon^Z}(e_\varepsilon^i(A)) \nabla^{\$_{g_\varepsilon^Z}, h_\varepsilon^Z}_{\eta(e_i^\varepsilon(A))}.$$

Remark that the map η for g_ε^Z etc. coincides with (1.4) for g^Z etc.

Let us next consider the identity

$$\mathrm{Tr}_{\pm}\left(\not{\partial}_{g^Z_\varepsilon}\not{\partial}_{h^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\right) = \mathrm{Tr}_{\mp}\left(\not{\partial}^*_{g^Z_\varepsilon,P'}\,\not{\partial}_{h^Z_\varepsilon,P}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right) \tag{1.8}$$

where we put $\not{\partial}^*_{g^Z_\varepsilon,P'}\,\not{\partial}_{h^Z_\varepsilon,P}\,\varphi_1(P)\boxtimes\varphi_2(P') \equiv \not{\partial}_{h^Z_\varepsilon,P}\,\varphi_1(P)\boxtimes\not{\partial}_{g^Z_\varepsilon,P'}\,\varphi_2(P')$. Conveniently the right hand side contains only derivatives up to the first order for each variables P, P'. First we intend to state the behavior of $\not{\partial}^*_{g^Z_\varepsilon}\not{\partial}_{h^Z_\varepsilon} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P) = \not{\partial}^*_{g^Z_\varepsilon,P'}\not{\partial}_{h^Z_\varepsilon,P} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P, P')|_{P=P'}$ (when $\varepsilon \to 0$) regarded as an element of the third side of the identification

$$\begin{aligned}\Gamma(\$_{g^Z_\varepsilon}\otimes\$_{g^Z_\varepsilon}) &= \Gamma(\$_{g^Z_\varepsilon}\otimes\$^*_{g^Z_\varepsilon}) = \Gamma(\wedge T^*Z\otimes\mathbb{C}),\\ s(e^\varepsilon_*(A))\otimes s(e^\varepsilon_*(A)) &\leftrightarrow s(e^\varepsilon_*(A))\otimes s(e^\varepsilon_*(A))^*,\ \rho_{g^Z_\varepsilon}(e^I_\varepsilon(A)) \leftrightarrow e^I_\varepsilon(A)\end{aligned} \tag{1.9}$$

where $s(e^\varepsilon_*(A))$ is the $SU(2^{n/2+1})$-frame of $\$_{g^Z_\varepsilon}$ induced from $e^\varepsilon_*(A)$, and $I = (I^b, I^f)$ is a multi-index with $I^b = (i^b_1 < \cdots < i^b_{|I^b|})$ and $I^f = (i^f_1 < \cdots < i^f_{|I^f|})$, and we put $e^{I^b}_{b\varepsilon} = e^{i^b_1}_{b\varepsilon}\wedge\cdots\wedge e^{i^b_{|I^b|}}_{b\varepsilon}$, $e^{I^f}_f(A) = e^{i^f_1}_f(A)\wedge\cdots\wedge e^{i^f_{|I^f|}}_f(A)$ and $e^I_\varepsilon(A) = e^{I^b}_{b\varepsilon}\wedge e^{I^f}_f(A)$. Let us take now a (globally defined) tensor field

$$T_A = \frac{1}{2}\sum\left\{[e^b_i, e^b_j](A) - [e^b_i(A), e^b_j(A)]\right\}\otimes e^i_b\wedge e^j_b \equiv \sum e^f_k\otimes T^k_A \tag{1.10}$$

where $[e^b_i, e^b_j](A)$ is the $\mathcal{H}$-horizontal lift ($\in\mathcal{H}$) of the bracket $[e^b_i, e^b_j]$ and the difference $[e^b_i, e^b_j](A) - [e^b_i(A), e^b_j(A)]$ is vertical ($\in\mathcal{V}$). And consider an elliptic operator acting on $\Gamma(\wedge T^*_pM\otimes\$_{g^\mathcal{V}}|Z_p)$ $(Z_p = \pi^{-1}(p))$

$$\mathcal{A}^2 = \not{\partial}^2_{g^\mathcal{V}} - \frac{1}{2}\sum T^k_A\wedge\cdot 1\otimes\nabla^{\$_{g^\mathcal{V}}}_{e^f_k} + \frac{1}{16}\left(\sum T^k_A\wedge\cdot\rho_{g^Z}(e^k_f(A))\right)^2 \tag{1.11}$$

where we put $\not{\partial}_{g^\mathcal{V}} = \sum\rho_{g^Z}(e^k_f(A))\nabla^{\$_{g^\mathcal{V}}}_{e^f_k}$, $\rho_{g^Z}(e^k_f(A)) = (-1)^\ell\otimes\rho_{g^\mathcal{V}}(e^k_f)$ for ℓ-forms in the M-direction and $T^k_A(P) = (1/2)\sum(e^i_b\wedge e^j_b)(p)\cdot T^k_{A,ij}(P)$. This generates a (C^0)-semi-group with C^∞-kernel which belongs to $\Gamma(\wedge T^*_pM\otimes(\$_{g^\mathcal{V}}|Z_p\boxtimes\$^*_{g^\mathcal{V}}|Z_p))$. Its value at (P,P) can be canonically regarded as an element of $\wedge(\pi^*T^*M)_P\otimes\wedge\mathcal{V}^*(A)_P\otimes\mathbb{C} = \wedge T^*_PZ\otimes\mathbb{C}$ (see (1.9)), which we denote by $\exp\left(-t\mathcal{A}^2\right)(P)$. Then we have

Proposition 1.1. *When $\varepsilon\to 0$, there exists a formal series expansion*

$$\not{\partial}^*_{g^Z_\varepsilon}\not{\partial}_{h^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P) = \sum_{m=-2}^{\infty}\varepsilon^{m/2}\, D_{(m/2)}(t, P\colon \not{\partial}_{h^Z}/\not{\partial}_{g^Z}) \quad \textit{with} \tag{1.12}$$

$$\begin{aligned}D_{(-2/2)}(t,P\colon\not{\partial}_{h^Z}/\not{\partial}_{g^Z}) &= -\theta^\wedge\frac{1}{2t}\left\langle e_b\,\middle|\,\eta^b\frac{tR^{g^M}}{2}\left\{\coth\frac{tR^{g^M}}{2} - 1\right\}\middle|\, e_b\right\rangle(p)\\ &\quad\times\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{tR^{g^M}/2}{\sinh(tR^{g^M}/2)}\right)(p)\,\exp\left(-t\mathcal{A}^2\right)(P)\end{aligned} \tag{1.13}$$

where we set $p = \pi(P)$, $\theta^\wedge \omega = (-1)^j \omega$ *for* j*-form* ω, *and* $R^{g^M}(p)$ *is an anti-symmetric matrix whose* (i,j)*-entries are equal to* $R^{g^M}_{ij}(p) = (1/2) \sum g^M(F(\nabla^{g^M})(e^b_i, e^b_j)e^b_{i_1}, e^b_{i_2})(p)$ $(e^{i_1}_b \wedge e^{i_2}_b)(p)$. *(See Lemma 2.4 for further informations for the coefficients.)*

Now let us state the main assertions.

Theorem 1.2. *In the definitions of* $\mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})$ *(see (0.3)), the functions to be differentiated by* s*, that is,* $\dfrac{1}{\Gamma(s)}\displaystyle\int_0^\infty t^s\, \mathrm{Tr}_\pm\Big(\partial\!\!\!/_{g^Z_\varepsilon}\, \partial\!\!\!/_{h^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\Big)\, dt$, *are absolutely integrable if* $\mathrm{Re}(s) > n/2+2$ *and have the meromorphic extensions to* $\mathbb{C}\,(\ni s)$ *which are analytic at* $s=0$. *When* $\varepsilon \to 0$, *then there exist the asymptotic expansions*

$$\begin{aligned}
(1.14)\quad & \mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) \\
&= \sum_{m=-2}^{\infty} \varepsilon^{m/2}\, \frac{\mp 2^{n/2}}{(\sqrt{-1})^{n/2+1}}\, \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s \int_Z D_{(m/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) \\
&\quad + \sum_{m=-(n+2)}^{\infty} \varepsilon^{m/2}\, 2^{n/2}\, \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s \int_Z D_{((m+n)/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) \wedge dg^Z(P),
\end{aligned}$$

$$\begin{aligned}
(1.15)\quad & \mathrm{STr}(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) \equiv \mathrm{Tr}_+(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) - \mathrm{Tr}_-(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) \\
&= -\sum_{m=-2}^{\infty} \varepsilon^{m/2}\, \frac{2^{n/2+1}}{(\sqrt{-1})^{n/2+1}}\, \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s \int_Z D_{(m/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})
\end{aligned}$$

where the functions differentiated by s *are also all absolutely integrable if* $\mathrm{Re}(s) > n/2+2$ *and have the meromorphic extensions to* $\mathbb{C}$ *which are analytic at* $s=0$. *In particular, as for (1.15), the coefficients of* $\varepsilon^{m/2}$ *with* $m<0$ *are all pure imaginary.*

As for the infinitesimally deformed chiral anomalies, we have

Corollary 1.3. *In the definitions of* $\log\det(\delta_{X_\varepsilon}\partial\!\!\!/_{g^Z_\varepsilon})^\pm$ *(see (0.5)), the functions to be differentiated by* s *are absolutely integrable if* $\mathrm{Re}(s) > n/2+2$ *and have the meromorphic extensions to* $\mathbb{C}$ *which are analytic at* $s=0$. *And set*

$$(1.16)\qquad CH_{(m/2)}(t,P\colon \delta_X\partial\!\!\!/_{g^Z}) = \frac{d}{du}\Big|_{u=0} D_{(m/2)}(t,P\colon \partial\!\!\!/_{g^Z_{(u)}}/\partial\!\!\!/_{g^Z}),\quad \textit{hence}$$

$$\begin{aligned}
(1.17)\qquad CH_{(-2/2)}(t,P\colon \delta_X\partial\!\!\!/_{g^Z}) &= \theta^\wedge \frac{1}{4t}\Big\langle e_b \Big| X\, \frac{tR^{g^M}}{2}\Big\{\coth\frac{tR^{g^M}}{2} - 1\Big\} \Big| e_b \Big\rangle(p) \\
&\quad \times \frac{1}{(4\pi t)^{n/2}} \det{}^{1/2}\left(\frac{tR^{g^M}/2}{\sinh(tR^{g^M}/2)}\right)(p)\, \exp\!\big(-t\mathcal{A}^2\big)(P).
\end{aligned}$$

Then we have the asymptotic expansions when $\varepsilon \to 0$

$$(1.18)\quad \log\det(\delta_{X_\varepsilon}\partial\!\!\!/_{g^Z_\varepsilon})^\pm$$

$$= \sum_{m=-2}^{\infty} \varepsilon^{m/2} \frac{\mp 2^{n/2}}{(\sqrt{-1})^{n/2+1}} \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt \cdot t^s \int_Z CH_{(m/2)}(t, P\colon \delta_X \partial\!\!\!/_{g^Z})$$

$$+ \sum_{m=-(n+2)}^{\infty} \varepsilon^{m/2}\, 2^{n/2} \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt \cdot t^s \int_Z CH_{((m+n)/2)}(t, P\colon \delta_X \partial\!\!\!/_{g^Z}) \wedge dg^Z(P),$$

$$(1.19)\quad S\text{-}\log\det(\delta_{X_\varepsilon} \partial\!\!\!/_{g^Z_\varepsilon}) \equiv \log\det(\delta_{X_\varepsilon} \partial\!\!\!/_{g^Z_\varepsilon})^+ - \log\det(\delta_{X_\varepsilon} \partial\!\!\!/_{g^Z_\varepsilon})^-$$

$$= -\sum_{m=-2}^{\infty} \varepsilon^{m/2} \frac{2^{n/2+1}}{(\sqrt{-1})^{n/2+1}} \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt \cdot t^s \int_Z CH_{(m/2)}(t, P\colon \delta_X \partial\!\!\!/_{g^Z}).$$

So as in Theorem 1.2 are the functions differentiated by s and the coefficients of $\varepsilon^{m/2}$ with $m < 0$ at (1.19).

2 Proofs of Theorem 1.2 and Corollary 1.3

First let us show

Lemma 2.1. *There exists a constant $\lambda_0 > 0$ satisfying* $\mathrm{Spec}(\partial\!\!\!/^2_{g^Z_\varepsilon}) \geq \lambda_0$ *for any ε with $0 < \varepsilon \leq \varepsilon_0$. And we have*

$$(2.1)\qquad \mathrm{Tr}_\pm\left(\partial\!\!\!/_{g^Z_\varepsilon} \partial\!\!\!/_{h^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right) = \mathrm{Tr}_\mp\left(\partial\!\!\!/_{h^Z_\varepsilon} \partial\!\!\!/_{g^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)$$

$$= \overline{\mathrm{Tr}_\mp\left(\partial\!\!\!/_{g^Z_\varepsilon} \partial\!\!\!/_{h^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)} + \mathrm{Tr}_\mp\Big(\sum_i \frac{\varepsilon^{1/2}\eta(e^b_i)(\det \eta^b)}{\det \eta^b}\, \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\, \partial\!\!\!/_{g^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\Big)$$

and further there exists a constant $C > 0$ satisfying

$$(2.2)\qquad \left|\mathrm{Tr}_\pm\left(\partial\!\!\!/^*_{g^Z_\varepsilon, P'}\, \partial\!\!\!/_{h^Z_\varepsilon, P}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P, P')\right)\right| \leq C\, e^{-t\lambda_0/3}\, \mathrm{Tr}_\pm\left(e^{-(t/6)\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)$$

$$(0 < \forall\varepsilon \leq \varepsilon_0 \text{ and } 0 < \forall t < \infty).$$

Proof. The assertion concerning the spectrum of $\partial\!\!\!/^2_{g^Z_\varepsilon}$ comes from the invertibility of $\partial\!\!\!/_{g^\mathcal{V}}$ ([10, (5.15)]) and [3, Proposition 4.41]. Namely, first consider a connection $\nabla^{g^\mathcal{V}} = P^\mathcal{V} \circ \nabla^{g^Z}$ of $\mathcal{V}$ where $P^\mathcal{V} : TZ = \mathcal{H} \oplus \mathcal{V} \to \mathcal{V}$ is the projection. This together with the Levi-Civita one $\nabla^{g^M_\varepsilon}$ gives a new connection $\nabla^{g^Z_\varepsilon \oplus} \equiv \pi^*\nabla^{g^M_\varepsilon} \oplus \nabla^{g^\mathcal{V}}$ of $TZ = \mathcal{H} \oplus \mathcal{V}$, which is compatible with g^Z_ε and whose torsion is equal to T_A given at (1.10) ([11, Lemma 3.1]). Denote by $\nabla^{\$_{g^Z_\varepsilon}\oplus}$ the associated connection on $\$_{g^Z_\varepsilon}$ and set $T^\sharp_A = \sum e^k_f(A) \otimes T^k_A = \frac{1}{2}\sum T^\sharp_{A,ij} \wedge e^i_b \wedge e^j_b$. Then we have

$$(2.3)\ \partial\!\!\!/_{g^Z_\varepsilon} = \varepsilon^{1/2} \sum \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}) \Big\{\nabla^{\$_{g^Z_\varepsilon}\oplus}_{e^b_i(A)} + \frac{\varepsilon^{1/2}}{8} \rho_{g^Z_\varepsilon}(\sum T^\sharp_{A,ij} \wedge e^j_{b\varepsilon})\Big\} + \partial\!\!\!/_{g^\mathcal{V}} \equiv \varepsilon^{1/2}\tilde{\partial\!\!\!/}_\varepsilon + \partial\!\!\!/_{g^\mathcal{V}},$$

$$\partial\!\!\!/^2_{g^Z_\varepsilon} = \varepsilon\tilde{\partial\!\!\!/}^2_\varepsilon + \partial\!\!\!/^2_{g^\mathcal{V}} + \varepsilon^{1/2}\Big\{\tilde{\partial\!\!\!/}_\varepsilon \circ \partial\!\!\!/_{g^\mathcal{V}} + \partial\!\!\!/_{g^\mathcal{V}} \circ \tilde{\partial\!\!\!/}_\varepsilon\Big\} = \varepsilon\tilde{\partial\!\!\!/}^2_\varepsilon + \partial\!\!\!/^2_{g^\mathcal{V}}$$

$$+\varepsilon^{1/2}\Big\{\sum \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}\wedge e^k_f(A))\,\nabla^{\$_{g^Z_\varepsilon}\oplus}_{[e^b_i(A),e^f_k]}-\frac{\varepsilon^{1/2}}{8}\rho_{g^Z_\varepsilon}(\sum T^\sharp_{A,ij}\wedge e^i_{b\varepsilon}\wedge e^j_{b\varepsilon})\circ\partial\!\!\!/_{g^V}$$
$$-\partial\!\!\!/_{g^V}\circ\frac{\varepsilon^{1/2}}{8}\rho_{g^Z_\varepsilon}(\sum T^\sharp_{A,ij}\wedge e^i_{b\varepsilon}\wedge e^j_{b\varepsilon})\Big\}.$$

Let $\|\cdot\|_{p,1}$ be now the Sobolev H^1-norm of elements of $\Gamma(\$_{g^Z_\varepsilon})$ restricted to Z_p with metric $g^Z_\varepsilon|Z_p$. Then there exist constants $C>0$, $C'>0$ such that for any $p\in M$, $\psi\in\Gamma(\$_{g^Z_\varepsilon})$ we have

$$\Big|\int_{Z_p}\langle\{\tilde{\partial\!\!\!/}_\varepsilon\circ\partial\!\!\!/_{g^V}+\partial\!\!\!/_{g^V}\circ\tilde{\partial\!\!\!/}_\varepsilon\}\psi,\psi\rangle_{\$_{g^Z_\varepsilon}}dg^Z_\varepsilon|Z_p\Big|\le C\|\psi\|_{p,1},\ C'\|\partial\!\!\!/_{g^V}\psi\|_{p,1}\ge\|\psi\|_{p,1}$$

where $\langle\cdot,\cdot\rangle_{\$_{g^Z_\varepsilon}}$ denotes the pointwise inner product which $\Gamma(\$_{g^Z_\varepsilon})$ has. The first estimate comes from the fact that $\tilde{\partial\!\!\!/}_\varepsilon\circ\partial\!\!\!/_{g^V}+\partial\!\!\!/_{g^V}\circ\tilde{\partial\!\!\!/}_\varepsilon$ is a first order differential operator on Z_p and the second comes from the fact that $\partial\!\!\!/_{g^V}$ is invertible. Since further $\tilde{\partial\!\!\!/}_\varepsilon$ is self-adjoint and $\tilde{\partial\!\!\!/}^2_\varepsilon$ is nonnegative, finally we have

$$\partial\!\!\!/^2_{g^Z_\varepsilon}\ge\partial\!\!\!/^2_{g^V}+\varepsilon^{1/2}\Big\{\tilde{\partial\!\!\!/}_\varepsilon\circ\partial\!\!\!/_{g^V}+\partial\!\!\!/_{g^V}\circ\tilde{\partial\!\!\!/}_\varepsilon\Big\}\ge(1-\varepsilon^{1/2}CC')\partial\!\!\!/^2_{g^V}.$$

We have thus shown the assertion concerning the spectrum. As for the equalities (2.1): To simplify the description, let us assume $\varepsilon=1$. Set $e^{-t\partial\!\!\!/^2_{g^Z}}=\sum e^{-t\lambda_j}\phi_j\boxtimes\phi^*_j\in\Gamma(\$^+_{g^Z}\boxtimes\$^{+*}_{g^Z})$ as usual (refer to the argument following (0.3)). Because of $\partial\!\!\!/^2_{g^Z}\partial\!\!\!/_{g^Z}\phi_j/\sqrt{\lambda_j}=\lambda_j\cdot\partial\!\!\!/_{g^Z}\phi_j/\sqrt{\lambda_j}\in\Gamma(\$^-_{g^Z})$ and $\langle\partial\!\!\!/_{g^Z}\phi_j/\sqrt{\lambda_j},\partial\!\!\!/_{g^Z}\phi_i/\sqrt{\lambda_i}\rangle_{L^2}=\delta_{ji}$, we have

$$\begin{aligned}\mathrm{Tr}_-\Big(\partial\!\!\!/_{g^Z}\partial\!\!\!/_{h^Z}\,e^{-t\partial\!\!\!/^2_{g^Z}}\Big)&=\sum e^{-t\lambda_j}\int\langle\partial\!\!\!/_{g^Z}\partial\!\!\!/_{h^Z}\partial\!\!\!/_{g^Z}\phi_j/\sqrt{\lambda_j},\,\partial\!\!\!/_{g^Z}\phi_j/\sqrt{\lambda_j}\rangle_{\$_{g^Z}}dg^Z(P)\\&=\sum e^{-t\lambda_j}\lambda_j^{-1}\int\langle\partial\!\!\!/_{h^Z}\partial\!\!\!/_{g^Z}\phi_j,\,\partial\!\!\!/^2_{g^Z}\phi_j\rangle_{\$_{g^Z}}dg^Z(P)\\&=\sum e^{-t\lambda_j}\int\langle\partial\!\!\!/_{h^Z}\partial\!\!\!/_{g^Z}\phi_j,\phi_j\rangle_{\$_{g^Z}}dg^Z(P)=\mathrm{Tr}_+\Big(\partial\!\!\!/_{h^Z}\partial\!\!\!/_{g^Z}\,e^{-t\partial\!\!\!/^2_{g^Z}}\Big).\end{aligned}$$

Thus the first equality at (2.1) was proved. Next let us prove the second one. We have

$$\partial\!\!\!/^*_{h^Z}=\det\eta^b\circ\partial\!\!\!/_{h^Z}\circ\det(\eta^b)^{-1}=\partial\!\!\!/_{h^Z}-\sum\frac{\eta(e^b_i)(\det\eta^b)}{\det\eta^b}\rho_{g^Z}(e^i_b)\tag{2.4}$$

because (1.6) implies

$$\begin{aligned}\int\langle\partial\!\!\!/_{h^Z}\psi,\phi\rangle_{\$_{g^Z}}dg^Z&=\int\langle\sum\rho_{g^Z}(e^i(A))\nabla^{\$_{g^Z},h^Z}_{\eta(e_i(A))}\psi,\det(\eta^b)^{-1}\cdot\phi\rangle_{\$_{g^Z}}dh^Z\\&=\int\langle\sum\rho_{h^Z}(\eta(e^i(A)))\nabla^{\$_{h^Z}}_{\eta(e_i(A))}\eta(\psi),\det(\eta^b)^{-1}\cdot\eta(\phi)\rangle_{\$_{h^Z}}dh^Z\\&=\int\langle\eta(\psi),\sum\rho_{h^Z}(\eta(e^i(A)))\nabla^{\$_{h^Z}}_{\eta(e_i(A))}\det(\eta^b)^{-1}\cdot\eta(\phi)\rangle_{\$_{h^Z}}dh^Z\\&=\int\langle\psi,\det\eta^b\sum\rho_{g^Z}(e^i(A))\nabla^{\$_{g^Z},h^Z}_{\eta(e_i(A))}\det(\eta^b)^{-1}\cdot\phi\rangle_{\$_{g^Z}}dg^Z.\end{aligned}$$

Hence, using the above expression of $\mathrm{Tr}_+\left(\partial\!\!\!/_{h^Z}\partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\right)$, we have

$$\begin{aligned}
&\mathrm{Tr}_+\left(\partial\!\!\!/_{h^Z}\partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\right) = \sum e^{-t\lambda_j}\int \langle \partial\!\!\!/_{g^Z}\phi_j(P), \partial\!\!\!/_{h^Z}\phi_j(P)\rangle_{\$_{g^Z}}\, dg^Z(P)\\
&-\sum e^{-t\lambda_j}\int \langle \partial\!\!\!/_{g^Z}\phi_j(P), \sum \frac{\eta(e_i^b)(\det\eta^b)}{\det\eta^b}\,\rho_{g^Z}(e_b^i)\phi_j(P)\rangle_{\$_{g^Z}}\, dg^Z(P)\\
&= \sum e^{-t\lambda_j}\int \overline{\langle \partial\!\!\!/_{h^Z}\phi_j(P), \partial\!\!\!/_{g^Z}\phi_j(P)\rangle_{\$_{g^Z}}}\, dg^Z(P)\\
&+\sum e^{-t\lambda_j}\int \langle \sum \frac{\eta(e_i^b)(\det\eta^b)}{\det\eta^b}\,\rho_{g^Z}(e_b^i)\,\partial\!\!\!/_{g^Z}\phi_j(P), \phi_j(P)\rangle_{\$_{g^Z}}\, dg^Z(P).
\end{aligned}$$

Thus we have proved the second equality. Last, as for the estimate (2.2): Assume $\varepsilon = 1$ and remember the above expression of $e^{-t\partial\!\!\!/^2_{g^Z}}$. Then we have

$$\begin{aligned}
&\left|\mathrm{Tr}_+\left(\partial\!\!\!/^*_{g^Z}\partial\!\!\!/_{h^Z} e^{-t\partial\!\!\!/^2_{g^Z}}\right)\right| \le \sum e^{-t\lambda_j}\left|\int \langle \partial\!\!\!/_{h^Z}\phi_j(P), \partial\!\!\!/_{g^Z}\phi_j(P)\rangle_{\$_{g^Z}}\, dg^Z(P)\right|\\
&\le \sum e^{-t\lambda_j}\, \|\partial\!\!\!/_{h^Z}\phi_j\|_{L^2}\, \|\partial\!\!\!/_{g^Z}\phi_j\|_{L^2} \le \sum e^{-t\lambda_j}\,(C_1\lambda_j^{1/2}+C_0)\lambda_j^{1/2}\\
&\le C_2\sum e^{-t\lambda_j/2} \le C_2\, e^{-t\lambda_0/3}\sum e^{-t\lambda_j/6} = C_2\, e^{-t\lambda_0/3}\,\mathrm{Tr}_+(e^{-(t/6)\partial\!\!\!/^2_{g^Z}}).
\end{aligned}$$

Thus (2.2) with $\varepsilon = 1$ was proved. And, remembering the estimate $\mathrm{Spec}(\partial\!\!\!/^2_{g^Z_\varepsilon}) \ge \lambda_0$ for any ε with $0 < \varepsilon \le \varepsilon_0$, obviously we know that the above estimation holds also for general ε.

Before we give the proofs of the three assertions stated in the previous section, we will make some preparatory arguments. Take a point $P^0 \in Z$. Though we have taken a g^Z-$SO(n+2)$-frame $e_*(A)$ around P^0 with no specific condition, now it is convenient for the proofs to take such a frame in the following specific way. First fix $e_*(A)(P^0) = (e^b(A)(P^0), e^f(P^0))$. Then let $e_*(A) = (e^b(A), e^f)$ be $\nabla^{g^Z\oplus}$-parallel along the $\nabla^{g^Z\oplus}$-geodesics from P^0 and be equal to the fixed one at P^0, and, further, let $e^*(A) = (e_b, e_f(A))$ be its dual. Remark that $\nabla^{g^Z\oplus} (= \nabla^{g^Z_1\oplus})$ is compatible with the metric g^Z so that $e_*(A)$ is certainly a g^Z-$SO(n+2)$-frame. Note also that $e^b(A)$ coincides with the $\mathcal{H}$-horizontal lift of the g^M-$SO(n)$-frame e^b on a neighborhood $U^b\,(\subset M)$ which is ∇^{g^M}-parallel along the ∇^{g^M}-geodesics from $p^0 = \pi(P^0)$ and is equal to the given $e^b(p^0)$ at p^0. Also take such a $g^{\mathcal{V}}$-$SO(2)$-frame on $U^f\,(\subset Z_{p^0})$ which coincides with the given $e^f(P^0)$ at P^0 and then spread it on a neighborhood $U\,(\subset Z)$ by the the $\mathcal{H}$-parallel displacement along the ∇^{g^M}-geodesics from p^0. The frame on U thus obtained is certainly equal to the above e^f. Further, let us take the $\nabla^{g^Z\oplus}$-normal coordinate neighborhood $(U = U^b \times U^f, x = (x^b, x^f))$ with $(\partial/\partial x)_{P^0} = e_*(A)(P^0)$. Similarly to the above, $x^b(P)$ are ∇^{g^M}-normal coordinates of $\pi(P)$ and $x^f(P)$ are $\nabla^{\mathcal{V}}$-normal coordinates of the image ($\in Z_{p^0}$) of the point P by the $\mathcal{H}$-parallel displacement. Hence we have

$$
(2.5) \qquad \begin{aligned} &e_i^b(x^b) = \sum (\partial/\partial x_j^b)_{x^b} \cdot v_{ji}^b(x^b), \quad v_{ji}^b(x^b) = \delta_{ji} + \mathcal{O}(|x^b|^2), \\ &C(\nabla^{g^M})_{i_2 i_1}(e_i^b) \equiv g^M(\nabla^{g^M}_{e_i^b} e_{i_1}^b, e_{i_2}^b) = \mathcal{O}(|x^b|), \quad A(e_i^b) = \mathcal{O}(|x^b|), \end{aligned}
$$

etc. Hereafter we will use the coordinates and the frames thus given and of course the g_ε^Z-$SO(n+2)$-frame $e_*^\varepsilon(A) = (e^{b\varepsilon}(A), e^f) = (\varepsilon^{1/2}e^b(A), e^f)$ and its dual $e_\varepsilon^*(A) = (e_{b\varepsilon}, e_f(A)) = (\varepsilon^{-1/2}e_b, e_f(A))$ (see (1.7)) are assumed to be defined by using such frames. Now first let us show

Lemma 2.2. *On the coordinate neighborhood (U, x), we have*

$$
\begin{aligned}
(2.6) \quad \not{\partial}_{g_\varepsilon^Z} &= \sum \partial/\partial x_k^f \cdot \rho_{g_\varepsilon^Z}(e_f^k(A)) + \sum \varepsilon^{1/2}\partial/\partial x_i^b \cdot \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^i) \\
&\quad - \sum \varepsilon^{2/2}\, \frac{1}{8} T_{A,i_1 i_2}^k(0) \cdot \rho_{g_\varepsilon^Z}(e_f^k(A)) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1}) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2}) + \mathcal{O}(|x|), \\
(2.7) \quad \not{\partial}_{h_\varepsilon^Z} &= \sum \partial/\partial x_k^f \cdot \rho_{g_\varepsilon^Z}(e_f^k(A)) + \sum \varepsilon^{1/2}\eta_{ji}^b(0)\, \partial/\partial x_j^b \cdot \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^i) \\
&\quad - \sum \varepsilon^{2/2}\, \eta_{j_1 i_1}^b(0)\, \eta_{j_2 i_2}^b(0)\, \frac{1}{8} T_{A,j_1 j_2}^k(0) \cdot \rho_{g_\varepsilon^Z}(e_f^k(A)) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1}) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2}) + \mathcal{O}(|x|).
\end{aligned}
$$

Proof. Remark that $\nabla^{g_\varepsilon^M} = \nabla^{g^M}$ and $\nabla^{h_\varepsilon^M} = \nabla^{h^M}$. Referring to (2.3) we have

$$
\begin{aligned}
\not{\partial}_{g_\varepsilon^Z} &= \sum \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^i)\, \varepsilon^{1/2} \Big\{ e_i^b(A) + \frac{1}{4} \sum C(\nabla^{g^M})_{i_2 i_1}(e_i^b)\, \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1}) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2}) \\
&+ \frac{1}{4} \sum C(\nabla^{\mathcal{V}})_{k_2 k_1}(e_i^b(A))\, \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \Big\} \\
&+ \sum \rho_{g_\varepsilon^Z}(e_f^k(A)) \Big\{ e_k^f + \frac{1}{4} \sum C(\nabla^{\mathcal{V}})_{k_2 k_1}(e_k^f)\, \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \Big\} \\
&- \frac{\varepsilon}{8} \sum T_{A,i_1 i_2}^k\, \rho_{g_\varepsilon^Z}(e_f^k(A)) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1}) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2}).
\end{aligned}
$$

Hence using (2.5) we obtain (2.6). Next, put $C(\nabla^{h^M})_{i_2 i_1}(\eta(e_i^b)) = h^M(\nabla^{h^M}_{\eta(e_i^b)}\eta(e_{i_1}^b), \eta(e_{i_2}^b))$. Then we have

$$
\begin{aligned}
\not{\partial}_{h_\varepsilon^Z} &= \sum \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^i)\, \varepsilon^{1/2} \Big\{ \eta(e_i^b(A)) + \frac{1}{4} \sum C(\nabla^{h^M})_{i_2 i_1}(\eta(e_i^b))\, \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1}) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2}) \\
&+ \frac{1}{4} \sum C(\nabla^{\mathcal{V}})_{k_2 k_1}(\eta(e_i^b(A)))\, \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \Big\} \\
&+ \sum \rho_{g_\varepsilon^Z}(e_f^k(A)) \Big\{ e_k^f + \frac{1}{4} \sum C(\nabla^{\mathcal{V}})_{k_2 k_1}(e_k^f)\, \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \rho_{g_\varepsilon^Z}(e_f^{k_1}(A)) \Big\} \\
&- \frac{\varepsilon}{8} \sum T_A^k(\eta(e_{i_1}^b), \eta(e_{i_2}^b))\, \rho_{g_\varepsilon^Z}(e_f^k(A)) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1}) \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2}).
\end{aligned}
$$

Hence using (1.2) and (2.5) we obtain (2.7).

