ポアソン多様体および接触多様体にかかわる幾何学の大域的研究

平成１１～１３年度科学研究費補助金（基盤研究（C)(2))

(課題番号：１１６４００６０)

研究成果報告書

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平成 14 年 3 月

平成11～１３年度科学研究費補助金（基盤研究（C)(2))

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作成年月　平成14年3月

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研究経費

平成11年度	１４００千円
平成12年度	１０００千円
平成13年度	１２００千円
合計	３６００千円

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1 Nambu-Jacobi Manifold の葉層構造

「接触構造，線叢，微分方程式」(熱海) 2000 年 1 月 14 日

2 Foliations Associated with Nambu-Jacobi Manifolds,

"International Conference FOLIATIONS: GEOMETRY AND DYNAMICS ", 2000 年 6 月 (ワルシャワ Banach Center)

3 Γ-構造と葉層構造について

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4 Foliation の symplectic 幾何「葉層構造の位相的研究」2001 年 10 月 23 日　(日本大学八海山セミナーハウス)

奥村正文

1 CR submanifolds of maximal CR dimension of the complex projective space

2000 年 9 月，(ワルシャワ，Banach Center)

2 The second fundamental form and CR submanifolds of maximal CR dimension.

"International Conference on Geometry and Applications" 2001 年 8 月 (Varna, Bulgaria)

研究成果

研究分担者は，それぞれの専門分野での研究を行ったが，ここでは研究代表者および研究分担者 長瀬正義の研究成果を中心に述べる

水谷忠良 (研究代表者)

平成 11 年度：平成 11 年度は Nambu-Jacobi 多様体の特徴づけとそれに同伴する葉層構造について研究した。P-M の拡張概念として J-M がある。これは，P-J M が $C^\infty(M)$ に括弧積 $\{f,g\}$ を持ち Lie 代数の構造を定めることのほかにライプニッツ則

$$\{f,gh\}=g\{f,g\}+\{f,g\}$$

を満たすことが要請されるのに対して，この要請をはずした構造をもつ多様体として定義される。P-J は 2-ベクトル場 Λ と通常のベクトル場 E を与えることによって得られる。ただし，これらは次の条件を満たさなければならない。

$$[\Lambda,\Lambda]=-2E\wedge\Lambda,\quad [E,A]=0.$$

括弧積との関係は

$$\{f,g\}=\Lambda(df,dg)-fE(g)+gE(f)$$

で与えられる。

一方，q 次の 南部ポアソン多様体 は **fundamental identity**

$$\{f_1,\ldots,f_{q-1},\{g_1,\ldots,g_q\}\}=\sum_{i=1}^{q}\{g_1,\ldots,\{f_1,\ldots,f_{q-1},g_i\},\ldots,g_q\}. \tag{1}$$

と ライプニッツ則 を満たす括弧積を持つものとして定義されるが，ライプニッツ則の条件をはずして fundamental identity だけを要求したものが 南部ヤコビ多様体である。本研究では，この種の多様体について q-次の 南部ヤコビ多様体 がどのような 多重ベクトル場

によって特徴付けられるのかを調べ，さらに自然に対応する葉層構造についての結果を得た。

これらの結果は秋田大学の三上健太郎との共著として論文
Foliations assocaited with Nambu-Jacobi structures
にまとめ，その結果をワルシャワにおける研究集会で発表した。その後類似する選考する結果があることが判明して出版にいたっていない。しかし，この方向の研究では基本的な結果であり，記述にも特徴があり意義のある結果であると確信している。

平成12年度，平成13年度：南部ポアソン多様体 には Leibniz 代数が付随する。Leibniz 代数とはベクトル空間 $\mathbf{g}$ で $[\,,\,] : \mathbf{g}\times\mathbf{g}\to\mathbf{g}$ が

$$[g_1,[g_2,g_3]] = [[g_1,g_2],g_3] + [g_2,[g_1,g_3]].$$

を満たすものであり，Lie 代数 の性質から 積 の反対称性の条件を除いたものである。Nambu-Poisson 構造に関連して，積分可能な(分解可能)p-次微分形式が与えられると (p+1)-次ベクトル場全体が Leibniz 代数 の構造を持つことがわかる。12年度には，このような状況で Lie 代数になる Leibniz 代数の例を探したが，そのような例はまれであることがわかっていた。13年度には，このような Leibniz 代数の同型類は葉層構造の同型類に対して定まり，さらに 葉層に沿うベクトル場の作る Lie 代数 の中心拡大となっていることを観察した。さらに，この中心拡大と Leibniz 代数 の2次元 Leibniz コホモロジー元との対応を明らかにした．その結果は次の論文としてまとめた。今後の課題として興味深いのは，特異点を持つ場合に具体的な切断を記述してその結果として得られる2次元コサイクルの表示を求めることである
Y. Hagiwara-Tmizutani " Leibniz algebras associated with foliations"
平成13年度にはこの他に Pfaff 系の幾何学的性質についての考察も行った より具体的に述べると，次のような考察である 接触多様体にはよく知られているようにシンプレクティック化が付随しているこれは標準的な symp 多様体 T^*M の symp 部分多様体であるがこれは接触構造の完全積分不可能性を忠実に反映している これをモデルにして，一般の接平面場の積分可能性と接平面場を定義する Pfaff 系の T^*M の部分多様体としての性質との関係を調べようとするものである 接平面場が積分可能であれば対応する Pfaff 系（T^*M の部分バンドル）は coisotropic 部分多様体となり，symp 直交する接平面場はこの部分多様体上で積分可能になる この接平面場は最初の接平面場＝葉層構造の Bott 接続から得られるものになっており，葉層構造の特性類との関係が注目される この研究では，コホモロジーを多重ベクトル場が定義し，コホモロジーの積を多重ベクトル場の交叉であると見直すことによって Godbillon-Vey 類を多重ベクトル場の交叉で捉える試みを行った これは，特異点のある葉層構造の不変量の記述への応用が期待される

長瀬正義（研究分担者）

平成１１年度: 主に，Spin 構造の変形物である Spin^q 構造に付随するトウィスター空間のカイラルアノマリー（右手系か左手系かに関する異常項）の初歩的研究に取り組んだ。

anomaly（異常項）は物理学者の多用する用語である。特に global (gravitational) anomaly, covariant anomaly, chiral anomaly は，宇宙の幾何的構造と密接に関係しており，物理学的には非常に重要な概念であることが知られている。我々幾何学者に取っては「宇宙＝多様体」であるから，それら概念は我々にとってもまた非常に重要であろう。実際，global anomaly はη-不変量，ホロノミーの理論に対応し，covariant anomaly は Quillen 計量の理論に対応している，と考えられる。こうした少々荒い考察の後，当研究期間においては，特に，数学的 chiral anomaly “$\log\det(\partial)$” の定式化に取り組みほぼ満足の行く結果を得た。

平成１２年度: 主に，Spin 構造の変形物である Spin^q 構造に付随するトウィスター空間の無限小カイラルアノマリーについての基礎的研究に取り組んだ。物理学サイドの言うカイラルアノマリーについてはそれをどの様に数学的に設定するかさえ曖昧であった。この研究では，前年度得たそれの妥当な数学的設定の持つ様々な問題点について考察を重ねた。

第一の問題点は計量に依存して変化するスピノール束をある固定されたスピノール束となんらかの手段で“標準的に同一視する”ことであり，この研究では Bourguignon-Gauduchon の手法を参考にその標準的同一視を与え，そのトウィスター空間に付随するある方向

への無限小ディラック作用素の公式を導いた。次にその無限小ディラック作用素の与える無限小カイラルアノマリーの研究の足場となる“ディラック作用素の二乗に関係するある初期値問題の核”の基本的性質について考察した。実際には，その初期値問題をどの様に設定するのか，その核の何をどの様な手段で考察するのか，本当に充分役立つ情報が最終的に得られるのか，といった問題が絡み合っており即断はできないが，ほぼ満足のいくところまで到達しており“無限小カイラルアノマリーの本質的部分”を引き出せる段階にまで至っている。ただし，それが真に本質的部分であるのかといった問題も絡んでいることも記しておく。カイラルアノマリーについては数学サイドから何を研究すべきかも曖昧であり，この研究ではその捕らえ所のない対象を何らかの意味で捕らえてみようとしている。

平成１３年度: 主に Spin^q 構造（四元数 Spin 構造）に付随するトウィスター空間の無限小カイラルアノマリーの本質的部分の抽出に取り組んだ。

前年度の数学的定義や意味付けについての考察の結果，そのアノマリー（ $\log\det\delta_X\partial$ と記される）を（計量 g^Z の与える）ディラック作用素 ∂ に付随する半群の C^∞-核 $e^{-t\partial^2}$ のある種の２階微分 $\delta_X\partial^2 e^{-t\partial_\varepsilon^2}$ のトレース $\mathrm{Tr}(\delta_X\partial^2 e^{-t\partial^2})$ を使って定義することとし，特にそのトレースの $t\to 0$ の場合の挙動の研究が重要であると認識した。今年度はそれの本質的部分と思われる断熱極限 $\lim_{\varepsilon\to 0}\log\det\delta_X\partial_\varepsilon$ の研究に取り組んだ。ここで ∂_ε はそのトウィスター空間の底空間方向に ε^{-1} だけ引き延ばした計量 g_ε^Z の与えるディラック作用素であり，対応してトレース $\mathrm{Tr}(\delta_X\partial_\varepsilon^2 e^{-t\partial_\varepsilon^2})$ の $t\to 0$ & $\varepsilon\to 0$ の場合の挙動についての考察が重要となる。そして当然その極限操作は t,ε 達の減少スピードの相互関係をどう設定するかによって結果は大きく異なると予想され，本研究では $0<t<\varepsilon^a$ $(a>0)$ なる状態でのその極限操作による極限値について考察した。研究分担者は，本質的と思われる断熱極限の真に本質的な部分はこの状態での極限操作より得られ他の状態での極限は消えてしまうと予想しており，更にその状態での極限値については予想（充分な準備が必要なので厳密な記述は避ける）を持っている。

奥村正文（研究分担者）

極大 CR 次元を持つ CR 部分多様体には特別な性質を持つ法線ベクトルが定義でき，この法線を用いて概接触構造が部分多様体上に自然に導入される。今回は，複素射影空間の極大 CR 次元を持つ CR 部分多様体の概接触構造が接触構造となる場合について研究し，このような極大 CR 次元を持つ CR 部分多様体を複素射影空間のなかで完全に決定した。

阪本邦夫（研究分担者）

極小曲面と Willmore 曲面，更に法曲率テンソルに関する変分問題の臨界部分多様体について研究した。特にコンパクト部分多様体の次元が４で法接続が自己双対又は反自己双対であるとき，臨界的であることを示した。更に，曲面の場合を考察し，法曲率ベクトルの長さが０でない定数で curvature ellipse が円という条件下では，臨界的であることと Willmore surface であることとは同値であることを示した。これに関連して S-Willmore point の対数的留数についての公式を得た。この公式は留数と共形不変量及びオイラー数との関係の研究も必要となり以前に田代氏によって得られた結果を一般化した。

竹内喜佐雄（研究分担者）

一次元および高次元有界対称領域に作用する不連続群のうちで、特に数論的に定義された群について研究をおこなった。特に，９次以下である代数体 k について，その判別式 $d(k)$ の小さいもののリストを作り，それぞれについて整数底を具体的に決定した。

矢野 環（研究分担者）

「山上宗二記シンポジューム」（五島美術館，平成７年１１月）において，およそ５０件の異本がいくつかの系統に分類して紹介された。それを聞いて，より重要な室町時代の美術書と言う

能であることを確信した。その研究結果が [1] である。また，その手法を更に発展させ，生物の種の分化の如く，名物記の生成構造図を得ることに成功した。

福井敏純（研究分担者）

[1] では写像のジェット空間におけるトム・ボードマン多様体の特異点がシチギーの幾何学的構成法を使って解析されることを示した。[2] では位相自明化を構成するイソトピー補題について考察した重み付き (w) 正則性の概念を導入しその基本的性質を明らかにした。

江頭信二（研究分担者）

コンパクト多様体上の C^2 級より弱い微分可能性における余次元 1 葉層構造の定性的理論を研究した。これと平行して Denjoy の定理において C^2 級の微分可能性がある場合，C^1 級と比較して定性的に大きな制限が生じることがわかっているが，この制限を生じさせる微分可能性の " 下限 " を探る研究も行った。また，葉層構造の GV 不変量と定性的理論の関係，エントロピーと定性的理論の関係，例外型局所極小集合の横断的ハウスドルフ次元やハウスドルフ測度について研究を行った。特に余次元 1 葉層構造の例外型局所極小集合の中で典型的に生成されるタイプの集合の場合，横断的ハウスドルフ次元が 0 になることが予想されるが，まだ解決に至っていない。

収録論文

1 Kentaro MIKAMI and Tadayoshi MIZUTANI;
Foliations associated with Nambu-Jacobi structures.

2 Yohsuke HAGIWARA and Tadayoshi MIZUTANI;
Leibniz algebras associated with foliations.

3 Kunio SAKAMOTO;
Variational Problem of normal curvature tensor and concircular scalar fields

Foliations associated with Nambu-Jacobi structures

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Jan 28, 2001

Abstract

We define a Nambu-Jacobi structure as a bracket of several functions which satisfies the Fundamental Identity. Then we express the Nambu-Jacobi structure in terms of two tensor fields and show the necessary and sufficient conditions they should satisfy. We investigate the foliations associated with a Nambu-Jacobi structure. This allows us to give many examples of Nambu-Jacobi manifolds.

1 Introduction and Definitions

It is well-known that a Poisson manifold has its associated foliation. It is a generalized foliation in the sense of Stefan and Sussmann whose leaves are immersed symplectic manifolds. In the case of a Jacobi manifold(in the sense of Lichnerowicz), we have also a generalized foliation whose leaves are either symplectic or a contact manifolds. In this paper, we consider the case of Nambu-Poisson and Nambu-Jacobi manifolds. First, we review briefly the definition of a Nambu-Poisson manifold and its natural generalization 'Nambu-Jacobi' manifold.

Let $C^\infty(M)$ be the set of smooth functions on a manifold M.

DEFINITION 1 $\{\dots\}$ is called a *Nambu-Poisson bracket* of degree q if it is a skew-symmetric q-linear map

$$\{\dots\} : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{q} \longrightarrow C^\infty(M)$$

which satisfies the following:

(1) **(Fundamental Identity or Generalized Jacobi Identity)**

$$\{f_1,\dots,f_{q-1},\{g_1,\dots,g_q\}\} = \sum_{i=1}^{q}\{g_1,\dots,\{f_1,\dots,f_{q-1},g_i\},\dots,g_q\}$$

where $f_1,\dots,f_{q-1}, g_1,\dots,g_q \in C^\infty(M)$.

(2) **(Leibniz rule)** For each argument of the bracket, the usual derivation rule holds, that is, for $f_1,\dots,f_{q+1} \in C^\infty(M)$

$$\{f_1,\dots,f_{q-1},f_qf_{q+1}\} = \{f_1,\dots,f_{q-1},f_q\}f_{q+1} + f_q\{f_1,\dots,f_{q-1},f_{q+1}\}$$

holds.

Remark Sometimes it is more convenient to write the Fundamental Identity in the following form.

$$\begin{aligned}\{\{f_1,\dots,f_q\},g_2,\dots,g_q\} &= \{\{f_1,g_2,\dots,g_q\},f_2,\dots,f_q\}\\ &+ \{f_1,\{f_2,g_2,\dots,g_q\},f_2,\dots,f_q\}\\ &+\cdots\cdots\\ &+ \{f_1,f_2,\dots,f_{q-1},\{f_q,g_2,\dots,g_q\}\}.\end{aligned}$$

In Appendix we use this formulation which is equivalent to condition **(1)**.

As in the case of usual Poisson bracket, this is also equivalent to the existence of a q-vector field η on M satisfying

$$\eta(df_1,\dots,df_q) = \{f_1,\dots,f_q\} \quad \text{for} \quad f_1,\dots,f_q \in C^\infty(M),$$

$$[\eta(d\boldsymbol{f}_{q-1}),\eta] = L_{\eta(d\boldsymbol{f}_{q-1})}\eta = 0 \quad \text{for} \quad f_1,\dots,f_{q-1} \in C^\infty(M),$$

where $d\boldsymbol{f}_{q-1}$ is the abbreviation of $df_1 \wedge \dots \wedge df_{q-1}$, $\eta(d\boldsymbol{f}_{q-1})$ stands for the vector field $\eta(df_1,\dots,df_{q-1},\cdot)$ on M, which is the Hamiltonian vector field determined by several functions (cf. [1]), and $L_{\eta(d\boldsymbol{f}_{q-1})}\eta$ is a Lie derivative.

DEFINITION 2 η is called a *Nambu-Poisson tensor* of degree q.

A Nambu-Poisson tensor has the following striking property.

THEOREM 1 (P. Gautheron[2] K. Mikami[3], N. Nakanishi[9]) *If q is greater than 2, Nambu-Poisson tensor of degree q is locally decomposable. This means that if η is non-zero at a point, then on a neighbourhood of the point there exist vector fields $Y_1,\dots,Y_q$ so that η can be written as*

$$\eta = Y_1 \wedge \cdots \wedge Y_q.$$

Now we define a Nambu-Jacobi manifold and see that the bracket of a Nambu-Jacobi manifold is described by a pair of two multi-vector fields. Let us begin with a general definition.

Let M be a C^∞ manifold and $C^\infty(M)$ the algebra of smooth functions on M. For an integer $q \geq 1$, we consider an alternating q-linear map

$$\mathcal{A} : \underbrace{C^\infty(M)\times\cdots\times C^\infty(M)}_{q} \longrightarrow C^\infty(M)$$

We call the map $\mathcal{A}$ a *bracket of degree* q and often write $\{f_1,\dots,f_q\}$ for $\mathcal{A}(f_1,\dots,f_q)$. We always assume a bracket satisfies the following conditions:

(a) The map $\mathcal{A}$ is continuous with respect to C^∞ topology, and

(b) $\mathrm{supp}\{f_1,\dots,f_q\} \subset \mathrm{supp} f_1 \cap \cdots \cap \mathrm{supp} f_q$.

When $q=1$, we understand $\mathcal{A}$ is just a linear map. By a theorem of Peetre[10], these conditions assure that the bracket is a differential operator and the resulting function is written in terms of the derivatives of the argument functions of $\mathcal{A}$.

DEFINITION 3 If $\mathcal{A}$ is a bracket of degree q on M which satisfies the Fundamental Identity in Definition 1, we call $\mathcal{A}$ a *Nambu-Jacobi bracket*. A smooth manifold with a Nambu-Jacobi bracket is called a *Nambu-Jacobi manifold*.

We remark that when $q=2$, a Nambu-Jacobi manifold reduces to a usual Jacobi manifold. The Leibniz rule in the definition of a Nambu-Poisson bracket clearly implies our condition (a) and (b) on bracket. Thus a Nambu-Poisson manifold is a special case of a Nambu-Jacobi manifold.

We first remark that a Nambu-Jacobi bracket is a differential operator of order *at most* 1. Indeed we have the following:

THEOREM 2 *Let M be a smooth C^∞ manifold of dimension n and $\mathcal{A}$ a Nambu-Jacobi bracket of degree $q \geq 2$ on M. Then on a coordinate neighbourhood $(U,x_1,\dots,x_n)$ of M, $\mathcal{A}(f_1,\dots,f_q)$ is given by the formula*

$$\sum_{|I_1|\leq 1,|I_2|\leq 1\cdots|I_q|\leq 1} C_{I_1I_2\cdots I_q}(\partial^{I_1}f_1)(\partial^{I_2}f_2)\cdots(\partial^{I_q}f_q), \tag{1.1}$$

where $C_{I_1I_2\cdots I_q}$ is a function, I_α denotes a multi-index $(i_1^{(\alpha)},\dots,i_n^{(\alpha)})$ and $|I_\alpha| = i_1^{(\alpha)}+\cdots+i_n^{(\alpha)}$. ∂^I stands for $(\partial_1)^{i_1}\cdots(\partial_n)^{i_n}$, $\partial_i = \partial/\partial x_i$.

In fact, this will be shown along the line of the proof of Kirillov (see for example [6]).

This theorem allows us to describe a Nambu-Jacobi bracket in terms of a pair of two multi-vector fields on M. Let P be a p-vector field on M. P naturally defines a bracket of degree p by $(f_1, \dots, f_p) \mapsto P(df_1, \dots, df_p)$. We denote this bracket by $\{f_1, \dots, f_p\}^P$ or sometimes by $P(f_1, \dots, f_p)$ when there is no danger of confusion. This notation is analogous to the notation $X(f)$ to denote the derivative of a function f by a vector field X. We define a new bracket $1 \wedge P$ of degree $p+1$ by the formula

$$(1 \wedge P)(f_1, \dots, f_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} f_i P(df_1, \dots, \hat{df_i}, \dots, df_{p+1}) \tag{1.2}$$

Then we have the following observation on a Nambu-Jacobi bracket.

LEMMA 3 *Let $\mathcal{A}$ be a Nambu-Jacobi bracket of degree q ($q \geq 2$). Then there exist uniquely a q-vector field Q and a $(q-1)$-vector field P which are Nambu-Poisson tensors such that $\mathcal{A} = Q + (1 \wedge P)$ holds.*

Proof Let $p = q - 1$ and put $\mathcal{B}(f_1, \dots, f_p) = \mathcal{A}(1, f_1, \dots, f_p)$. Then it is easily seen that $\mathcal{B}$ is a bracket of degree p and satisfies the Fundamental Identity. From the skewness of $\mathcal{A}$, $\mathcal{B}(f_1, \dots, f_p) = 0$ if one of the arguments is a constant function. Thus, the order of $\mathcal{B}$ as a differential operator is exactly equal to 1. This means $\mathcal{B}$ is defined by a p-vector field P. Now put $\mathcal{C} = \mathcal{A} - 1 \wedge P$. Then by the same reason $\mathcal{C}$ is also a bracket defined by a q-vector field. Denoting it by Q, we obtain Lemma 3. Uniqueness is verified easily.

DEFINITION 4 We call the pair (Q, P) a *Nambu-Jacobi pair* if $Q + 1 \wedge P$ defines a Nambu-Jacobi bracket.

Notation: In the sequel, we frequently use the contraction of tensor fields. For example, let Q be a q-vector field and $\alpha = \alpha_1 \wedge \dots \wedge \alpha_p$ a p-form ($p \leq q$). We denote the contraction Q and α by the following various notations, interchangeably. $i_\alpha Q = Q(\alpha) = Q(\alpha, \cdot) = Q(\alpha_1 \wedge \dots \wedge \alpha_p, \cdot)$.

2 What the Fundamental Identity means

We consider the bracket $\mathcal{A}$ defined by $(p+1)$-vector field Q and p-vector field P, by the equality $\mathcal{A} = Q + 1 \wedge P$. We now look for the conditions on P and Q under which $\mathcal{A}$ satisfies the Fundamental Identity. Namely the conditions that make (Q, P) a Nambu-Jacobi pair. When $\deg P = 1$ and $\deg Q = 2$, the Nambu-Jacobi pair $Q + 1 \wedge P$ is a usual Jacobi bracket and the conditions on P and Q are well known.

Namely, they satisfy $[P, Q] = 0$, $[Q, Q] = -2P \wedge Q$ if and only if the bracket satisfies the Jacobi identity, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket (cf. (3.1).) Therefore our interest is on the case $\deg P \geq 2$.

To proceed our calculations, we introduce the following notations.