Next let us consider the identification

$$
(2.8) \qquad \Gamma(\$_{g_\varepsilon^Z}|U \boxtimes \$_{g_\varepsilon^Z}^*|U) = C^\infty(U \times U, \wedge T_{p^0}^* Z)
$$

given by $s(e^\varepsilon_*(A))(x) \otimes s(e^\varepsilon_*(A))^*(x') \cdot \phi(x,x') \leftrightarrow ((x,x'),\ s(e^\varepsilon_*(A))(0) \otimes s(e^\varepsilon_*(A))^*(0) \cdot \phi(x,x')) \in U \times U \times \$_{g^Z_\varepsilon}|_{P^0} \otimes \$^*_{g^Z_\varepsilon}|_{P^0} \ni ((x,x'), \rho_{g^Z_\varepsilon}(e^I_\varepsilon(A)) \leftrightarrow ((x,x'), e^I_\varepsilon(A)(P^0))$. The Clifford action $\rho_{g^Z_\varepsilon}(e^i_\varepsilon(A))$ acting on the left hand side can be expressed on the right hand side as

$$\rho_{g^Z_\varepsilon}(e^i_\varepsilon(A)) = e^i_\varepsilon(A) \wedge - e^i_\varepsilon(A) \vee \tag{2.9}$$

and the operator $\partial\!\!\!/^*_{g^Z_\varepsilon, P'}$ given at (1.8) can be expressed on the right hand side as

$$\begin{aligned} \partial\!\!\!/^*_{g^Z_\varepsilon} &= \sum \rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A)) \cdot e^\varepsilon_i(A)(P') \\ &+ \frac{1}{4} \sum \rho^*_{g^Z_\varepsilon}(e^{i_2}_\varepsilon(A)) \rho^*_{g^Z_\varepsilon}(e^{i_1}_\varepsilon(A)) \rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A)) \cdot C(\nabla^{g^Z_\varepsilon})(e^\varepsilon_i(A))_{i_2 i_1}(P') \quad \text{with} \\ & \rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A)) = \theta^\wedge(e^i_\varepsilon(A) \wedge + e^i_\varepsilon(A) \vee) \end{aligned} \tag{2.10}$$

Let us then regard the kernel $e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}$ as an element of the right hand side of (2.8) and set $e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x') \equiv \sum e^I(A)(P^0) \cdot \left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\right)_I$, and moreover define its differentiations as

$$\partial^\alpha_x \partial^{\alpha'}_{x'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x') \equiv \sum e^I(A)(P^0) \cdot \partial^\alpha_x \partial^{\alpha'}_{x'} \left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\right)_I \tag{2.11}$$

$$\text{with} \quad \left|\partial^\alpha_x \partial^{\alpha'}_{x'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\right|_{g^Z} \equiv \left\{\sum \left|\partial^\alpha_x \partial^{\alpha'}_{x'} \left(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\right)_I\right|^2\right\}^{1/2} \tag{2.12}$$

where $\alpha = (\alpha^b, \alpha^f) = (\alpha^b_1, \cdots, \alpha^b_n, \alpha^f_1, \alpha^f_2)$ is a multi-index and we put $\partial^\alpha_x = (\partial/\partial x)^\alpha = (\partial/\partial x^b)^{\alpha^b} (\partial/\partial x^f)^{\alpha^f} = (\partial/\partial x^b_1)^{\alpha^b_1} \cdots (\partial/\partial x^b_n)^{\alpha^b_n} (\partial/\partial x^f_1)^{\alpha^f_1} (\partial/\partial x^f_2)^{\alpha^f_2}$, etc. Then we have

Lemma 2.3 *(The general adiabatic expansion theorem as to $e^{-t\partial\!\!\!/^2_{g^Z}}$: [11, Theorems 1.2, 1.3 and the proof of Proposition 2.2 for $E(t,\varepsilon)$ with t small]).*

(1) For any integer $m_0 \geq 0$, there exist C^∞-functions $K_{(m/2)}(t, P^0, x, x')$ ($m = 0, 1, \cdots, m_0$), $K_{((m_0+1)/2, \varepsilon^{1/2})}(t, P^0, x, x')$ belonging to the right hand side of (2.8), which are also C^∞ with respect to the variable P^0 (and $\varepsilon^{1/2}$), and satisfying the following condition: For any α and α', (2.11) with $(x,x') = (0,0)$ has the series expansion

$$\begin{aligned} \partial^\alpha_x \partial^{\alpha'}_{x'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0, P^0) &= \sum_{m=0}^{m_0} \varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+m/2}\, \partial^\alpha_x \partial^{\alpha'}_{x'} K_{(m/2)}(t, P^0) \\ &+ \varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+(m_0+1)/2}\, \partial^\alpha_x \partial^{\alpha'}_{x'} K_{((m_0+1)/2, \varepsilon^{1/2})}(t, P^0) \end{aligned} \tag{2.13}$$

where we put $|\alpha^b| = \sum \alpha^b_i$ etc. and $\partial^\alpha_x \partial^{\alpha'}_{x'} K_{(m/2)}(t, P^0)$ etc. mean $\partial^\alpha_x \partial^{\alpha'}_{x'} K_{(m/2)}(t, P^0, x, x')|_{x=x'=0}$ etc. Further, there exist constants $\lambda > 0$, $C > 0$ and an integer $N > 0$

satisfying

$$(2.14)\quad \begin{gathered} \left|\partial_x^{\alpha}\partial_{x'}^{\alpha'}K_{(m/2)}(t,P^0)\right|_{g^Z} \le C\,e^{-t\lambda}\,t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big), \\ \left|\partial_x^{\alpha}\partial_{x'}^{\alpha'}K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0)\right|_{g^Z} \le C\,t^{1/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\Big) \\ (\,0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall P^0\in Z\,). \end{gathered}$$

And, if $|\alpha|+|\alpha'|\le 2$, *then, given* $T_0>0$, *we have the series expansion*

$$(2.15)\ \partial_x^{\alpha}\partial_{x'}^{\alpha'}K_{(m/2,\cdot)}(t,P^0)=\frac{1}{(4\pi t)^{(n+2)/2}}\Big\{\sum_{i=-\delta_{0m}}^{i_0} t^i\,\partial_x^{\alpha}\partial_{x'}^{\alpha'}K_{(m/2,\cdot)}(i:P^0)+\mathcal{O}(t^{i_0+1})\Big\}$$
$$(\forall i_0\ge 0,\ 0\le\forall m\le m_0+1,\ 0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t\le T_0,\ \forall P^0\in Z\,).$$

(2) The top term $K_{(0)}(t,P^0,x,x')$ *can be written as*

$$(2.16)\qquad K_{(0)}(t,P^0,x,x')=K_M(t,P^0,x^b,x'^b)\ \exp\Big(-t\mathcal{A}^2\Big)(x^f,x'^f)\cdot\det v^b(x'^b)$$

where we set

$$(2.17)\ K_M(t,P^0,x^b,x'^b)=\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{tR^{g^M}(p^0)/2}{\sinh\big(tR^{g^M}(p^0)/2\big)}\right)$$
$$\cdot\exp\Big(-\frac{1}{4t}\Big\langle(x^b-x'^b)\Big|\frac{tR^{g^M}(p^0)}{2}\coth\frac{tR^{g^M}(p^0)}{2}\Big|(x^b-x'^b)\Big\rangle+\frac{1}{4}\Big\langle x^b\Big|R^{g^M}(p^0)\Big|x'^b\Big\rangle\Big)$$

and $\exp\Big(-t\mathcal{A}^2\Big)$, *see (1.11) around, is here regarded as an element of the right hand side of* $\Gamma(\wedge T^*_{p^0}M\otimes(\$_{g^V}|U^f\boxtimes\$^*_{g^V}|U^f))=C^\infty(U^f\times U^f,\wedge T^*_{P^0}Z)$, *and we have* $\det v^b(x'^b)=\det(g^M(\partial/\partial x_i^b,\partial/\partial x_j^b)(x'^b))^{-1/2}=1+\mathcal{O}(|x'^b|^2)$ *(see (2.5)).*

Here let us prove Proposition 1.1.

Proof of Proposition 1.1. (2.10), (2.13) and Lemma 2.2 imply the formal series expansion

$$\begin{aligned}(2.18)\quad &\not{\partial}^*_{g^Z_\varepsilon}\not{\partial}_{h^Z_\varepsilon}e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)\equiv\not{\partial}^*_{g^Z_\varepsilon,P'}\not{\partial}_{h^Z_\varepsilon,P}e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')|_{P=P'=P^0}\\
&=\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{i'}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i}_{b\varepsilon})\,\eta^b_{ji}(0)\,(\partial/\partial x'^b_{i'})(\partial/\partial x^b_j)K_{(m/2)}(t,P^0)\\
&+\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{i}_{b\varepsilon})\,\eta^b_{ji}(0)\,(\partial/\partial x'^f_{k'})(\partial/\partial x^b_j)K_{(m/2)}(t,P^0)\\
&+\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{i'}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{k}_f(A))(\partial/\partial x'^b_{i'})(\partial/\partial x^f_k)K_{(m/2)}(t,P^0)\\
&+\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{k}_f(A))(\partial/\partial x'^f_{k'})(\partial/\partial x^f_k)K_{(m/2)}(t,P^0)\\
&+\sum\varepsilon^{2/2+m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{i'}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{k}_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\\
&\qquad\cdot\eta^b_{j_1i_1}(0)\,\eta^b_{j_2i_2}(0)\Big(-\frac{1}{4}\nu^k(F_{Aj_1j_2})\Big)(0)\,(\partial/\partial x'^b_{i'})K_{(m/2)}(t,P^0)\end{aligned}$$

$$+\sum \varepsilon^{2/2+m/2} \sum (\rho_{g_\varepsilon^Z}(e_f^{k'}(A))\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i'_1})\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i'_2}))^* \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i})$$
$$\cdot \eta_{ji}^b(0)\Big(-\frac{1}{4}\nu^{k'}(F_{A,i'_1 i'_2})\Big)(0)\,(\partial/\partial x_j^b)K_{(m/2)}(t,P^0)$$
$$+\sum \varepsilon^{2/2+m/2} \sum \rho^*_{g_\varepsilon^Z}(e_f^{k'}(A))\rho_{g_\varepsilon^Z}(e_f^{k}(A))\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1})\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2})$$
$$\cdot \eta_{j_1 i_1}^b(0)\,\eta_{j_2 i_2}^b(0)\Big(-\frac{1}{4}\nu^{k}(F_{A,j_1 j_2})\Big)(0)(\partial/\partial x_{k'}^{\prime f})K_{(m/2)}(t,P^0)$$
$$+\sum \varepsilon^{2/2+m/2} \sum (\rho_{g_\varepsilon^Z}(e_f^{k'}(A))\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i'_1})\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i'_2}))^* \rho_{g_\varepsilon^Z}(e_f^{k}(A))$$
$$\cdot \Big(-\frac{1}{4}\nu^{k'}(F_{A,i'_1 i'_2})\Big)(0)\,(\partial/\partial x_k^f)K_{(m/2)}(t,P^0)$$
$$+\sum \varepsilon^{4/2+m/2} \sum (\rho_{g_\varepsilon^Z}(e_f^{k'}(A))\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i'_1})\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i'_2}))^* \rho_{g_\varepsilon^Z}(e_f^{k}(A))\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_1})\rho_{g_\varepsilon^Z}(e_{b\varepsilon}^{i_2})$$
$$\cdot \eta_{j_1 i_1}^b(0)\,\eta_{j_2 i_2}^b(0)\Big(-\frac{1}{4}\nu^{k'}(F_{A,i'_1 i'_2})\Big)(0)\Big(-\frac{1}{4}\nu^{k}(F_{A,j_1 j_2})\Big)(0)K_{(m/2)}(t,P^0).$$

Hence, observing (2.10), we know that (2.18) can be expanded as in (1.12). And (2.16) implies further

$$(2.19)\quad D_{(-2/2)}(t,P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z) = -\theta^\wedge \sum e_b^{i'}\wedge e_b^{i}\wedge \eta_{ji}^b(0)\,(\partial/\partial x_{i'}^{\prime b})(\partial/\partial x_j^b)K_{(0)}(t,P^0)$$
$$= -\theta^\wedge \frac{1}{2t}\Big\langle e_b\,\Big|\,\eta^b\,\frac{tR^{g^M}}{2}\Big\{\coth\frac{tR^{g^M}}{2}-1\Big\}\,\Big|\,e_b\Big\rangle(p^0)\,K_{(0)}(t,P^0).$$

Thus we have obtained the formula (1.13).

Further, as for the coefficients in (1.12), Lemmata 2.2 and 2.3 say

Lemma 2.4. *For any integer* $m_0 \ge 0$, *put* $D_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z) \equiv \partial\!\!\!/^*_{g_\varepsilon^Z}\partial\!\!\!/_{h_\varepsilon^Z}\, e^{-t\partial\!\!\!/^2_{g_\varepsilon^Z}}(P^0,P^0) - \sum_{m=-2}^{m_0}\varepsilon^{m/2}D_{(m/2)}(t,P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z)$. *Then there exist constants* $\lambda>0$, $C>0$ *and an integer* $N>0$ *satisfying*

$$(2.20)\quad \begin{aligned} &\big|D_{(m/2)}(t,P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z)\big|_{g^Z} \le C\,e^{-t\lambda}\,t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+2)/2}}+1\Big)\quad (m\le m_0),\\ &\big|D_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z)\big|_{g^Z} \le C\,t^{1/2}\Big(\frac{1}{t^{(n+2)/2}}+t^N\Big)\\ &\qquad (0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall P^0\in Z).\end{aligned}$$

Further, for given $T_0>0$, *we have the series expansion*

$$(2.21)\quad \begin{aligned} &D_{(m/2,\cdot)}(t,P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z)\\ &= \frac{1}{(4\pi t)^{(n+2)/2}}\Big\{\sum_{i=-\delta_{0m}}^{i_0} t^i\,D_{(m/2,\cdot)}(i:P^0:\partial\!\!\!/_h z/\partial\!\!\!/_g z)+O(t^{i_0+1})\Big\}\\ &(\forall i_0\ge 0,\ 0\le\forall m\le m_0+1,\ 0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t\le T_0,\ \forall P^0\in Z).\end{aligned}$$

As our last preparation let us investigate the pointwise trace $\mathrm{tr}_\pm(\rho_{g^Z}(e^I(A)))$.

Lemma 2.5. *We have*

$$\text{(2.22)}\qquad \begin{aligned}&\operatorname{tr}_\pm(\rho_{g^Z}(e^\emptyset(A))) = 2^{n/2}, \quad \operatorname{tr}_\pm(\rho_{g^Z}(e^{(1,\cdots,n+2)}(A))) = \pm\frac{2^{n/2}}{(\sqrt{-1})^{n/2+1}},\\ &\operatorname{tr}_\pm(\rho_{g^Z}(e^I(A))) = 0 \ (\textit{otherwise}),\end{aligned}$$

$$\text{(2.23)}\qquad \begin{aligned}\Omega^\pm(\varepsilon, P) &\equiv \sum e^I(A)(P)\cdot\varepsilon^{-(n-|I^b|)/2}\operatorname{tr}_\pm(\rho_{g^Z}(e^I(A)))\\ &= \varepsilon^{-n/2}\,2^{n/2} \pm \frac{2^{n/2}}{(\sqrt{-1})^{n/2+1}}\,dg^Z(P).\end{aligned}$$

Proof. The first two equalities at (2.22) and the equality $\operatorname{tr}_\pm(\rho_{g^Z}(e^I(A))) = 0$ ($|I|$ is odd), and moreover $\operatorname{tr}_+(\rho_{g^Z}(e^I(A))) = \operatorname{tr}_-(\rho_{g^Z}(e^I(A))) = (1/2)\operatorname{tr}(\rho_{g^Z}(e^I(A)))$ ($|I|$ is even and $|I| < 2n$) are all obvious. Hence we have only to prove

$$\text{(2.24)}\qquad \operatorname{tr}(\rho_{g^Z}(e^I(A))) = 0 \quad (0 < |I| = 2m < n+2).$$

Take the standard frame $(e^1, \cdots, e^{2r}) = (e_1, Je_1, \cdots, e_r, Je_r)$ of $\mathbb{R}^{2r}$ where J is the standard complex structure, and let us prove (2.24) for the standard Clifford action $\rho : \mathbb{C}l(\mathbb{R}^{2r}) \to \operatorname{End}(\wedge^*\mathbb{C}^r)$, i.e., $\rho(e^{2\ell-1}) = e_\ell\wedge - e_\ell\vee$, $\rho(e^{2\ell}) = \sqrt{-1}(e_\ell\wedge + e_\ell\vee)$ and hence $\rho(e^{2\ell-1}\circ e^{2\ell}) = \sqrt{-1}(e_\ell\wedge e_\ell\vee - e_\ell\vee e_\ell\wedge)$. Assume $I = ((2i_1-1, 2i_1), \cdots, (2i_m-1, 2i_m))$ ($\mathbb{I} \equiv (i_1 < \cdots < i_m)$, $0 < m < r$) and set $(1, 2, \cdots, r) = \mathbb{I}\cup\mathbb{J}$ with $\mathbb{J} = (j_1 < \cdots < j_{r-m})$. Then we have

$$\begin{aligned}&\rho(e^I) = (\sqrt{-1})^m\prod_{\ell=1}^m(e_{i_\ell}\wedge e_{i_\ell}\vee - e_{i_\ell}\vee e_{i_\ell}\wedge),\\ &\mathbb{I} \supset \mathbb{K} = (k_1 < \cdots < k_{|\mathbb{K}|}),\ \rho(e^I)\,e_{\mathbb{K}}\wedge e_{\mathbb{J}} = (\sqrt{-1})^m(-1)^{m-|\mathbb{K}|}\,e_{\mathbb{K}}\wedge e_{\mathbb{J}},\\ &\operatorname{tr}(\rho(e^I)) = (\sqrt{-1})^m\sum_{\mathbb{K}}(-1)^{m-|\mathbb{K}|} = 0.\end{aligned}$$

Thus (2.24) for such a type of I was proved. And it will obviously holds if I is not of such a type.

Now let us prove Theorem 1.2.

Proof of Theorem 1.2. Let us set $D_{(m/2)}(t, P^0) = D_{(m/2)}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})$, etc. to simplify the description, and put $D_{(m/2)}(t, P^0) = \sum e^I(A)(P^0)\cdot D_{(m/2)}(t, P^0)_I$ as in (2.11). Then we have

$$\text{(2.25)}\qquad \begin{aligned}\operatorname{Tr}_\pm^{g_\varepsilon^Z}(D_{(m/2)}(t)) &\equiv \int_Z \operatorname{tr}_\pm^{g_\varepsilon^Z}(D_{(m/2)}(t, P^0))\,dg_\varepsilon^Z(P^0)\\ &= \int_Z\sum \operatorname{tr}_\pm(\rho_{g_\varepsilon^Z}(e_\varepsilon^I(A)))\cdot\varepsilon^{|I^b|/2}\Big(D_{(m/2)}(t, P^0)\Big)_I\,e_\varepsilon^I(A)\wedge\star_{g_\varepsilon^Z}e_\varepsilon^I(A)\\ &= \int_Z D_{(m/2)}(t, P^0)\wedge\star_{g^Z}\Omega^\pm(\varepsilon, P^0)\end{aligned}$$

where $\mathrm{tr}_{\pm}^{g_\varepsilon^Z}(D_{(m/2)}(t,P^0))$ mean the pointwise traces of $D_{(m/2)}(t,P^0)$ regarded as an element of $\$_{g_\varepsilon^Z,P^0}\otimes\$^*_{g_\varepsilon^Z,P^0}$ and $\star_{g_\varepsilon^Z}$ is the star operator associated to the metric g_ε^Z. Hence, setting $\Omega^{\pm}(\varepsilon,P^0)=\sum_{-n\le\ell\le 0}\varepsilon^{\ell/2}\,\Omega^{\pm}(\ell/2\colon P^0)$ (see (2.23)), (1.12) and the above give the formal series expansion

$$
\begin{aligned}
(2.26)\qquad \mathrm{Tr}_{\pm}\left(\not{\partial}^*_{g_\varepsilon^Z}\not{\partial}_{h_\varepsilon^Z}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}\right) &= \sum_{m=-2}^{\infty}\varepsilon^{m/2}\,\mathrm{Tr}_{\pm}^{g_\varepsilon^Z}(D_{(m/2)}(t))\\
&= \sum_{m=-(n+2)}^{\infty}\varepsilon^{m/2}\int_Z\sum_{m=m_1+m_2}D_{(m_1/2)}(t,P^0)\wedge\star_{g^Z}\Omega^{\pm}(m_2/2\colon P^0)\\
&\equiv \sum_{m=-(n+2)}^{\infty}\varepsilon^{m/2}\int_Z D(m/2:t,P^0).
\end{aligned}
$$

Thus, observing (2.23), we find out that $\mathrm{Tr}_{\pm}(\not{\partial}_{h_\varepsilon^Z}/\not{\partial}_{g_\varepsilon^Z})$ can be expanded into (1.14) (still not asymptotically but) formally. Further the first estimate at (1.17) and the series expansion (1.18) imply that, for given $n_0>0$, if $m\le n_0$, then the function (to be differentiated by s)

$$
(2.27)\qquad \frac{1}{\Gamma(s)}\int_0^{\infty}dt\cdot t^s\int_Z D(m/2:t,P^0)
$$

is absolutely integrable if $\mathrm{Re}(s)>n/2+1$ and has a meromorphic extension to $\mathbb{C}\,(\ni s)$ which is analytic at $s=0$. Hence, to finish the proof of the assertions concerning $\mathrm{Tr}_{\pm}(\not{\partial}_{h_\varepsilon^Z}/\not{\partial}_{g_\varepsilon^Z})$, we have only to show that so is (2.27) with $D(m/2:t,P^0)$ replaced by the remainder term $D((n_0+1)/2,\varepsilon^{1/2}:t,P^0)=\mathrm{tr}_{\pm}^{g_\varepsilon^Z}\left(\not{\partial}^*_{g_\varepsilon^Z}\not{\partial}_{h_\varepsilon^Z}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P^0,P^0)\right)dg_\varepsilon^Z(P^0)-\sum_{m=-(n+2)}^{n_0}\varepsilon^{m/2}D(m/2:t,P^0)$. To prove it let us now investigate the remainder term for t large. That is, fix $T_0>0$ and assume $t\ge T_0$. Then there exists a constant $C=C(T_0)>0$ such that, for any $t\,(\ge T_0)$, we have

$$
(2.28)\qquad \left|\mathrm{Tr}_{\pm}\left(\not{\partial}^*_{g_\varepsilon^Z}\not{\partial}_{h_\varepsilon^Z}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}\right)\right|\le C\,\varepsilon^{-n/2}\,e^{-t\lambda_0/4}.
$$

Actually (2.14) with $\alpha=\alpha'=\emptyset$ implies

$$
\left|\mathrm{Tr}_{\pm}(e^{-t\not{\partial}^2_{g_\varepsilon^Z}})\right|=\left|\int_Z e^{-t\not{\partial}^2_{g_\varepsilon^Z}}(P^0,P^0)\wedge\star_{g^Z}\Omega^{\pm}(\varepsilon,P^0)\right|\le C'\,\varepsilon^{-n/2}\,t^N,
$$

which, combined with (2.2), gives the estimate (2.28). Next let $m_0>0$ be the integer appearing in (2.20). Then (2.28) and the first estimate at (2.20) imply

$$
\begin{aligned}
&\left|\varepsilon^{(m_0+1)/2}\int_Z D((m_0+1)/2,\varepsilon^{1/2}:t,P^0)\right|\\
&=\left|\mathrm{Tr}_{\pm}\left(\not{\partial}^*_{g_\varepsilon^Z}\not{\partial}_{h_\varepsilon^Z}e^{-t\not{\partial}^2_{g_\varepsilon^Z}}\right)-\sum_{m=-(n+2)}^{m_0}\varepsilon^{m/2}\int_Z D(m/2:t,P^0)\right|\le C_1\,\varepsilon^{-(n+2)/2}\,e^{-t\lambda_0/4},
\end{aligned}
$$

which, combined with the second estimate at (2.20), yields

$$\text{(2.29)} \qquad \Big|\varepsilon^{(m_0+1)/2}\int_Z D((m_0+1)/2,\varepsilon^{1/2}:t,P^0)\Big| \\ \le C_2\,\varepsilon^{-(n+2)/4}\,e^{-t\lambda_0/8}\varepsilon^{(m_0+1)/4}\,t^{N/2} \le C_3\,\varepsilon^{(m_0-n-1)/4}\,e^{-t\lambda_0/9}.$$

Hence we may take $m_0 > 0$ so large that we have

$$\text{(2.30)} \qquad \Big|\int_Z D((n_0+1)/2,\varepsilon^{1/2}:t,P^0)\Big| \le C_4\,e^{-t\lambda_0/9}.$$

Actually, since we have

$$\int_Z D((n_0+1)/2,\varepsilon^{1/2}:t,P^0) = \int_Z\Big\{\sum_{m=n_0+1}^{m_0}\varepsilon^{(m-n_0-1)/2}D(m/2:t,P^0) \\ +\varepsilon^{(m_0-n_0)/2}D((m_0+1)/2,\varepsilon^{1/2}:t,P^0)\Big\}$$

and (2.29) yields

$$\Big|\varepsilon^{(m_0-n_0)/2}\int_Z D((m_0+1)/2,\varepsilon^{1/2}:t,P^0)\Big| \le C_3\,\varepsilon^{(m_0-2n_0-n-3)/4}\,e^{-t\lambda_0/9},$$

we have only to take m_0 satisfying $m_0-2n_0-n-3\ge 0$. The estimate (2.30) with t large and the series expansion (2.21) with $m = m_0+1$ (and here $m_0 = n_0$) now imply the desired assertion about (2.27) for the remainder term.

Let us show the remained assertions concerning the difference (1.15). (1.15) is obvious. And (1.1) says

$$\text{(2.31)} \quad \mathrm{Re}\Big(\mathrm{STr}(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})\Big) \\ = \frac{1}{2}\Big(\mathrm{Tr}_+(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) - \overline{\mathrm{Tr}_-(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})}\Big) - \frac{1}{2}\Big(\mathrm{Tr}_-(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) - \overline{\mathrm{Tr}_+(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})}\Big) \\ = -\frac{d}{ds}\Big|_{s=0}\frac{1}{2\Gamma(s)}\int_0^\infty t^s\,\mathrm{STr}\Big(\sum_i\frac{\varepsilon^{1/2}\eta(e_i^b)(\det\eta^b)}{\det\eta^b}\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\partial\!\!\!/_{g^Z_\varepsilon}\,e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\Big)\,dt$$

and (2.6) implies

$$\sum\frac{\varepsilon^{1/2}\eta(e_i^b)(\det\eta^b)}{\det\eta^b}\,\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\partial\!\!\!/_{g^Z_\varepsilon}\,e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}} \\ = \varepsilon^{0/2}\sum\frac{\eta(e_i^b)(\det\eta^b)}{\det\eta^b}(0)\,e_b^i\wedge\Big\{\sum e_b^j\wedge(\partial/\partial x_j^b)K_{(1/2)}(t,P^0) \\ +\sum\rho_{g^Z}(e_f^k(A))(\partial/\partial x_k^f)K_{(0/2)}(t,P^0) \\ +\sum\Big(-\frac{1}{8}\,T^k_{A,j_1j_2}\Big)(0)\,\rho_{g^Z}(e_f^k(A))\,e_b^{j_1}\wedge e_b^{j_2}\wedge K_{(0/2)}(t,P^0)\Big\}+\cdots.$$

Thus the series expansion of (2.31) has no term with $\varepsilon^{m/2}$ $(m<0)$.

Last, let us prove Corollary 1.3.

Proof of Corollary 1.3. We have $\eta^b_{(u)} \equiv (g^M_{(u)}(e^b_i, e^b_j))^{-1/2} = (E + uX)^{-1/2}$ where E is the unit matrix (compare with (2.1)), which implies

$$\frac{d}{du}\Big|_{u=0} \eta^b_{(u)} = -\frac{1}{2}X. \tag{2.32}$$

Hence, referring to (2.7) with h^Z replaced by $g^Z_{(u)} = \pi^* g^M_{(u)} + g^{\mathcal{V}}$, we have

$$\begin{aligned} \delta_{X_\varepsilon} \not{\partial}_{g^Z_\varepsilon} &= \frac{d}{du}\Big|_{u=0} \not{\partial}_{g^Z_{(u)\varepsilon}} = -\varepsilon^{1/2} \sum \frac{1}{2} X_{ji}(0)\, \partial/\partial x^b_j \cdot \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}) \\ &+ \varepsilon^{2/2} \sum \{X_{j_1 i_1}\delta_{j_2 i_2} + \delta_{j_1 i_1} X_{j_2 i_2}\}(0)\, \frac{1}{16} T^k_{A, j_1 j_2}(0) \cdot \rho_{g^Z_\varepsilon}(e^k_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon}) \\ &+ \mathcal{O}(|x|). \end{aligned} \tag{2.33}$$

Thus obviously Theorem 1.2 implies the corollary.

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On the infinitesimally deformed super chiral anomaly of Dirac operators and the gauge transformation of twistor spaces

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Abstract. We investigate the adiabatic expansion of the super chiral anomaly associated to the Bourguignon and Gauduchon sense infinitesimal deformation of Dirac operator in the direction of a cross-section of the adjoint bundle. Further its top term is explicitly described.