DEFINITION 5 For a p-vector field P and a q-vector field Q ($p \geq 2, q \geq 1$) which are both considered as brackets, we define a map

$$J^P Q : \underbrace{C^\infty(M) \times \dots \times C^\infty(M)}_{p+q-1} \longrightarrow C^\infty(M)$$

by

$$\begin{aligned}
&(J^P Q)(f_1, \dots, f_{p-1}; g_1, \dots, g_q) \\
&= P(f_1, \dots, f_{p-1}, Q(g_1, \dots, g_q)) \\
&\quad - (Q(P(f_1, \dots, f_{p-1}, g_1), g_2, \dots, g_q) - Q(g_1, P(f_1, \dots, f_{p-1}, g_2), g_3, \dots, g_q) \\
&\quad - \dots - Q(g_1, \dots, g_{q-1}, P(f_1, \dots, f_{p-1}, g_q)) \qquad (2.1) \\
&\qquad\qquad f_1, \dots, f_{p-1}, g_1, \dots, g_q \in C^\infty(M)
\end{aligned}$$

Note that $J^P Q$ can be considered as a contravariant tensor field but *does not* define a bracket since it is not fully alternating with respect to the argument functions. Note that $J^P P = 0$ means that P satisfies the Fundamental Identity. The same formula as $J^P Q$ can be defined for any brackets(not necessarily given by multi-vector fields). In the present case, where the initial brackets are defined by multi-vector fields (P and Q), we have the following equivalent expression.

$$J^P Q(f_1, \dots, f_{p-1}; g_1, \dots, g_q) = [P(d\boldsymbol{f}_{p-1}, \cdot), Q](dg_1, \dots, dg_q) \tag{2.2}$$

where $d\boldsymbol{f}_{p-1} = df_1 \wedge \dots \wedge df_{p-1}$ as before, $P(d\boldsymbol{f}_{p-1}, \cdot) = P(df_1, \dots, df_{p-1}, \cdot)$ is a vector field and $[P(d\boldsymbol{f}_{p-1}, \cdot), Q] = L_{P(d\boldsymbol{f}_{p-1}, \cdot)} Q$ is a Lie derivation. Thus $J^P Q = 0$ is equivalent to that the 'Hamiltonian vector fields' preserve Q .

We also need the following map

DEFINITION 6 For a p-vector field P and a q-vector field Q ($p \geq 2, q \geq 1$), we define a map

$$P \vdash Q : \underbrace{C^\infty(M) \times \dots \times C^\infty(M)}_{p+q} \longrightarrow C^\infty(M)$$

by

$$(P \vdash Q)(f_1, \dots, f_{p-1}; g_0, \dots, g_q) = (P(d\boldsymbol{f}_{p-1}, \cdot) \wedge Q)(dg_0, \dots, dg_q) \tag{2.3}$$

where $d\boldsymbol{f}_{p-1} = df_1 \wedge \dots \wedge df_{p-1}$.

Also, $P \vdash Q$ is not a bracket in general. Note that $P \vdash P = 0$ if and only if P is a locally decomposable.

In order to get the relation between P and Q in $\mathcal{A} = Q + 1 \wedge P$, we need to calculate $J^{\mathcal{A}}\mathcal{A}$ since the condition that $\mathcal{A}$ is to be a Nambu-Jacobi bracket is

$$J^{\mathcal{A}}\mathcal{A}(f_1, \dots, f_p; g_1, \dots, g_{p+1}) = 0 \tag{2.4}$$

By a direct computation we can express the left hand side of this equation (2.4) so that a certain sum of multiples of functions consisting of $\{\dots\}^P$, $\{\dots\}^Q$, and f_i, g_j which are outside of the brackets $\{\dots\}^P$ or $\{\dots\}^Q$.

Although it is straightforward, the computation is lengthy. We will do it in Appendix separately.

The relations of P and Q which we obtain are in the following:

PROPOSITION 4 *Let $\mathcal{A} = Q + (1 \wedge P)$ be a bracket of degree $q = p + 1$ defined by p-vector field P and q-vector field Q. Then a necessary and sufficient condition for the bracket $\mathcal{A}$ to be a Nambu-Jacobi bracket is that P and Q satisfy the following four identities.*

(1) $J^P P = 0$,

(2) $J^P Q = 0$,

(3) $$J^Q P(d\boldsymbol{f}_p; \cdots) + (-1)^{p+1} Q(d(P(d\boldsymbol{f}_p)), \cdots) + \sum_{i=1}^{p} (-1)^i (P \vdash P)(df_1 \cdots \hat{df_i} \cdots df_p; df_i, \cdots) = 0,$$

(4) $$J^Q Q(d\boldsymbol{f}_p; \cdots) + \sum_{i=1}^{p} (-1)^i (P \vdash Q)(df_1 \cdots \hat{df_i} \cdots df_p; df_i, \cdots) = 0,$$

where $d\boldsymbol{f}_p$ stands for differential form $df_1 \wedge \ldots \wedge df_p$.

Proof As is stated above, this is done in Appendix by a direct but a long calculation . □

To simplify the above identities we need the following two lemmas.

LEMMA 5 *Let P and Q be multi-vector fields of degree p and q, respectively. If $P \vdash Q$ is also a multi-vector field (i.e. $P \vdash Q$ is a skew symmetric tensor field) and $p \geq 3$ and $q \geq 1$ then $P \vdash Q$ vanishes identically.*

Proof Considering the equation at each point of the manifold, we may assume the case when P and Q are alternating multi-linear maps on finite dimensional vector

space. Put $B = P \vdash Q$. It suffices to show that $B_{|V} = 0$ for arbitrary $(p+q)$-dimensional subspace V. From the definition, if $Q_{|W} = 0$ for any q-dimensional subspace $W \subset V$, then clearly $B_{|V} = 0$. So assume $Q_{|W} \neq 0$ for a q-dimensional subspace W. Then there exist $y_1, \dots, y_q \in W$ satisfying

$$Q(y_1, \dots, y_q) \neq 0.$$

claim We can find an element $x \in V$ such that $x \wedge y_1 \wedge \cdots \wedge y_q \neq 0$ and

$$Q(y_1, \dots, \underbrace{x}_{i}, \dots, y_q) \neq 0 \quad \text{for some} \quad i.$$

(proof of claim) Consider the following linear functional on V

$$\varphi : V \longrightarrow \boldsymbol{R} : x \mapsto \sum_{j=1}^{q} Q(y_1, \dots, \underbrace{x}_{j}, \dots, y_q).$$

We have

$$\varphi(y_1) = \cdots = \varphi(y_q) = Q(y_1, \dots, y_q) \neq 0$$

and

$$y_2 - y_1, \dots, y_q - y_1 \in \operatorname{Ker} \varphi$$

Since $\dim \operatorname{Ker} \varphi = p + q - 1$ and $p \geq 1$, we find an element $z \in \operatorname{Ker} \varphi$ so that

$$z, y_2 - y_1, \dots, y_q - y_1 \in \operatorname{Ker} \varphi$$

are linearly independent. Then it can be seen that the set

$$z + y_1, y_1, y_2, \dots, y_q$$

is linearly independent and $x = z + y_1$ is an element with desired property. Indeed, we have

$$\sum_{j=1}^{q} Q(y_1, \dots, \underbrace{x}_{j}, \dots, y_q) = \varphi(x) = \varphi(z + y_1) = \varphi(y_1) = Q(y_1, \dots, y_q) \neq 0.$$

This means there exists some i such that

$$Q(y_1, \dots, \underbrace{x}_{i}, \dots, y_q) \neq 0.$$

(end of the proof of claim).

For x and $Y = (y_1, \cdots, y_q)$ which we found above and for any $(p-2)$-tuple T, we have

$$\begin{aligned}
0 &= B(T, x; x, Y) \\
&= (P(T, x, \cdot) \wedge Q)(x, Y) \\
&= P(T, x, x)Q(Y) - \sum_{j=1}^{q} P(T, x, y_j)Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q) \\
&= -P(T, x, \sum_{j=1}^{q} Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q)y_j)
\end{aligned}$$

If we put $u = \sum_{j=1}^{q} Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q)y_j$, this shows

$$P(x, u, T) = 0 \quad \text{for all } (p-2)\text{-tuple } T.$$

For any $(p-3)$-tuple of vectors T' and q-tuple of vectors T'' in V, we have

$$B(x, u, T'; T'') = (P(x, u, T', \cdot) \wedge Q)(T'') = 0$$

($p \geq 3$ is necessary here).

Since we are assuming B is a multi-vector and $\{x, u\}$ are linearly independent, this clearly shows $B_{|V} = 0$. □

The next lemma shows that in our case, $P \vdash P$, $P \vdash Q$ and $Q \vdash Q$ are proved to be multi-vector fields and we can apply Lemma 5 to the identities in Proposition 4.

LEMMA 6 Let $\mathcal{A} = Q + (1 \wedge P)$ be a Nambu-Jacobi bracket determined by q-vector field and p-vector field $P, (q = p + 1)$. Then $P \vdash P$, $P \vdash Q$ and $Q \vdash Q$ are fully alternating and hence they are multi-vector fields.

Proof To prove $P \vdash Q$ is a multi-vector field, we have only to verify skewness of $P \vdash Q$, namely the identity

$$\begin{aligned}
&(P \vdash Q)(h_1, f_2, \cdots, f_{p-1}; h_2, g_1, \cdots, g_{p+1}) \\
&+(P \vdash Q)(h_2, f_2, \cdots, f_{p-1}; h_1, g_1, \cdots, g_{p+1}) = 0
\end{aligned} \tag{2.5}$$

for arbitrary functions $h_1, h_2, f_2, \cdots, f_{p-1}, g_1, \cdots, g_{p+1}$.

For this, we calculate

$$\begin{aligned}
&J^P Q(h_1h_2, f_2, \cdots, f_{p-1}; g_1, \cdots, g_{p+1}) \\
&\quad = [P(d(h_1h_2), df_2, \cdots, df_{p-1}, \cdot), Q](dg_1, \cdots, dg_{p+1})
\end{aligned}$$

which is identically equal to 0 by Proposition 4. Thus we have

$$\begin{aligned}
0 &= [P(d(h_1h_2), df_2, \cdots, df_{p-1}, \cdot), Q] = [P(h_1dh_2 + h_2dh_1, df_2, \cdots, df_{p-1}, \cdot), Q] \\
&= [h_1P(dh_2, df_2, \cdots, df_{p-1}, \cdot), Q] + [h_2P(dh_1, df_2, \cdots, df_{p-1}, \cdot), Q] \\
&= h_1[P(dh_2, df_2, \cdots, df_{p-1}, \cdot), Q] - (P(dh_2, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_1)) \\
&\quad + h_2[P(dh_1, df_2, \cdots, df_{p-1}, \cdot), Q] - (P(dh_1, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_2)) \\
&= -(P(dh_2, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_1)) - (P(dh_1, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_2))
\end{aligned}$$

¿From this it is easy to see the identity (2.5) holds.

The cases of $P \vdash P$ and $Q \vdash Q$ are proved by similar calculations using (1) and (4) in Proposition 4. □

Remark In a similar way, what we obtain from (3) in Proposition 4 is the following relation.
The function

$$(Q \vdash P)(f_1, \ldots, f_p; g_1, \ldots, g_{p+1}) + (-1)^{p+1}P(df_1, \ldots, df_p)Q(dg_1, \ldots, dg_{p+1}) \tag{2.6}$$

is skew symmetric with respect to the all arguments. In particular, if $f_1 = g_1$ the above function vanishes.

By the above two lemmas, we have most part of the following:

PROPOSITION 7 *If (Q, P) is a Nambu-Jacobi pair and* $\deg P = p \geq 2$, *then* $P \vdash P = 0$, $P \vdash Q = 0$ *and* $Q \vdash Q = 0$.

Proof For $p > 2$, the statement is obvious from Lemma 5 and Lemma 6. The case $p = 2$ is treated separately.

Proof when $p = 2$. Assume $\mathcal{A}$ is a Nambu-Jacobi bracket. First, we prove P is decomposable 2-vector at a point where $Q \neq 0$. As before, we consider P and Q are 2-vector and 3-vector of a vector space V. Since $\deg Q = 3$, by Lemma 5 and Lemma 6, we have $Q \vdash Q = 0$ that is Q is decomposable. The condition $P \vdash Q$ is fully skew symmetric means that

$$P(x, \cdot) \wedge Q(x, \cdot, \cdot) = 0 \quad \text{for } x \in V^*. \tag{2.7}$$

Taking the value at (y, z, w) we have

$$P(x, y)Q(x, z, w) - P(x, z)Q(x, y, w) + P(x, w)Q(x, y, z) = 0. \tag{2.8}$$

Regard Q as a linear map $V^* \to \wedge^2 V$ and fix a direct sum decomposition

$$V^* = K \oplus L \tag{2.9}$$

where $K = \ker Q$ and L is a complementary subspace which is isomorphic to $\mathrm{Im}\, Q$. If $w \in K$, from the above relation we have

$$P(x,w)Q(x,y,z) = 0. \tag{2.10}$$

Given $0 \neq x \in L$, we can choose y, z such that $Q(x,y,z) \neq 0$. Thus, we have

$$P(x,w) = 0 \quad \text{for any} \quad x \in L, w \in K.$$

If we replace x by $x + v$, $(v \in K)$, in (2.10), we have

$$(P(x,w) + P(v,w))Q(x,y,z) = P(v,w)Q(x,y,z) = 0 \quad w \in K. \tag{2.11}$$

From this we see that

$$P(x,y) = 0 \quad \text{unless} \quad x, y \in L. \tag{2.12}$$

Since $\mathrm{rank}\, Q = \dim L = 3$, $\mathrm{rank}\, P$ must be 2 and P is a decomposable 2-vector hence $P \vdash P = 0$.

Next we prove $P \vdash Q$. Regard Q and P as linear maps $\wedge^2 V^* \to V$, $V^* \to V$, respectively. Since we have

$$P(x,y)Q(x,z,\cdot) - P(x,z)Q(x,y,\cdot) + P(x,\cdot)Q(x,y,z) = 0, \tag{2.13}$$

and as we saw above P maps K to 0. This means $\mathrm{Im}\, P \subset \mathrm{Im}\, Q$. ¿From this we have $P(x,\cdot) \wedge Q = 0$ for any $x \in V^*$. This shows $P \vdash Q = 0$.

If at a point a, $Q = 0$ and a is in the closure of the set where $Q \neq 0$, the semi-continuity of the rank assures that $\mathrm{rank}\, P \leq 2$ and we have $P \vdash P = 0, P \vdash Q = 0$ in this case too.

Finally, we consider the point where Q vanishes identically on some neighbourhood of the point. In this case Proposition 4 (3) says P is a 2-vector satisfying

$$P(x,y)P - P(x,\cdot) \wedge P(y,\cdot) = 0 \quad \text{for} \quad x, y \in V^*$$

This clearly shows that P is decomposable.

Thus we proved $P \vdash P = 0$ and $P \vdash Q = 0$ hold everywhere. This finishes the proof in the case where $p = 2$. □

By the above Proposition 7, the identities in Proposition 4 are simplified as in the following form for $p \geq 2$.

THEOREM 8 *Let $\mathcal{A} = Q + (1 \wedge P)$ be a bracket on a manifold M, which is given by a $(p+1)$-vector field Q and a p-vector field P where $p \geq 2$ and assume*

$\mathrm{rank}\, P \leq 2$ is when $p = 2$. Then $\mathcal{A}$ is a Nambu-Jacobi bracket if and only if the following conditions are satisfied.

For any functions $f_1, f_2, \dots, f_p \in C^\infty(M)$,

$$\begin{aligned}
&(1) \quad [P(d\boldsymbol{f}_{p-1},\cdot), P] = 0,\\
&(2) \quad [P(d\boldsymbol{f}_{p-1},\cdot), Q] = 0,\\
&(3) \quad [Q(d\boldsymbol{f}_p;\cdot), P] = (-1)^p Q(d(P(d\boldsymbol{f}_p)),\cdot),\\
&(4) \quad [Q(d\boldsymbol{f}_p,\cdot), Q] = 0.
\end{aligned}$$

hold, where $d\boldsymbol{f}_{p-1}$ stands for $df_1 \wedge \ldots \wedge df_{p-1}$.

Proof By Proposition 7, the conditions in Proposition 4 reduce to the above formulas. Conversely, assume that P and Q satisfy the above formulas. Then we have the same conclusion as those of Lemma 6. Thus if $p > 2$, we have $P \vdash P = 0, P \vdash Q = 0, Q \vdash Q = 0$ by Lemma 5. If we assume P is decomposable when $p = 2$, we can get the same conclusion by the argument of Lemma 7. Consequently, P and Q satisfy the conditions in Proposition 4. □

Since $P \vdash Q = 0$ means $P(d\boldsymbol{f}_{p-1},\cdot) \wedge Q = 0$, in the case when P is non-zero, Q is a multiple of P. Thus we have a vector field v satisfying $Q = v \wedge P$. It is desirable, in this case, to find the conditions on P and v which imply the Fundamental Identity. This will be done in the next section.

3 Associated Foliations

In this section, we investigate some geometric structure of Nambu-Jacobi manifold, namely the associated foliation which is given by the characteristic distributions of the structure. As is well-known, the Jacobi identity of a Poisson manifold implies the integrability of the characteristic distribution of the Poisson structure. This leads us to the foliation by symplectic leaves. This foliation is singular in general in the sense that the dimension of the leaves varies from point to point. Similarly on a Nambu-Poisson manifold we have a foliation and a contravariant *volume tensor*(multi-vector field of highest degree on a manifold) on each leaf. Theorem 9 below may be considered as a geometric characterization of a Nambu-Poisson manifold. We mean by the characteristic distribution of a p-vector field η the image of the bundle map $B_\eta : \wedge^{p-1}T^*M \to TM$ where $B_\eta(\alpha) = \eta(\alpha,\cdot)$.

Recall that *the generalized divergence* of η is defined as follows. Let ∇ be a torsion free affine connection on TM. ∇ gives a map $\nabla : \Gamma(\wedge^p TM) \to \Gamma(T^*M) \otimes \Gamma(\wedge^p TM)$. Let

$$c : \Gamma(T^*M) \otimes \Gamma(\wedge^p TM) \to \Gamma(\wedge^{p-1}TM)$$

be the map given by the contraction of 1-forms and p-vector fields. The generalized divergence $\operatorname{Div}\eta$ associated with ∇ is defined by

$$\operatorname{Div}\eta = c(\nabla(\eta)).$$

One of the definition of the Schouten bracket of multi-vector fields is given by the formula

$$[P,Q] = \operatorname{Div}(P\wedge Q) - (\operatorname{Div}P)\wedge Q - (-1)^p P\wedge \operatorname{Div}Q \tag{3.1}$$

where p is the degree of P. It is independent of the choice of connections. In what follows, we choose once and for all a Riemannian connection on TM and the Div will be the one which is associated with this connection (See also [4]).

THEOREM 9 *Let η be a decomposable C^∞ p-vector field on a C^∞-manifold M. Then the following statements are equivalent.*

(1) *The bracket $\{f_1,\dots,f_p\}^\eta = \eta(df_1,\dots,df_p)$ satisfies the Fundamental Identity.*

(2) *The characteristic distribution of η is integrable (in the sense of Sussmann and Stefan).*

(3) *On the open set U where η is non-zero, there exists a smooth 1-form γ which satisfies the equality*

$$\operatorname{Div}\eta = i_\gamma\eta.$$

Proof (1) $\Rightarrow$ (2). By an integrability theorem of Sussmann-Stefan, it is sufficient to prove that on a neighbourhood of each point in M, there exists a set of vector fields $\mathfrak{X}$ such that

(a) at each point of M, $\mathfrak{X}$ spans $\operatorname{Im}B_\eta$.

(b) the local 1-parameter subgroup of diffeomorphisms generated by a vector field belonging to $\mathfrak{X}$ leaves $\operatorname{Im}B_\eta$ invariant.

We discuss every thing locally. Let $\varphi_t(-\varepsilon<t<\varepsilon)$ be a local 1-parameter subgroup of diffeomorphisms generated by a vector field $\eta(df_1,\dots,df_{p-1},\cdot)$. By (1), we have

$$L_{\eta(df_1,\dots,df_{p-1},\cdot)}\eta = 0.$$

This means $(\varphi_t)_*\eta = \eta$. Thus for $\eta(dg_1,\dots,dg_{p-1},\cdot)$ in $\operatorname{Im}B_\eta$ we have

$$\begin{aligned}&(\varphi_t)_*(\eta(dg_1,\dots,dg_{p-1},\cdot))\\&= ((\varphi_t)_*\eta)((\varphi_t^*)^{-1}(dg_1),\dots,(\varphi_t^*)^{-1}(dg_{p-1}),\cdot)\\&= \eta(d(g_1\circ\varphi_t^{-1}),\dots,d(g_{p-1}\circ\varphi_t^{-1}),\cdot).\end{aligned}$$

This shows φ_t preserves $\operatorname{Im}B_\eta$. Clearly, the set of vector fields $\eta(df_1,\dots,df_{p-1},\cdot)$ for various functions span $\operatorname{Im}B_\eta$. Thus we may choose the $\{\eta(df_1,\dots,df_{p-q},\cdot)\}$ to be $\mathfrak{X}$.

(2) $\Rightarrow$ (3). Since η is decomposable p-vector field, on a neighbourhood of each point $a\in U$, we can choose a set of vector fields $X_1,X_2,\dots,X_p$ $X_i\in\Gamma(ImB_\eta)$such that

$$\eta = X_1\wedge X_2\wedge\dots\wedge X_p.$$

Then we have

$$\begin{aligned}\operatorname{Div}\eta &= \sum_{i=1}^{p}(-1)^{i-1}X_1\wedge\dots\wedge(\operatorname{Div}X_i)\wedge\dots\wedge X_p\\&+\sum_{i<j}(-1)^{i+j-1}[X_i,X_j]\wedge\dots\hat{X}_i\wedge\dots\wedge\hat{X}_j\wedge\dots\wedge X_p\end{aligned}$$

Since $\operatorname{Im}B_\eta$ is integrable, $[X_i,X_j]$ is a linear combination of $X_1,X_2,\dots,X_p$ at each point of U. Thus $\operatorname{Div}\eta$ is a $(p-1)$-vector field which is generated by $X_1,X_2,\dots,X_p$ and hence a cross-section of the bundle $\wedge^{p-1}(\operatorname{Im}B_\eta)$. Define

$$J_\eta : (\operatorname{Im}B_\eta)^* \to \wedge^{p-1}(\operatorname{Im}B_\eta)$$

to be the bundle map given by $J_\eta(\alpha) = \eta(\alpha,\cdot) = i_\alpha\eta$. Clearly it is a bundle isomorphism on U where η is non-zero. This assures that there exists a 1-form γ on U such that $i_\gamma\eta = \operatorname{Div}\eta$.

(3)$\Rightarrow$(1). First we note the following formula.