1 Introduction

Let us assume that an even dimensional compact oriented Riemannian manifold $M = (M, g^M)$ is equipped with a Spin^q structure introduced in [11]

$$\Xi^q : P_{\mathrm{Spin}^q(n)}(M) = P_{\mathrm{Spin}(n)}(M) \times_{\mathbf{Z}_2} P_{Sp(1)} \to P_{SO(n)}(M) \times P_{SO(3)} \tag{1.1}$$

where $P_{SO(n)}(M)$ $(n = \dim M)$ is the reduced structure bundle consisting of $SO(n)$-frames of TM and $P_{SO(3)}$, $P_{\mathrm{Spin}^q(n)}(M)$ are some principal bundles with structure groups $SO(3)$, $\mathrm{Spin}^q(n) \equiv \mathrm{Spin}(n) \times_{\mathbf{Z}_2} Sp(1)$, respectively. Remark that $P_{\mathrm{Spin}(n)}(M)$, $P_{Sp(1)}$ are locally defined bundles. Then, using the canonical action of $\mathrm{Spin}^q(n)$ or $Sp(1)$ on $\mathrm{Spin}^q(n)/\mathrm{Spin}^c(n) = Sp(1)/U(1)$ and the identification $Sp(1)/U(1) = \mathbb{C}P^1$ through the representation $r_H : Sp(1) \to GL_{\mathbf{C}}(H) = GL_{\mathbf{C}}(\mathbb{C}^2)$ with $r_H(\alpha + j\beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, we get a $\mathbb{C}P^1$-fibration

$$\pi : Z = P_{\mathrm{Spin}^q(n)}(M) \times_{\mathrm{can}} \mathbb{C}P^1 = P_{Sp(1)} \times_{\mathrm{can}} \mathbb{C}P^1 \to M. \tag{1.2}$$

Let us now take an $Sp(1)$-connection A of $P_{Sp(1)}$, so that the twistor space Z possesses a canonical Spin structure ([12], [13], [14]). Namely, the connection induces a splitting of TZ into horizontal and vertical components, $TZ = \mathcal{H} \oplus \mathcal{V}$, with natural orientation and with the metric $g^Z = \pi^* g^M + g^{\mathcal{V}}$ ($\pi^* g^M = g^Z|\mathcal{H}$) where $g^{\mathcal{V}}$ is a Riemannian metric on $\mathcal{V}$ induced from the Fubini-Study one of $\mathbb{C}P^1$. Further we have the locally defined spinor bundle $\$_{g^M}$ associated to $P_{\mathrm{Spin}(n)}(M)$ and a locally defined hermitian vector bundle $\mathbb{H} = P_{Sp(1)} \times_{r_H} H$, which together produce the globally defined vector bundle $\pi^* \$_{g^M} \otimes \pi^* \mathbb{H} \equiv \pi^* \$_{g^M} \otimes \$_{g^{\mathcal{V}}} \equiv \$_{g^Z}$ on Z, whose rank is certainly equal to $2^{\dim Z}$. Then, the locally defined Clifford action ρ_{g^M} of $\mathbb{C}l(T^*M, g^M)$ on $\$_{g^M}$, together with the action $\rho_{g^{\mathcal{V}}}$ of $\mathbb{C}l(\mathcal{V}^*, g^{\mathcal{V}})$ on $\$_{g^{\mathcal{V}}}$ induced from the fiberwise globally defined canonical Spin structure, gives the globally defined action ρ_{g^Z} of $\mathbb{C}l(T^*Z, g^Z)$ on $\$_{g^Z}$, i.e., $\rho_{g^Z}(\pi^*\xi_b) = \pi^*\rho_{g^M}(\xi_b) \otimes 1$ ($\xi_b \in T^*M$) and $\rho_{g^Z}(\xi_f) = \pi^*\rho_{g^M}(\tau_{g^M}) \otimes \rho_{g^{\mathcal{V}}}(\xi_f)$ ($\xi_f \in \mathcal{V}^*$) where τ_{g^M} is the complex volume element of (M, g^M). Thus (Z, g^Z) has a canonical Spin structure, which gives the Dirac operator $\partial\!\!\!/_{g^Z}^{(\pm)} : \Gamma(\$_{g^Z}^{(\pm)}) \to \Gamma(\$_{g^Z}^{(\mp)})$. Note that the canonical splittings $\$_{g^M} = \$_{g^M}^+ \oplus \$_{g^M}^-$, $\$_{g^{\mathcal{V}}} = \pi^*\mathbb{H} = \$_{g^{\mathcal{V}}}^+ \oplus \$_{g^{\mathcal{V}}}^- = \{([v], cv) \in \pi^*\mathbb{H}\} \oplus (\$_{g^{\mathcal{V}}}^+)^\perp$ where $(\$_{g^{\mathcal{V}}}^+)^\perp$ is the orthogonal complement induce the canonical one $\$_{g^Z} = \$_{g^Z}^+ \oplus \$_{g^Z}^-$.

Concisely to state, the purpose of the paper is to **investigate the adiabatic behavior of the super chiral anomaly associated to the Bourguignon and Gauduchon sense infinitesimal deformation $\delta_X \partial\!\!\!/_{g^Z}$ of $\partial\!\!\!/_{g^Z}$ in the direction of a cross-section X of the (globally defined) adjoint bundle $\mathfrak{sp}(P_{Sp(1)}) = P_{Sp(1)} \times_{\mathrm{Ad}} \mathfrak{sp}(1)$**

$$\text{(1.3)} \qquad \text{S-log}\det(\delta_X \partial\!\!\!/_{g^Z}) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s\, \mathrm{STr}\Big(\partial\!\!\!/_{g^Z}\, \delta_X \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\Big)\, dt$$

$$\text{with the equality } \mathrm{STr}\Big(\partial\!\!\!/_{g^Z}\, \delta_X \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\Big) = -\mathrm{STr}\Big(\delta_X \partial\!\!\!/_{g^Z}\, \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\Big)$$

where we set $\mathrm{STr}(\cdots) = \mathrm{Tr}_+(\cdots) - \mathrm{Tr}_-(\cdots)$ and $\mathrm{Tr}_\pm(\cdots)$ are the global traces of the operator $\partial\!\!\!/_{g^Z}\, \delta_X \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}$ acting on $\Gamma(\$^\pm_{g^Z})$. (The equality at the second line will be obvious by describing $e^{-t\partial\!\!\!/^2_{g^Z}}$ with the eigenvalues and the corresponding eigen-cross-sections of $\partial\!\!\!/^2_{g^Z}$.) First of all it will be proper to explain here the operator $\delta_X \partial\!\!\!/_{g^Z}$ exactly. To do so, let us take the curve $a \mapsto h_a = \exp(aX) \in \Gamma(Sp(P_{Sp(1)})) = \Gamma(P_{Sp(1)} \times_{\mathrm{Ad}} Sp(1))$ hence with $X = (d/da)|_{a=0} h_a$. Each h_a gives **a gauge transformation of the twistor space** Z (over the identity map on M), $P \mapsto h_a(P) = h_a([\phi, \tilde{P}]) = [h_a(\phi), \tilde{P}]$. We have hence the pull-back metric $h_a^* g^Z$, which defines a Dirac operator $\partial\!\!\!/_{h_a^* g^Z}$ acting on $\Gamma(\$_{h_a^* g^Z})$. In a naive sense the operator $\delta_X \partial\!\!\!/_{g^Z}$ is its differentiation by a at $a = 0$. But such a differentiation is obviously impossible because the spinor bundle changes with the parameter a. Here by applying the Bourguignon and Gauduchon's method ([4], [5]) let us make $\partial\!\!\!/_{h_a^* g^Z}$ act on $\Gamma(\$_{g^Z})$ so as to be possible. Consider the projection from the set $F^+(T_PZ)$ of positively oriented frames on T_PZ to the set $I(T_PZ)$ of inner products on T_PZ, given by $e \mapsto$ " the inner product $\langle\cdot,\cdot\rangle_e$ which has e as an orthonormal frame", has a structure of (trivial) principal $SO(n+2)$-bundle. And the tangent space $T_eF^+(T_PZ) \cong \mathfrak{gl}(n+2)$, $(d/da)|_{a=0}(e\cdot B_a) \leftrightarrow (d/da)|_{a=0}B_a$, has a subspace $\mathcal{H}_e(F^+(T_PZ)) \cong \{B \in \mathfrak{gl}(n+2) \mid B = {}^tB\}$ which is projected onto $T_{\langle\cdot,\cdot\rangle_e}I(T_PZ)$ isomorphically. Clearly the distribution $e \mapsto \mathcal{H}_e(F^+(T_PZ))$ gives then a connection for the bundle, which induces a parallel displacement $\eta_a^Z : P_{SO(n+2)}(Z)_P \cong P_{SO(n+2)}(Z, h_a^*g^Z)_P$ along the segment from g_P^Z to $(h_a^*g^Z)_P$. Gathering such displacements we get the bundle isomorphism $\eta_a^Z : P_{SO(n+2)}(Z) \cong P_{SO(n+2)}(Z, h_a^*g^Z)$, which induces the vector bundle isomorphism

$$(1.4)\qquad \eta_a^Z : TZ \cong (TZ, h_a^*g^Z),\quad \eta_a^Z([e,v]) = [\eta_a^Z(e), v].$$

Now we take a local g^Z-$SO(n+2)$-frame $e = (e_1,\cdots,e_{n+2})$ and its dual $(e^1,\cdots,e^{n+2})$, denote by $\nabla^{h_a^*g^Z}$ the Levi-Civita connection associated to the metric $h_a^*g^Z$ and set

$$(1.5)\qquad \begin{aligned}&\partial\!\!\!/_{(h_a^*g^Z)} = \sum \rho_{g^Z}(e^i)\nabla^{\$_{g^Z},h_a}_{\eta_a^Z(e_i)} : \Gamma(\$_{g^Z}) \to \Gamma(\$_{g^Z}) \quad\text{with}\\ &\nabla_v^{\$_{g^Z},h_a} = v + \frac{1}{4}\sum (h_a^*g^Z)(\nabla_v^{h_a^*g^Z}\eta_a^Z(e_{i_1}), \eta_a^Z(e_{i_2}))\,\rho_{g^Z}(e^{i_1})\rho_{g^Z}(e^{i_2}).\end{aligned}$$

This is the desired one called the Bourguignon and Gauduchon sense deformation of $\partial\!\!\!/_{g^Z}$ and

$$(1.6)\qquad \delta_X\partial\!\!\!/_{g^Z} = \frac{d}{da}\Big|_{a=0}\partial\!\!\!/_{(h_a^*g^Z)}$$

is the infinitesimally deformed one appearing at (1.3).

The concept of (super) chiral anomaly was of course introduced physically and in [15] we formulated its mathematical definition according to its interpretation offered by I.M. Singer ([16, Appendix]). Our investigation in the paper is an attempt to extract some intrinsic values from the anomaly by such an operation as replacing g^Z by $g_\varepsilon^Z = \varepsilon^{-1}\pi^*g^M + g^\mathcal{V}$ and taking the parameter ε up to 0, that is, adiabatically blowing up the metric in the base space direction (refer to [3], [17], [15]). For the latter metric we can take canonically a Spin^q structure with the same $P_{SO(3)}$ as in (1.1) and whose twistor space is equal to the one given at (1.2), and precisely we want to investigate the series expansion of S-log det $(\delta_X\partial\!\!\!/_{g_\varepsilon^Z})$ when $\varepsilon \to 0$ using the general adiabatic expansion theory concerning the kernel $e^{-t\partial\!\!\!/^2_{g_\varepsilon^Z}}$ ([14]).

2 The Main Theorems

First we want to state

Theorem 2.1. *In the definition (1.3), the function to be differentiated by s, that is,* $\frac{1}{\Gamma(s)}\int_0^\infty t^s\,\mathrm{STr}\Big(\not{\partial}_{g^z}\delta_X\not{\partial}_{g^z}\,e^{-t\not{\partial}_{g^z}^2}\Big)\,dt$*, is absolutely integrable (i.e.* $\frac{1}{\Gamma(s)}\int_0^\infty\Big|t^s\mathrm{STr}\Big(\not{\partial}_{g^z}\delta_X\not{\partial}_{g^z}\,e^{-t\not{\partial}_{g^z}^2}\Big)\Big|\,dt<\infty$*) if* $\mathrm{Re}(s)>(n+2)/2$ *and has a meromorphic extension to* $\mathbb{C}$ *which is analytic at* $s=0$*. When* $\varepsilon^{1/2}\to 0$ *we have then a Taylor expansion*

$$S\text{-}\log\det\,(\delta_X\not{\partial}_{g^z_\varepsilon})=\sum_{m=0}^{\infty}\varepsilon^{m/2}\,S\text{-}\log\det\,(m/2:\delta_X\not{\partial}_{g^z}). \tag{2.1}$$

Subsequently to some preparations we will describe the top term S-log det $(0/2:\delta_X\not{\partial}_{g^z})$ explicitly.

We fix a point p^0 of M arbitrarily and take a local trivialization of $P_{Sp(1)}$ over an open neighborhood U^b of p^0 by using a local cross-section ϕ with $\nabla^A_{\partial/\partial r^b}\phi=0$ where r_b is the distance function from p^0, which gives a local trivialization $\pi^{-1}(U^b)=U^b\times Z_{p^0}$ of Z with $Z_{p^0}\equiv\pi^{-1}(p^0)$. Let us take a local coordinate neighborhood $(U=U^b\times U^f, x=(x^b,x^f))$ at $P^0\in Z_{p^0}$ as follows: (U^b,x^b) is a g^M-normal coordinate neighborhood at p^0 on M with $(\partial/\partial x^b)(p^0)=(\partial/\partial x^b_1,\cdots,\partial/\partial x^b_n)(p^0)\in P_{SO(n)}(M)_{p^0}$ and (U^f,x^f) is a $g^{\mathcal{V}}$-normal coordinate one at P^0 on Z_{p^0} with $(\partial/\partial x^f)(P^0)\in P_{SO(2)}(Z_{p^0})_{P^0}$. Further let $e^f=(e^f_1,e^f_2)$ be a local $g^{\mathcal{V}}$-$SO(2)$-frame of $\mathcal{V}|Z_{p^0}$ which is $g^{\mathcal{V}}$-parallel along the geodesics from P^0 and is equal to $(\partial/\partial x^f)$ at P^0, and, for each generator $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the Lie algebra $\mathfrak{sp}(1)$, let $\nu(\boldsymbol{i})^\natural=\sum\nu^k(\boldsymbol{i})\,e^f_k$ etc. be the uniquely determined cross-sections of $\mathcal{V}|Z_{p^0}$ defined as follows: For any connection $B=\boldsymbol{i}\otimes B^{(\boldsymbol{i})}+\boldsymbol{j}\otimes B^{(\boldsymbol{j})}+\boldsymbol{k}\otimes B^{(\boldsymbol{k})}$ of $P_{Sp(1)}$ and for any $v\in T_{p^0}M$, the B-horizontal lift of v to a point $P\in Z_{p^0}$ near P^0 can be given by $v(B)(P)=v-2\{B^{(\boldsymbol{i})}(v)\,\nu(\boldsymbol{i})^\natural(P)+B^{(\boldsymbol{j})}(v)\,\nu(\boldsymbol{j})^\natural(P)+B^{(\boldsymbol{k})}(v)\,\nu(\boldsymbol{k})^\natural(P)\}\equiv v-2\,\nu(B(v))^\natural\equiv v-2\sum\nu^k(B(v))\,e^f_k\in\mathcal{H}_B$. Using these, for general $\mathfrak{sp}(P_{Sp(1)})$-valued differential form $F=\boldsymbol{i}\otimes F^{(\boldsymbol{i})}+\boldsymbol{j}\otimes F^{(\boldsymbol{j})}+\boldsymbol{k}\otimes F^{(\boldsymbol{k})}$ we put $\nu^k(F(p^0))\equiv\nu^k(\boldsymbol{i})F^{(\boldsymbol{i})}(p^0)+\nu^k(\boldsymbol{j})F^{(\boldsymbol{j})}(p^0)+\nu^k(\boldsymbol{k})F^{(\boldsymbol{k})}(p^0)$. In particular, we often use $\nu^k(F_A)$, $\nu^k(d_AX)$, $\nu(F_A)^\natural=\sum e^f_k\otimes\nu^k(F_A)$, etc., where F_A is the curvature 2-form of A and d_AX is the covariant exterior derivative of X by A.

Now let us set $Z(p^0)=M(p^0)\times Z_{p^0}$ with $M(p^0)=(\mathbb{R}^n,x^b)\,(\supset(U^b,x^b)$, canonically) and consider the bundle $\wedge T^*M(p^0)$ on $M(p^0)$ and the standard spinor bundle $\$_{g^{\mathcal{V}},p^0}=\$_{g^{\mathcal{V}}}|Z_{p^0}$ on $(Z_{p^0},g^{Z_{p^0}}=g^{\mathcal{V}}|Z_{p^0})$. We denote by $\wedge T^*M(p^0)\otimes_\pi\$_{g^{\mathcal{V}}}$ the tensor product of their pull-backs to $Z(p^0)$, consider the bundle

$$(\wedge T^*M(p^0)\otimes_\pi \$_{g^\nu})\boxtimes(\wedge T^*M(p^0)\otimes_\pi \$^*_{g^\nu}) \tag{2.2}$$

over $Z(p^0)\times Z(p^0)$ $(\ni (P,P')=((x^b,P^f),(x'^b,P'^f)))$ and take its cross-section

$$E(t,p^0,P,P')=E_M(t,p^0,x^b,x'^b)\,\exp(-t\mathcal{A}^2_{p^0})(P,P') \tag{2.3}$$

as follows: Let $R^{g^M}_{i_2i_1ij}(p^0)=g^M(F(\nabla^{g^M})(\partial/\partial x^b_i,\partial/\partial x^b_j)\partial/\partial x^b_{i_1},\partial/\partial x^b_{i_2})(p^0)$ be the curvature coefficients of ∇^{g^M} at p^0, and let us denote by $R^{g^M}(p^0)=R^{g^M}(p^0)(x^b)$ the skew-symmetric matrix whose (i,j)-entries are equal to $R^{g^M}_{ij}(p^0)=R^{g^M}_{ij}(p^0)(x^b)=\frac{1}{2}\sum R^{g^M}_{i_2i_1ij}(p^0)\,dx^b_{i_1}(x^b)\wedge dx^b_{i_2}(x^b)$ and set $\left\langle x^b\middle|R^{g^M}(p^0)\middle|x'^b\right\rangle=\sum x^b_i x'^b_j R^{g^M}_{ij}(p^0)$, etc., and now put

$$\begin{aligned}
K_M(t,p^0,x^b,x'^b)&=\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{tR^{g^M}(p^0)/2}{\sinh(tR^{g^M}(p^0)/2)}\right)\\
&\quad\cdot\exp\Big(-\frac{1}{4t}\Big\langle x^b-x'^b\Big|\frac{tR^{g^M}(p^0)}{2}\coth\frac{tR^{g^M}(p^0)}{2}\Big|x^b-x'^b\Big\rangle+\frac{1}{4}\Big\langle x^b\Big|R^{g^M}(p^0)\Big|x'^b\Big\rangle\Big)\\
&\equiv\sum(dx^b)^I(x^b)\cdot K_M(t,p^0,x^b,x'^b)_I,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
E_M(t,p^0,x^b,x'^b)&=\sum(dx^b)^I(x^b)\wedge(dx^b)^J(x^b)\otimes(dx^b)^J(x'^b)\cdot K_M(t,p^0,x^b,x'^b)_I\\
&\equiv\sum(dx^b)^I(x^b)\otimes(dx^b)^{I'}(x'^b)\cdot E_M(t,p^0,x^b,x'^b)_{II'}
\end{aligned} \tag{2.5}$$

where the multi-index I is lined up in increasing order, i.e., $I=(i_1<i_2<\cdots<i_{|I|})$ and $(dx^b)^I$ denotes $dx^b_{i_1}\wedge\cdots\wedge dx^b_{i_{|I|}}$. Next let $e_f=(e^1_f,e^2_f)$ be the dual of e^f and let us take the Dirac operator $\not{\partial}_{g^\nu}=\rho_{g^\nu}(e^k_f)\nabla^{\$_{g^\nu}}_{e^f_k}$ acting on $\Gamma(\$_{g^\nu,p^0})$ and consider an elliptic operator

$$\mathcal{A}^2_{p^0}=1\otimes\not{\partial}^2_{g^\nu}-\sum\nu^k(F_A(p^0))\wedge\cdot1\otimes\nabla^{\$_{g^\nu}}_{e^f_k}-\frac{1}{4}\sum\nu^k(F_A(p^0))^2\wedge \tag{2.6}$$

acting on the cross-sections of $\wedge T^*_{p^0}M(p^0)\otimes\$_{g^\nu}$, which generates a (C^0) semi-group with C^∞-kernel $\sum(dx^b)^I(p^0)\cdot\exp(-t\mathcal{A}^2_{p^0})(P^f,P'^f)_I$. Then we set

$$\begin{aligned}
\exp(-t\mathcal{A}^2_{p^0})(P,P')&=\sum(dx^b)^I(P)\,(dx^b)^J(P)\otimes(dx^b)^J(P')\cdot\exp(-t\mathcal{A}^2_{p^0})(P^f,P'^f)_I\\
&=\sum(dx^b)^I(P)\otimes(dx^b)^{I'}(P')\cdot\exp(-t\mathcal{A}^2_{p^0})(P^f,P'^f)_{II'}.
\end{aligned} \tag{2.7}$$

(Strictly speaking we should distinguish $(dx^b)^I(x^b)$ and $(dx^b)^I(P)=(\pi^*(dx^b)^I)(P)$ but, to simplify the description, we will use them without distinction.)

For two elements $E_1(P,Q) = \sum (dx^b)^I(P) \otimes (dx^b)^J(Q) \cdot E_1(P,Q)_{IJ}$, $E_2(Q,P') = \sum (dx^b)^J(Q) \otimes (dx^b)^{I'}(P') \cdot E_2(Q,P')_{JI'}$ of (2.2), let us set

$$\text{(2.8)} \qquad \begin{aligned} &\langle E_1(P,Q), E_2(Q,P')\rangle_{g^{(p^0)}(Q)} \\ &\equiv \sum (dx^b)^I(P) \otimes (dx^b)^{I'}(P') \cdot \langle E_1(P,Q)_{IJ}, E_2(Q,P')_{JI'}\rangle_{\$_{g^{\mathcal{V}},Q}}, \end{aligned}$$

which is an element of (2.2) at (P,P'). Here $\langle E_1(P,Q)_{IJ}, E_2(Q,P')_{JI'}\rangle_{\$_{g^{\mathcal{V}},Q}}$ is the pairing of the $\$^*_{g^{\mathcal{V}},Q}$-component of $E_1(P,Q)_{IJ}$ and the $\$_{g^{\mathcal{V}},Q}$-component of $E_2(Q,P')_{JI'}$ and $g^{(p^0)}$ is the metric given by

$$\text{(2.9)} \quad g^{(p^0)}(Q) = \sum dx_i^b(y^b) \otimes dx_i^b(y^b) + \sum e_f^k(Q^f) \otimes e_f^k(Q^f) \equiv g_{(p^0)}(y^b) + g^{\mathcal{V}}(Q^f).$$

Then the second assertion is as follows.

Theorem 2.2. *We set*

$$\begin{aligned} \text{(2.10)} \quad & s\text{-}\log\det(\delta_X \not{\partial}_g z)(t, P^0) = \int_0^t d\tau \int_{Z(p^0)\ni Q} dg^{(p^0)}(Q) \Big\langle \mathrm{chi}(\delta_X \not{\partial}_g z)\, E(t-\tau, p^0, P^0, Q), \\ & \quad \mathrm{chi}(\not{\partial}_g^2 z)\Big\{\Big\langle x^b \Big| \frac{R^{g^M}(p^0)}{4}\Big(1+\coth\frac{\tau R^{g^M}(p^0)}{2}\Big)\Big| dx^b\Big\rangle(Q)\, E(\tau, p^0, Q, P^0)\Big\}\Big\rangle_{g^{(p^0)}(Q)} \\ & = \sum (dx^b)^I(P^0) \otimes (dx^b)^{I'}(P^0) \otimes (dx^f)^J(P^0) \cdot s\text{-}\log\det(\delta_X \not{\partial}_g z)(t,P^0)_{(I,I'),J} \\ & \quad \in \text{“(2.2) at } (P^0,P^0)\text{”} = (\wedge T^*_{p^0}M(p^0) \otimes \wedge T^*_{p^0}M(p^0)) \otimes \wedge T^*_{P^0}Z_{p^0} \otimes \mathbb{C}, \end{aligned}$$

where the operators $\mathrm{chi}(\delta_X \not{\partial}_g z)$, *etc. (acting at* $P = (x^b, P^f)$*) are given as follows:*

$$\text{(2.11)} \quad \mathrm{chi}(\delta_X \not{\partial}_g z) = \sum \nu^k((d_A X)(p^0)) \wedge \Big(e_k^f + \frac{1}{2}\nu^k(F_A(p^0))\wedge\Big),$$

$$\begin{aligned} \text{(2.12)} \quad \mathrm{chi}(\not{\partial}_g^2 z) &= \frac{1}{6}\sum x_{j_1}^b x_{j_2}^b \frac{\partial R^{g^M}_{i_2 i_1 i j_1}}{\partial x^b_{j_2}}(p^0)\, dx^b_{i_1} \wedge dx^b_{i_2} \wedge \Big\{\frac{\partial}{\partial x_i^b} + \frac{1}{4}\sum x_j^b R^{g^M}_{ji}(p^0)\wedge\Big\} \\ & \quad - \sum x_j^b \nu^k\Big(\frac{\partial F_A}{\partial x_j^b}(p^0)\Big) \wedge \Big\{1 \otimes \nabla^{\$_{g^{\mathcal{V}}}}_{e_k^f} + \frac{1}{2}\nu^k(F_A(p^0))\wedge\Big\}. \end{aligned}$$

The value (2.10) does not depend on the choice of the coordinates x *at* P^0, *the double integral is absolutely integrable and we have the formula*

$$\begin{aligned} \text{(2.13)} \quad & S\text{-}\log\det(0/2 : \delta_X \not{\partial}_g z) = \Big(\frac{2}{\sqrt{-1}}\Big)^{(n+2)/2} \\ & \quad \times \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^s \int_Z dg^Z(P^0)\, s\text{-}\log\det(\delta_X \not{\partial}_g z)(t,P^0)_{((1,\cdots,n),\emptyset),(1,2)}. \end{aligned}$$

Here the above function to be differentiated by s *is also absolutely integrable if* $\mathrm{Re}(s) > (n+2)/2$ *and has a meromorphic extension to* $\mathbb{C}\,(\ni s)$ *which is analytic at* $s = 0$.

3 The general adiabatic expansion of $e^{-t\partial\!\!\!/^2_{g^Z}}$

We put $g = g^{(p^0)}$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_g$ (see (2.8)), $|\cdot| = |\cdot|_g$ (the pointwise norm), $r(P,P') = r_g(P,P')$ (the distance between P and P' with respect to the metric g) and $r(P) = r(P,P^0)$ to simplify the description. Let us start our argument with reviewing the general adiabatic expansion theory ([14]) concerning the semi-group with C^∞-kernel $e^{-t\partial\!\!\!/^2_{g^Z}}$.

Let us consider a connection $\nabla^{g^{\mathcal{V}}} = P^{\mathcal{V}} \circ \nabla^{g^Z}$ of $\mathcal{V}$ where $P^{\mathcal{V}} : TZ = \mathcal{H} \oplus \mathcal{V} \to \mathcal{V}$ is the projection. This together with the Levi-Civita one ∇^{g^M} gives a new connection $\nabla^{g^Z \oplus} \equiv \pi^*\nabla^{g^M} \oplus \nabla^{g^{\mathcal{V}}}$ of $TZ = \mathcal{H}\oplus\mathcal{V}$, which is compatible with g^Z and whose torsion is equal to $2\nu(F_A)^\natural$ ([14, Lemma 3.1]). Note that the coordinates x are the $\nabla^{g^Z\oplus}$-normal ones. Let us take then a local g^Z-$SO(n+2)$-frame $e_*(A) = (e^b(A), e^f)$ which is $\nabla^{g^Z\oplus}$-parallel along the $\nabla^{g^Z\oplus}$-geodesics from P^0 and is equal to $(\partial/\partial x) = ((\partial/\partial x^b), (\partial/\partial x^f))$ at P^0 and let us denote its dual by $e^*(A) = (e_b, e_f(A))$. Take a local g^M-$SO(n)$-frame e^b which is ∇^{g^M}-parallel along the ∇^{g^M}-geodesics from p^0 and is equal to $(\partial/\partial x^b)$ at p^0, hence, with

$$e^b_i(x^b) = \sum (\partial/\partial x^b_j)_{x^b} \cdot v^b_{ji}(x^b)\,,\ v^b_{ji}(x^b) = \delta_{ji} + O(|x^b|^2), \tag{3.1}$$

then its A-horizontal lift clearly coincides with $e^b(A)$, that is, we have $e^b_i(A) = e^b_i - 2\sum \nu^k(A(e^b_i))\, e^f_k = e^b_i - 2\nu(A(e^b_i))^\natural$. We set $\nu(F_A) = \sum e^k_f(A)\wedge\nu^k(F_A)$ and denote by $\nabla^{\$_{g^Z}\oplus}$ the spinor connection associated to $\nabla^{g^Z}\oplus$. Then we have

$$\nabla^{\$_{g^Z}}_{e^b_i(A)} = \nabla^{\$_{g^Z}\oplus}_{e^b_i(A)} + \frac{1}{2}\rho_{g^Z}(\nu(e^i_b \vee F_A))\,,\quad \nabla^{\$_{g^Z}}_{e^f_k} = \nabla^{\$_{g^Z}\oplus}_{e^f_k} + \frac{1}{2}\rho_{g^Z}(\nu^k(F_A)), \tag{3.2}$$

$$\partial\!\!\!/_{g^Z} = \sum \rho_{g^Z}(e^i_b)\,\nabla^{\$_{g^Z}\oplus}_{e^b_i(A)} + \sum \rho_{g^Z}(e^k_f(A))\,\nabla^{\$_{g^Z}\oplus}_{e^f_k} - \frac{1}{2}\rho_{g^Z}(\nu(F_A)) \tag{3.3}$$

where $\vee$ denotes the interior product, hence, we have $e^i_b \vee F_A = \sum F_A(e^b_i, e^b_j)\cdot e^j_b$.