LEMMA 10 *Let β be a $(p-1)$-form and η a decomposable p-vector field on M. Then we have the following equality.*

$$\begin{aligned}[\eta(\beta,\cdot),\eta] &= (-1)^p\eta(d\beta)\eta\\&+\eta(\beta,\cdot)\wedge\operatorname{Div}\eta + (-1)^p(\operatorname{Div}\eta)(\beta)\eta\end{aligned} \tag{3.2}$$

Proof of lemma: Taking the contraction on both sides of

$$\nabla(\eta(\beta,\cdot)) = (\nabla\eta)(\beta,\cdot) + \eta(\nabla\beta,\cdot),$$

we have

$$\operatorname{Div}(\eta(\beta,\cdot)) = (-1)^{p-1}(\operatorname{Div}\eta)(\beta) + (-1)^{p-1}\eta(d\beta). \tag{3.3}$$

Thus we have

$$\begin{aligned}[\eta(\beta,\cdot),\eta] &= \operatorname{Div}(\eta(\beta,\cdot)\wedge\eta) - (\operatorname{Div}\eta(\beta,\cdot))\eta + \eta(\beta,\cdot)\wedge\operatorname{Div}\eta\\&= (-1)^p(\operatorname{Div}\eta)(\beta)\eta + (-1)^p\eta(d\beta)\eta + \eta(\beta,\cdot)\wedge\operatorname{Div}\eta.\end{aligned}$$

Note that $\eta(\beta,\cdot)\wedge\eta = 0$ holds by the decomposability. □

We continue the proof of Theorem. Since $\eta\wedge\eta(df_1,\dots,df_{p-1},\cdot) = 0$ on U, we have

$$\begin{aligned}
0 &= i_\gamma(\eta\wedge\eta(df_1,\dots,df_{p-1},\cdot)) \\
&= i_\gamma(\eta)\wedge\eta(df_1,\dots,df_{p-1},\cdot) + (-1)^p\eta(df_1,\dots,df_{p-1},\gamma)\eta \\
&= (\mathrm{Div}\,\eta)\wedge\eta(df_1,\dots,df_{p-1}\cdot) - ((\mathrm{Div}\,\eta)(df_1,\dots,df_{p-1}))\eta \\
&= (-1)^{p-1}[\eta(df_1,\dots,df_{p-1},\cdot),\eta].
\end{aligned}$$

We used Lemma above for $\beta = df_1\wedge\ldots\wedge df_{p-1}$.

If $a\in M\setminus U$, $\eta_{|a} = 0$ and the right hand side of the above lemma is equal to 0 and thus $[\eta(df_1,\dots,df_{p-1},\cdot),\eta]_{|a} = 0$.

Consequently, we have

$$[\eta(df_1,\dots,df_{p-1},\cdot),\eta] = 0$$

on the whole M and the bracket $\{\dots\}^\eta$ satisfies the Fundamental Identity. □

Now we are going to investigate the foliation associated with a Nambu-Jacobi structure. Let $\mathcal{A} = Q + (1\wedge P)$ be a Nambu-Jacobi bracket on a manifold M, which is given by a $(p+1)$-vector field Q and a p-vector field P. Then by Theorem 8 (1) of preceding section, $[P(df_1,\dots,df_{p-1},\cdot),P] = 0$. Thus P is a Nambu-Poisson tensor and its characteristic distribution is integrable, giving a generalized foliation(Theorem 9). We denote this foliation by $\mathcal{F}_P$. Exactly the same thing holds for the $(p+1)$-vector field Q. Thus we have two foliations $\mathcal{F}_P$ and $\mathcal{F}_Q$ of M. First we restrict our attention to the case when P is non-zero or it may be said that we consider the foliations of the open set of M where P is non-zero. By Proposition 7, $P\vdash Q = 0$. This is equivalent to

$$P(df_1,\dots,df_{p-1},\cdot)\wedge Q = 0,$$

for any $(p-1)$ functions $df_1,\dots,df_{p-1}$. On a neighbourhood of a point where $P\neq 0$, we have functions $df_1,\dots,df_p$ such that $P(df_1,\dots,df_p)\neq 0$. Thus the set of vector fields $\{X_1,\dots,X_p\}$ where $X_i = P(df_1,\dots,\hat{df_i},\dots,df_p)$ is linearly independent at each point. From the above relation, Q is a multiple of X_i's and consequently, there is a vector field v such that $Q = v\wedge P$. A partition of unity argument assures that we may consider v a global one.

PROPOSITION 11 *The vector field v preserves the associated foliation $\mathcal{F}_P$. In fact, there exists a function φ such that $L_vP = \varphi P$ holds.*

Proof By Theorem 8, we have

$$\begin{aligned}
0 &= [P(d\boldsymbol{f}_{p-1},\cdot),Q] = [P(d\boldsymbol{f}_{p-1},\cdot),v\wedge P] \\
&= [P(d\boldsymbol{f}_{p-1},\cdot),v]\wedge P + v\wedge[P(d\boldsymbol{f}_{p-1},\cdot),P] \\
&= [P(d\boldsymbol{f}_{p-1},\cdot),v]\wedge P
\end{aligned}$$

because $[P(d\boldsymbol{f}_{p-1}),P] = 0$. But $[P(d\boldsymbol{f}_{p-1},\cdot),v]\wedge P = 0$ is expressed as

$$-(L_vP)(d\boldsymbol{f}_{p-1},\cdot))\wedge P - (P(L_vd\boldsymbol{f}_{p-1},\cdot))\wedge P = 0$$

and $(P(L_vd\boldsymbol{f}_{p-1},\cdot))\wedge P = 0$ because of the decomposability of P. Thus we have

$$((L_vP)(d\boldsymbol{f}_{p-1},\cdot))\wedge P = 0$$

for arbitrary $(p-1)$-form $d\boldsymbol{f}_{p-1}$. Again by the decomposability of P, we see L_vP is a multiple of P. Thus we have a function φ satisfying $L_vP = \varphi P$. The equation

$$[v,P(d\boldsymbol{f}_{p-1},\cdot)] = \varphi P(d\boldsymbol{f}_{p-1},\cdot) + P(L_vd\boldsymbol{f}_{p-1})$$

shows that v preserves foliation $\mathcal{F}_P$. □

We have the converse.

THEOREM 12 *Let P be a regular Nambu-Poisson tensor of degree $p\geq 2$, which we assume decomposable when $p = 2$. Suppose that there exists a vector field v which satisfies*

$$L_vP = \varphi P$$

for some smooth function φ. Define $(p+1)$-vector field Q by $Q = v\wedge P$. Then the pair (Q,P) is a Nambu-Jacobi pair, namely the bracket

$$\mathcal{A} = Q + 1\wedge P$$

defines a Nambu-Jacobi structure.

Proof Since we are assuming the decomposability of P, it is sufficient to verify the conditions (1)–(4) of Theorem 8.

Condition (1) is our assumption. Condition (2) asserts that $[P(d\boldsymbol{g},\cdot),Q] = 0$ holds for any $d\boldsymbol{g} := dg_1\wedge\cdots\wedge dg_{p-1}$. This is easily verified as follows by using $[P(d\boldsymbol{g},\cdot),P] = 0$ and the decomposability of P;

$$\begin{aligned}
[P(d\boldsymbol{g},\cdot),v\wedge P] &= [P(d\boldsymbol{g},\cdot),v]\wedge P + v\wedge[P(d\boldsymbol{g},\cdot),P] \\
&= -(\varphi P(d\boldsymbol{g},\cdot) + P(L_v(d\boldsymbol{g}),\cdot))\wedge P = 0.
\end{aligned}$$

We verify Condition (4), *first*.

Since $Q = v \wedge P$ is decomposable, from the view point of Theorem 9, it is enough to see the integrability of the characteristic distribution $\mathcal{F}_Q$ of Q on open set where $Q \neq 0$. Locally, we can write P and Q as follows.

$$P = hX_1 \wedge \cdots \wedge X_p, \qquad Q = hv \wedge X_1 \wedge \cdots \wedge X_p$$

where X_i is a local vector field of the form $P(df_1 \wedge \cdots \wedge \hat{df_i} \wedge \cdots \wedge df_p)$ and h is a function. The vector fields $X_1, \cdots, X_p, v$ generate the distribution $\mathcal{F}_Q$ and they form a involutive system since $\mathcal{F}_P$ is integrable by assumption and since we have the following:

$$\begin{aligned}[v, P(d\boldsymbol{g}, \cdot)] &= L_v((P(d\boldsymbol{g}, \cdot)) \\ &= (L_vP)(d\boldsymbol{g}, \cdot)) + P(L_vd\boldsymbol{g}, \cdot) \\ &= P(\varphi d\boldsymbol{g} + L_vd\boldsymbol{g}, \cdot).\end{aligned}$$

Thus $\mathcal{F}_Q$ is integrable and we have $[Q, Q] = 0$.
To verify Condition (3), we must prove the equality

$$[Q(d\boldsymbol{f}; \cdot), P](\cdots) = (-1)^p Q(d(P(d\boldsymbol{f})), \cdots) \tag{3.4}$$

for $d\boldsymbol{f} = df_1 \wedge \cdots \wedge df_p$.
First we calculate the left hand side of this equality. Using

$$\begin{aligned}Q(d\boldsymbol{f}, \cdot) &= (v \wedge P)(d\boldsymbol{f}) \\ &= P(i_v(d\boldsymbol{f}), \cdot) + (-1)^p P(d\boldsymbol{f})v\end{aligned}$$

and a general formula

$$\mathrm{Div}(P(\alpha)) = (-1)^{\deg \alpha}(\mathrm{Div}\, P)(\alpha) + (-1)^{\deg \alpha} P(d\alpha), \tag{3.5}$$

we calculate as follows:

$$\begin{aligned}&[Q(d\boldsymbol{f}, \cdot), P] \\ &= [P(i_v(d\boldsymbol{f}), \cdot), P] + (-1)^p [P(d\boldsymbol{f})v, P] \\ &= \mathrm{Div}(P(i_v(d\boldsymbol{f}, \cdot)) \wedge P) - \mathrm{Div}(P(i_v(d\boldsymbol{f})), \cdot)P + P(i_v(d\boldsymbol{f}), \cdot) \wedge \mathrm{Div}\, P \\ &\quad + (-1)^p P(d\boldsymbol{f})[v, P] + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdots) \\ &= (-1)^p((\mathrm{Div}\, P)(i_v(d\boldsymbol{f})))P + (-1)^p P(d(i_v(d\boldsymbol{f})))P + P(i_v(d\boldsymbol{f}), \cdot) \wedge \mathrm{Div}\, P \\ &\quad + (-1)^p P(d\boldsymbol{f})[v, P] + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot).\end{aligned} \tag{3.6}$$

Now we use the assumption that there exits a 1-form such that $\mathrm{Div}\, P = i_\gamma P$ (Theorem 9). Then the above (3.6) is equal to

$$\begin{aligned}&(-1)^p(i_\gamma P)(i_v(d\boldsymbol{f})))P + (-1)^p P(d(i_v(d\boldsymbol{f})))P + P(i_v(d\boldsymbol{f})) \wedge (i_\gamma P) \\ &\quad + (-1)^p P(d\boldsymbol{f})[v, P] + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot) \\ &= -i_\gamma(P(i_v(d\boldsymbol{f})))P + (-1)^p P(d(i_v(d\boldsymbol{f})))P + P(i_v d\boldsymbol{f}) \wedge (i_\gamma P) \\ &\quad + (-1)^p P(d\boldsymbol{f})[v, P] + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot) \\ &= -i_\gamma(P(i_v d\boldsymbol{f}) \wedge P) + (-1)^p P(di_v(d\boldsymbol{f}))P \\ &\quad + (-1)^p P(d\boldsymbol{f})\varphi P + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot) \\ &= (-1)^p P(L_v(d\boldsymbol{f}))P + (-1)^p P(d\boldsymbol{f})\varphi P + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot) \\ &= (-1)^p L_v(P(d\boldsymbol{f})P + (-1)^{p+1}(L_vP)(d\boldsymbol{f})P \\ &\quad (-1)^p P(d\boldsymbol{f})(\varphi P) + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot) \\ &= (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot) + (-1)^p L_v(P(d\boldsymbol{f}))P.\end{aligned}$$

This can be seen to be equal to the right hand side of (3.4), since we have

$$\begin{aligned}(-1)^p Q(d(P(d\boldsymbol{f})), \cdot) &= (-1)^p(v \wedge P)(d(P(d\boldsymbol{f})), \cdots) \\ &= (-1)^p v(d(P(d\boldsymbol{f})))P + (-1)^{p+1} v \wedge P(d(P(d\boldsymbol{f})), \cdot).\end{aligned}$$

Thus, the pair $(Q = v \wedge P, P)$ satisfy the conditions (1)–(4) and the bracket is a Nambu-Jacobi pair. □

Next we consider a Nambu-Jacobi structure $Q + 1 \wedge P$ where Q is regular, that is Q is nowhere zero. In this case we obtain the following:

THEOREM 13 *Let Q be a Nambu Poisson tensor or degree $q(\geq 2)$. We assume when $q = 2$, Q is decomposable. Let α be a 1-form which is closed on the leaves of $\mathfrak{F}_Q$. That is $Q(d\alpha, \cdot) = 0$. Put $P = Q(\alpha, \cdot)$. Then (Q, P) makes a Nambu-Jacobi pair.*

Conversely, if (Q, P) is a Nambu-Jacobi pair and Q is regular, there exists a 1-form α which is closed along the leaves of Q such that $P = Q(\alpha, \cdot)$.