Now let us take the g^Z_ε-$SO(n+2)$-frames $e^\varepsilon_*(A) = (e^{b\varepsilon}(A), e^f) = (\varepsilon^{1/2}e^b(A), e^f)$, $e^*_\varepsilon(A) = (e_{b\varepsilon}, e_f(A)) = (\varepsilon^{-1/2}e_b, e_f(A))$ and consider the inclusion

$$\Gamma(\$_{g^Z_\varepsilon}|\pi^{-1}(U^b) \boxtimes \$^*_{g^Z_\varepsilon}||\pi^{-1}(U^b)) \subset \Gamma((\wedge T^*U^b \otimes_\pi \$_{g^{\mathcal{V}}}) \boxtimes (\wedge T^*U^b \otimes_\pi \$^*_{g^{\mathcal{V}}})). \tag{3.4}$$

That is, denoting by $s(e^\varepsilon_*(A))$ the local frame of $\$_{g^Z_\varepsilon}$ associated to $e^\varepsilon_*(A)$, we have $s(e^\varepsilon_*(A))(P) \boxtimes s(e^\varepsilon_*(A))^*(P')\cdot\phi \leftrightarrow s(e^\varepsilon_*(A))(p^0, P^f) \boxtimes s(e^\varepsilon_*(A))^*(p^0, P'^f)\cdot\phi = s(e^{b\varepsilon}(A))(p^0)$ $\otimes s(e^{b\varepsilon}(A))^*(p^0)\cdot s(e^f)(P^f) \boxtimes s(e^f)^*(P'^f)\cdot\phi \in C^\infty(\pi^{-1}(U^b)\times\pi^{-1}(U^b)), (\$_{g^M_\varepsilon,p^0} \otimes \$^*_{g^M_\varepsilon,p^0}) \otimes_\pi$ $(\$_{g^{\mathcal{V}},p^0} \boxtimes \$^*_{g^{\mathcal{V}},p^0})) \ni \rho_{g^M_\varepsilon}(e^I_{b\varepsilon})\cdot s(e^f)(P^f) \boxtimes s(e^f)^*(P'^f)\cdot\phi \leftrightarrow e^I_{b\varepsilon}(p^0)\cdot s(e^f)(P^f) \boxtimes s(e^f)^*(P'^f)\cdot$

$\phi = (dx^b)^I(0) \cdot s(e^f)(P^f) \boxtimes s(e^f)^*(P'^f) \cdot \varepsilon^{-|I|/2}\phi \leftrightarrow (dx^b)^I(x^b) \cdot s(e^f)(P^f) \boxtimes s(e^f)^*(P'^f) \cdot \varepsilon^{-|I|/2}\phi\,(\in C^\infty(U^b, \Gamma((\wedge T^*U^b \otimes_\pi \$_{g^V}) \boxtimes \$^*_{g^V,p^0}))) \mapsto \sum (dx^b)^I(x^b) \wedge (dx^b)^J(x^b) \otimes (dx^b)^J (x'^b) \cdot s(e^f)(P^f) \boxtimes s(e^f)^*(P'^f) \cdot \varepsilon^{-|I|/2}\phi$. Regarding $e^{-t\not{\partial}^2_{g^Z_\varepsilon}}$ as an element of the right hand side of (3.4), i.e., $e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P') \equiv \sum (dx^b)^I(x^b) \otimes (dx^b)^{I'}(x'^b) \cdot \left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right)_{II'}$ (compare with (2.5)), we set

$$(3.5)\quad \partial_x^\alpha \partial_{x'}^{\alpha'} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P') \equiv \sum (dx^b)^I(x^b) \otimes (dx^b)^{I'}(x'^b) \cdot \partial_x^\alpha \partial_{x'}^{\alpha'} \left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right)_{II'}$$
$$\equiv \sum ((dx^b)^I \otimes s(e^f)_k)(P) \otimes ((dx^b)^{I'} \otimes s(e^f)^*_{k'})(P') \cdot \partial_x^\alpha \partial_{x'}^{\alpha'} \left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right)_{II'}^{(k,k')}$$

$$(3.6)\quad \text{with}\quad \left|\partial_x^\alpha \partial_{x'}^{\alpha'} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right| \equiv \left\{\sum \left|\partial_x^\alpha \partial_{x'}^{\alpha'} \left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right)_{II'}\right|^2_{\$_{g^V}}\right\}^{1/2}$$
$$\equiv \left\{\sum \left|\partial_x^\alpha \partial_{x'}^{\alpha'} \left(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P,P')\right)_{II'}^{(k,k')}\right|^2\right\}^{1/2}$$

where we put $\partial_x^\alpha \equiv (\partial/\partial x)^\alpha(x) = (\partial/\partial x^b)^{\alpha^b}(x)\,(\partial/\partial x^f)^{\alpha^f}(x) = (\partial/\partial x_1^b)^{\alpha_1^b} \cdots (\partial/\partial x_n^b)^{\alpha_n^b}$ $(\partial/\partial x_1^f)^{\alpha_1^f}(\partial/\partial x_2^f)^{\alpha_2^f}$, etc. Then we have

Proposition 3.1 *([14, Theorem 1.2]). For any integer $m_0 \geq 0$, there exist C^∞-cross-sections $E^{(m/2)}(t,P^0,P,P')$ $(m = 0,1,\cdots,m_0)$ and $E^{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0,P,P')$ belonging to the right hand side of (3.4), which are also C^∞ with respect to the variable P^0 (and $\varepsilon^{1/2}$), and satisfying the following condition: For any α and α', when $\varepsilon^{1/2} \to 0$, (3.5) with $(P,P') = (P^0,P^0)$ has the series expansion*

$$(3.7)\quad \partial_x^\alpha \partial_{x'}^{\alpha'} e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0) = \sum_{m=0}^{m_0} \varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+m/2}\, \partial_x^\alpha \partial_{x'}^{\alpha'} E^{(m/2)}(t,P^0)$$
$$+\,\varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+(m_0+1)/2}\, \partial_x^\alpha \partial_{x'}^{\alpha'} E^{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0)$$

where we put $|\alpha^b| = \sum \alpha_i^b$ etc. and $\partial_x^\alpha \partial_{x'}^{\alpha'} E^{(m/2)}(t,P^0)$ etc. mean $\partial_x^\alpha \partial_{x'}^{\alpha'} E^{(m/2)}(t,P^0,P,P')|_{P=P'=0}$ etc. Further, there exist constants $\lambda > 0$, $C > 0$ and an integer $N > 0$ satisfying

$$(3.8)\quad \begin{aligned} &\left|\partial_x^\alpha \partial_{x'}^{\alpha'} E^{(m/2)}(t,P^0)\right| \leq C\, e^{-t\lambda}\, t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}} + 1\Big),\\ &\left|\partial_x^\alpha \partial_{x'}^{\alpha'} E^{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0)\right| \leq C\, t^{1/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}} + t^N\Big)\\ &\quad (0 \leq \forall m \leq m_0,\ 0 < \forall \varepsilon^{1/2} \leq \varepsilon_0^{1/2},\ 0 < \forall t < \infty,\ \forall P^0 \in Z). \end{aligned}$$

Next let us examine the terms $E^{(m/2)}(t,P^0,P,P')$ closely. We take a metric $g^{M(p^0)}$ on $M(p^0)\,(\supset U^b)$ so that its restriction to U^b is equal to g^M, outside some open set $\tilde{U}^b\,(\supset$

$\tilde{U}^b$) it is trivial, and, moreover, the coordinates x^b are the $g^{M(p^0)}$-normal ones all over $M(p^0)$. Also we will take a connection A of $M(p^0) \times Sp(1)$ ($\supset U^b \times Sp(1) = P_{Sp(1)}|U^b$) which coincides with the given one on U^b and vanishes outside $\tilde{U}^b$. Consequently we have the trivial Spin^q structure and the trivial $\mathbb{C}P^1$-fibration

$$\begin{aligned}(3.9)\qquad &\Xi^q(p^0) : P_{\mathrm{Spin}^q(n)}(M(p^0)) = P_{\mathrm{Spin}(n)}(M(p^0)) \times_{\mathbf{Z}_2} P_{Sp(1)}(M(p^0)) \\ &\qquad \to P_{SO(n)}(M(p^0), g^{M(p^0)}) \times P_{SO(3)}(M(p^0)), \\ &\pi(p^0) : Z(p^0) = P_{\mathrm{Spin}^q(n)}(M(p^0)) \times_{\mathrm{can}} \mathbb{C}P^1 = M(p^0) \times Z_{p^0} \to M(p^0)\end{aligned}$$

which coincide with (1.1), (1.2) on U^b. Take then another metric $g^{Z(p^0)} = \pi(p^0)^* g^{M(p^0)} + g^{\mathcal{V}}$ on $Z(p^0)$ (compare with (2.9)) and let us use the same symbols as for Z, i.e., denote by $e_*(A) = (e^b(A), e^f)$, etc. " a local $g^{Z(p^0)}$-$SO(n+2)$-frame over $M(p^0) \times U^f$ which is $\nabla^{g^{Z(p^0)}\oplus}$-parallel along the $\nabla^{g^{Z(p^0)}\oplus}$-geodesics from P^0 and is equal to $(\partial/\partial x)$ at P^0," etc. We consider further the transformation of $Z(p^0)$

$$(3.10)\qquad \iota_\varepsilon : Z(p^0) \cong Z(p^0), \quad P = (x^b, P^f) \mapsto \iota_\varepsilon(P) = (\varepsilon^{1/2} x^b, P^f)$$

and set

$$\begin{aligned}(3.11)\qquad & e_*(\varepsilon, \iota_\varepsilon^* A) = \iota_\varepsilon^* e_*^\varepsilon(A) = (e_1(\varepsilon, \iota_\varepsilon^* A), \cdots) = (e^b(\varepsilon, \iota_\varepsilon^* A), e^f), \\ &\text{hence, with } e_i^b(\varepsilon, \iota_\varepsilon^* A)(P) = (\iota_\varepsilon^* e_i^{b\varepsilon}(A))(P) \\ &= \sum (\partial/\partial x_j^b)_P \cdot v_{ji}^b(\iota_\varepsilon(P)) - \varepsilon^{1/2}\, 2 \sum \nu^k(A(e_i^b))(\iota_\varepsilon(P))\, e_k^f(P),\end{aligned}$$

which is a local $SO(n+2)$-frame with respect to the metric $\iota_\varepsilon^* g_\varepsilon^{Z(p^0)} = \pi_{\iota_\varepsilon^* A}^* \iota_\varepsilon^* g_\varepsilon^{M(p^0)} + g^{\mathcal{V}} \equiv \sum e_b^i(\varepsilon) \otimes e_b^i(\varepsilon) + e_f^k(\varepsilon, \iota_\varepsilon^* A) \otimes e_f^k(\varepsilon, \iota_\varepsilon^* A)$. We have now the (globally defined) elliptic differential operator acting on the cross-sections of $\wedge T^* M(p^0) \otimes_\pi \$_{g^{\mathcal{V}}}$

$$\begin{aligned}(3.12)\qquad & (\not{\partial}^{(\varepsilon)})^2 = -\sum \Big(\nabla^{(\varepsilon)}_{e_i(\varepsilon, \iota_\varepsilon^* A)} \nabla^{(\varepsilon)}_{e_i(\varepsilon, \iota_\varepsilon^* A)} - \nabla^{(\varepsilon)}_{\nabla^{\iota_\varepsilon^* g_\varepsilon^{Z(p^0)}}_{e_i(\varepsilon,\iota_\varepsilon^* A)} e_i(\varepsilon, \iota_\varepsilon^* A)} \Big) \\ &\quad + \frac{1}{4}\Big\{ \varepsilon \kappa_{g^{M(p^0)}} + 2 - \varepsilon^2 \sum \nu^k(F_A(e_i^b, e_j^b))^2 \Big\}(\iota_\varepsilon(\cdot)) \quad \text{with} \\ & \rho^{(\varepsilon)}(e_b^i) = dx_i^b \wedge - \varepsilon dx_i^b \vee, \ \rho^{(\varepsilon)}(e_f^k(A)) = \rho_{g^{Z(p^0)}}(e_f^k(A)), \\ & \nabla^{(\varepsilon)}_{e_i^b(\varepsilon, \iota_\varepsilon^* A)} = e_i^b(\varepsilon, \iota_\varepsilon^* A) + \frac{\varepsilon^{-1/2}}{4} \sum C(\nabla^{g^{M(p^0)}})(e_i^b)_{i_2 i_1}(\iota_\varepsilon(\cdot))\, \rho^{(\varepsilon)}(e_b^{i_1}) \rho^{(\varepsilon)}(e_b^{i_2}) \\ &\quad + \varepsilon^{1/2} \Big(e_1^f + C(\nabla^{g^{\mathcal{V}}})(e_2^f)_{12} \Big) \nu^2(A(e_i^b))(\iota_\varepsilon(\cdot))\, \rho^{(\varepsilon)}(e_f^1(A) \wedge e_f^2(A)) \\ &\quad + \frac{\varepsilon^{1/2}}{2}\, \rho^{(\varepsilon)}(\nu(e_b^i \vee F_A))(\iota_\varepsilon(\cdot)) \quad (\nabla^{g^{M(p^0)}}_{e_i^b} e_{i_1}^b = \sum C(\nabla^{g^{M(p^0)}})(e_i^b)_{i_2 i_1} e_{i_2}^b, \text{ etc.}), \\ & \nabla^{(\varepsilon)}_{e_k^f} = e_k^f - \frac{1}{2} C(\nabla^{g^{\mathcal{V}}})(e_k^f)_{12}\, \rho^{(\varepsilon)}(e_f^1(A) \wedge e_f^2(A)) + \frac{1}{2} \rho^{(\varepsilon)}(\nu^k(F_A))(\iota_\varepsilon(\cdot))\end{aligned}$$

where $\kappa_{g^{M(p^0)}}$ is the scalar curvature of $g^{M(p^0)}$ and we put $\rho^{(\varepsilon)}(\nu(e_b^i \vee F_A))(\iota_\varepsilon(\cdot)) = \sum \nu^k(F_A(e_i^b, e_j^b))(\iota_\varepsilon(\cdot))\, \rho^{(\varepsilon)}(e_f^k(A) \wedge e_b^j)$, etc. Note that this is the pull-back of the square of the Dirac operator $\not{\partial}_{\iota_\varepsilon^* g_\varepsilon^{Z(p^0)}}$ by the bundle isomorphism $\wedge T^*M(p^0) \otimes_\pi \$_{g^\mathcal{V}} \cong \wedge T^*M(p^0) \otimes_\pi \$_{g^\mathcal{V}}$, $(dx^b)^I(x) \otimes h(x) \mapsto (\varepsilon^{1/2} e_b(\varepsilon))^I(x) \otimes h(x)$ (see (3.2), (3.3) and [14, (2.26)]). Hence, if we write (3.12) as $-\sum \dfrac{\partial}{\partial x_i} a_{ij}(\varepsilon, x) \dfrac{\partial}{\partial x_j} + \sum a_i(\varepsilon, x) \dfrac{\partial}{\partial x_i} + c(\varepsilon, x)$ $(a_{ij} = a_{ji})$ and fix the parameter $\varepsilon > 0$, then the finite-times derivatives of the coefficients $a_{ij}(\varepsilon, x)$, $a_i(\varepsilon, x)$, $c(\varepsilon, x)$ are all bounded on $Z(p^0)$. The Yosida's theorem ([18, Chapter IX]) says now that the parabolic equation with the initial condition

$$(3.13) \qquad \Big(\frac{\partial}{\partial t} + (\not{\partial}^{(\varepsilon)})^2\Big)\psi = 0, \quad \psi|_{t=0} = \psi_0 \in L^2\Gamma(\wedge T^*M(p^0) \otimes_\pi \$_{g^\mathcal{V}}, g)$$

has a (C^0) semi-group with C^∞-kernel $E^{(\varepsilon)}(t, p^0, P, P')$, which is a cross-section of (2.2).

Proposition 3.2 *([14, Theorem 1.3, Corollary 2.3 and the proof of Proposition 2.2 for $E(t,\varepsilon)$ with t small]). When $\varepsilon^{1/2} \to 0$, the kernel has a Taylor expansion*

$$(3.14) \qquad E^{(\varepsilon)}(t, p^0, P, P') = \sum_{m=0}^{\infty} \varepsilon^{m/2}\, E^{(m/2:)}(t, p^0, P, P')$$

$$(3.15) \qquad \text{with } \; E^{(0/2:)}(t, p^0, P, P') = E(t, p^0, P, P')$$

and, as for $E^{(m/2)}$ in Proposition 3.1, we may take

$$(3.16) \qquad E^{(m/2)}(t, P^0, P, P') = E^{(m/2:)}(t, p^0, P, P')\, \det v^b(P')$$

where we set $\det v^b(P') = \det v^b(x'^b) = \det(g^M(\partial/\partial x_i^b, \partial/\partial x_j^b)(x'^b))^{-1/2} = 1 + O(|x'^b|^2)$ *(see (3.1)). For given integer $m_0 \geq 0$, $E^{(m/2:)}(t, p^0, P, P')$ $(0 \leq m \leq m_0)$ and the remainder term* $E^{((m_0+1)/2:\varepsilon^{1/2})}(t, p^0, P, P') = \varepsilon^{-(m_0+1)/2}\{E^{(\varepsilon)}(t, p^0, P, P') - \sum_{m=0}^{m_0} \varepsilon^{m/2}$ $E^{(m/2:)}(t, p^0, P, P')\}$ *have the asymptotic expansions, when $t \to 0$,*

$$(3.17) \quad E^{(m/2:\cdot)}(t, p^0, P, P') = \frac{e^{-r(P,P')^2/4t}}{(4\pi t)^{(n+2)/2}} \sum_{-(m+2)/3 \leq i}^{i \neq 0\ (if\ m \geq 1)} t^i E^{(m/2:\cdot)}(i : p^0, P, P') \quad \textit{with}$$

$$\Big|E^{(m/2:\cdot)}(i : p^0, P, P')\Big| \leq C_{m_0}(1+r(P))^m(1+r(P'))^{n+m} \begin{cases} 1 \;:\; i \geq 0, \\ r(P, P')^{3|i|} \;:\; i < 0, \end{cases}$$

$$\Big|\partial_x^\alpha \partial_{x'}^{\alpha'} E^{(m/2:\cdot)}(t, p^0, P, P')\Big|$$
$$\leq C_{m_0}\,(1+r(P))^m(1+r(P'))^{n+m} t^{-(n+2+|\alpha|+|\alpha'|)/2+(1-\delta_{0m})/2}\, e^{-r(P,P')^2/5t}$$
$$(0 \leq \forall m \leq m_0+1\,,\; 0 < \forall \varepsilon^{1/2} \leq \varepsilon_0^{1/2}\,,\; 0 < \forall t \leq T_0\,,\; \forall P, \forall P' \in Z(p^0))$$

where we set $E^{(m/2:\cdot)} \equiv E^{(m/2:)}$ $(m \leq m_0)$ *and* $E^{((m_0+1)/2:\cdot)} \equiv E^{((m_0+1)/2:\varepsilon^{1/2})}$.

Here let us investigate $E^{(1/2:)}(t,p^0,P,P')$ closely.

Lemma 3.3 . *(3.12) has the Taylor expansion*

$$(\not{\partial}^{(\varepsilon)})^2 = \sum_{m\geq 0} \varepsilon^{m/2}\,(\not{\partial}^2)^{(m/2:)} \tag{3.18}$$

with

$$(\not{\partial}^2)^{(0/2:)} = -\sum\Big\{\frac{\partial}{\partial x_i^b} + \frac{1}{4}\sum x_j^b R_{ji}^{g^M}(p^0)\wedge\Big\}^2 + \mathcal{A}_{p^0}^2, \tag{3.19}$$

$$\begin{aligned}(\not{\partial}^2)^{(1/2:)} = &\sum\Big\{\frac{1}{6}\sum x_{j_1}^b x_{j_2}^b \frac{\partial R_{i_2 i_1 i j_1}^{g^M}}{\partial x_{j_2}^b}(p^0)\, dx_{i_1}^b\wedge dx_{i_2}^b\wedge \\ &-\rho^{(0)}(\nu(e_b^i\vee F_A))(\iota_0(\cdot))\Big\}\Big\{\frac{\partial}{\partial x_i^b} + \frac{1}{4}\sum x_j^b R_{ji}^{g^M}(p^0)\wedge\Big\} \\ &-\sum x_j^b\,\nu^k\big(\frac{\partial F_A}{\partial x_j^b}(p^0)\big)\wedge\Big\{1\otimes\nabla_{e_k^f}^{\$_g\nu} + \frac{1}{2}\nu^k(F_A(p^0))\wedge\Big\}.\end{aligned} \tag{3.20}$$

Proof. Set $R_{ji}^{g^M} = R_{ji}^{g^M}(p^0)$, etc. and refer to (3.12). We have

$$\nabla_{e_i(\varepsilon,\iota_\varepsilon^* A)}^{(\varepsilon)} = \sum_{m\geq 0}\varepsilon^{m/2}\nabla_i^{(m/2:)},\ \nabla_i^{(m/2:)}\equiv\nabla_{b,i}^{(m/2:)}\ (i\leq n),\ \nabla_{n+k}^{(m/2:)}\equiv\nabla_{f,k}^{(m/2:)} \text{ with} \tag{3.21}$$

$$\nabla_{b,i}^{(0/2:)} = \frac{\partial}{\partial x_i^b} + \frac{1}{4}\sum x_j^b\, R_{ji}^{g^M}\wedge,\quad \nabla_{f,k}^{(0/2:)} = 1\otimes\nabla_{e_k^f}^{\$_g\nu} + \frac{1}{2}\nu^k(F_A)\wedge,$$

$$\nabla_{b,i}^{(1/2:)} = -\frac{1}{12}\sum x_{j_1}^b x_{j_2}^b\frac{\partial R_{i_2 i_1 i j_1}^{g^M}}{\partial x_{j_2}^b}\, dx_{i_1}^b\wedge dx_{i_2}^b\wedge + \frac{1}{2}\,\rho^{(0)}(\nu(e_b^i\vee F_A))(\iota_0(\cdot))$$

$$= -\frac{1}{12}\sum x_{j_1}^b x_{j_2}^b\frac{\partial R_{i_2 i_1 i j_2}^{g^M}}{\partial x_{j_1}^b}\, dx_{i_1}^b\wedge dx_{i_2}^b\wedge + \frac{1}{2}\,\rho^{(0)}(\nu(e_b^i\vee F_A))(\iota_0(\cdot)),$$

$$\nabla_{f,k}^{(1/2:)} = \frac{1}{2}\sum x_j^b\,\nu^k\big(\frac{\partial F_A}{\partial x_j^b}\big)\wedge$$

and

$$\nabla_{e_i^b(\varepsilon,\iota_\varepsilon^* A)}^{\iota_\varepsilon^* g_\varepsilon^{Z(p^0)}} e_i^b(\varepsilon,\iota_\varepsilon^* A) = \sum C(\nabla^{\iota_\varepsilon^* g_\varepsilon^{M(p^0)}})(e_i^b(\varepsilon,\iota_\varepsilon^* A))_{ji} e_j^b(\varepsilon,\iota_\varepsilon^* A)$$

$$= \sum\varepsilon^{1/2} C(\nabla^{g^{M(p^0)}})(e_i^b)_{ji}(\iota_\varepsilon(\cdot))\, e_j^b(\varepsilon,\iota_\varepsilon^* A),$$

$$\nabla_{e_k^f}^{\iota_\varepsilon^* g_\varepsilon^{Z(p^0)}} e_k^f = \nabla_{e_k^f}^{g^\nu} e_k^f = \sum C(\nabla^{g^\nu})(e_k^f)_{k'k}\, e_{k'}^f,$$

$$\nabla^{(\varepsilon)}_{\nabla_{e_i^b(\varepsilon,\iota_\varepsilon^* A)}^{\iota_\varepsilon^* g_\varepsilon^{Z(p^0)}} e_i^b(\varepsilon,\iota_\varepsilon^* A)} = \sum\varepsilon^{1/2} C(\nabla^{g^{M(p^0)}})(e_i^b)_{ji}(\iota_\varepsilon(\cdot))\,\nabla_{e_j^b(\varepsilon,\iota_\varepsilon^* A)}^{(\varepsilon)} = \mathcal{O}(\varepsilon^{2/2}), \tag{3.22}$$

$$\nabla^{(\varepsilon)}_{\nabla_{e_k^f}^{\iota_\varepsilon^* g_\varepsilon^{Z(p^0)}} e_k^f} = \sum C(\nabla^{g^\nu})(e_k^f)_{k'k}\,\nabla_{e_{k'}^f}^{(\varepsilon)} \tag{3.23}$$

and

$$\frac{1}{4}\Big\{\varepsilon\kappa_{g^{M(p^0)}} + 2 - \varepsilon^2\sum\nu^k(F_A(e_i^b, e_j^b))^2\Big\}(\iota_\varepsilon(\cdot)) = \frac{1}{2} + \mathcal{O}(\varepsilon^{2/2}). \tag{3.24}$$

Hence, certainly we have the Taylor expansion (3.18) and the formula (3.19). (3.21) implies

$$\begin{aligned}
(\not{\partial}^2)^{(1/2:)} &= -\sum\Big(\nabla_i^{(1/2:)}\nabla_i^{(0/2:)} + \nabla_i^{(0/2:)}\nabla_i^{(1/2:)}\Big) + \sum C(\nabla^{g^\nu})(e_k^f)_{k'k}\nabla_{f,k'}^{(1/2:)} \\
&= -2\sum\nabla_i^{(1/2:)}\nabla_i^{(0/2:)} - \frac{1}{2}\sum x_j^b\, e_k^f \nu^k(\frac{\partial F_A}{\partial x_j^b})\wedge + \sum C(\nabla^{g^\nu})(e_{k'}^f)_{kk'}\nabla_{f,k}^{(1/2:)} \\
&= -2\sum\nabla_i^{(1/2:)}\nabla_i^{(0/2:)},
\end{aligned}$$

which yields (3.20).

Let us consider then the Fréchet space consisting of rapidly decreasing cross-sections

$$\mathcal{S} \equiv \{\psi\in\Gamma(\wedge T^*M\otimes_\pi \$_{g^\nu}) \mid \lim_{r(P)\to\infty}|(1+r(P))^\ell\partial^\alpha\psi(P)| = 0\ ({}^\forall\ell,{}^\forall\alpha)\} \tag{3.25}$$

$$\text{with semi-norms } p_{\ell,k}(\psi) \equiv \sup_{P\in Z(p^0),|\alpha|\le 2k}|(1+r(P))^\ell\partial^\alpha\psi(P)|$$

and the parabolic equation with the initial condition

$$\Big(\frac{\partial}{\partial t} + (\not{\partial}^2)^{(0/2:)}\Big)\psi = 0\,,\ \psi|_{t=0} = \psi_0\in\mathcal{S}. \tag{3.26}$$

Proposition 3.4 *([14, Lemmata 5.2, 5.3]).*

(1) Set $p_{\ell,k}^{(0)}(\psi) \equiv \sup_{P\in Z(p^0),k'\le k}|(1+r(P))^\ell((\not{\partial}^2)^{(0/2:)})^{k'}\psi(P)|$. *Then the two kinds of families of semi-norms* $\{p_{\ell,k}^{(0)}\}$ *and* $\{p_{\ell,k}\}$ *are equivalent to each other.*

(2) $\{E(t,p^0)\}_{0<t<\infty}$ *defines an equicontinuous* (C^0) *semi-group with* C^∞*-kernel associated to the parabolic equation (3.26). Hence, in particular, we have*

$$\Big(\frac{\partial}{\partial t} + (\not{\partial}^2)^{(0/2:)}\Big)E(t,p^0,P,P') = 0, \tag{3.27}$$

and, for any $T_0 > 0$ *and any semi-norm* $p_{\ell,k}$*, there exists a constant* $C>0$ *and a semi-norm* $p_{\ell',k'}$ *satisfying*

$$p_{\ell,k}(E(t,p^0)\psi - \psi) \le C\,t^{1/2}\,p_{\ell',k'}(\psi)\quad (0<\forall t\le T_0,\ \forall\psi\in\mathcal{S}). \tag{3.28}$$

Now we have

Proposition 3.5. *We have*

$$\begin{aligned}
&E^{(1/2:)}(t,p^0,P,P') \\
&= -\int_0^t d\tau\int_{Z(p^0)} dg(Q)\,\langle E(t-\tau,p^0,P,Q),(\not{\partial}^2)^{(1/2:)}E(\tau,p^0,Q,P')\rangle.
\end{aligned} \tag{3.29}$$

Here the double integral on the right hand side is absolutely integrable.

Remark. Similarly to the following proof, we can show that generally we have

$$E^{(m/2:)}(t,p^0,P,P')$$
$$= -\int_0^t d\tau \int_{Z(p^0)} dg(Q)\,\langle E(t-\tau,p^0,P,Q), \sum_{m_1+m_2=m}^{m_2<m} (\not{\partial}^2)^{(m_1/2:)} E^{(m_2/2:)}(\tau,p^0,Q,P')\rangle.$$

But the double integral on the right hand side may not be absolutely integrable.

Proof. We abbreviate $E(t,p^0,P,P')$ to $E(t,P,P')$. The right hand side denoted $\mathcal{E}(t,P,P')$ is absolutely integrable and there exists a constant $C>0$ satisfying

$$\text{(3.30)}\qquad \begin{aligned} \left|\mathcal{E}(t,P,P')\right| &\le \int_0^t d\tau \int_{Z(p^0)} dg(Q) \left|\langle E(t-\tau,P,Q), (\not{\partial}^2)^{(1/2:)} E(\tau,Q,P')\rangle\right| \\ &\le C\, t^{-(n+2)/2+1/2} e^{-r(P,P')^2/7t} \quad (0<\forall t\le T_0,\ \forall P,\forall P'\in Z(p^0)). \end{aligned}$$

Actually, observing (3.17) and (3.20), we have

$$\text{(3.31)}\qquad \begin{aligned} &\int_{Z(p^0)} dg(Q) \left|\langle E(t-\tau,P,Q), (\not{\partial}^2)^{(1/2:)} E(\tau,Q,P')\rangle\right| \\ &\le \int_{Z(p^0)} dg(Q) \left|E(t-\tau,P,Q)\right|\cdot\left|(\not{\partial}^2)^{(1/2:)} E(\tau,Q,P')\right| \\ &\le C_1\,(1+r(P')^N)(t-\tau)^{-(n+2)/2}\tau^{-(n+2)/2-1/2} \\ &\qquad \cdot \int_{Z(p^0)} dg(Q)\,(1+r(Q)^N) e^{-r(P,Q)^2/5(t-\tau)} e^{-r(Q,P')^2/5\tau} \\ &\le C_2\,(1+r(P')^{2N})(t-\tau)^{-(n+2)/2}\tau^{-(n+2)/2-1/2} \\ &\qquad \cdot \int_{Z(p^0)} dg(Q)\,(1+r(Q,P')^N) e^{-r(P,Q)^2/5(t-\tau)} e^{-r(Q,P')^2/5\tau} \\ &\le C_3\,(1+r(P')^{2N})(t-\tau)^{-(n+2)/2}\tau^{-(n+2)/2-1/2} \\ &\qquad \cdot \int_{Z(p^0)} dg(Q)\,(1+\tau^{N/2}) e^{-r(P,Q)^2/5(t-\tau)} e^{-r(Q,P')^2/6\tau} \\ &\le C_4\,(1+r(P')^{2N})(t-\tau)^{-(n+2)/2}\tau^{-(n+2)/2-1/2} \\ &\qquad \cdot \int_{Z(p^0)} dg(Q)\, e^{-r(P,Q)^2/6(t-\tau)-r(Q,P')^2/6\tau} \end{aligned}$$

and

$$\text{(3.32)}\qquad \begin{aligned} &\int_0^{t/2} d\tau (t-\tau)^{-(n+2)/2}\tau^{-(n+2)/2-1/2} \int_{Z(p^0)} dg(Q)\, e^{-r(P,Q)^2/6(t-\tau)-r(Q,P')^2/6\tau} \\ &\le C_1\, t^{-(n+2)/2} e^{-r(P,P')^2/7t} \int_0^{t/2} d\tau\, \tau^{-(n+3)/2} \\ &\qquad \cdot \int_{Z(p^0)} dg(Q)\, e^{-r(P,Q)^2/7(t-\tau)-r(Q,P')^2/7\tau} \\ &\le C_1\, t^{-(n+2)/2} e^{-r(P,P')^2/7t} \int_0^{t/2} d\tau\, \tau^{-(n+3)/2} \int_{Z(p^0)} dg(Q)\, e^{-r(Q,P')^2/7\tau} \\ &\le C_2\, t^{-(n+2)/2+1/2} e^{-r(P,P')^2/7t}, \end{aligned}$$

$$(3.33)\qquad \int_{t/2}^{t} d\tau\,(t-\tau)^{-(n+2)/2}\tau^{-(n+2)/2-1/2}\int_{Z(p^0)} dg(Q)\, e^{-r(P,Q)^2/6(t-\tau)-r(Q,P')^2/6\tau}$$
$$= \int_0^{t/2} d\tau\,\tau^{-(n+2)/2}(t-\tau)^{-(n+2)/2-1/2} \cdot \int_{Z(p^0)} dg(Q)\, e^{-r(P,Q)^2/6\tau-r(Q,P')^2/6(t-\tau)}$$
$$\le C_1\, t^{-(n+2)/2-1/2} e^{-r(P,P')^2/7t}\int_0^{t/2} d\tau\,\tau^{-(n+2)/2}\int_{Z(p^0)} dg(Q)\, e^{-r(P,Q)^2/7\tau}$$
$$\le C_2\, t^{-(n+2)/2+1/2} e^{-r(P,P')^2/7t}.$$

It will now suffice for the proof of the proposition to show that, for any $\varphi \in \mathcal{S}$, we have

$$(3.34)\qquad \int_{Z(p^0)} dg(P')\,\langle E^{(1/2:)}(t,P,P'),\varphi(P')\rangle = \int_{Z(p^0)} dg(P')\,\langle \mathcal{E}(t,P,P'),\varphi(P')\rangle.$$

(Remark that the both hand sides are absolutely integrable because of (3.30) and (3.17) with $m = 1$, $\alpha = \alpha' = \emptyset$.) To show it, we want to prove the following assertion: The left hand side $(E^{(1/2:)}(t)\varphi)(P)$ and the right hand side $(\mathcal{E}(t)\varphi)(P)$ are both of class C^∞ with respect to P, (at least) of class C^1 with respect to t and satisfy

$$(3.35)\qquad \begin{aligned} &\Big(\frac{\partial}{\partial t} + (\not\partial^2)^{(0/2:)}\Big)(E^{(1/2:)}(t)\varphi)(P) + (\not\partial^2)^{(1/2:)}(E(t)\varphi)(P) = 0,\\ &\Big(\frac{\partial}{\partial t} + (\not\partial^2)^{(0/2:)}\Big)(\mathcal{E}(t)\varphi)(P) + (\not\partial^2)^{(1/2:)}(E(t)\varphi)(P) = 0, \end{aligned}$$

and, for any $T_0 > 0$ and any semi-norm $p_{\ell,k}$ (at (3.25)), there exists a constant $C > 0$ and a semi-norm $p_{\ell',k'}$ satisfying

$$(3.36)\qquad p_{\ell,k}(E^{(1/2:)}(t)\varphi) \le t\,C\,p_{\ell',k'}(\varphi),\ \ p_{\ell,k}(\mathcal{E}(t)\varphi) \le t\,C\,p_{\ell',k'}(\varphi) \quad (0 < \forall t \le T_0,\ \forall\varphi \in \mathcal{S}).$$

Assume that this holds and take $\varphi \in \mathcal{S}$. Then $\psi = (E^{(1/2:)}(t) - \mathcal{E}(t))\varphi \in \mathcal{S}$ is a solution of (3.26) with $\psi_0 = 0$. Hence we have $\psi = 0$ because of Proposition 3.4 **(2)**. That is, we obtain the formula (3.34). Accordingly let us prove the above assertion. First, as for the assertion for $(\mathcal{E}(t)\varphi)(P)$: Since $\left|\langle E(t-\tau,P,Q), \langle(\not\partial^2)^{(1/2:)}E(\tau,Q,P'),\varphi(P')\rangle\rangle\right|$ is integrable on $(0,t] \times Z(p^0) \times Z(p^0)\,(\ni (\tau,Q,P'))$ (see (3.30), etc.), one may reverse the order of the integrals of $(\mathcal{E}(t)\varphi)(P)$. Hence we may put

$$\tilde\varphi(\tau,Q) \equiv ((\not\partial^2)^{(1/2:)}E(\tau)\varphi)(Q) = \int_{Z(p^0)} dg(P')\,\langle(\not\partial^2)^{(1/2:)}E(\tau,Q,P'),\varphi(P')\rangle,$$
$$\phi(t,\tau,P) \equiv (E(t-\tau)\tilde\varphi(\tau))(P),\quad (\mathcal{E}(t)\varphi)(P) = -\int_0^t d\tau\,\phi(t,\tau,P).$$

Clearly $\tilde{\varphi}(\tau)$ and $\phi(t,\tau)$ belong to $\mathcal{S}$ and (3.20), (3.28) imply

$$(3.37)\qquad p_{\ell,k}(\phi(t,\tau)) \le C_1\, p_{\ell_1,k_1}(\tilde{\varphi}(\tau)) \le C_2\, p_{\ell_2,k_2}(E(\tau)\varphi) \le C_3\, p_{\ell_3,k_3}(\varphi).$$

Hence we obtain the second inequality in (3.36). Further (3.37) implies

$$(3.38)\qquad (\not{\partial}^2)^{(0/2:)}(\mathcal{E}(t)\varphi)(P) = -\int_0^t d\tau\, (\not{\partial}^2)^{(0/2:)}\phi(t,\tau,P),$$

$$(3.39)\qquad |(\partial/\partial t)\phi(t,\tau,P)| = \left|(\not{\partial}^2)^{(0/2:)}\phi(t,\tau,P)\right| \le C_4\, p_{\ell_4,k_4}(\varphi) \quad (0<\forall\tau<t).$$

Set $\phi(t,t,P) = \tilde{\varphi}(t,P)$. Then the function $\phi(t,\tau,P)$ of τ is continuous on $[\delta,t]$ $(\delta>0)$ and has the estimate (3.39). Hence we have

$$\begin{aligned}(3.40)\qquad (\partial/\partial t)(\mathcal{E}(t)\varphi)(P) &= (\partial/\partial t)\Big\{-\int_\delta^t d\tau\,\phi(t,\tau,P) - \int_0^\delta d\tau\,\phi(t,\tau,P)\Big\}\\ &= \Big\{-\phi(t,t,P) - \int_\delta^t d\tau\,(\partial/\partial t)\phi(t,\tau,P)\Big\} - \int_0^\delta d\tau\,(\partial/\partial t)\phi(t,\tau,P)\\ &= -\phi(t,t,P) - \int_0^t d\tau\,(\partial/\partial t)\phi(t,\tau,P),\end{aligned}$$

which shows that $(\mathcal{E}(t)\varphi)(P)$ is of class C^1 with respect to t and, combined with (3.27) and (3.38), implies the second equality in (3.35). Second, as for the assertion for $(\boldsymbol{E}^{(1/2:)}(t)\varphi)(P)$: The first equality in (3.35) will be obvious. Let us show the first inequality in (3.36) in the following. By observing (3.17), the inequality with $\ell=k=0$ holds obviously. As for the inequality with $k=0$, we have

$$\begin{aligned}\left|(1+r(P))^\ell(\boldsymbol{E}^{(1/2:)}(t)\varphi)(P)\right| &\le \left|(\boldsymbol{E}^{(1/2:)}(t)(1+r(\cdot))^\ell\varphi)(P)\right|\\ &+ \left|(1+r(P))^\ell(\boldsymbol{E}^{(1/2:)}(t)\varphi)(P) - (\boldsymbol{E}^{(1/2:)}(t)(1+r(\cdot))^\ell\varphi)(P)\right|\end{aligned}$$

and the second term on the right hand side can be estimated as

$$\begin{aligned}&\left|\int dg(P')\,\langle((1+r(P))^\ell - (1+r(P'))^\ell)\boldsymbol{E}^{(1/2:)}(t,P,P'),\varphi(P')\rangle\right|\\ &\le C_1 \sum_{\ell_1+\ell_2=\ell}^{\ell_1>0}\int dg(P')\left|(r(P)-r(P'))^{\ell_1}(1+r(P'))^{\ell_2}\boldsymbol{E}^{(1/2:)}(t,P,P')\right|\,|\varphi(P')|\\ &\le C_2\sum\int dg(P')\left|r(P,P')^{\ell_1}(1+r(P'))^{\ell_2}\boldsymbol{E}^{(1/2:)}(t,P,P')\right|\,|\varphi(P')|\\ &\le C_3\sum\int dg(P')\,r(P,P')^{\ell_1}(1+r(P))(1+r(P'))^{\ell_2+n+1}\\ &\qquad\qquad \cdot t^{-(n+2)/2+1/2}e^{-r(P,P')^2/5t}\,|\varphi(P')| \quad \text{(by (3.17))}\\ &\le C_4\sum_{\ell_1+\ell_2=\ell+1}^{\ell_1>0}\int dg(P')\,r(P,P')^{\ell_1}(1+r(P'))^{\ell_2+n+1}\,t^{-(n+2)/2+1/2}e^{-r(P,P')^2/5t}\,|\varphi(P')|\\ &\le t\,C_5\,p_{N,\emptyset}(\varphi).\end{aligned}$$

Thus the inequality with $k=0$ holds. Last, consider the series expansion of $(\not{\partial}^{(\varepsilon)})^2 E^{(\varepsilon)} = E^{(\varepsilon)}(\not{\partial}^{(\varepsilon)})^2$ with respect to $\varepsilon^{1/2}$. We have

$$
\begin{aligned}
&(\not{\partial}^2)^{(0/2:)}E(t) = E(t)(\not{\partial}^2)^{(0/2:)},\\
&(\not{\partial}^2)^{(0/2:)}E^{(1/2:)}(t) = E^{(1/2:)}(t)(\not{\partial}^2)^{(0/2:)} + E(t)(\not{\partial}^2)^{(1/2:)} - (\not{\partial}^2)^{(1/2:)}E(t)\\
&= E^{(1/2:)}(t)(\not{\partial}^2)^{(0/2:)} - [(\not{\partial}^2)^{(1/2:)}, E(t)-1]
\end{aligned}
$$

and inductively we have

$$
((\not{\partial}^2)^{(0/2:)})^k E^{(1/2:)}(t) = E^{(1/2:)}(t)((\not{\partial}^2)^{(0/2:)})^k + P_0\,(E(t)-1)\,Q_0 \tag{3.41}
$$

where P_0 and Q_0 are some polynomials consisting of $(\not{\partial}^2)^{(m/2:)}$ $(m=0,1)$. Hence Proposition 3.4 (**1**), (3.36) with $k=0$ and (3.28) imply the general inequality.

4 The local expression of the operator $\mathrm{adia}(\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon})$

If we denote by $\{\mu_j\}$ and $\{\varphi_j\}$ the spectrum consisting of eigenvalues and the corresponding orthonormal eigen-cross-sections of $\not{\partial}^2_{g^Z_\varepsilon}$ acting on $\Gamma(\$^{(\pm)}_{g^Z_\varepsilon})$, we have

$$
\begin{aligned}
\mathrm{STr}\Big(\not{\partial}_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big) &= \sum e^{-t\mu_j}\,\mathrm{STr}\Big(\not{\partial}_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\,\varphi_j(P)\boxtimes\varphi_j(P')\Big)\\
&= -\sum e^{-t\mu_j}\,\mathrm{STr}\Big(\delta_X\not{\partial}_{g^Z_\varepsilon}\,\varphi_j(P)\boxtimes\not{\partial}_{g^Z_\varepsilon}\varphi_j(P')\Big) \equiv -\mathrm{STr}\Big(\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big)
\end{aligned} \tag{4.1}
$$

where, as an operator acting on the left hand side of (3.4), $\not{\partial}^*_{g^Z_\varepsilon}$ is given as

$$
\begin{aligned}
\not{\partial}^*_{g^Z_\varepsilon} &= \sum \rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A))\cdot e^\varepsilon_i(A)(P')\\
&\quad+\frac{1}{4}\sum \rho^*_{g^Z_\varepsilon}(e^{i_2}_\varepsilon(A))\rho^*_{g^Z_\varepsilon}(e^{i_1}_\varepsilon(A))\rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A))\cdot C(\nabla^{g^Z_\varepsilon})(e^\varepsilon_i(A))_{i_2 i_1}(P')\quad\text{with}\\
&\rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A))\,(s(e^\varepsilon_*)(P)\boxtimes s(e^\varepsilon_*)^*(P')) \equiv s(e^\varepsilon_*)(P)\boxtimes s(e^\varepsilon_*)^*(P')\,(-\rho_{g^Z_\varepsilon}(e^i_\varepsilon(A))).
\end{aligned} \tag{4.2}
$$

We will investigate (4.1) by examining $\mathrm{STr}\Big(\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big)$ in the following because the latter has the merit that for each parameter only a first order differential at the most appears. Since we want to expand it into a series (see (5.1) and (5.2)) by using the series expansion (3.7), in the section we will explicitly write down the operator $\mathrm{adia}(\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon})$ which is defined as follows: Express the operator $\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}$ acting on the right hand side of (3.4) by using the coordinates x on U at P^0, replace the differentials $\partial/\partial x^b_i$, $\partial/\partial x^f_k$ at P by $\varepsilon^{-1/2}\partial/\partial x^b_i$, $\partial/\partial x^f_k$ (at P) and replace the differentials $\partial/\partial x^b_i$, $\partial/\partial x^f_k$ at P' by $\varepsilon^{-1/2}\partial/\partial x'^b_i$, $\partial/\partial x'^f_k$ (at P').

Let our argument start with studying the operator $\delta_X\not{\partial}_{g^Z}$ acting on $\Gamma(\$_{g^Z})$ (compare with (3.3)).

Lemma 4.1. *We denote by $\eta_a = (\eta_{a,ij})$ the matrix expression of (1.4) with respect to the g^Z-$SO(n+2)$-frame $e_*(A)$, i.e., $\eta_a^Z(e_*(A)) = e_*(A)\cdot\eta_a$. Then we have $\eta_a = ((h_a^* g^Z)(e_i(A), e_j(A)))^{-1/2}$ and, if we set $\dot\eta = (\dot\eta_{ij}) = \frac{d}{da}\Big|_{a=0}\eta_a$, then we have*

$$\dot\eta_{ij} = \dot\eta_{n+k,n+k'} = 0,\ \dot\eta_{n+k,j} = \dot\eta_{j,n+k} = -\nu^k(\nabla^A_{e_j^b}X)\quad (1\le i,j\le n) \tag{4.3}$$

where $\nabla^A = d + [A,\cdot]$ is the covariant derivative for the bundle $\mathfrak{sp}(P_{Sp(1)})$.

Proof. Let us consider a segment $t\mapsto g^Z_{a,P}(t) = (1-t)g^Z_P + t(h_a^* g^Z)_P$ in $I(T_P Z)$ and a curve $t\mapsto \eta_a^Z(t)(e_*(A)) = e_*(A)\cdot(g^Z_{a,P}(t)(e_i(A), e_j(A)))^{-1/2}$ in $F^+(T_P Z)$. Then $\eta_a^Z(t)(e_*(A))$ is a $g^Z_{a,P}(t)$-$SO(n+2)$-frame and, moreover, $\partial\eta_a^Z(t)(e_*(A))/\partial t$ belongs to $\mathcal{H}_{\eta_a^Z(t)(e_*(A))}(F^+(T_P Z))$ ([4], [5]). Hence we have $\eta_a = ((h_a^* g^Z)(e_i(A), e_j(A)))^{-1/2}$. In general a gauge transformation $(\Gamma(Sp(P_{Sp(1)}))\ni)\, h : Z\cong Z$ gives a bundle isomorphism

$$\begin{aligned} h_* : TZ = \mathcal{H}_A\oplus\mathcal{V} &\cong TZ = \mathcal{H}_{h_*A}\oplus\mathcal{V}\\ e_i^b(A)(P),\, e_k^f(P) &\mapsto e_i^b(h_*A)(h(P)),\, e_k^f(h(P)) \end{aligned} \tag{4.4}$$

and we have $e_i^b(h_*A) = e_i^b(A) - 2\sum\nu^k((h_*A - A)(e_i^b))\, e_k^f$ at $h(P)$. Therefore we have

$$\begin{aligned} &(h^*g^Z)(e_i^b(A), e_j^b(A)) = g^Z(e_i^b(h_*A), e_j^b(h_*A))\\ &= \delta_{ij} + 4\sum\nu^k((h_*A - A)(e_i^b))\nu^k((h_*A-A)(e_j^b)),\\ &(h^*g^Z)(e_j^b(A), e_k^f) = g^Z(e_j^b(h_*A), e_k^f) = -2\sum\nu^k((h_*A-A)(e_j^b)),\\ &(h^*g^Z)(e_k^f, e_{k'}^f) = g^Z(e_k^f, e_{k'}^f) = \delta_{kk'} \end{aligned}$$

and certainly we have $(\partial/\partial a)|_{a=0}(h_{a*}A - A)_{\pi(h_a(P))} = (\partial/\partial a)|_{a=0}(h_{a*}A - A)_{\pi(P)} = -d_A X\in\Omega^1(\mathfrak{sp}(P_{Sp(1)}))$. The lemma was thus proved.

Denoting by Δ_A the covariant Laplacian, i.e., $\Delta_A X = -\sum\Big(\nabla^A_{e_i^b}\nabla^A_{e_i^b} - \nabla^A_{\nabla^M_{e_i^b}e_i^b}\Big)X$, we know

Lemma 4.2. *We have*

$$\begin{aligned} \delta_X \not{D}_{g^Z} = &-\sum\rho_{g^Z}(\nu(\nabla^A_{e_i^b}X))\,\nabla^{\$_{g^Z}\oplus}_{e_i^b(A)} - \sum\rho_{g^Z}(\nu^k(d_AX))\,\nabla^{\$_{g^Z}\oplus}_{e_k^f}\\ &+\frac{1}{2}\rho_{g^Z}\Big(\nu(\Delta_AX) - \sum\nu^k(d_AX)\wedge\nu^k(F_A) - \sum\nu(\nabla^A_{e_i^b}X)\wedge\nu(e_b^i\vee F_A)\Big). \end{aligned} \tag{4.5}$$

Proof. We refer to [4], [5]. Set $\dot\eta^Z(e_i(A)) \equiv (\partial/\partial a)|_{a=0}\,\eta_a^Z(e_i(A)) = \sum e_j(A)\cdot\dot\eta_{ji}$. Then (1.5) and (1.6) imply

$$\delta_X\not{D}_{g^Z} = \frac{\partial}{\partial a}\Big|_{a=0}\sum\rho_{g^Z}(e^i(A))\Big\{\nabla^{\$_{g^Z}}_{\eta_a^Z(e_i(A))} + \Big(\nabla^{\$_{g^Z},h_a} - \nabla^{\$_{g^Z}}\Big)_{\eta_a^Z(e_i(A))}\Big\} \tag{4.6}$$

$$= \sum \rho_{g^Z}(e^i(A)) \nabla^{\$_{g^Z}}_{\dot\eta^Z(e_i(A))} + \frac{\partial}{\partial a}\Big|_{a=0} \sum \rho_{g^Z}(e^i(A))\, \eta_{a,ji} \Big(\nabla^{\$_{g^Z},h_a} - \nabla^{\$_{g^Z}}\Big)_{e_j(A)}$$
$$= \sum \rho_{g^Z}(e^i(A)) \nabla^{\$_{g^Z}}_{\dot\eta^Z(e_i(A))} + \sum \rho_{g^Z}(e^i(A))\, \frac{\partial}{\partial a}\Big|_{a=0} \nabla^{\$_{g^Z},h_a}_{e_i(A)}$$

And (3.2), (4.3) imply

$$\begin{aligned}
(4.7)\quad & \sum \rho_{g^Z}(e^i(A))\, \nabla^{\$_{g^Z}}_{\dot\eta^Z(e_i(A))} = -\sum \nu^k(\nabla^A_{e^b_i} X)\Big\{\rho_{g^Z}(e^k_f(A))\, \nabla^{\$_{g^Z}}_{e^b_i(A)} + \rho_{g^Z}(e^i_b)\, \nabla^{\$_{g^Z}}_{e^f_k}\Big\} \\
&= -\sum \rho_{g^Z}(\nu(\nabla^A_{e^b_i} X))\, \nabla^{\$_{g^Z}}_{e^b_i(A)} - \sum \rho_{g^Z}(\nu^k(d_A X))\, \nabla^{\$_{g^Z}}_{e^f_k} \\
&= -\sum \rho_{g^Z}(\nu(\nabla^A_{e^b_i} X))\, \nabla^{\$_{g^Z}\oplus}_{e^b_i(A)} - \sum \rho_{g^Z}(\nu^k(d_A X))\, \nabla^{\$_{g^Z}\oplus}_{e^f_k} \\
&\quad -\frac{1}{2} \sum \rho_{g^Z}(\nu(\nabla^A_{e^b_i} X))\, \rho_{g^Z}(\nu(e^i_b \vee F_A)) - \frac{1}{2}\sum \rho_{g^Z}(\nu^k(d_A X))\, \rho_{g^Z}(\nu^k(F_A)) \\
&= -\sum \rho_{g^Z}(\nu(\nabla^A_{e^b_i} X))\, \nabla^{\$_{g^Z}\oplus}_{e^b_i(A)} - \sum \rho_{g^Z}(\nu^k(d_A X))\, \nabla^{\$_{g^Z}\oplus}_{e^f_k} \\
&\quad -\frac{1}{2}\, \rho_{g^Z}\Big(\sum \nu^k(d_A X) \wedge \nu^k(F_A) + \sum \nu(\nabla^A_{e^b_i} X) \wedge \nu(e^i_b \vee F_A) \\
&\quad - 2 \sum \nu^k(\nabla^A_{e^b_i} X)\, \nu^k(e^i_b \vee F_A)\Big).
\end{aligned}$$

Further, let us regard $\dot\eta^Z \in \Gamma(\mathrm{End}(TZ))$ as a symmetric 2-tensor, i.e.,

$$(4.8)\quad \dot\eta^Z \equiv \sum \dot\eta_{ij}\, e^i(A) \otimes e^j(A) = -\sum \Big\{\nu^k(d_A X) \otimes e^k_f(A) + e^k_f(A) \otimes \nu^k(d_A X)\Big\}.$$

Then we have

$$\begin{aligned}
(4.9)\quad & \sum \rho_{g^Z}(e^i(A))\, \frac{\partial}{\partial a}\Big|_{a=0} \nabla^{\$_{g^Z},h_a}_{e_i(A)} \\
&= \sum \rho_{g^Z}(e^i(A))\, \frac{1}{4}\Big\{(\nabla^{g^Z}_{e_j(A)} \dot\eta^Z)(e_j(A), e_i(A)) + (\nabla^{g^Z}_{e_j(A)} \dot\eta^Z)(e_i(A), e_j(A))\Big\} \\
&= \frac{1}{2}\, \rho_{g^Z}\Big(\nu(\Delta_A X) - 2 \sum \nu^k(\nabla^A_{e^b_i} X)\, \nu^k(e^i_b \vee F_A)\Big).
\end{aligned}$$

Thus we obtain (4.5).

Now let us write down $\mathrm{adia}(\partial\!\!\!/^*_{g^Z_\varepsilon}\, \delta_X \partial\!\!\!/_{g^Z_\varepsilon})$ explicitly. Through the canonical identification

$$(4.10)\qquad \$_{g^Z_\varepsilon} \otimes \$_{g^Z_\varepsilon}\ (\cong \$_{g^Z_\varepsilon} \otimes \$^*_{g^Z_\varepsilon}) \cong \wedge T^* Z \otimes \mathbb{C},$$

the Clifford action $\rho_{g^Z_\varepsilon}(e^{b\varepsilon}_i)$ acting on $\varphi_j(P)$ at (4.1) and the one $\rho_{g^Z_\varepsilon}(e^{b\varepsilon}_i)$ acting on $\varphi_j(P')$ (i.e., $\rho^*_{g^Z_\varepsilon}(e^{b\varepsilon}_i)$ at (4.2)) induce the actions on the right hand side of (4.10)

$$(4.11)\quad \rho_{g^Z_\varepsilon}(e^{b\varepsilon}_i) = \varepsilon^{-1/2} dx^b_i \wedge - \varepsilon^{1/2} dx^b_i \vee,\ \ \rho^*_{g^Z_\varepsilon}(e^{b\varepsilon}_i) = \theta^\wedge(\varepsilon^{-1/2} dx^b_i \wedge + \varepsilon^{1/2} dx^b_i \vee)$$

with $\theta^\wedge \omega \equiv (-1)^p \omega$ for $\omega \in \wedge^p T^* Z \otimes \mathbb{C}$.

Lemma 4.3. *We have*

$$\text{(4.12)}\quad \operatorname{adia}(\not\partial^*_{g^Z_\varepsilon}\,\delta_X\not\partial_{g^Z_\varepsilon}) = \sum_{m=-1}^{10}\varepsilon^{m/2}\operatorname{adia}(\not\partial^*_{g^Z}\delta_X\not\partial_{g^Z} : m/2) + \mathcal{O}(r(P)) + \mathcal{O}(r(P'))$$

$$= \varepsilon^{-1/2}\Big\{-\theta^\wedge\sum dx^b_{i'}\wedge\nu^k(d_AX)\wedge\frac{\partial}{\partial x'^b_{i'}}\Big(\frac{\partial}{\partial x^f_k}+\frac{1}{2}\nu^k(F_A)\wedge\Big)\Big\}(P^0)$$

$$+\varepsilon^{0/2}\Big\{-\theta^\wedge\sum dx^b_{i'}\wedge\rho_{g^Z}(\nu(\nabla^A_{e^b_i}X))\,\frac{\partial}{\partial x'^b_{i'}}\frac{\partial}{\partial x^b_i}$$

$$-\sum\rho^*_{g^Z}(e^{k'}_f(A))\,\nu^k(d_AX)\wedge\Big(\frac{\partial}{\partial x'^f_{k'}}+\frac{1}{2}\,\nu^{k'}(F_A)\wedge\Big)\Big(\frac{\partial}{\partial x^f_k}+\frac{1}{2}\,\nu^k(F_A)\wedge\Big)\Big\}(P^0)$$

$$+\mathcal{O}(\varepsilon^{1/2})+\mathcal{O}(r(P))+\mathcal{O}(r(P'))$$

Proof. Let us describe $\operatorname{adia}(\not\partial^*_{g^Z_\varepsilon})$, $\operatorname{adia}(\delta_X\not\partial_{g^Z_\varepsilon})$ clearly. Referring to (3.3) and (4.11), we have

$$\not\partial_{g^Z_\varepsilon,P'} = \sum\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\nabla^{\$_{g^Z_\varepsilon}\oplus}_{e^{b\varepsilon}_i(A)}+\sum\rho_{g^Z_\varepsilon}(e^k_f(A))\,\nabla^{\$_{g^Z_\varepsilon}\oplus}_{e^f_k}-\frac{1}{2}\,\rho_{g^Z_\varepsilon}(\nu(F_A))$$

$$=\sum\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\varepsilon^{1/2}\Big\{e^b_i(A)+\frac{1}{4}\sum C(\nabla^{g^M})_{i_2i_1}(e^b_i)\,\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\,\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})$$

$$+\frac{1}{4}\sum C(\nabla^{g^V})_{k_2k_1}(e^b_i(A))\,\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\,\rho_{g^Z_\varepsilon}(e^{k_2}_f(A))\Big\}$$

$$+\sum\rho_{g^Z_\varepsilon}(e^k_f(A))\Big\{e^f_k+\frac{1}{4}\sum C(\nabla^{g^V})_{k_2k_1}(e^f_k)\,\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\,\rho_{g^Z_\varepsilon}(e^{k_2}_f(A))\Big\}$$

$$-\sum\frac{1}{4}\,\rho_{g^Z_\varepsilon}(e^k_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\,\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\,\varepsilon^{2/2}\nu^k(F_A(e^b_{i_1},e^b_{i_2}))$$

$$=\varepsilon^{1/2}\sum\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\partial/\partial x^b_i+\sum\rho_{g^Z_\varepsilon}(e^k_f(A))\,\partial/\partial x^f_k$$

$$-\varepsilon^{2/2}\sum\frac{1}{4}\,\rho_{g^Z_\varepsilon}(e^k_f(A))\,\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\,\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\,\nu^k(F_A(e^b_{i_1},e^b_{i_2}))+\mathcal{O}(r(P')),$$

$$\not\partial^*_{g^Z_\varepsilon,P'}=\varepsilon^{1/2}\sum\rho^*_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\partial/\partial x^b_i+\sum\rho^*_{g^Z_\varepsilon}(e^k_f(A))\,\partial/\partial x^f_k$$

$$-\varepsilon^{2/2}\sum\frac{1}{4}\,\rho^*_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\,\rho^*_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\,\rho^*_{g^Z_\varepsilon}(e^k_f(A))\,\nu^k(F_A(e^b_{i_1},e^b_{i_2}))+\mathcal{O}(r(P'))$$

$$=\sum\theta^\wedge\,(dx^b_i\wedge+\varepsilon^{2/2}dx^b_i\vee)\,\partial/\partial x^b_i+\sum\rho^*_{g^Z}(e^k_f(A))\,\partial/\partial x^f_k+\sum\frac{1}{4}\,(dx^b_{i_2}\wedge$$

$$+\varepsilon^{2/2}dx^b_{i_2}\vee)(dx^b_{i_1}\wedge+\varepsilon^{2/2}dx^b_{i_1}\vee)\,\rho^*_{g^Z}(e^k_f(A))\,\nu^k(F_A(e^b_{i_1},e^b_{i_2}))+\mathcal{O}(r(P'))$$

and, hence, we have

$$\text{(4.13)}\qquad \operatorname{adia}(\not\partial^*_{g^Z_\varepsilon})=\varepsilon^{-1/2}\,\theta^\wedge\sum dx^b_i\wedge\partial/\partial x'^b_i$$

$$+\sum\rho^*_{g^Z}(e^k_f(A))\Big(\partial/\partial x'^f_k+\frac{1}{2}\nu^k(F_A)\wedge\Big)+\mathcal{O}(\varepsilon^{1/2})+\mathcal{O}(r(P')).$$