Proof We first verify the condition $J^P Q = 0$. Namely, we prove

$$[P(d\boldsymbol{f}_{p-1}, \cdot), Q] = [Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot), Q] = 0.$$

Using the decomposability of Q and the formula (3.5) for $\mathrm{Div}(Q(\alpha))$, we calculate as follows:

$$[Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot), Q] = \mathrm{Div}(Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q)$$
$$- \mathrm{Div}(Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot)) \wedge Q + Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge \mathrm{Div}\, Q$$
$$= (-1)^{p+1}(\mathrm{Div}\, Q)(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q$$
$$+ (-1)^{p+1}Q(d\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q + Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge \mathrm{Div}\, Q.$$

Clearly, this is equal to 0 where $Q = 0$. On the other hand, on the open set where $Q \neq 0$, we have a 1-form γ such that $\mathrm{Div}\, Q = Q(\gamma, \cdot)$ and the above is equal to

$$(-1)^{p+1}Q(\gamma \wedge \alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q + Q(\alpha \wedge d\boldsymbol{f}_{p-1}) \wedge Q(\gamma)$$
$$= -i_\gamma(Q(\alpha \wedge d\boldsymbol{f}_{p-1}) \wedge Q) = 0.$$

Thus we proved $J^P Q = 0$.
Secondly, we prove

$$[P(d\boldsymbol{f}_{p-1}, \cdot), P] = 0$$

for any functions $f_1, \dots, f_{p-1}$. We use the abbreviated notations that $p = q - 1$ and $d\boldsymbol{f}_{p-1} = df_1 \wedge \dots \wedge df_{p-1}$ as before. Then we calculate as follows;

$$[P(d\boldsymbol{f}_{p-1}, \cdot), P] = [P(d\boldsymbol{f}_{p-1}, \cdot), Q(\alpha, \cdot)]$$
$$= [P(d\boldsymbol{f}_{p-1}, \cdot), Q](\alpha) + Q(L_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha). \tag{3.7}$$

As we showed above, $[P(d\boldsymbol{f}_{p-1}, \cdot), Q](\alpha) = 0$ and $Q(L_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha) = 0$ is verified as follows.

$$Q(L_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha, \cdot)$$
$$= Q(di_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha + i_{P(d\boldsymbol{f}_{p-1}, \cdot)}d\alpha, \cdot) = Q(i_{P(d\boldsymbol{f}_{p-1}, \cdot)}d\alpha) = Q(d\alpha(P(d\boldsymbol{f}_{p-1}, \cdot), \cdot).$$

The most right term vanishes since if write $Q = X_1 \wedge \dots \wedge X_q$, this is equal to

$$\sum_{i=1}^{q}(-1)^{i-1}d\alpha(P(d\boldsymbol{f}_{p-1}, \cdot), X_i)X_1 \wedge \dots, \wedge \hat{X}_i \wedge \dots \wedge X_q$$

and since α is closed on $\mathrm{Im}\, Q$.

Next, we prove

$$[Q(d\mathbf{g}, \cdot), P] = (-1)^p Q(d(P(d\mathbf{g}), \cdot) \qquad \text{for} \quad d\mathbf{g} = dg_1 \wedge \dots \wedge dg_p.$$

This is shown as follows.

$$[Q(d\mathbf{g}, \cdot), Q(\alpha)] = [Q(d\mathbf{g}, \cdot), Q](\alpha) + Q(L_{Q(d\mathbf{g}, \cdot)}\alpha, \cdot)$$
$$= Q(di_{Q(d\mathbf{g}, \cdot)}\alpha + i_{Q(d\mathbf{g}, \cdot)}d\alpha) = Q(di_{Q(d\mathbf{g}, \cdot)}\alpha)$$
$$= Q(d(Q(d\mathbf{g}, \alpha))) = (-1)^p Q(d(P(d\mathbf{g})).$$

$Q(i_{Q(d\mathbf{g}, \cdot)}d\alpha) = 0$ follows from $d\alpha = 0$ on $\mathfrak{F}_Q$.

Now we prove the converse. Namely, assuming (Q, P) is a Nambu-Jacobi pair on M and Q is non-singular, we prove that there exists a 1-form α such that $P = Q(\alpha, \cdot)$ and $Q(d\alpha) = 0$. By assumption (Q, P) satisfies the following:

(1) $[P(d\boldsymbol{f}_{p-1}, \cdot), P] = 0$,
(2) $[P(d\boldsymbol{f}_{p-1}, \cdot), Q] = 0$,
(3) $[Q(d\boldsymbol{f}_p; \cdot), P] = (-1)^p Q(d(P(d\boldsymbol{f}_p)), \cdot)$,
(4) $[Q(d\boldsymbol{f}_p, \cdot), Q] = 0$.

If we consider Q as a bundle map $\wedge^p T^*M \to TM$, $\mathrm{Im}\, Q$ is a $(p+1)$-dimensional sub-bundle of TM. Q is also considered as a non-zero cross section of $\wedge^{p+1}\mathrm{Im}\, Q$ and gives a natural isomorphism $(\mathrm{Im}\, Q)^* \to \wedge^p \mathrm{Im}\, Q$. Let $B_Q : \wedge^p \mathrm{Im}\, Q \to (\mathrm{Im}\, Q)^*$ denote the inverse isomorphism. Since we have $P(df_1 \wedge \dots \wedge df_{p-1}, \cdot) \wedge Q = 0$, $\mathrm{Im}\, P \subset \mathrm{Im}\, Q$ (see Proposition 7). Thus P is a cross section of the bundle $\wedge^p \mathrm{Im}\, Q$. Put $\alpha' = B_Q(P)$ and choose a 1-form α so that α projects to α' under the natural surjection $T^*M \to (\mathrm{Im}\, Q)^*$. Then we can see that

$$Q(\alpha, \cdot) = Q(\alpha', \cdot) = P.$$

Now by a characterization of Nambu-Poisson tensor field, there exists a 1-form on M, satisfying $\mathrm{Div}\, Q = Q(\gamma, \cdot)$. Since $Q(\alpha, \cdot) = P$ is also a Nambu-Poisson tensor field, there exists a 1-form λ on the open set where $Q(\alpha, \cdot) \neq 0$, satisfying $\mathrm{Div}(Q(\alpha, \cdot)) = Q(\alpha, \lambda, \cdot)$. By the condition $[Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot), Q] = 0$, and the decomposability of Q, we have

$$0 = -(\mathrm{Div}(Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot))Q + Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot) \wedge \mathrm{Div}\, Q.$$

But we have the following

$$\mathrm{Div}(Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot))$$
$$= (-1)^{p-1}\mathrm{Div}(Q(\alpha, \cdot))(d\boldsymbol{f}_{p-1}) + (-1)^{p-1}Q(\alpha, dd\boldsymbol{f}_{p-1})$$
$$= (-1)^{p-1}Q(\alpha, \lambda, d\boldsymbol{f}_{p-1}).$$

Thus we have

$$(-1)^{p-1}Q(\alpha, \lambda, d\boldsymbol{f}_{p-1})Q = Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot) \wedge \mathrm{Div}\, Q$$
$$= Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot) \wedge Q(\gamma, \cdot)$$
$$= -i_\gamma(Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot) \wedge Q) + Q(\alpha, d\boldsymbol{f}_{p-1}, \gamma)Q.$$

Since $Q(\alpha, d\boldsymbol{f}_{p-1}, \cdot) \wedge Q = 0$ by the decomposability, this means $Q(\alpha, \lambda - \gamma, \cdot) = 0$. If we use the formula

$$\mathrm{Div}(Q(\alpha, \cdot)) = -(\mathrm{Div}\, Q)(\alpha, \cdot) - Q(d\alpha, \cdot),$$

we have $Q(d\alpha,\cdot) = -Q(\alpha,\gamma-\lambda,\cdot) = 0$. This is what we wanted to show. □

Examples

By Theorem 12 and Theorem 13, we obtain concrete examples of Nambu-Jacobi manifolds. Here, we have a few of them.

1. We consider the Reeb foliation of S^3 as the underlying foliation. There exists a 2-vector field P which is non-singular on each leaf and tangent to it. We can assume every thing is invariant under the natural S^1-action on S^3. Let v be the vector field on S^3 which is obtained from this action. Then $L_v P = 0$ and $(Q = v \wedge P, P)$ is a Nambu-Jacobi pair by Theorem 12. Q vanishes exactly along the toral leaf.

2. Let $\mathfrak{F}$ be the Anosov foliation on the circle bundle over a closed surface of genus $g \geq 2$. The leaves are diffeomorphic to either $\boldsymbol{R}^2$ or cylinder $S^1 \times \boldsymbol{R}$. Since both types of leaves are dense, there is no non-trivial vector field transverse to $\mathfrak{F}$. Therefore the only possible Jacobi pair is trivial one, namely it is $(0, P)$.

3. Let $A : T^n \to T^n$ be a hyperbolic toral automorphism. The mapping torus M_A of A has a foliation foliated by the weak unstable manifolds. Let Q denote a natural tensor field which gives a volume tensor field along each leaf. Let α be the 1-form on $M_A = T^n \times [0,1]/\sim \to S^1$, which is the pull-back of $d\theta$ by the projection $M_A \to S^1$. Then α is closed and $(Q, P = Q(\alpha,\cdot))$ is a Nambu-Jacobi pair. P defines a foliation foliated by strong unstable manifolds.

4. For any Nambu-Poisson structure Q on M, $(Q, \operatorname{Div} Q)$ is a Nambu-Jacobi pair. Here Div is a divergence associated with a connection which preserves a volume form of M. If $\operatorname{Div} Q = Q(\gamma,\cdot)$, we have $Q(d\gamma,\cdot) = -\operatorname{Div}^2 Q$ and $\operatorname{Div}^2 = 0$ since we assumed the connection preserves a volume form. Thus by Theorem 13, we have the result.

On a Nambu-Jacobi manifold for which the tensor fields P and Q are both non-singular, we have a regular foliation $\mathcal{F}_Q$ and its subfoliation $\mathcal{F}_P$. By our theorem, on each leaf of $\mathcal{F}_Q$ there exits a non-singular vector field and the subfoliation $\mathcal{F}_P$ is defined by a closed 1-form on the leaf. These impose a rather strong restriction on the foliated structure of such a Nambu-Jacobi structure. It seems an interesting topological question to find which manifold has such a foliated structure.

4 Appendix

In this appendix, we prove Proposition 4.

We denote the bracket defined by a p-vector field P by $\{\dots\}^P$. Namely, $\{f_1,\dots,f_p\}^P = P(df_1,\dots,df_p)$. The bracket $\{\dots\} := \{\dots\}^{Q+1\wedge P} = Q + 1 \wedge P$ determined by a $q(=p+1)$-vector field Q and a p-vector field P, is by definition is

the following:

$$\{f_1,\dots,f_q\} = \{f_1,\dots,f_q\}^Q + \sum_{j=1}^{q}(-1)^{j-1} f_j \{f_1,\dots,\hat{f_j},\dots,f_q\}^P.$$
$$= Q(df_1,\dots,df_q) + \sum_{j=1}^{q}(-1)^{j-1} f_j P(df_1,\dots,\widehat{df_j},\dots,df_q)$$

We would like to write down the Fundamental Identity for this bracket in terms of the brackets of Q and P and find the relations which Q and P satisfy.