Next, referring to the above calculation and (4.5), we have

$$\delta_X\not\partial_{g^Z_\varepsilon}=-\sum\rho_{g^Z_\varepsilon}(e^k_f(A))\,\nu^k(\nabla^A_{e^{b\varepsilon}_i}X)\nabla^{\$_{g^Z_\varepsilon}\oplus}_{e^{b\varepsilon}_i(A)}-\sum\rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\nu^k(\nabla^A_{e^{b\varepsilon}_i}X)\nabla^{\$_{g^Z_\varepsilon}\oplus}_{e^f_k}$$

$$
\begin{aligned}
&+\frac{1}{2}\rho_{g^Z_\varepsilon}\Big(\sum e^k_f(A)\,\varepsilon^{2/2}\nu^k(\Delta_A X) - \sum e^{i_1}_{b\varepsilon}\wedge e^{i_2}_{b\varepsilon}\wedge e^{i_2}_{b\varepsilon}\,\frac{1}{2}\,\nu^k(\nabla^A_{e^{b\varepsilon}_{i_1}}X)\,\nu^k(F_A(e^{b\varepsilon}_{i_2},e^{b\varepsilon}_{i_2}))\\
&-\sum e^{k_1}_f(A)\wedge e^{k_2}_f(A)\wedge e^j_{b\varepsilon}\,\nu^{k_1}(\nabla^A_{e^{b\varepsilon}_i}X)\,\nu^{k_2}(F_A(e^{b\varepsilon}_i,e^{b\varepsilon}_j))\Big)\\
&=-\sum \rho_{g^Z_\varepsilon}(e^k_f(A))\,\varepsilon^{2/2}\nu^k(\nabla^A_{e^b_i}X)\,\partial/\partial x^b_i - \sum \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\varepsilon^{1/2}\nu^k(\nabla^A_{e^b_i}X)\,\partial/\partial x^f_k\\
&+\frac{1}{2}\rho_{g^Z_\varepsilon}\Big(\sum e^k_f(A)\,\varepsilon^{2/2}\nu^k(\Delta_A X) - \sum e^{i_1}_{b\varepsilon}\wedge e^{i_2}_{b\varepsilon}\wedge e^{i_3}_{b\varepsilon}\,\frac{1}{2}\,\nu^k(\nabla^A_{e^{b\varepsilon}_{i_1}}X)\,\nu^k(F_A(e^{b\varepsilon}_{i_2},e^{b\varepsilon}_{i_3}))\\
&-\sum e^{k_1}_f(A)\wedge e^{k_2}_f(A)\wedge e^j_{b\varepsilon}\,\nu^{k_1}(\nabla^A_{e^{b\varepsilon}_i}X)\,\nu^{k_2}(F_A(e^{b\varepsilon}_i,e^{b\varepsilon}_j))\Big) + \mathcal{O}(r(P))\\
&=-\varepsilon^{2/2}\sum \rho_{g^Z}(\nu(\nabla^A_{e^b_i}X))\,\partial/\partial x^b_i - \sum \nu^k(d_A X)\,\partial/\partial x^f_k\\
&+\varepsilon^{2/2}\sum dx^b_i\vee\nu^k(\nabla^A_{e^b_i}X)\,\partial/\partial x^f_k + \varepsilon^{2/2}\,\frac{1}{2}\,\rho_{g^Z}(\nu(\Delta_A X)) - \frac{1}{2}\sum \nu^k(d_A X)\nu^k(F_A)\\
&+\varepsilon^{2/2}\frac{1}{4}\sum\Big(dx^b_{i_1}\wedge dx^b_{i_2}\vee dx^b_{i_3}\wedge + dx^b_{i_1}\vee dx^b_{i_2}\wedge dx^b_{i_3}\wedge\\
&+dx^b_{i_1}\wedge dx^b_{i_2}\wedge dx^b_{i_3}\vee\Big)\,\nu^k(\nabla^A_{e^b_{i_1}}X)\,\nu^k(F_A(e^b_{i_2},e^b_{i_3}))\\
&-\varepsilon^{4/2}\frac{1}{4}\sum\Big(dx^b_{i_1}\vee dx^b_{i_2}\vee dx^b_{i_3}\wedge + dx^b_{i_1}\vee dx^b_{i_2}\wedge dx^b_{i_3}\vee\\
&+dx^b_{i_1}\wedge dx^b_{i_2}\vee dx^b_{i_3}\vee\Big)\,\nu^k(\nabla^A_{e^b_{i_1}}X)\,\nu^k(F_A(e^b_{i_2},e^b_{i_3}))\\
&+\varepsilon^{6/2}\frac{1}{4}\sum dx^b_{i_1}\vee dx^b_{i_2}\vee dx^b_{i_3}\vee\nu^k(\nabla^A_{e^b_{i_1}}X)\,\nu^k(F_A(e^b_{i_2},e^b_{i_3}))\\
&-\varepsilon^{2/2}\frac{1}{2}\sum \rho_{g^Z}(e^{k_1}_f(A)\wedge e^{k_2}_f(A))\,dx^b_j\wedge\nu^{k_1}(\nabla^A_{e^b_i}X)\,\nu^{k_2}(F_A(e^b_i,e^b_j))\\
&+\varepsilon^{4/2}\frac{1}{2}\sum \rho_{g^Z}(e^{k_1}_f(A)\wedge e^{k_2}_f(A))\,dx^b_j\vee\nu^{k_1}(\nabla^A_{e^b_i}X)\,\nu^{k_2}(F_A(e^b_i,e^b_j)) + \mathcal{O}(r(P))
\end{aligned}
$$

and, hence, we have

$$
\begin{aligned}
\text{(4.14)}\qquad \operatorname{adia}(\delta_X \not{\partial}_{g^Z_\varepsilon}) &= -\sum \nu^k(d_A X)\wedge\Big(\partial/\partial x^f_k + \frac{1}{2}\nu^k(F_A)\wedge\Big)\\
&-\varepsilon^{1/2}\sum \rho_{g^Z}(\nu(\nabla^A_{e^b_i}X))\,\partial/\partial x^b_i + \mathcal{O}(\varepsilon^{2/2}) + \mathcal{O}(r(P)).
\end{aligned}
$$

(4.13) and (4.14) then imply (4.12).

5 The proofs of Theorems 2.1, 2.2

As is well-known, the pointwise super trace $\operatorname{str}(\rho_{g^Z_\varepsilon}(e^I(A))) = \operatorname{tr}_+(\rho_{g^Z_\varepsilon}(e^I(A))) - \operatorname{tr}_-(\rho_{g^Z_\varepsilon}(e^I(A)))$ is equal to $(2/\sqrt{-1})^{(n+2)/2}$ if $I = (1,\cdots,n+2)$ and equal to 0 if $I \neq (1,\cdots,n+2)$ (see [15, Lemma 2.5]). Hence we have

$$
\begin{aligned}
\text{(5.1)}\qquad \operatorname{STr}\Big(\not{\partial}_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big) &= -\operatorname{STr}\Big(\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big)\ \ (\text{see (4.1)})\\
&= -\Big(\frac{2}{\sqrt{-1}}\Big)^{(n+2)/2}\int_Z \not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)\\
\Big(&= -\Big(\frac{2}{\sqrt{-1}}\Big)^{(n+2)/2}\int_Z dg^Z(P^0)\,\Big(\not{\partial}^*_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)\Big)_{\max}
\end{aligned}
$$

where we regard $\not{\partial}^*_{g^Z_\varepsilon}\,\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)$ as an element of the right hand side of (4.10) and $dg^Z(P^0)\Big(\not{\partial}^*_{g^Z_\varepsilon}\,\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)\Big)_{\max}$ is its term with degree $n+2$. Note that $(dx^f)^{(1,2)}(P^0)\Big(\not{\partial}^*_{g^Z_\varepsilon}\,\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)\Big)_{\max}$ is equal to the term with degree 2 of $\Big(\not{\partial}^*_{g^Z_\varepsilon}\,\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P^0,P^0)\Big)_{(1,\cdots,n),\emptyset}\in \$_{g^V,P^0}\otimes\$^*_{g^V,P^0}=\wedge T^*_{P^0}Z_{P^0}\otimes\mathbb{C}$ (see (3.4) and (3.5)). Therefore, by observing (3.7), (3.16), (4.12), we find out that there exists a Taylor expansion

$$
\begin{aligned}
(5.2)\quad &\mathrm{STr}\Big(\not{\partial}_{g^Z_\varepsilon}\delta_X\not{\partial}_{g^Z_\varepsilon}\,e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big)\\
&=-\Big(\frac{2}{\sqrt{-1}}\Big)^{(n+2)/2}\Big\{\sum_{m=-1}^{m_0}\varepsilon^{m/2}\int_Z\sum_{m_1+m_2=m}\mathrm{adia}(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}:m_1/2)\,E^{(m_2/2:)}(t,P^0)\\
&\qquad\qquad+\varepsilon^{(m_0+1)/2}\int_Z(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}E)^{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0)\Big\}\\
&\equiv-\Big(\frac{2}{\sqrt{-1}}\Big)^{(n+2)/2}\sum_{m=-1}^{m_0+1}\varepsilon^{m/2}\int_Z(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}E)^{(m/2,\cdot)}(t,P^0).
\end{aligned}
$$

Futher (3.8) says that there exist constants $\lambda>0$, $C>0$ and an integer $N>0$ satisfying

$$
(5.3)\qquad
\begin{aligned}
&\Big|(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}E)^{(m/2)}(t,P^0)\Big|\le C\,e^{-t\lambda}\,t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+4)/2}}+1\Big),\\
&\Big|(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}E)^{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0)\Big|\le C\,t^{1/2}\Big(\frac{1}{t^{(n+4)/2}}+t^N\Big)\\
&(\,0\le\forall m\le m_0,\ 0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall P^0\in Z\,)
\end{aligned}
$$

and (3.17) says that for given $T_0>0$ there exists a series expansion

$$
\begin{aligned}
(5.4)\qquad &(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}E)^{(m/2,\cdot)}(t,P^0)\\
&=\frac{1}{(4\pi t)^{(n+2)/2}}\Big\{\sum_{i=-\delta_{0m}}^{i_0}t^i(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}E)^{(m/2,\cdot)}(i:P^0)+O(t^{i_0+1})\Big\}\\
&(\,\forall i_0\ge0\,,\,0\le\forall m\le m_0+1\,,\,0<\forall\varepsilon^{1/2}\le\varepsilon_0^{1/2}\,,\,0<\forall t\le T_0\,,\,\forall P^0\in Z\,).
\end{aligned}
$$

Lemma 5.1. *We have*

$$
(5.5)\qquad \int_Z\mathrm{adia}(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}:m/2)\,E^{(0/2:)}(t,P^0)=0\quad(m=-1,0)
$$

and the double integral at the right hand side of (2.10) is absolutely integrable and we have

$$
(5.6)\qquad \int_Z\mathrm{adia}(\not{\partial}^*_{g^Z}\delta_X\not{\partial}_{g^Z}:-1/2)\,E^{(1/2:)}(t,P^0)=-\int_Z\text{s-log}\det(\delta_X\not{\partial}_{g^Z})(t,P^0).
$$

Remark. By examining (4.12) in detail we know

$$\int_Z \mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : 2/2)\, E^{(0/2:)}(t, P^0)$$
$$= \int_Z \frac{1}{2}\,\nu(\Delta_A X) \wedge \sum \rho_{gz}^*(e_f^k(A))\Big(\frac{\partial}{\partial x_k'^f} + \frac{1}{2}\,\nu^k(F_A)\wedge\Big) E(t, P^0),$$
$$\int_Z \mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : m/2)\, E^{(0/2:)}(t, P^0) = 0 \quad \text{(otherwise)}.$$

Proof. Refer to (2.3) and (4.12). As for (5.5): Since $\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2)$ is a first order differential in the M-direction, (5.5) holds if $m = -1$. Since $\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : 0/2)$ is an exterior product of odd degree in the M-direction, (5.5) holds also if $m = 0$. As for (5.6): We abbreviate $E(t, p^0, P, P')$ to $E(t, P, P')$. (3.29) says

$$\text{(5.7)} \quad \mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2) E^{(1/2:)}(t, P^0)$$
$$= -\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2) \int_0^t d\tau \int_{Z(p^0)} dg(Q)\, \langle E(t-\tau, P^0, Q), (\not{\partial}^2)^{(1/2:)} E(\tau, Q, P^0)\rangle$$

but it may not permitted to reverse the order of $\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2)$ and the double integral. Consider then the term of even degree in the M-direction

$$\text{(5.8)} \quad \text{even-}\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2) E^{(1/2:)}(t, P^0)$$
$$= -\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2) \int_0^t d\tau \int_{Z(p^0)} dg(Q)\, \langle E(t-\tau, P^0, Q), \mathrm{chi}(\not{\partial}_{gz}^2) E(\tau, Q, P^0)\rangle$$

(refer to (2.12) and (3.20)). On its right hand side, fortunately we can reverse the order as is proved hereafter. Let us denote by $e^{i'} \in M(p^0)$ the fundamental vector whose i'-th entry is equal to 1. Then (2.3), (2.4) and (4.12) imply

$$\text{(5.9)} \quad -\mathrm{adia}(\not{\partial}_{gz}^* \delta_X \not{\partial}_{gz} : -1/2) \langle E(t-\tau, P^0, Q), \mathrm{chi}(\not{\partial}_{gz}^2) E(\tau, Q, P^0)\rangle$$
$$= \theta^\wedge \sum dx_{i'}^b(P^0) \wedge \langle \mathrm{chi}(\delta_X \not{\partial}_{gz}) E(t-\tau, P^0, Q), \frac{\partial}{\partial x_{i'}'^b} \mathrm{chi}(\not{\partial}_{gz}^2) E(\tau, Q, P^0)\rangle$$
$$= \theta^\wedge \sum dx_{i'}^b(P^0) \wedge \langle \mathrm{chi}(\delta_X \not{\partial}_{gz}) E(t-\tau, P^0, Q),$$
$$\mathrm{chi}(\not{\partial}_{gz}^2) \Big\langle y^b \Big| \frac{R^{g^M}(p^0, y^b)}{4} (1 + \coth \frac{\tau R^{g^M}(p^0, y^b)}{2}) \Big| e^{i'} \Big\rangle E(\tau, Q, P^0)\,\rangle$$
$$= -\theta^\wedge \langle \mathrm{chi}(\delta_X \not{\partial}_{gz}) E(t-\tau, P^0, Q), \sum dx_{i'}^b(y^b) \wedge$$
$$\mathrm{chi}(\not{\partial}_{gz}^2) \Big\langle y^b \Big| \frac{R^{g^M}(p^0, y^b)}{4} (1 + \coth \frac{\tau R^{g^M}(p^0, y^b)}{2}) \Big| e^{i'} \Big\rangle E(\tau, Q, P^0)\,\rangle$$
$$= -\theta^\wedge \langle \mathrm{chi}(\delta_X \not{\partial}_{gz}) E(t-\tau, P^0, Q),$$
$$\mathrm{chi}(\not{\partial}_{gz}^2) \Big\langle y^b \Big| \frac{R^{g^M}(p^0, y^b)}{4} (1 + \coth \frac{\tau R^{g^M}(p^0, y^b)}{2}) \Big| dx^b(y^b) \Big\rangle E(\tau, Q, P^0)\,\rangle.$$

Remark that by observing (2.4) the second equality above comes from the formula

$$\frac{\partial}{\partial x_{i'}'^b} K_M(t, x^b, 0)$$

$$= \Big\{ \frac{1}{2t} \sum x_i^b \big(\frac{tR^{g^M}(p^0)}{2} \coth \frac{tR^{g^M}(p^0)}{2} \big)_{ii'} + \frac{1}{4} \sum x_i^b R^{g^M}(p^0)_{ii'} \Big\} K_M(t, x^b, 0)$$

$$= \sum x_i^b \Big(\frac{R^{g^M}(p^0)}{4} (1 + \coth \frac{tR^{g^M}(p^0)}{2}) \Big)_{ii'} K_M(t, x^b, 0).$$

Similarly to the estimation (3.31), then we have

$$\begin{aligned} (5.10) \qquad & \int_{Z(p^0)} dg(Q) \, \Big| \langle \operatorname{chi}(\delta_X \not{\partial}_{g} z) E(t-\tau, P^0, Q), \\ & \operatorname{chi}(\not{\partial}_{g}^2 z) \Big\langle y^b \Big| \frac{R^{g^M}(p^0, y^b)}{4} (1 + \coth \frac{\tau R^{g^M}(p^0, y^b)}{2}) \Big| dx^b(y^b) \Big\rangle E(\tau, Q, P^0) \rangle \Big| \\ & \leq C\, (t-\tau)^{-(n+2)/2-1/2} \tau^{-(n+2)/2-1/2} \int_{Z(p^0)} dg(Q)\, e^{-r(Q)^2/6(t-\tau) - r(Q)^2/6\tau}. \end{aligned}$$

(Note that (5.7) may not have such an estimate.) And, by the same argument as (3.32)-(3.33), we find out that (5.10) is integrable over the interval $(0, t)\,(\ni \tau)$. Thus we can reverse the order of $\operatorname{adia}(\not{\partial}_{g}^* z \delta_X \not{\partial}_{g} z : -1/2)$ and the double integral in (5.8) and the double integral at the right hand side of (2.10) is certainly absolutely integrable. Further (5.8) and (5.9) imply the formula (5.6).

Now let us prove Theorems 2.1, 2.2.

Proofs of Theorems 2.1, 2.2 (compare with [15, Proof of Theorem 1.2]). Take an integer $n_0 > 0$. Then the first estimate at (5.3) and (5.4) imply that, if $-1 \leq m \leq n_0$, the function (to be differentiated by s)

$$(5.11) \qquad \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^s \int_Z (\not{\partial}_{g}^* z \delta_X \not{\partial}_{g} z E)^{(m/2)}(t, P^0)$$

is absolutely integrable if $\operatorname{Re}(s) > (n+2)/2$ and has a meromorphic extension to $\mathbb{C}\,(\ni s)$ which is analytic at $s = 0$. Hence, to finish the proof of Theorem 2.1, we have only to show the assertion that so is (5.11) with $(\not{\partial}_{g}^* z \delta_X \not{\partial}_{g} z E)^{(m/2)}$ replaced by $(\not{\partial}_{g}^* z \delta_X \not{\partial}_{g} z E)^{((n_0+1)/2, \varepsilon^{1/2})}$. And, to show it, it will suffice to prove that, for given $T_0 > 0$, there exist constants $C > 0$, $\lambda_0 > 0$ satisfying

$$(5.12) \qquad \Big| \int_Z (\not{\partial}_{g}^* z \delta_X \not{\partial}_{g} z E)^{((n_0+1)/2, \varepsilon^{1/2})}(t, P^0) \Big| \leq C\, e^{-t\lambda_0} \quad (\forall t \geq T_0).$$

Actually, if this holds, then this, together with (5.4) (with $m = m_0 + 1 = n_0 + 1$), clearly implies the above assertion for the remainder term. Now let us show the estimate (5.12). First, since $\not{\partial}_{g} v$ is invertible ([13, (5.15)]), there exists a constant $\lambda_1 > 0$ satisfying $\operatorname{Spec}(\not{\partial}_{g_\varepsilon}^2 z) \geq \lambda_1\, (> 0)$ for any ε with $0 < \varepsilon \leq \varepsilon_0$ ([3, Proposition 4.41], [15, Lemma 2.1]). Hence we have

$$\begin{aligned} (5.13) \qquad & \Big| \operatorname{STr}\Big(\not{\partial}_{g_\varepsilon}^* z \delta_X \not{\partial}_{g_\varepsilon} z\, e^{-t \not{\partial}_{g_\varepsilon}^2 z} \Big) \Big| \leq C_1\, e^{-t\lambda_1/3} \operatorname{Tr}\Big(e^{-(t/6) \not{\partial}_{g_\varepsilon}^2 z} \Big) \\ & \leq C_2\, e^{-t\lambda_1/3} \varepsilon^{-n/2} t^N \leq C\, \varepsilon^{-n/2} e^{-t\lambda_1/4} \end{aligned}$$

(see [15, (2.26)]). The first inequality comes from the standard elliptic estimate and the boundedness of the spectrum from below (see [15, (2.2)]), and the second one comes from (3.8). Let m_0 be the integer appearing at (5.3). Then (5.13) and the first estimate at (5.3) imply

$$\begin{aligned} &\left|\varepsilon^{(m_0+1)/2}\int_Z (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{((m_0+1)/2,\varepsilon^{1/2})}(t, P^0)\right| \\ &= \left|\left(\frac{\sqrt{-1}}{2}\right)^{(n+2)/2} \mathrm{STr}\left(\partial\!\!\!/_{g_\varepsilon^Z}^* \delta_X \partial\!\!\!/_{g_\varepsilon^Z}\, e^{-t\partial\!\!\!/_{g_\varepsilon^Z}^2}\right) - \sum_{m\le m_0} \varepsilon^{m/2}\int_Z (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{(m/2,\cdot)}(t, P^0)\right| \\ &\le C_1\, \varepsilon^{-n/2}\, e^{-t\lambda_1/4}, \end{aligned}$$

which, combined with the second estimate at (5.3), yields

$$\begin{aligned} &\left|\varepsilon^{(m_0+1)/2}\int_Z (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{((m_0+1)/2,\varepsilon^{1/2})}(t, P^0)\right| \\ &\le C_2\, \varepsilon^{-n/4}\, e^{-t\lambda_1/8}\, \varepsilon^{(m_0+1)/4}\, t^{N/2} \le C_3\, \varepsilon^{(m_0+1-n)/4}\, e^{-t\lambda_1/9}. \end{aligned} \tag{5.14}$$

Now consider the formula

$$\begin{aligned} \int_Z (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{((n_0+1)/2,\varepsilon^{1/2})}(t, P^0) = \int_Z \Big\{ &\sum_{m=n_0+1}^{m_0} \varepsilon^{(m-n_0-1)/2} (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{(m/2)}(t, P^0) \\ &+ \varepsilon^{(m_0-n_0)/2} (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{((m_0+1)/2,\varepsilon^{1/2})}(t, P^0) \Big\}. \end{aligned}$$

(5.14) implies

$$\left|\varepsilon^{(m_0-n_0)/2}\int_Z (\partial\!\!\!/_{gz}^* \delta_X \partial\!\!\!/_{gz} E)^{((m_0+1)/2,\varepsilon^{1/2})}(t, P^0)\right| \le C_3\, \varepsilon^{(m_0-2n_0-n-1)/4}\, e^{-t\lambda_1/9}.$$

Hence, by taking m_0 with $m_0 - 2n_0 - n - 1 \ge 0$, this and the first estimate at (5.3) imply (5.12). Thus we finished the proof of Theorem 2.1. And (5.2), Lemma 5.1 imply now Theorem 2.2.

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ISOLATED SINGULARITIES OF BINARY DIFFERENTIAL EQUATIONS OF DEGREE n

T. FUKUI AND J.J. NUÑO-BALLESTEROS

ABSTRACT. We study isolated singularities of binary differential equations of degree n which are totally real. This means that at any regular point, the associated algebraic equation of degree n has exactly n different real roots (this generalizes the so called positive quadratic differential forms when $n = 2$). We introduce the concept of index for isolated singularities and generalize Poincaré-Hopf theorem and Bendixon formula. Moreover, we give a classification of phase portraits of the n-web around a generic singular point. We show that there are only three types, which generalize the Darbouxian umbilics D_1, D_2 and D_3.

1. INTRODUCTION

The study of the principal foliations near an isolated umbilic point of a surface M immersed in $\mathbb{R}^3$ leads us to the consideration of quadratic binary differential equations (BDE) of the form

$$a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2 = 0,$$

where $a(x,y), b(x,y), c(x,y)$ are smooth functions in some open subset $U \subset \mathbb{R}^2$ which are defined, after taking a parametrization of M, by means of the coefficients of the first and second fundamental form of M. Since the principal lines are orthogonal in the induced metric of M, we have that the discriminant $\Delta = b(x,y)^2 - a(x,y)c(x,y) \geq 0$, with equality if and only if (x,y) corresponds to an umbilic of M, so that $a(x,y) = b(x,y) = c(x,y) = 0$ and hence, (x,y) is a singularity of the BDE. It was Darboux [5] who classified the generic singularities and discovered there are only three topological types, known as the Darbouxian umbilics D_1, D_2 and D_3 (see [1] and [12] for a modern and precise study of this classification).

In fact, we can consider quadratic BDE of this type for general functions $a(x,y)$, $b(x,y)$ and $c(x,y)$, with the discriminant property: $\Delta \geq 0$ with equality if and only if $a(x,y) = b(x,y) = c(x,y) = 0$. The quadratic forms with this property are called *positive* and have been studied by many authors [2, 6, 9, 11, 13]. A positive quadratic differential form defines a pair of transverse foliations in the region of regular points. Moreover, Guíñez showed has that in this more general situation, the only generic singularities are again the Darbouxian umbilics D_1, D_2 and D_3.

The aim of this paper is to generalize this to degree n BDE of the form

$$a_0(x,y)dx^n + a_1(x,y)dx^{n-1}dy + \cdots + a_n(x,y)dy^n = 0,$$

where $a_i(x,y)$ are smooth functions defined on $U \subset \mathbb{R}^2$ such that for any $(x,y) \in U$, either it is a singular point (that is, $a_i(x,y) = 0$ for any $i = 1,\ldots,n$) or the associated algebraic

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equation has exactly n different real roots. If the functions $a_i(x,y)$ have this property, then we say that the symmetric differential n-form $\omega = \sum_{i=1}^n a_i(x,y)dx^{n-i}dy^i$ is *totally real.*

When $n = 1$, a differential n-form is always totally real and it induces an oriented foliation in the plane with singularities. For $n = 2$, totally real is equivalent to positive in the Guíñez sense and hence, the BDE defines a pair of transverse (non oriented) foliations. However, for $n \geq 3$, the corresponding BDE induces locally a n-web in the regular region (that is, a set of n foliations $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ which are pairwise transverse). It seems that isolated singularities of n-webs in the plane have not been considered previously in the literature. Moreover, we feel that the use of degree n BDE is a good approach to treat this subject.

The topological configuration of a n-web ($n \geq 3$) can be extremely complicated, even in the regular case. When $n = 3$, the curvature of the web is a function which is a topological invariant. Hence, even for regular webs we find that the topological classification has functional moduli. It is known that a regular 3-web is parallelizable or hexagonal (that is, equivalent to three families of parallel straight lines) if and only if the curvature is zero. We should also mention that because of the rigidity of webs (any homeomorphism between two regular webs is in fact a diffeomorphism [7]) the topological and differentiable classifications are the same.

We show here that for $n \geq 3$, the classification of generic singularities of totally real differential n-forms gives again only three types, which we call the generalized Darbouxian D_1, D_2 and D_3. Here, generic means a generic choice of coefficients in the linear part of the functions $a_i(x,y)$. Moreover, the classification has to be understood not as a topological classification, but just as a description of the phase portrait of the foliations around the singular point.

One of the main ingredients of the classification is the index of an isolated singular point. It is defined as a rational number of the form k/n, where $k \in \mathbb{Z}$ and it can be interpreted as the rotation number of a continuously chosen vector tangent to the leaves, when we make a trip around the singular point. We also show the generalization of the Poincaré-Hopf theorem: if M is a compact surface and ω is a totally real n-form with a finite number of singular points, then the sum of the indices is equal to the Euler characteristic $\chi(M)$.

Another important point in the paper is the use of complex coordinates. By setting $z = x+iy$ and $\overline{z} = x-iy$, we can express any n-form as $\omega = A_0dz^n + A_1dz^{n-1}d\overline{z} + \cdots + A_nd\overline{z}^n$, where $A_j = \overline{A}_{n-j}$ are differentiable functions. Then the index of an isolated singular point is equal to $-\deg(A_0)/n$, where $\deg(A_0)$ is the mapping degree of A_0. This implies that generically, the index is $\pm 1/n$.

The final ingredient for the classification is the use of the polar blow-up method to study singularities with a non degenerate principal part (see [3] and [11] for related results for vector fields or quadratic forms). We obtain a generalization of the Bendixon formula, which says that the index is equal to $1 + (e-h)/2n$ where e, h are the number of elliptic and hyperbolic sectors respectively. On the other hand, for a non degenerate singularity, the blow-up produces a n-form which has only singularities of saddle/node type. The configuration of these singularities gives a description of the phase portrait of the foliations around the singular point.

We finish the paper with a section dedicated to higher order principal lines and umbilics of surfaces M immersed in some Euclidean space $\mathbb{R}^N$. This was the original motivation of the authors to study singularities of differential n-forms. Other geometrical motivations of the same kind can be found also in [15] or [10].

2. Totally real differential forms

Definition 2.1. Let M be a C^∞ surface. A *(symmetric) differential n-form* on M is a differentiable section of the symmetric tensor fiber bundle $S^n(T^*M)$. If we take coordinates x, y on some open subset $U \subset M$, any differential n-form can be written in a unique way as

$$w = \sum_{i=0}^{n} f_i dx^i dy^{n-i},$$

where $f_i : U \to \mathbb{R}$ are smooth functions.

We will say that $p \in M$ is a *singular point* of ω if $\omega(p) = 0$. We will denote by $\mathrm{Sing}(\omega)$ the set of singular points of ω.

In general, if $p \in M$, $\omega(p) : T_pM \to \mathbb{R}$ is a form of degree n. Let $p \in M \setminus \mathrm{Sing}(\omega)$, we say that ω is *totally real at p* if there are n linear forms $\lambda_1, \dots, \lambda_n \in T_pM^*$ which are pairwise linearly independent and such that $\omega(p) = \lambda_1 \dots \lambda_n$. We say that ω is *totally real* if it is totally real at any point $p \in M \setminus \mathrm{Sing}(\omega)$.

A linear differential form ($n = 1$) is always totally real. In the case $n = 2$, a quadratic differential form is totally real if it is *positive* in the sense of [9]. Take local coordinates x, y defined on some open subset $U \subset M$ and assume that ω is given by

$$\omega = Adx^2 + 2Bdxdy + Cdy^2,$$

for some smooth functions $A, B, C : U \to \mathbb{R}$. Then ω is totally real in U if and only if for any $p \in U$, either $A(p) = B(p) = C(p) = 0$ or $B^2(p) - A(p)C(p) > 0$.

Definition 2.2. A (1-dimensional) *n-web* on a surface M is a set of n (1-dimensional) foliations $\mathcal{W} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ on M such that they are pairwise transverse at any point of M.

If ω is a totally real differential n-form on M, then we can locally associate a n-web on $M \setminus \mathrm{Sing}(\omega)$ in the following way. For each $p \in M \setminus \mathrm{Sing}(\omega)$, there are pairwise linearly independent linear forms $\lambda_1, \dots, \lambda_n \in T_pM^*$ such that $\omega(p) = \lambda_1 \dots \lambda_n$. Moreover, it is possible to choose these linear forms so that they depend smoothly on p (and hence define differential linear forms) on some open neighbourhood $U \subset M$. Then, the n-web is just defined by taking $\mathcal{F}_i$ as the foliation determined by λ_i on U (that is, the tangent vectors to $\mathcal{F}_i$ are the null vectors of λ_i).

Note that in general, it is not possible to extend this to a global n-web on $M \setminus \mathrm{Sing}(\omega)$ (unless it is simply connected). Moreover, two totally real differential n-forms ω_1 and ω_2 define the same n-web on U if and only if there is a non-zero smooth function $f : U \to \mathbb{R}$ such that $\omega_1 = f\omega_2$ on U.

Remember that if ω is a differential n-form on N and $f : M \to N$ is a differentiable map between surfaces, then $f^*\omega$ is the n-form on M given by $f^*\omega(p)(X) = \omega(f(p))(f_*X)$ for any $p \in M$ and $X \in T_pM$, and being $f_* : T_pM \to T_{f(p)}N$ the differential of f at the point p.

Definition 2.3. Let ω_1, ω_2 be two totally real differential n-forms defined on surfaces M, N respectively. We say that they are *C^∞-equivalent* (resp. *topologically equivalent*) if there is a C^∞ diffeomorphism (resp. homeomorphism) $\phi : M \to N$ such that

(1) $\phi(\mathrm{Sing}(\omega_1)) = \mathrm{Sing}(\omega_2)$,
(2) $\phi : M \setminus \mathrm{Sing}(\omega_1) \to N \setminus \mathrm{Sing}(\omega_2)$ preserves locally the leaves of the foliations of the n-webs defined by ω_1, ω_2.

It is obvious that if ϕ is a C^∞ diffeomorphism, then condition (2) is equivalent to the existence of a nonzero smooth function $f : M \setminus \mathrm{Sing}(\omega_1) \to \mathbb{R}$ such that $\phi^*(\omega_2) = f\omega_1$ on $M \setminus \mathrm{Sing}(\omega_1)$.

3. The index of an isolated singular point

We will define an index for isolated singular points of totally real differential forms, which generalize the index in the case of linear or quadratic forms.

Definition 3.1. Let ω be a totally real differential n-form on a surface M and $p \in M$ an isolated singular point. Assume that M is orientable and choose some orientation. Moreover, we choose a Riemannian metric g on M and orthogonal coordinates x, y on some open neighbourhood U of p in M, compatible with the orientation. Now, let $\alpha : [0, \ell] \to M$ be a simple, closed and piecewise regular curve, such that $\alpha([0, \ell]) \subset U$ is the boundary of a simple region R, which contains p as the only singular point in the interior. Moreover, we assume that α goes through the boundary of R in positive sense. Since α is a closed curve, it is obvious that we can extend it to $\alpha : \mathbb{R} \to M$, with $\alpha(t + \ell) = \alpha(t)$.

For each $t \in \mathbb{R}$ we choose a unit tangent vector $X(t)$ which is a solution of the equation $\omega(\alpha(t))(X) = 0$ at the point $\alpha(t)$. Since it is an algebraic equation of degree n, we can choose $X(t)$ so that it defines a differentiable unit vector field along α.

If we start with $t = 0$, after a complete turn, $X(\ell)$ must coincide with one of the $2n$ unit vectors which are solution of $\omega(\alpha(0))(X) = 0$. Because of transversality, after $2n$ turns in positive sense, we must return to the initial vector, that is, $X(2n\ell) = X(0)$. Now, let $\theta(t)$ be a differentiable determination of the angle from $\frac{\partial}{\partial x}|_{\alpha(t)}$ to$X(t)$. Then, $\theta(2n\ell)$ and $\theta(0)$ differ in an integer multiple of 2π. We define the *index* of ω in p by

$$\mathrm{ind}(\omega, p) = \frac{\theta(2n\ell) - \theta(0)}{4\pi n}.$$

It follows from the definition that the index is always a rational number of the form $s/2n$, with $s \in \mathbb{Z}$.

Lemma 3.2. *The index* $\mathrm{ind}(\omega, p)$ *does not depend on the choice of:*

(1) *the determination of the angle* θ,
(2) *the vector field* X,
(3) *the coordinates* x, y,
(4) *the curve* α,
(5) *the Riemannian metric* g,
(6) *the orientation of* M.

Proof. Note that two determinations of the angle must differ in an integer multiple of 2π. Thus, it is clear that the index does not depend on the determination.

We show now that the index does not depend on the vector field X. Suppose that we consider two vector fields $X_1(t), X_2(t)$. Note that they are solutions of an algebraic equation of degree n and they are differentiable with respect to the parameter t. Thus if $X_1(t) = \pm X_2(t)$ at some point of the curve, then this should be true for any point. In this case, the corresponding determinations of the angles should differ in an integer multiple of π, giving the same index.

Thus, we can assume that $X_1(t) \neq \pm X_2(t)$, for all $t \in \mathbb{R}$. Then, we can choose the determinations so that

$$0 < |\theta_1(t) - \theta_2(t)| < \pi,$$

for all $t \in \mathbb{R}$. Moreover, suppose that

$$\frac{\theta_i(2n\ell) - \theta_i(0)}{4\pi n} = \frac{s_i}{2n}$$

with $s_1, s_2 \in \mathbb{Z}$. Then,

$$|s_1 - s_2| = \frac{1}{2\pi} |\theta_1(2n\ell) - \theta_1(0) - \theta_2(2n\ell) + \theta_2(0)| < 1,$$

and necessarily $s_1 = s_2$.

To show that the index does not depend on the coordinates, let $Y_0 \in T_{\alpha(0)}M$ be any nonzero tangent vector. Let us denote by $Y(t)$ the parallel transport of Y_0 along $\alpha(t)$ and let $\psi(t)$ be a determination of the angle from $\frac{\partial}{\partial x}|_{\alpha(t)}$ to $Y(t)$. Following [4, Eq (2), page 271], we have that

$$\psi(\ell) - \psi(0) = \int_R K d\sigma,$$

where K denotes the gaussian curvature of M and $d\sigma$ is the area element. From this we deduce

$$2n \int_R K d\sigma - 2\pi s = (\psi - \theta)(2n\ell) - (\psi - \theta)(0), \tag{1}$$

being $s/2n$ the index. Since the angle $\psi - \theta$ does not depend on the coordinates x, y, we get that the index does not depend either.

Let now α and β be two curves satisfying the conditions of the definition of the index. We will show that the index given by both curves is the same. Suppose first that the curves are disjoint. Then it is obvious that we can construct a family of curves α_t, with $t \in [0, 1]$, depending continuously on t, which verify the conditions of the definition of index and such that $\alpha_0 = \alpha$ and $\alpha_1 = \beta$. Taking into account that it is possible to express the index by means of an integral expression, we deduce that the index with respect to α_t depends continuously on t. Since the index can only take rational values, we deduce that it must be constant. In the case that the curves α and β are not disjoint, we can take a third curve small enough so that it is disjoint with α and β and then apply the above argument.

The independence with respect to the Riemannian metric g has an analogous argument. In fact, if g and h are two Riemannian metrics, we can consider the family of Riemannian metrics $g_t = (1 - t)g + th$ so that $g_0 = g$ and $g_1 = h$. Again by means of an integral expression of the index, we see that the index with respect to the metric g_t depends continuously on t and hence, it must be constant.

Finally, it only remains to show that it does not depend on the orientation. In fact, if we change the orientation, we have to change α by $\tilde{\alpha}(t) = \alpha(\ell - t)$ and θ by $\tilde{\theta}(t) = -\theta(\ell - t)$. Then,

$$\tilde{\theta}(2n\ell) - \tilde{\theta}(0) = -\theta(\ell - 2n\ell) + \theta(\ell) = -\theta(0) + \theta(2n\ell).$$

□

As a consequence of this lemma, we deduce that the index is well defined and it only depends on the differential form ω. Moreover, the definition can be extended to the case that M is not orientable, just by taking a local orientation in a neighbourhood of the singular point.

On the other hand, the definition of index can be also extended to the case that p is a regular point, although in such case the index is always zero. In fact, we can take coordinates in such a way that $\partial/\partial x$ coincides with X along α and hence, $\theta(t) \equiv 0$.

Finally, another immediate consequence of the above lemma is that the index is invariant by equivalence. Let ω_1, ω_2 be two totally real differential n-forms defined on surfaces M, N respectively, which are equivalent through the diffeomorphism $\phi : M \to N$. Then, for each $p \in \mathrm{Sing}(\omega_1)$,

$$\mathrm{ind}(\omega_1, p) = \mathrm{ind}(\omega_2, \phi(p)).$$

Remark 3.3. We give here a formula which can be very useful to compute the index. Let us denote by $X_1(t), \dots, X_{2n}(t)$ the unit vector fields along α which are solution of $\omega(\alpha(t))(X) = 0$. We assume that they are ordered so that

$$\theta_1(t) < \theta_2(t) < \cdots < \theta_{2n}(t) < \theta_1(t) + 2\pi,$$

where $\theta_j(t)$ denotes the determination of the angle of each vector field $X_j(t)$. In particular, we have that

$$\theta_1(\ell) = \theta_i(0) + 2\pi m,$$

for some $m \in \mathbb{Z}$ and $i \in \{1, \dots, 2n\}$. Then, the index is given by

$$\mathrm{ind}(\omega, p) = m + \frac{i-1}{2n}.$$

In fact, we introduce the notation $\theta_{2n+1}(t) = \theta_1(t) + 2\pi$, $\theta_{2n+2}(t) = \theta_2(t) + 2\pi$, and in general, $\theta_{2qn+j}(t) = \theta_j(t) + 2q\pi$, for any $q \in \mathbb{Z}$ and $j \in \{1, \dots, 2n\}$. Then,

$$\begin{aligned}
\theta_1(\ell) &= \theta_i(0) + 2\pi m, \\
\theta_1(2\ell) &= \theta_i(\ell) + 2\pi m = \theta_{2i-1}(0) + 4\pi m, \\
&\dots \\
\theta_1(2n\ell) &= \theta_{2n(i-1)+1}(0) + 4\pi n m = \theta_1(0) + 2\pi(2nm + i - 1).
\end{aligned}$$

From this, we arrive to

$$\mathrm{ind}(\omega, p) = \frac{\theta_1(2n\ell) - \theta_1(0)}{4\pi n} = m + \frac{i-1}{2n}.$$

We finish this section by showing the generalization of the well known Poincaré-Hopf Theorem for vector fields or quadratic differential forms [14, 4].

Theorem 3.4. *Let M a compact surface and let ω be a totally real differential n-form with a finite number of singular points $p_1, \dots, p_m$. Then,*

$$\chi(M) = \sum_{i=1}^{m} \mathrm{ind}(\omega, p_i),$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of M.

Proof. The proof given here is just an adaptation of the proof given in [4, page 279] for the case of vector fields. We show first the theorem in the case that M is orientable.

We choose some orientation and a Riemannian metric on M. Let $\{\varphi_i : U_i \to \mathbb{R}^2\}_{i \in I}$ an atlas on M so that each chart is orthogonal and compatible with the orientation. Moreover, we take a triangulation $\mathcal{T}$ such that:

(1) Each triangle $T \in \mathcal{T}$ is contained in some coordinate neighbourhood.

(2) Each triangle $T \in \mathcal{T}$ contains at most one singular point p_T. (In the triangles with no singular points we choose any interior point p_T.)
(3) The boundary of each triangle $T \in \mathcal{T}$ has no singular points and is positively oriented.

Let X_T be a vector field along the boundary of each triangle $T \in \mathcal{T}$ which is a solution of equation $\omega(X) = 0$. Moreover, we choose it in such a way that if T_1, T_2 are adjacent triangles, then X_{T_1}, X_{T_2} coincide along the common edge. From Equation (1) we obtain

$$\int_T K d\sigma - 2\pi \operatorname{ind}(\omega, p_T) = \frac{\Delta_T}{2n},$$

for any $T \in \mathcal{T}$, where Δ_T denotes the variation of the angle from X_T to some parallel vector field after going through the boundary of T $2n$ times in positive sense.

Now, summing up for any $T \in \mathcal{T}$ and taking into account that each edge is common to two triangles with opposite orientations, we arrive to

$$\int_M K d\sigma - 2\pi \sum_{T\in\mathcal{T}} \operatorname{ind}(\omega, p_T) = \sum_{T\in\mathcal{T}} \frac{\Delta_T}{2n} = 0.$$

Finally, the result is a consequence of the Gauss-Bonnet Theorem:

$$\int_M K d\sigma = 2\pi\chi(M).$$

In the case that the surface M is not orientable, we consider $\pi : \tilde{M} \to M$ a double covering, where $\tilde{M}$ is an orientable and compact surface. Then $\chi(\tilde{M}) = 2\chi(M)$ and since π is a local diffeomorphism, each singular point p_i of ω gives exactly two singular points of the induced n-form $\pi^*\omega$ with the same index. Thus, this case is a consequence of the orientable case. □

4. Differential forms in complex coordinates

We identify $\mathbb{R}^2 = \mathbb{C}$ and use the following notation

$$z = x + iy, \qquad \bar{z} = x - iy,$$
$$dz = dx + idy, \qquad d\bar{z} = dx - idy,$$
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With this notation, it is obvious that any differential n-form on an open subset $U \subset \mathbb{C}$ can be written in a unique way in this coordinates as

$$\omega = A_0 dz^n + A_1 dz^{n-1} d\bar{z} + \cdots + A_n d\bar{z}^n,$$

for some differentiable functions $A_j : U \to \mathbb{C}$ such that $A_j = \overline{A}_{n-j}$ for all $j = 0, \ldots, n$.

The following theorem is a generalization of [14, VII.2.3] in the case $n = 2$.

Theorem 4.1. *Let ω be a totally real differential n-form on an open subset $U \subset \mathbb{C}$ and let $p \in U$ be an isolated singular point. Then, p is an isolated zero of A_0 and*

$$\operatorname{ind}(\omega, p) = -\frac{\deg(A_0, p)}{n},$$

where $\deg(A_0, p)$ denotes the local degree of A_0 at p.

Proof. Let $\delta > 0$ small enough and let $\alpha(t) = p + \delta e^{it}$, for $t \in \mathbb{R}$. We denote by $X_1(t), \dots, X_n(t)$ unit vector fields along α which are pairwise linearly independent and are solution of the equation $\omega(\alpha(t))(X) = 0$. We also denote by $\theta_j(t)$ a differentiable determination of the angle of $X_j(t)$, so that

$$X_j(t) = e^{i\theta_j(t)}\frac{\partial}{\partial z} + e^{-i\theta_j(t)}\frac{\partial}{\partial \overline{z}}.$$

It is obvious that $X_j(t)$ annihilates the linear form $\lambda_j(t)$ along α given by

$$\lambda_j(t) = e^{i\phi_j(t)}dz + e^{-i\phi_j(t)}d\overline{z},$$

being $\phi_j(t) = \pi/2 - \theta_j(t)$. Thus, by using elementary properties of the algebraic equations of degree n, we deduce that along α it is possible to factor ω as

$$\omega(\alpha(t)) = f(t)\lambda_1(t)\dots\lambda_n(t),$$

for some non vanishing function $f : \mathbb{R} \to \mathbb{R}$.

On the other hand, by comparing the coefficient of dz^n in the above expression, we have that

$$A_0(\alpha(t)) = f(t)e^{i(\phi_1(t)+\dots+\phi_n(t))}.$$

From this we see that $A_0(\alpha(t)) \neq 0$, for all $t \in \mathbb{R}$, which shows the first statement. Moreover, a differentiable determination of the angle of $A_0(\alpha(t))$ is given by

$$\beta(t) = \phi_1(t) + \dots + \phi_n(t) + \pi q,$$

for some $q \in \mathbb{Z}$.

Finally,

$$\deg(A_0, p) = \frac{\beta(4\pi n) - \beta(0)}{4\pi n} = \sum_{j=1}^{n} \frac{\phi_j(4\pi n) - \phi_j(0)}{4\pi n}$$
$$= -\sum_{j=1}^{n} \frac{\theta_j(4\pi n) - \theta_j(0)}{4\pi n} = -n\,\mathrm{ind}(\omega, p).$$

□

Corollary 4.2. *The index of any isolated singular point of a totally real differential n-form on a surface M has the form s/n, with $s \in \mathbb{Z}$. Moreover, for each $s \in \mathbb{Z}$ there is a totally real differential n-form with an isolated singular point of index s/n.*

Proof. The first part is an immediate consequence of the above theorem. To see the second part, just consider $M = \mathbb{C}$, $p = 0$ and

$$\omega = \begin{cases} z^s dz^n + \overline{z}^s d\overline{z}^n, & \text{if } s \geq 0, \\ \overline{z}^{|s|} dz^n + z^{|s|} d\overline{z}^n, & \text{if } s < 0. \end{cases}$$

□

According to Definition 3.1, an isolated singular point of an n-web will have an index of the form $s/2n$, with $s \in \mathbb{Z}$. The above corollary says that in the case that the n-web is induced from a totally real differential n-form, the index will be of the form s/n, with $s \in \mathbb{Z}$. This can be interpreted as some kind of orientability condition for the n-web defined by a differential n-form.

For instance, when $n = 1$, a linear differential form in M induces an orientable foliation in a neighbourhood of each point of M. In this case, the index of an isolated singular point is an integer. For $n = 2$, a positive quadratic differential form induces a pair of (non necessarily orientable) transverse foliations in a neighbourhood of each point of M. The index associated to each one of the foliations is the same (because of transversality) and it is a half-integer (see [14, VII.2.2]).

Corollary 4.3. *Let ω be a totally real differential n-form on a surface M and $p \in M$ an isolated singular point. Let $\alpha : [0, \ell] \to M$ be a curve satisfying the conditions for the definition of the index and let $X(t)$ be a unit vector field along α, solution of $\omega(\alpha(t))(X) = 0$. Then $X(n\ell) = X(0)$ and*

$$\operatorname{ind}(\omega, p) = \frac{\theta(n\ell) - \theta(0)}{2\pi n},$$

where $\theta(t)$ denotes a determination of the angle of $X(t)$.

Proof. This is consequence of the above corollary and Remark 3.3. Let us denote by $X_1(t), \dots, X_{2n}(t)$ the unit vector fields along α which are solution of $\omega(\alpha(t))(X) = 0$, being $X(t) = X_1(t)$. We suppose that they are ordered so that

$$\theta_1(t) < \theta_2(t) < \cdots < \theta_{2n}(t) < \theta_1(t) + 2\pi,$$

where $\theta_j(t)$ is the determination of the angle of each vector field $X_j(t)$. Then,

$$\operatorname{ind}(\omega, p) = m + \frac{i-1}{2n},$$

where $\theta_1(\ell) = \theta_i(0) + 2\pi m$, with $m \in \mathbb{Z}$ and $i \in \{1, \dots, 2n\}$. Moreover, we introduce the notation $\theta_{2qn+j}(t) = \theta_j(t) + 2q\pi$, for any $q \in \mathbb{Z}$ and $j \in \{1, \dots, 2n\}$.

From the above corollary we see that $i - 1$ must be even and hence, we can write $i - 1 = 2q$, with $q \in \mathbb{Z}$. Thus,

$$\theta_1(n\ell) = \theta_{n(i-1)+1}(0) + 2\pi mn = \theta_1(0) + 2\pi(mn + q),$$

giving $X_1(n\ell) = X_1(0)$. □

Definition 4.4. We say that a singular point p of a totally real differential n-form ω is *simple* if the linear part of ω at p is itself a totally real differential n-form having p as an isolated singular point. Suppose that in complex coordinates

$$\omega = A_0 dz^n + A_1 dz^{n-1} d\bar{z} + \cdots + A_n d\bar{z}^n,$$

for some differentiable functions $A_i : U \to \mathbb{C}$. We also assume, for simplicity, that $p = 0$. Then, each one of these functions A_i has a Taylor expansion at the origin

$$A_i = a_i z + b_i \bar{z} + \dots$$

with $a_i, b_i \in \mathbb{C}$. The linear part of ω at p is the differential n-form

$$\omega_1 = (a_0 z + b_0 \bar{z}) dz^n + (a_1 z + b_1 \bar{z}) dz^{n-1} d\bar{z} + \cdots + (a_n z + b_n \bar{z}) d\bar{z}^n.$$

Corollary 4.5. *Any simple singular point of a totally real differential n-form on a surface M has index $\pm 1/n$.*

Proof. We take complex coordinates, suppose that $p = 0$ and the linear part of ω at p is

$$\omega_1 = (a_0 z + b_0 \bar{z}) dz^n + (a_1 z + b_1 \bar{z}) dz^{n-1} d\bar{z} + \cdots + (a_n z + b_n \bar{z}) d\bar{z}^n.$$

If ω_1 is totally real and p is an isolated singular point, by Theorem 4.1, p is an isolated zero of the linear function $a_0 z + b_0 \bar{z}$ and hence, such linear function is regular. Since it is the linear part of the function A_0, p is a regular point of A_0. Thus, $\deg(A_0, p) = \pm 1$ and $\operatorname{ind}(\omega, p) = \pm 1/n$. □

5. Non degenerate differential forms

Let ω be a totally real differential n-form on some open subset $U \subset \mathbb{C}$ and let $p \in U$ be an isolated singular point. We can extend the notation introduced in the above section and denote by ω_k the homogeneous part of degree k of ω. That is, each one of the coefficients A_j admits a Taylor expansion at p and ω_k is the n-form whose coefficients are the homogeneous parts of degree k in the expansion of the A_j.

Definition 5.1. We say that ω is *semi-homogeneous* at p if there is $k \geq 1$ such that $\omega_i = 0$ for $i = 1, \dots, k-1$ and ω_k is a totally real differential n-form having p as an isolated singular point. Note that when $k = 1$, this is equal to the definition of simple singular point.

Assume for simplicity that $p = 0$ and let

$$\omega_k = A_0^k dz^n + A_1^k dz^{n-1} d\overline{z} + \cdots + A_n^k d\overline{z}^n,$$

where A_i^k are homogeneous polynomials of degree k. We define the *characteristic polynomial of* ω as the (real) homogeneous polynomial of degree $k+n$

$$P_\omega = A_0^k z^n + A_1^k z^{n-1}\overline{z} + \cdots + A_n^k \overline{z}^n.$$

Let us denote by $\pi : \mathbb{R}^2 \to \mathbb{C}$ the *polar blow-up*, that is, $\pi(r,t) = re^{it}$. We fix $\delta > 0$ small enough such that $\pi((-\delta,\delta)\times\mathbb{R}) \subset U$ and $p = 0$ is the only singular point of ω in such set.

Lemma 5.2. *If ω is semi-homogeneous with principal part ω_k, then*

$$\tilde{\omega}(r,t) = \begin{cases} \dfrac{1}{r^k}\omega(re^{it}), & \text{if } r \neq 0, \\ \omega_k(e^{it}), & \text{if } r = 0, \end{cases}$$

defines a totally real differential n-form along π on $(-\delta,\delta)\times\mathbb{R}$ with no singular points.

Proof. Suppose that ω is given by

$$\omega = A_0 dz^n + A_1 dz^{n-1} d\overline{z} + \cdots + A_n d\overline{z}^n$$

and let us denote by A_j^k the homogeneous part of degree k of A_j. By the Hadamard Lemma it follows that

$$A_j(re^{it}) = r^k B_j(r,t),$$

for some differentiable functions $B_j : (-\delta,\delta)\times\mathbb{R} \to \mathbb{R}$ such that $B_j(0,t) = A_j^k(e^{it})$. In particular

$$\tilde{\omega}(r,t) = B_0(r,t)dz^n + B_1(r,t)dz^{n-1}d\overline{z} + \cdots + B_n(r,t)d\overline{z}^n.$$

□

As a consequence of the above lemma, if ω is semi-homogeneous, we can choose n unit vector fields $X_1(r,t), \dots, X_n(r,t)$ along π on $(-\delta,\delta)\times\mathbb{R}$ which are pairwise linearly independent and solution of $\tilde{\omega}(r,t)(X) = 0$. Moreover, we denote by $\theta_j(r,t)$ a differentiable determination of the angle of each vector field $X_j(r,t)$. Then we showed in the proof of Theorem 4.1, that it is possible to factor $\tilde{\omega}$ as

$$\tilde{\omega} = f\lambda_1 \dots \lambda_n,$$

being λ_j the linear forms given by

$$\lambda_j = e^{i\phi_j}dz + e^{-i\phi_j}d\overline{z},$$

with $\phi_j = \pi/2 - \theta_j$ and $f : (-\delta,\delta)\times\mathbb{R} \to \mathbb{R}$ a non vanishing function.

Definition 5.3. The pull-back through π of the n-form $\tilde{\omega}$ defines an n-form $\pi^*\tilde{\omega}$ on $(-\delta,\delta)\times\mathbb{R}$, which is called *the polar n-form of ω*. Analogously, we call *linear polar forms of ω* the linear forms $\pi^*\lambda_1,\ldots,\pi^*\lambda_n$, in such a way that

$$\pi^*\tilde{\omega} = f\pi^*\lambda_1\ldots\pi^*\lambda_n,$$

An easy computation gives that for each $j=1,\ldots,n$

$$\pi^*\lambda_j = 2(\cos\varphi_j dr - r\sin\varphi_j dt),$$

where $\varphi_j = \phi_j + t$. Thus, each one of these polar linear forms has singular points $(0,t)$ with $\varphi_j(0,t) = \pi/2 + q\pi$, $q\in\mathbb{Z}$.

Note that a point $(0,t)$ can be a singular point of only one of the polar linear forms. In fact, suppose that

$$\varphi_{j_1}(0,t) = \pi/2 + q_1\pi, \quad \varphi_{j_2}(0,t) = \pi/2 + q_2\pi,$$

for some $q_1,q_2\in\mathbb{Z}$. Then

$$\theta_{j_1}(0,t) - \theta_{j_2}(0,t) = (q_2-q_1)\pi,$$

which implies that the corresponding vector fields are linearly dependent and hence, $j_1 = j_2$.

Moreover, under some conditions it is possible to determine the topological type of these singular points. Let Λ_j the vector field given by

$$\Lambda_j = r\sin\varphi_j\frac{\partial}{\partial r} + \cos\varphi_j\frac{\partial}{\partial t}.$$

Then, the jacobian matrix at a singular point is

$$D\Lambda_j(0,t) = \pm\begin{pmatrix} 1 & -\frac{\partial\varphi_j}{\partial r} \\ 0 & -\frac{\partial\varphi_j}{\partial t}\end{pmatrix}.$$

As a consequence, we have that $(0,t)$ is a hyperbolic singular point of $\pi^*\lambda_j$ if and only if $\frac{\partial\varphi_j}{\partial t}\neq 0$. Moreover, $(0,t)$ is of *saddle type* when $\frac{\partial\varphi_j}{\partial t}>0$ and of *node type* when $\frac{\partial\varphi_j}{\partial t}<0$.

Lemma 5.4. *Let ω be a semi-homogeneous totally real differential n-form and $p=0$ an isolated singular point. Then $z=e^{it}$ is a root of the characteristic polynomial P_ω if and only if $(0,t)$ is a singular point of one of its polar linear forms. Moreover, it is a simple root if and only if $(0,t)$ is a hyperbolic singular point of such polar linear form.*

Proof. In general, we have that $\pi^*dz = e^{it}(dr + irdt)$ and $\pi^*d\bar{z} = e^{-it}(dr - irdt)$. In particular, restricted to $r=0$, we get

$$\pi^*\tilde{\omega}(0,t) = \left(\sum_{j=0}^{n} A_j^k(e^{it})(e^{it})^j(e^{-it})^{n-j}\right)dr^n = P_\omega(e^{it})dr^n.$$

On the other hand, by using the factor of $\pi^*\tilde{\omega}$ in the polar linear forms, we see that

$$\pi^*\tilde{\omega}(0,t) = f(0,t)\cos\varphi_1(0,t)\ldots\cos\varphi_n(0,t)dr^n,$$

which implies that

$$P_\omega(e^{it}) = 2^n f(0,t)\cos\varphi_1(0,t)\ldots\cos\varphi_n(0,t).$$

Thus, it is obvious that $z=e^{it}$ is a root of P_ω if and only if $(0,t)$ is a singular point of one of the polar linear forms.

Moreover, since P_ω is a homogeneous polynomial it is easy to check that z is a simple root if and only if $\frac{d}{dt}\left(P_\omega(e^{it})\right) \neq 0$. But if we differentiate in the above expression, we arrive to

$$\frac{d}{dt}\left(P_\omega(e^{it})\right) = \pm 2^n f(0,t)\frac{\partial \varphi_j}{\partial t}(0,t).$$

Therefore, it is a simple root if and only if $(0,t)$ is a hyperbolic singular point, by the above remark. □

Remark 5.5. Suppose that $z = e^{it}$ is a root of the characteristic polynomial P_ω. By the above lemma, $(0,t)$ is a singular point of one the polar linear forms, that is, $\varphi_j(0,t) = \pi/2 + q\pi$, for some $j \in \{1,\dots,n\}$, and $q \in \mathbb{Z}$. For each $p \in \mathbb{Z}$, $e^{i(t+p\pi)} = \pm z$ is also a root of P_ω and hence, there are $j_p \in \{1,\dots,n\}$, and $q_p \in \mathbb{Z}$ such that $\varphi_{j_p}(0,t+p\pi) = \pi/2 + q_p\pi$. This implies that

$$\varphi_j(0,t) - \varphi_{j_p}(0,t+p\pi) = (p-q_p)\pi,$$

for any $p \in \mathbb{Z}$. But looking at the way that the functions φ_j are constructed, if this is true for some point $t \in \mathbb{R}$, then it must be true for any $t \in \mathbb{R}$. Then, by taking derivatives with respect to t,

$$\frac{\partial \varphi_j}{\partial t}(0,t) = \frac{\partial \varphi_{j_p}}{\partial t}(0,t+p\pi).$$

Thus, $(0,t)$ is a singular point of $\pi^*\lambda_j$ of saddle or node type if and only if $(0,t+p\pi)$ is a singular point of $\pi^*\lambda_{j_p}$ of saddle or node type respectively. In conclusion, the singularity type only depends on the direction determined by $z = e^{it}$.

Definition 5.6. Let ω be a totally real differential n-form with an isolated singular point p. We say ω is *non degenerate* at p if it is semi-homogeneous and the characteristic polynomial has only simple roots.

Theorem 5.7. *Let ω be a totally real differential n-form with a non degenerate singular point p. Then,*

$$\operatorname{ind}(\omega,p) = 1 - \frac{S^+ - S^-}{n},$$

where S^+ and S^- denote the numbers of characteristic directions of saddle and node type respectively.

Proof. Denote by S_j^+ and S_j^- the numbers of singular points of saddle and node type respectively of the polar linear form $\pi^*\lambda_j$ in the interval $[0,2\pi n)$. Then,

$$\sum_{j=1}^{n} S_j^+ = 2nS^+, \quad \sum_{j=1}^{n} S_j^- = 2nS^-.$$

Remember that such points are given by the points $(0,t)$ such that $\varphi_j(0,t) = \pi/2 + q\pi$, with $q \in \mathbb{Z}$. Moreover, it is of saddle type when φ_j is increasing at such point and of node type when it is decreasing. This implies that

$$\varphi_j(0,2\pi n) - \varphi_j(0,0) = \pi(S_j^+ - S_j^-),$$

for all $j = 1,\dots,n$.

Now, by Corollary 4.3,

$$\begin{aligned}\operatorname{ind}(\omega,p) &= \frac{1}{n}\sum_{j=1}^{n}\frac{\theta_j(0,2\pi n)-\theta_j(0,0)}{2\pi n} = -\frac{1}{n}\sum_{j=1}^{n}\frac{\phi_j(0,2\pi n)-\phi_j(0,0)}{2\pi n} \\ &= -\frac{1}{n}\sum_{j=1}^{n}\frac{\varphi_j(0,2\pi n)-2\pi n-\varphi_j(0,0)}{2\pi n} = 1-\frac{1}{n}\sum_{j=1}^{n}\frac{\varphi_j(0,2\pi n)-\varphi_j(0,0)}{2\pi n} \\ &= 1-\frac{1}{n}\sum_{j=1}^{n}\frac{S_j^+-S_j^-}{2n} = 1-\frac{S^+-S^-}{n},\end{aligned}$$

since $\phi_j(0,t)=\frac{\pi}{2}-\theta_j(0,t)$ and $\varphi_j(0,t)=\phi_j(0,t)+t$. □

Definition 5.8. Let ω be a totally real differential n-form with a non degenerate singular point p. By a *sector* we mean each one of the regions bounded by two consecutive characteristic directions S_1 and S_2. We say a sector is

(1) *hyperbolic:* if both S_1 and S_2 are of saddle type;
(2) *parabolic:* if one of S_1 and S_2 is of saddle type and the other one is of node type;
(3) *elliptic:* if both S_1 and S_2 are of node type.