For the brackets $\{\cdots\}^P$ and $\{\cdots\}^Q$ of degree p and q, respectively, we defined $J^P Q$ and $P \vdash Q$ as follows.

$$\begin{aligned} &J^P Q(f_1,\dots,f_{p-1}; g_1,\dots,g_q) \\ &= \{f_1,\dots,f_{p-1},\{g_1,\dots,g_q\}^Q\}^P - \{\{f_1,\dots,f_{p-1},g_1\}^P, g_2,\dots,g_q\}^Q \\ &-\{g_1,\{f_1,\dots,f_{p-1},g_2\}^P, g_3,\dots,g_q\}^Q - \cdots - \{g_1,\dots,g_{q-1},\{f_1,\dots,f_{p-1},g_q\}^P\}^Q \\ &= [P(df_1\wedge\cdots\wedge df_{p-1},\cdot), Q](dg_1,\dots,dg_q) \end{aligned}$$

$$\begin{aligned} &(P\vdash Q)(f_1,\dots,f_{p-1}; g_0,\dots,g_q) \\ &= \sum_{j=0}^{q}(-1)^j\{f_1,\dots,f_{p-1},g_j\}^P\{g_0,\dots,\hat{g_j},\dots,g_q\}^Q \\ &= (P(df_1,\dots,df_{p-1},\cdot)\wedge Q)(dg_0,\dots,dg_q) \end{aligned}$$

The usual Fundamental Identity for $\{\dots\} = \{\dots\}^{Q+1\wedge P}$ is the following identity for any C^∞ functions $f_1,\dots,f_{q-1},g_1,\dots,g_q$ on M.

$$\begin{aligned} &\{f_1,\dots,f_{q-1},\{g_1,\dots,g_q\}\} \\ &= \sum_{i=1}^{q}(-1)^{i-1}\{\{f_1,\dots,f_{q-1},g_i\},g_1,\dots,\hat{g_i},\dots,g_q\} \end{aligned}$$

In this appendix, however, for our notational convenience, we adopt the following equivalent equation as the Fundamental Identity.

$$\begin{aligned} &\{\{f_1,\dots,f_q\},g_2,\dots,g_q\} \\ &= \sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\},f_{j+1},\dots,f_q\} \end{aligned} \tag{4.1}$$

We now start the computation. Since by definition,

$$\{f_1,\dots,f_q\} := \{f_1,\dots,f_q\}^Q + \sum_{j=1}^{q}(-1)^{j-1} f_j\{f_1,\dots,\hat{f_j},\dots,f_q\}^P,$$

the left hand side of (4.1) is calculated as follows.

$$
\begin{aligned}
&\{\{f_1,\dots,f_q\},g_2,\dots,g_q\} \\
&= \{\{f_1,\dots,f_q\},g_2,\dots,g_q\}^Q + \{f_1,\dots,f_q\}\{g_2,\dots,g_q\}^P \\
&\quad + \sum_{k=2}^{q}(-1)^{k-1} g_k\{\{f_1,\dots,f_q\},g_2,\dots,\hat{g}_k,\dots,g_q\}^P \\
&= \{\{f_1,\dots,f_q\}^Q + \sum_{i=1}^{q}(-1)^{i-1} f_i\{f_1,\dots,\hat{f}_i,\dots,f_q\}^P, g_2,\dots,g_q\}^Q \\
&\quad + \{f_1,\dots,f_q\}^Q\{g_2,\dots,g_q\}^P \\
&\quad + \sum_{i=1}^{q}(-1)^{i-1} f_i\{f_1,\dots,\hat{f}_i,\dots,f_q\}^P\{g_2,\dots,g_q\}^P \\
&\quad + \sum_{k=2}^{q}(-1)^{k-1} g_k\{\{f_1,\dots,f_q\}^Q,g_2,\dots,\hat{g}_k,\dots,g_q\}^P \\
&\quad + \sum_{k=2}^{q}\sum_{i=1}^{q}(-1)^{i+k} g_k\{f_i\{f_1,\dots\hat{f}_i\dots,f_q\}^P,g_2,\dots,\hat{g}_k,\dots,g_q\}^P
\end{aligned}
$$

$$
\begin{aligned}
&= \{\{f_1,\dots,f_q\}^Q,g_2,\dots,g_q\}^Q \\
&\quad + \sum_{i=1}^{q}(-1)^{i-1} f_i\{\{f_1,\dots,\hat{f}_i,\dots,f_q\}^P,g_2,\dots,g_q\}^Q \\
&\quad + \sum_{i=1}^{q}(-1)^{i-1} \{f_1,\dots,\hat{f}_i,\dots,f_q\}^P\{f_i,g_2,\dots,g_q\}^Q \\
&\quad + \{f_1,\dots,f_q\}^Q\{g_2,\dots,g_q\}^P \\
&\quad + \sum_{i=1}^{q}(-1)^{i-1} f_i\{f_1,\dots,\hat{f}_i,\dots,f_q\}^P\{g_2,\dots,g_q\}^P \\
&\quad + \sum_{k=2}^{q}(-1)^{k-1} g_k\{\{f_1,\dots,f_q\}^Q,g_2,\dots,\hat{g}_k,\dots,g_q\}^P \\
&\quad + \sum_{k=2}^{q}\sum_{i=1}^{q}(-1)^{i+k} g_k f_i\{\{f_1,\dots\hat{f}_i\dots,f_q\}^P,g_2,\dots,\hat{g}_k,\dots,g_q\}^P \\
&\quad + \sum_{k=2}^{q}\sum_{i=1}^{q}(-1)^{i+k} g_k\{f_1,\dots\hat{f}_i\dots,f_q\}^P\{f_i,g_2,\dots,\hat{g}_k,\dots,g_q\}^P
\end{aligned}
$$

This is the left hand side of (4.1) expressed in terms of f_i's, g_i's and their brackets with respect to $\{\dots\}^P$ and $\{\dots\}^Q$.

In a similar way, we calculate the right hand side of the Fundamental Identity (4.1).

$$
\begin{aligned}
&\sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\},f_{j+1},\dots,f_q\} \\
&= \sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,f_q\} \\
&\quad + \sum_{j=1}^{q}\{f_1,\dots,f_{j-1},f_j\{g_2,\dots,g_q\}^P,f_{j+1},\dots,f_q\} \\
&\quad + \sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\sum_{k=2}^{q}(-1)^{k-1} g_k\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,f_q\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,f_q\}^Q \\
&\quad + \sum_{j=1}^{q}\sum_{i=1}^{j-1}(-1)^{i-1} f_i\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,f_q\}^P \\
&\quad + \sum_{j=1}^{q}(-1)^{j-1}\{f_j,g_2,\dots,g_q\}^Q\{f_1,\dots,\hat{f}_j,\dots,f_q\}^P \\
&\quad + \sum_{j=1}^{q}\sum_{i=j+1}^{q}(-1)^{i-1} f_i\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{q}\{f_1,\dots,f_{j-1},f_j\{g_2,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^Q \\
&\quad + \sum_{j=1}^{q}\sum_{i=1}^{j-1}(-1)^{i-1} f_i\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},f_j\{g_2,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^P \\
&\quad + \sum_{j=1}^{q}(-1)^{j-1} f_j\{g_2,\dots,g_q\}^P\{f_1,\dots,\hat{f}_j,\dots,f_q\}^P \\
&\quad + \sum_{j=1}^{q}\sum_{i=j+1}^{q}(-1)^{i-1} f_i\{f_1,\dots,f_{j-1},f_j\{g_2,\dots,g_q\}^P,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P
\end{aligned}
$$

$$+\sum_{j=1}^{q}\sum_{k=2}^{q}(-1)^{k-1}\{f_1,\dots,f_{j-1},g_k\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^Q$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}\sum_{i=1}^{j-1}(-1)^{k+i}f_i\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},g_k\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}(-1)^{j+k}g_k\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P\{f_1,\dots,\hat{f}_j,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}\sum_{i=j+1}^{q}(-1)^{k+i}f_i\{f_1,\dots,f_{j-1},g_k\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P$$

Further by applying the Leibniz rule, these are calculated as follows:

$$=\sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,f_q\}^Q$$
$$+\sum_{j=1}^{q}\sum_{i=1}^{j-1}(-1)^{i-1}f_i\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}(-1)^{j-1}\{f_j,g_2,\dots,g_q\}^Q\{f_1,\dots,\hat{f}_j,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{i=j+1}^{q}(-1)^{i-1}f_i\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}f_j\{f_1,\dots,f_{j-1},\{g_2,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^Q$$
$$+\sum_{j=1}^{q}\{f_1,\dots,f_q\}^Q\{g_2,\dots,g_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{i=1}^{j-1}(-1)^{i-1}f_if_j\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},\{g_2,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{i=1}^{j-1}(-1)^{i-1}f_i\{g_2,\dots,g_q\}^P\{f_1,\dots,\hat{f}_i,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}(-1)^{j-1}f_j\{g_2,\dots,g_q\}^P\{f_1,\dots,\hat{f}_j,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{i=j+1}^{q}(-1)^{i-1}f_if_j\{f_1,\dots,f_{j-1},\{g_2,\dots,g_q\}^P,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P$$

$$+\sum_{j=1}^{q}\sum_{i=j+1}^{q}(-1)^{i-1}f_i\{g_2,\dots,g_q\}^P\{f_1,\dots,\hat{f}_i,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}(-1)^{k-1}g_k\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^Q$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}(-1)^{k-1}\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P\{f_1,\dots,f_{j-1},g_k,f_{j+1},\dots,f_q\}^Q$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}\sum_{i=1}^{j-1}(-1)^{k+i}f_ig_k\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,f_q\}^P$$

$$+\sum_{j=1}^{q}\sum_{k=2}^{q}\sum_{i=1}^{j-1}(-1)^{k+i}f_i\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P\{f_1,\dots,\hat{f}_i,\dots,f_{j-1},g_k,f_{j+1},\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}(-1)^{j+k}g_k\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P\{f_1,\dots,\hat{f}_j,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}\sum_{i=j+1}^{q}(-1)^{k+i}f_ig_k\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P$$
$$+\sum_{j=1}^{q}\sum_{k=2}^{q}\sum_{i=j+1}^{q}(-1)^{k+i}f_i\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P\{f_1,\dots,f_{j-1},g_k,f_{j+1},\dots,\hat{f}_i,\dots,f_q\}^P$$

We will not simplify these any further since from the computation we can obtain necessary conditions on P and Q for the bracket $\{\}$ to satisfy the Fundamental Identity.

First we note that the sums containing the product f_if_j cancel out because of the skewness of the bracket.

To get the conditions, we put $f_q = g_q \equiv 1 (q = p+1)$ and compare the left hand side and the right hand side of (4.1) which we computed above. Since $\{\dots,1\}^P$ and $\{\dots,1\}^Q$ are both constantly equal to 0, we obtain the following relations:

$$\{\{f_1,\dots,f_p\}^P,g_2,\dots,g_p\}^P$$
$$=\sum_{j=1}^{p}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_p\}^P,f_{j+1},\dots,f_p\}^P.$$

This is nothing but (1) of Proposition 4 and the Fundamental Identity for $\{\dots\}^P$. Namely we get the condition

$$J^PP = 0. \tag{4.2}$$

Putting this condition in our computation, we see that the terms containing $f_i g_k$ all cancel out.