Let S^+ and S^- denote the number of characteristic directions of saddle and node type respectively and let h and e denote the numbers of hyperbolic and elliptic sectors respectively. It is obvious that $e-h=2(S^- - S^+)$. Thus, we get the following immediate consequence of Theorem 5.7, which generalizes the well known Bendixson formula for the index when $n=1$.

Corollary 5.9. *Let ω be a totally real differential n-form with a non degenerate singular point p. Then,*

$$\operatorname{ind}(\omega,p)=1+\frac{e-h}{2n},$$

where e and h are the numbers of elliptic and hyperbolic sectors respectively.

Remark 5.10. When ω has a non degenerate principal part, it is possible to improve the formula for the index given in Remark 3.3. Let $X_1(r,t),\ldots,X_n(r,t)$ be unit vector fields along π on $(-\delta,\delta)\times\mathbb{R}$ which are pairwise linearly independent and solution of $\tilde{\omega}(r,t)(X)=0$. Moreover, we suppose that they are chosen so that

$$\theta_1(0,t)<\theta_2(0,t)<\cdots<\theta_n(0,t)<\theta_1(0,t)+\pi,$$

where $\theta_j(r,t)$ denotes the determination of the angle of each vector field $X_j(r,t)$. Note that for $r=0$, these vector fields are solution of an equation with homogeneous coefficients, which implies that

$$\theta_1(0,\pi)=\theta_i(0,0)+\pi m,$$

for some $m\in\mathbb{Z}$ and $i\in\{1,\ldots,n\}$. Then, it follows that

$$\operatorname{ind}(\omega,p)=m+\frac{i-1}{n}.$$

6. Phase portrait of non degenerate singular points

In general, the foliations of an n-form can present very complicated configurations around a singular point. In the case that ω has a non degenerate singular point, the n foliations are obtained as the image of the integral curves of the polar linear forms through the polar blow-up. Moreover, since the characteristic polynomial has only simple roots, then the problem is simpler, because the polar linear forms only have singularities of saddle or node type.

Definition 6.1. Let ω be a totally real differential n-form with a non degenerate singular point p. Let x be a point near p and let L one of the n leaves of the web passing through x. We say that L is

(1) *hyperbolic:* if p is not an accumulation point of L;
(2) *parabolic:* if p is an accumulation point on just one side of L;
(3) *elliptic:* if p is an accumulation point on both sides of L.

If the leaf L is hyperbolic (respectively parabolic, elliptic), then it corresponds to an integral curve of one of the polar linear forms with a saddle-saddle (respectively saddle-node, node-node) connection. In order to have a complete description, we need to know how many sectors the leaf is going to pass through when connecting the two singular directions (Figure 6.1).

Lemma 6.2. *Let x be a point near p and let L be a hyperbolic leaf through x connecting two saddles. Assume that L passes through k sectors containing n_1 saddles and n_2 nodes (so that $n_1 + n_2 = k - 1$). Then,*

$$k = n + 2n_2.$$

Proof. Let R be the union of the closed sectors that L passes trhough, which is bounded by the two saddles S_1 and S_2. Since R is simply connected, we can separate the web in R into n foliations $\mathcal{F}_1, \dots, \mathcal{F}_n$. We will assume that L is a leaf of $\mathcal{F}_1$. Then $\mathcal{F}_1$ also contains the saddles S_1, S_2 and all its other leaves of $\mathcal{F}_1$ are also hyperbolic.

Let $\mathcal{F}_i$ be one of the other foliations, with $i = 2, \dots, n$. We can use the leaves of $\mathcal{F}_i$ to define a continuous map $\phi_i : L \to S_1 \cup S_2$. Given $y \in L$, we take the leaf L_i of $\mathcal{F}_i$ passing through y. Because of transversality, either L_i intersects $S_1 \cup S_2$ in a single point which we define as $\phi_i(y)$ or p is an accumulation point of L_i, in which case we define $\phi_i(y) = p$ (see Figure 2).

Since the leaves of $\mathcal{F}_i$ are disjoint, we have two possibilities: either $\phi_i^{-1}(p)$ is just one point and $\mathcal{F}_i$ contains just one saddle, or $\phi_i^{-1}(p)$ is an interval, so that $\mathcal{F}_i$ contains one node and two saddles (see Figure 3).

Finally, assume there are a foliations of the first type and b of the second type, with $a + b = n - 1$. Then, $n_1 = a + 2b$ and $n_2 = b$, which gives the desired result. □

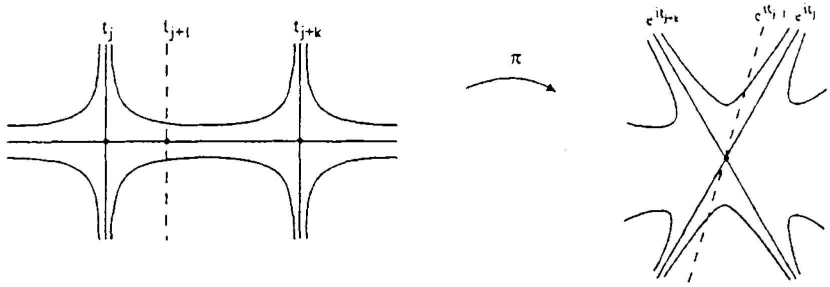
Lemma 6.3. *Let x be a point near p and let L be an elliptic leaf through x connecting two nodes. Assume that L passes through k sectors containing n_1 saddles and n_2 nodes (so that $n_1 + n_2 = k - 1$). Then,*

$$k = n + 2n_1.$$

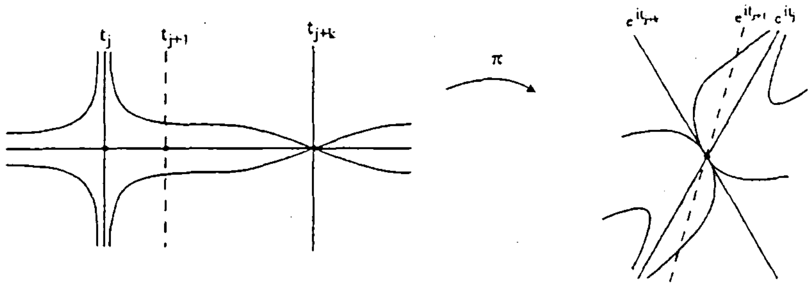
Proof. We assume that $p = 0$ and that ω has the following principal part

$$\omega_k = A_0^k dz^n + \cdots + A_n^k d\bar{z}^n,$$

(1) saddle-saddle;



(2) saddle-node;



(3) node-node.

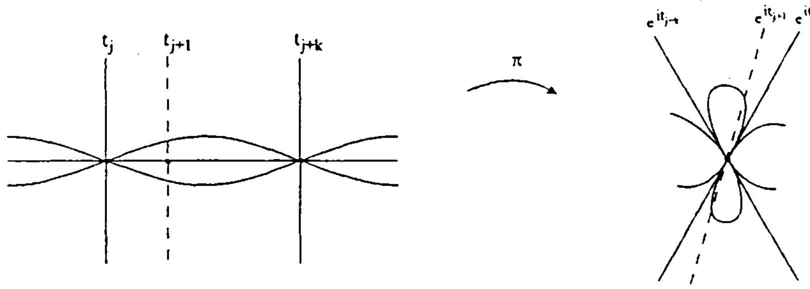


FIGURE 1

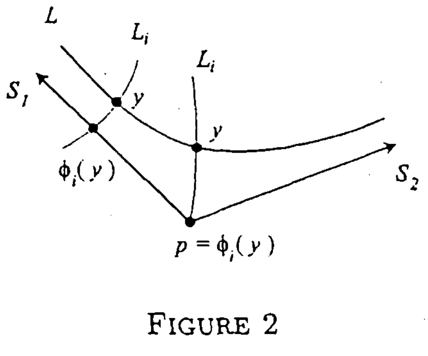


FIGURE 2

where A_i^k are homogeneous polynomials of degree k. We take now the inversion $z = 1/w$, which gives:

$$dz^n = -\frac{dw}{w^{2n}} - \frac{\overline{w}^{2n} dw}{(w\overline{w})^{2n}},$$

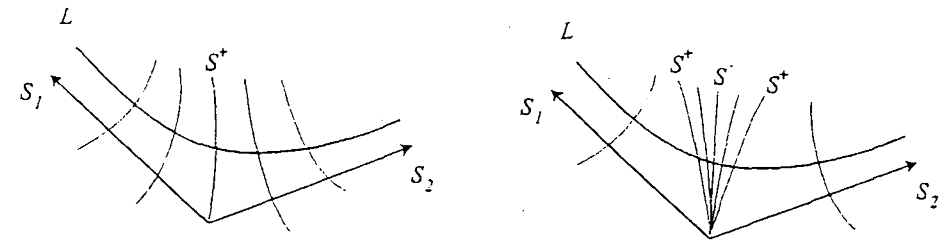


FIGURE 3

and

$$A_i^k(z) = A_i^k(\frac{1}{w}) = \frac{A_i^k(\overline{w})}{(w\overline{w})^k}.$$

Then we obtain that in $\mathbb{C} \setminus \{0\}$, ω_k is equivalent to the differential form

$$\sigma_k = A_0^k(\overline{w})\overline{w}^{2n}dw^n + \cdots + A_n^k(\overline{w})w^{2n}d\overline{w}^n).$$

Note that σ_k is also totally real with non degenerate principal part and the characteristic polynomial has the same roots as ω_k, although the inversion transforms saddles into nodes and nodes into saddles. Moreover, elliptic leaves of the foliations of ω_k are transformed into hyperbolic leaves of σ_k and vice versa. Thus, the result is a consequence of the above lemma. In Figure 4 we present the result of taking the inversion of Figure 3.

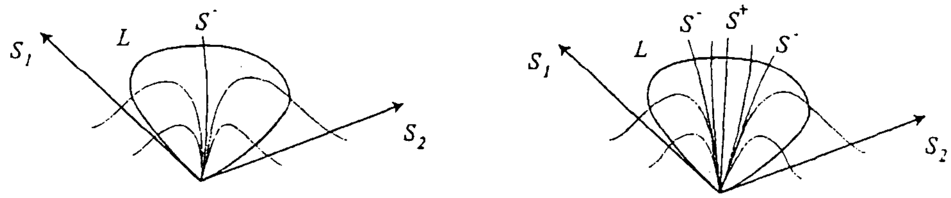


FIGURE 4

□

Lemma 6.4. *Let x be a point near p and let L be a parabolic leaf through x connecting a saddle and a node. Assume that L passes through k sectors containing n_1 saddles and n_2 nodes (so that $n_1 + n_2 = k - 1$). Then,*

$$k = 1 + 2n_2.$$

Proof. We follow a similar argument to that of the proof of Lemma 6.2. We denote by R the union of sectors containing the leaf L, which is bounded by the saddle S_1 and the node S_2. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be the n foliations determined by ω in R so that L is a leaf of $\mathcal{F}_1$.

For each one of the foliations $\mathcal{F}_i$, with $i = 2, \dots, n$ we have again two possibilities as listed in Figure 5. In one case $\mathcal{F}_i$ does not contain any characteristic direction, while in the other cased it contains one saddle and one node. If we denote by a, b the number of foliations of each type respectively, we have that $a+b = n-1$ and $n_1 = n_2 = b$. Therefore, we get $k = 1 + 2n_2$.

□

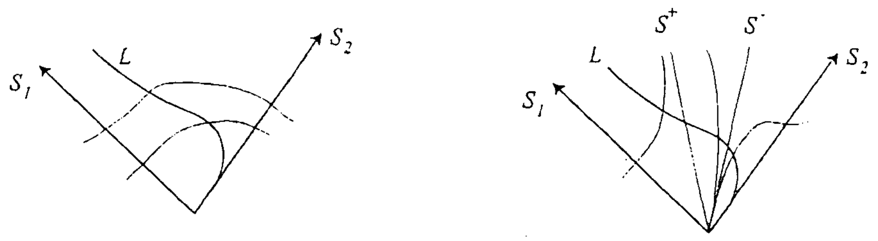


FIGURE 5

Remark 6.5. Once we know how many directions of saddle or of node type we have, as well as their relative position around the singular point p, the three above lemmas allow us to complete the phase portrait of all the leaves of the n web determined by ω. We call this the *phase portrait* of ω at p. When $n \leq 2$, it is well known that this is enough for topological classification, that is, if two differential n-forms have the same phase portrait at a point, then they are locally topologically equivalent. For $n \geq 3$, this is not true anymore because the curvature of the web is a topological invariant.

7. Phase portraits near hyperbolic singular points

In this section we give the possible phase portraits of "generic" singular points of totally real differential n-forms.

Definition 7.1. We say that p is a *hyperbolic* singular point of a totally real differential n-form ω if it is simple and the characteristic polynomial P_ω has only simple roots.

Theorem 7.2. *Let p be a hyperbolic singular point of a totally real differential n-form ω ($n \geq 2$). Then, there are only three possible phase portraits of the foliations of ω around p:*

(1) *Type D_1 or lemon: there are $n-1$ directions of saddle type with hyperbolic leaves passing through n sectors.*
(2) *Type D_2 or monstar: there are n directions of saddle type and one of node type; the hyperbolic leaves pass through $n+2$ sectors, while the parabolic leaves pass through one sector.*
(3) *Type D_3 or star: there are $n+1$ directions of saddle type with hyperbolic leaves passing through n sectors.*

Proof. Let S^+ and S^- be the numbers of directions of saddle and node type respectively. The sum $S^+ + S^-$ is the total number of roots of the characteristic polynomial P_ω, which has degree $n+1$. Since the roots are simple,

$$0 \leq S^+ + S^- \leq n+1, \quad S^+ + S^- \equiv n+1 \mod 2.$$

Assume that $S^+ + S^- = n+1$. If $S^- \geq 2$, then $S^+ \leq n-1$ and by Theorem 5.7,

$$\operatorname{ind}(\omega, p) = 1 - \frac{S^+ - S^-}{n} \geq 1 - \frac{n-1-2}{n} = \frac{3}{n}.$$

This is not possible, by Corollary 4.5, since the index can only be $\pm 1/n$. Thus, the only possibilities are $S^+ = n+1$, $S^- = 0$ or $S^+ = n$, $S^- = 1$ which correspond to the types D_3 and D_2 respectively. Note that the index in each case is $-1/n$ or $1/n$ respectively.

Next case is $S^+ + S^- = n-1$. As above, if we suppose that $S^- \geq 1$, then $S^+ \leq n-2$ and hence,

$$\operatorname{ind}(\omega,p) = 1 - \frac{S^+ - S^-}{n} \geq 1 - \frac{n-2-1}{n} = \frac{3}{n}.$$

The only possibility is $S^+ = n-1$, $S^- = 0$ which correspond to the type D_1 and has index $1/n$.

Finally, assume that $S^+ + S^- \leq n-3$. Then necessarily $S^- \geq 0$, $S^+ \leq n-3$ and hence,

$$\operatorname{ind}(\omega,p) = 1 - \frac{S^+ - S^-}{n} \geq 1 - \frac{n-3-0}{n} = \frac{3}{n}.$$

Therefore, it is clear that there are no more possibilities.

The discussion about the number of sectors of hyperbolic or parabolic leaves is a consequence of above lemmas. □

The above classification in the case $n = 2$ gives the classification obtained by Darboux for the curvature lines around generic umbilic points of an immersed surface in $\mathbb{R}^3$ (see [1] and [12]). A proof for the general case of hyperbolic singular points of quadratic forms can found in [9].

Example 7.3. Consider $\omega_1 = \bar{z}dz^n + zd\bar{z}^n$. By Theorem 4.3,

$$\operatorname{ind}(\omega_1, 0) = -\deg(\bar{z}, 0)/n = 1/n.$$

Moreover, the characteristic polynomial is

$$P_{\omega_1} = \bar{z}z^n + z\bar{z}^n = \bar{z}z(z^{n-1} + \bar{z}^{n-1}),$$

which has $n-1$ real simple roots. Thus, for any n, ω_1 has a hyperbolic singular point of type lemon or D_1.

Now, let $\omega_{2,\epsilon} = (iz - (1+\epsilon)i\bar{z})dz^n + (-i\bar{z} + (1+\epsilon)iz)d\bar{z}^n$, with $\epsilon > 0$. In this case, the index is again $1/n$ and the characteristic polynomial is

$$\begin{aligned} P_{\omega_{2,\epsilon}} &= (iz - (1+\epsilon)i\bar{z})z^n + (-i\bar{z} + (1+\epsilon)iz)\bar{z}^n \\ &= (iz - i\bar{z})(z^n + \bar{z}^n) + \epsilon i z\bar{z}(z^{n-1} - \bar{z}^{n-1}). \end{aligned}$$

Given n, it follows that for ϵ small enough, $P_{\omega_{2,\epsilon}}$ has exactly $n+1$ real simple roots. Then, $\omega_{2,\epsilon}$ has a hyperbolic singular point of type monstar or D_2.

Finally, we consider $\omega_3 = zdz^n + \bar{z}d\bar{z}^n$. The index is now $-1/n$ and $P_{\omega_3} = z^{n+1} + \bar{z}^{n+1}$. For any n, it has $n+1$ simple real roots, so that ω_3 is of type star or D_3.

In Figure 6, we can find pictures of the foliations for the three examples D_1, D_2, D_3 in the cases $n = 2$ (top) and $n = 3$ (bottom) obtained with *Mathematica* (D_1 and D_3) and with the program *Homogeneous equations lines* by A. Montesinos [16] (D_2 with $\epsilon = 1/2$).

8. Higher order principal lines and umbilics

Let $g : M \to \mathbb{R}^N$ be a C^∞ immersion of a surface M in Euclidean space $\mathbb{R}^N$. We consider the distance squared unfolding $D : \mathbb{R}^N \times M \to \mathbb{R}^N \times \mathbb{R}$ given by

$$D(x,p) = (x, d_x(p)) = (x, \frac{1}{2}\|x - g(p)\|^2).$$

We use Thom-Boardman notation for singularities. Then, it follows that $\Sigma^2(D)$ is the subset of $\mathbb{R}^N \times M$ of pairs (x,p) such that the jacobian matrix of d_x has kernel rank 2 at p, which is nothing but the *normal bundle* of M in $\mathbb{R}^N$.

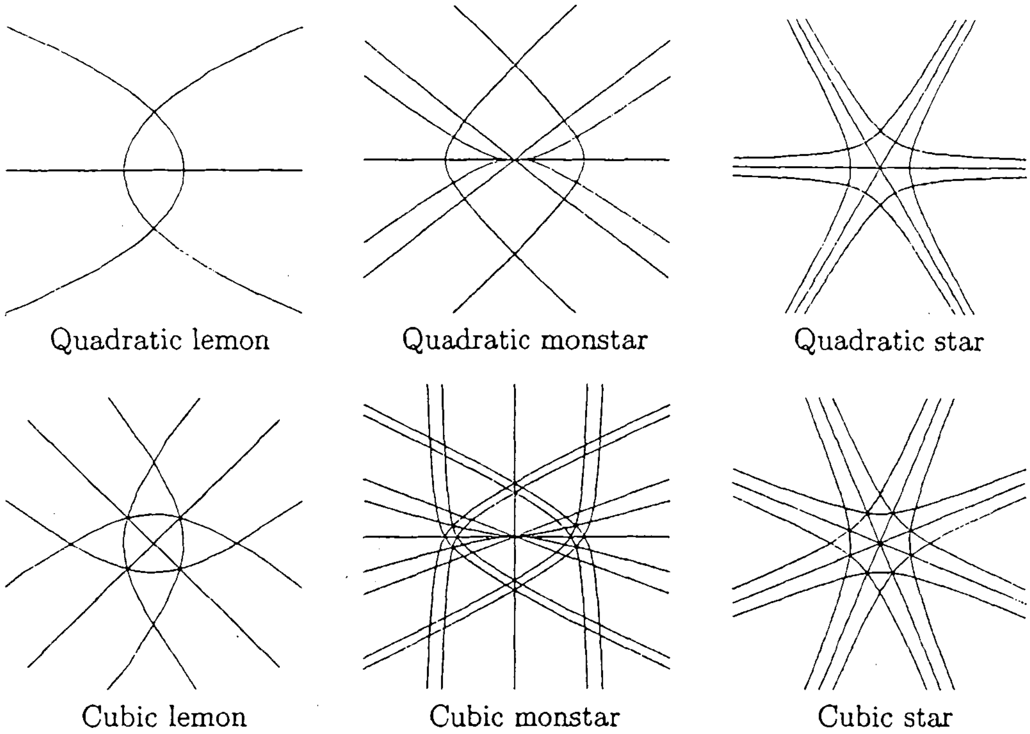


FIGURE 6

Assume $N = 3$. Then $\Sigma^{2,1}(D)$ is the subset of $\Sigma^2(D)$ given by pairs (x, p) such that the hessian matrix of d_x has kernel rank 1 at p. This is known as the *focal set* of M in $\mathbb{R}^3$ and corresponds to the subset of pairs (x, p) such that x is a centre of principal curvature at a non umbilic point $p \in M$. Moreover, we can also consider the *contact directions*, which are defined as the tangent directions $X \in T_pM$ such that $X \in \ker \operatorname{Hess}(d_x)_p$. When p is not parabolic, then these contact direction correspond to the principal lines of M (when p is parabolic, principal lines are in fact contact directions of the height function, in which case the sphere becomes a plane and x goes to infinity).

By taking local coordinates u, v in an open subset $U \subset M$, it is possible to find the differential equation of principal lines:

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0,$$

where E, F, G and L, M, N are respectively the coefficients of the first and second fundamental forms of M in $\mathbb{R}^3$. The singular points of this equation are the umbilic points of M where the surface has a contact of type $\Sigma^{2,2}$ with some sphere of $\mathbb{R}^3$ (if the umbilic is non flat). However, for our purposes, it is better to consider the following equivalent differential equation:

$$\begin{vmatrix} g_{1u} & g_{2u} & g_{3u} & 0 & 0 \\ g_{1v} & g_{2v} & g_{3v} & 0 & 0 \\ g_{1uu} & g_{2uu} & g_{3uu} & E & dv^2 \\ g_{1uv} & g_{2uv} & g_{3uv} & F & -dudv \\ g_{1vv} & g_{2vv} & g_{3vv} & G & du^2 \end{vmatrix} = 0.$$

This matrix (excluding the last column) was introduced in [8] to define the notion of k-rounding of an immersion $g: M \to \mathbb{R}^N$. It is a higher order generalization of umbilic point and for an appropriate choice of the ambient dimension N these points are generically isolated.

For instance, assume now that $k = 3$ and consider an immersion of a surface M in $\mathbb{R}^7$. We define the *third order contact directions* as the tangent directions $X \in T_pM$ such that $X \in \ker J^3(d_x)_p$ and $(x,p) \in \Sigma^{2,2,1}(D)$. Here J^3 is the operator defined in local coordinates

$$J^3(f) = \begin{pmatrix} f_{uuu} & f_{uuv} \\ f_{uuv} & f_{uvv} \\ f_{uvv} & f_{vvv} \end{pmatrix}.$$

Note that $f_u = f_v = f_{uu} = f_{uv} = f_{vv} = 0$, this definition does not depend on the coordinates.

Note that we can do the same construction by taking the height function unfolding $H: S^6 \times M \to S^6 \times \mathbb{R}$ given by

$$H(\mathrm{v},p) = (\mathrm{v}, h_v(p)) = (\mathrm{v}, \langle \mathrm{v}, g(p)\rangle).$$

We also include in the above definition of third order contact directions those $X \in T_pM$ such that $X \in \ker J^3(h_{\mathrm{v}})_p$ and $(\mathrm{v},p) \in \Sigma^{2,2,1}(H)$.

Assume that M is locally parameterized locally by a map $g: U \subset \mathbb{R}^2 \to \mathbb{R}^7$. We use the following notation: $\varphi_{\alpha\beta} = \langle g_\alpha, g_\beta\rangle$ are the coefficients of the first fundamental form and

$$\varphi_{\alpha\beta\gamma} = \langle g_{\alpha\beta}, g_\gamma\rangle + (\varphi_{\alpha\beta})_\gamma.$$

Theorem 8.1. *With the above notation, the differential equation for the third order contact directions is*

$$\begin{vmatrix} g_{1u} & \cdots & g_{7u} & 0 & 0 \\ g_{1v} & \cdots & g_{7v} & 0 & 0 \\ g_{1uu} & \cdots & g_{7uu} & \varphi_{uu} & 0 \\ g_{1uv} & \cdots & g_{7uv} & \varphi_{uv} & 0 \\ g_{1vv} & \cdots & g_{7vv} & \varphi_{vv} & 0 \\ g_{1uuu} & \cdots & g_{7uuu} & \varphi_{uuu} & dv^3 \\ g_{1uuv} & \cdots & g_{7uuv} & \varphi_{uuv} & -dudv^2 \\ g_{1uvv} & \cdots & g_{7uvv} & \varphi_{uvv} & du^2dv \\ g_{1vvv} & \cdots & g_{7vvv} & \varphi_{vvv} & -du^3 \end{vmatrix} = 0.$$

Proof. Let $X = ag_u + bg_v$ be a non-zero tangent vector at p with $a, b \in \mathbb{R}$. It follows from the definition that X is a third order contact direction if and only if $f_u = f_v = f_{uu} = f_{uv} = f_{vv} = 0$, $(f_{uuu}, f_{uuu}, f_{uuu}, f_{uuu}) \neq 0$ and

$$\begin{pmatrix} f_{uuu} & f_{uuv} \\ f_{uuv} & f_{uvv} \\ f_{uvv} & f_{vvv} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for either $f = d_x$ or $f = h_{\mathrm{v}}$. Note that this last equation is equivalent to

$$\begin{pmatrix} f_{uuu} \\ f_{uuv} \\ f_{uvv} \\ f_{vvv} \end{pmatrix} = \lambda \begin{pmatrix} b^3 \\ -ab^2 \\ a^2b \\ -a^3 \end{pmatrix},$$

for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$. In order to simplify the expressions we introduce the notation $\sigma_{\alpha\beta\gamma}$, which are defined by

$$\begin{pmatrix} \sigma_{uuu} \\ \sigma_{uuv} \\ \sigma_{uvv} \\ \sigma_{vvv} \end{pmatrix} = \begin{pmatrix} b^3 \\ -ab^2 \\ a^2b \\ -a^3 \end{pmatrix}.$$

Then we can express shortly our conditions by $f_\alpha = f_{\alpha\beta} = 0$ and $f_{\alpha\beta\gamma} = \lambda\sigma_{\alpha\beta\gamma}$.

On the other hand, we recall that for $f = d_x$ we have

$$\begin{aligned} f_\alpha &= -\langle g_\alpha, x - g\rangle, \\ f_{\alpha\beta} &= -\langle g_{\alpha\beta}, x - g\rangle + \varphi_{\alpha\beta}, \\ f_{\alpha\beta\gamma} &= -\langle g_{\alpha\beta\gamma}, x - g\rangle + \varphi_{\alpha\beta\gamma}, \end{aligned}$$

while for $f = h_{\mathrm{v}}$,

$$\begin{aligned} f_\alpha &= \langle g_\alpha, v\rangle, \\ f_{\alpha\beta} &= \langle g_{\alpha\beta}, v\rangle, \\ f_{\alpha\beta\gamma} &= \langle g_{\alpha\beta\gamma}, v\rangle. \end{aligned}$$

Assume we have a third order contact line with $f = d_x$. Then,

$$\begin{pmatrix} g_\alpha & 0 & 0 \\ g_{\alpha\beta} & \varphi_{\alpha\beta} & 0 \\ g_{\alpha\beta\gamma} & \varphi_{\alpha\beta\gamma} & \sigma_{\alpha\beta\gamma} \end{pmatrix} \begin{pmatrix} x - g \\ -1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which implies that the matrix has determinant equal to zero. For $f = h_{\mathrm{v}}$ we take $(\mathrm{v}, 0, \lambda)$ instead of $(x - g, -1, \lambda)$.

Conversely, if the determinant of the matrix is zero, then there is $(X, Y, Z) \neq 0$ such that

$$\begin{pmatrix} g_\alpha & 0 & 0 \\ g_{\alpha\beta} & \varphi_{\alpha\beta} & 0 \\ g_{\alpha\beta\gamma} & \varphi_{\alpha\beta\gamma} & \sigma_{\alpha\beta\gamma} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $Y \neq 0$, we take $x = -X/Y + g$ and $\lambda = -Z/Y$, which gives a third order contact line for $f = d_x$. Otherwise, $Y = 0$ implies, necessarily $X \neq 0$ so that we can define $\mathrm{v} = X/\|X\|$ and $\lambda = Z$. This gives a third order contact line for $f = h_{\mathrm{v}}$. □

We see that third order contact lines are defined by means of a cubic differential form and can be interpreted as some kind of "third order principal directions". The singular points corresponds to the "third order umbilics" (that is, points $p \in M$ where $g(M)$ has a third order contact $\Sigma^{2,2,2}$ with some hypersphere or hyperplane of $\mathbb{R}^7$). In general, this cubic differential form is not always totally real (as it happens with principal lines of a surface in $\mathbb{R}^3$). However, in the case that it is, we find that for a generic immersion the singularities are hyperbolic and the phase portrait of the 3-web is described in Theorem 7.2, in analogy with the classical Darbouxian classification of principal foliations near generic umbilics.

Corollary 8.2. *Let $g : M \to \mathbb{R}^7$ be a generic immersion. Let p be a third order umbilic such that the third order contact lines are defined by a totally real cubic differential form near p. Then, p is hyperbolic in the sense of Definition 7.1.*

Proof. Given a map $g : M \to \mathbb{R}^7$, we denote by $j^4 g : M \to J^4(M, \mathbb{R}^7)$ its 4-jet extension. We also denote by $\mathcal{U} \subset J^4(M, \mathbb{R}^7)$ with the following property: $p \in M$ is a third order umbilic of g if and only if $j^4 g(p) \in \mathcal{U}$. It follows that $\subset U$ is an algebraic subset of codimension 2 in $J^4(M, \mathbb{R}^7)$.

We also denote by $\mathcal{U}_1$ the subset of $\mathcal{U}$ such that $j^4 g(p) \in \mathcal{U}_1$ if and only if p is not a simple singularity of the cubic differential form which defines third order contact lines. Analogously, we define $\mathcal{U}_2$ as the subset of $\mathcal{U}$ where the characteristic polynomial of the cubic differential form has not simple roots.

In both cases, $\mathcal{U}_i$ is an algebraic subset of $J^4(M, \mathbb{R}^7)$ of codimension 3. In fact, the equations of $\mathcal{U}$ are functions which only depend on the derivatives of g up to order 3, whilst the equations of $\mathcal{U}_i$ involve in a non-trivial way the 4th order derivatives. This implies that $\operatorname{codim} \mathcal{U}_i > \operatorname{codim} \mathcal{U}$. The result follows now from Thom transversality theorem by requesting transversality to both $\mathcal{U}_1$ and $\mathcal{U}_2$. □

This construction can be generalized easily to any k. We just need to consider an immersion $g : M \to \mathbb{R}^N$, with $N = \frac{(k+2)(k+1)}{2} - 3$. Then, the k-th order contact lines are defined by means of a symmetric differential form of degree k, whose singularities correspond to the k-roundings of M in $\mathbb{R}^N$ (see [8] for more details).

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