Next, we put $g_q \equiv 1$, and by the same reason as before, we get the relation

$$\begin{aligned}&\{\{f_1,\dots,f_q\}^Q, g_2,\dots,g_p\}^P\\&=\sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_p\}^P,f_{j+1},\dots,f_q\}^Q.\end{aligned}$$

This shows

$$(-1)^p J^P Q(g_2,\dots,g_p;f_1,\dots,f_q)=0,$$

and we get

$$J^P Q = 0. \tag{4.3}$$

By this relation, we see that the terms which are the multiple of the function g_k all cancel out. Similarly, knowing the relations (4.2) and (4.3) and by putting $f_q \equiv 1$, we obtain the following relation

$$\begin{aligned}&\{\{f_1,\dots,f_p\}^P, g_2,\dots,g_q\}^Q\\&=\sum_{j=1}^{p}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1}\dots,f_p\}^P\\&\quad+(-1)^p\{f_1,\dots,f_p,\{g_2,\dots,g_q\}^P\}^Q+p\{f_1,\dots,f_p\}^P\{g_2,\dots,g_q\}^P\\&\quad+\sum_{j=1}^{p}\sum_{k=2}^{p+1}(-1)^{p+k+j+1}\{g_2,\dots,\hat{g}_k,\dots,g_q,f_j\}^P\{g_k,f_1,\dots,f_{j-1},f_{j+1},\dots,f_p\}^P.\end{aligned}$$

A little computation shows that this is equivalent to the following:

$$\begin{aligned}&J^Q P(g_2,\dots,g_q;f_1,\dots,f_p)\\&\quad=(-1)^p\{\{g_2,\dots,g_q\}^P f_1,\dots,f_p\}^Q\\&\qquad+\sum_{k=2}^{q}(-1)^k(P\vdash P)(g_2,\dots,\hat{g}_k,\dots,g_q;g_k,f_1,\dots,f_p)\end{aligned} \tag{4.4}$$

This is the relation equivalent to (3) of Proposition 4. Note that

$$\begin{aligned}&(P\vdash P)(g_2,\dots,\hat{g}_k,\dots,g_q;g_k,f_1,\dots,f_p)\\&=\sum_{j=1}^{p}(-1)^j\{g_2,\dots,\hat{g}_k,\dots,g_q,f_j\}^P\{g_k,f_1\dots,\hat{f}_j,\dots,f_p\}^P\\&\quad+(-1)^{q-k}\{g_2,\dots,g_q\}^P\{f_1,\dots,f_p\}^P.\end{aligned}$$

If P and Q satisfy the above condition (4.4), the terms of the form $f_j\{\dots\}$ cancel out.

Finally, in the same way, we obtain the following condition on Q.

$$\begin{aligned}&\{\{f_1,\dots,f_q\}^Q, g_2,\dots,g_q\}^Q\\&=\sum_{j=1}^{q}\{f_1,\dots,f_{j-1},\{f_j,g_2,\dots,g_q\}^Q,f_{j+1},\dots,f_q\}^Q\\&\quad+\sum_{j=1}^{q}\sum_{k=2}^{q}(-1)^{k-1}\{f_j,g_2,\dots,\hat{g}_k,\dots,g_q\}^P\{f_1,\dots,f_{j-1},g_k,f_{j+1},\dots,f_q\}^Q.\\&\quad+p\{f_1,\dots,f_q\}^Q\{g_2,\dots,g_q\}^P.\end{aligned}$$

This is expressed as

$$\begin{aligned}&J^Q Q(g_2,\dots,g_q;f_1,\dots,f_q)\\&\quad=\sum_{k=2}^{q}(-1)^k(P\vdash Q)(g_2,\dots,\hat{g}_k,\dots,g_q;g_k,f_1,\dots,f_q)\end{aligned} \tag{4.5}$$

This is nothing but (4) of Proposition 4.

We have shown that (1)—(4) of Proposition 4 are necessary conditions for $\{\dots\}^{Q+1\wedge P}$ satisfy the Fundamental Identity.

Conversely, from our computation, we can easily see that if the relations (4.2),(4.3),(4.4) and (4.5) on the brackets $\{\dots\}^P$ and $\{\dots\}^Q$ hold, the Fundamental Identity of the bracket $\{\dots\}=\{\dots\}^{Q+1\wedge P}$ is true.

Thus the relations (4.2),(4.3),(4.4) and (4.5) together are equivalent to the Fundamental Identity for $\{\dots\}^{Q+1\wedge P}$.

In this way, we obtained

PROPOSITION 4 *Let $\mathcal{A}=Q+(1\wedge P)$ be a Nambu-Jacobi bracket degree $Q=q=p+1\geq 3$. Then we have the following identities*

(1) $J^P P=0$,

(2) $J^P Q=0$,

(3) $$\begin{aligned}&J^Q P(d\boldsymbol{f}_p;\cdots)+(-1)^{p+1}Q(dP(d\boldsymbol{f}_p),\cdots)\\&\qquad+\sum_{i=1}^{p}(-1)^i(P\vdash P)(df_1\cdots\hat{df_i}\cdots df_p;df_i,\cdots)=0,\end{aligned}$$

(4) $$J^Q Q(d\boldsymbol{f}_p;\cdots)+\sum_{i=1}^{p}(-1)^i(P\vdash Q)(df_1\cdots\hat{df_i}\cdots df_p;df_i,\cdots)=0$$

These together are also sufficient for the bracket $\mathcal{A}=Q+(1\wedge P)$ to satisfy the Fundamental Identity.

Remark: When $p = 1$, if we interpret the formulas properly, the relation obtained from the above is expressed as

$$[P, Q] = 0, \qquad [Q, Q] = -2P \wedge Q, \tag{4.6}$$

which is the usual definition of Jacobi structure.

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Leibniz algebras associated with foliations

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Abstract

Certain types of singular foliations on a manifold have Leibniz algebra structures on the space of multivector fields. Each of them has a structure of a central extension of a Lie algebra in the sense of Leibniz algebra. To a specific Leibniz cohomology class, there corresponds an isomorphism class of central extension of a Leibniz algebra similarly as in the case of Lie algebra.

1 Introduction

Recently, a lot of interests have been taken in Leibniz algebra, which is introduced by Loday [10, 11] as a non-commutative variation of Lie algebra. A Leibniz algebra $\mathfrak{g}$ is an R-module, where R is a commutative ring, endowed with a bilinear map

$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[g_1, [g_2, g_3]] = [[g_1, g_2], g_3] + [g_2, [g_1, g_3]].$$

Note that we do not require the anti–symmetricity of $[\ ,\ ]$.

In this paper, we consider Leibniz algebra associated with a certain type of singular foliations on a manifold. More precisely, we observe that when an integrable and locally decomposable q-form ω on a manifold M is given, there yields a foliation $\mathcal{F}$ of M whose leaves are either of dimension $n - q$ or 0. Any transversely oriented regular foliation of codimension q is defined by such a q-form. We show that the bundle of $(q+1)$-vectors $\bigwedge^{q+1} TM$ on M has a Leibniz algebroid structure whose anchor map is a interior product by ω and whose bracket is given by

$$[\![X, Y]\!]_\omega = [\iota_\omega X, Y] + (-1)^q \langle X | d\omega \rangle\, Y$$

for any $X, Y \in \mathfrak{X}^{q+1}(M)$, where $[\ ,\ ]$ denotes the Schouten bracket, $\langle\ |\ \rangle$ the natural pairing and $\mathfrak{X}^{q+1}(M)$ the space of $(q+1)$-vector fields. We see that the isomorphism class of the algebra is determined by the foliation $\mathcal{F}$. It is not a Lie algebra in general unless $q = 0$ or $q = n - 2$. Considering the difference of $\mathfrak{X}^{q+1}(M)$ from Lie algebra, it is shown that $\mathfrak{X}^{q+1}(M)$ is, as a Leibniz algebra, a central extension of the

Lie algebra of vector fields tangent to $\mathcal{F}$.

As it is known, central extensions of a Lie algebra $\mathfrak{g}$ with the center A are described by $H^2_{\mathrm{Lie}}(\mathfrak{g}; A)$ where $H^*_{\mathrm{Lie}}(\mathfrak{g}; A)$ denotes the Lie algebra cohomology with coefficients in A. One can ask the question: how about the case of Leibniz algebras? We see that the "natural" cohomology of Leibniz algebra does not work, but a slightly different cohomology $H^*(\mathfrak{g}; A)$ makes a similar one–to–one correspondence between equivalent classes of central extensions and elements in $H^2(\mathfrak{g}; A)$. It means that, when $\mathfrak{g}'$ is a central extension of a Leibniz algebra $\mathfrak{g}$ with the center A, Leibniz algebra structures of $\mathfrak{g}'$ is determined by an element in $H^2(\mathfrak{g}; A)$. Applying it to Leibniz algebras associated with foliations, we can obtain a lot of geometric examples of central extensions of Leibniz algebras.

The (co)homology of Leibniz algebra is studied by Loday and Pirashvili [12]. Lodder [14] extends the Leibniz cohomology from a Lie algebra invariant to an invariant for a differential manifold. The notion of Leibniz algebroid over a manifold was defined in [9] as a vector bundle with certain additional conditions as in the case of Lie algebroid, and it was proved that the bundle of $(p-1)$-forms on a Nambu-Poisson manifold has a Leibniz algebroid structure. In [6], one of the author discovered an alternative Leibniz algebroid structure which is a natural generalization of the Lie algebroid associated with a Poisson manifold. Description of all Leibniz algebras of dimension three is given in [1].

2 Leibniz algebras and cohomologies

First we review the notion of Leibniz algebra defined by Loday [10, 11, 12]. Let R be a commutative ring and $\mathfrak{g}$ an R-module endowed with a bilinear map $[\ ,\ ]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying

$$[g_1,[g_2,g_3]]=[[g_1,g_2],g_3]+[g_2,[g_1,g_3]] \tag{2.1}$$

for $g_1,g_2,g_3\in\mathfrak{g}$. The map $[\ ,\ ]$ is called the Leibniz bracket on $\mathfrak{g}$ and (2.1) the Leibniz identity. We remark that if $[\ ,\ ]$ is additionally skew-symmetric, then the Leibniz identity is just the Jacobi identity and $(\mathfrak{g},[\ ,\ ])$ is a Lie algebra. Therefore, a Leibniz algebra is a non-commutative variant of Lie algebra.

Now we consider the cohomology of a Leibniz algebra with values in a module [12]. Suppose that $(\mathfrak{g},[\ ,\ ])$ is a Leibniz algebra and A an R-module equipped with bilinear actions of $\mathfrak{g}$

$$[\ ,\ ]:\mathfrak{g}\times A\to A,\quad [\ ,\ ]:A\times\mathfrak{g}\to A$$

such that

$$[a,[g_1,g_2]]=[[a,g_1],g_2]+[g_1,[a,g_2]] \tag{2.2}$$

$$[g_1,[a,g_2]]=[[g_1,a],g_2]+[a,[g_1,g_2]] \tag{2.3}$$

$$[g_1,[g_2,a]]=[[g_1,g_2],a]+[g_2,[g_1,a]] \tag{2.4}$$

for $g_1,g_2\in\mathfrak{g}$ and $a\in A$. We also use the notations $ga=l_g(a)=[g,a]$ and $ag=r_g(a)=[a,g]$. The condition (2.2) – (2.4) above is equivalent to that

$$l_{[g_1,g_2]}=[l_{g_1},l_{g_2}] \tag{2.5}$$

$$r_{[g_1,g_2]}=[l_{g_1},r_{g_2}] \tag{2.6}$$

$$r_{g_2}\circ l_{g_1}=-r_{g_2}\circ r_{g_1} \tag{2.7}$$

where $[\ ,\ ]$ in the right-hand side of (2.5) and (2.6) denotes the commutator of operators.

The Leibniz cohomology of $\mathfrak{g}$ with coefficients in A is the homology of the cochain complex $C^k(\mathfrak{g};A)=\mathrm{Hom}_R(\otimes^k\mathfrak{g},A)$ $(k\geq 0)$ whose coboundary operator $\partial^k:C^k(\mathfrak{g};A)\to$

$C^{k+1}(\mathfrak{g}; A)$ is defined by

$$\partial^k c^k(g_1, \dots, g_{k+1})$$
$$= \sum_{i=1}^{k} (-1)^{i-1} g_i \left(c^k(g_1, \dots, \widehat{g_i}, \dots, g_{k+1})\right) + (-1)^k \left(c^k(g_1, \dots, g_k)\right) g_{k+1}$$
$$+ \sum_{1 \le i < j \le k+1} (-1)^i c^k(g_1, \dots, \widehat{g_i}, \dots, g_{j-1}, [g_i, g_j], g_{j+1}, \dots, g_{k+1}) \quad (2.8)$$

where $(g_1, \dots, g_{k+1})$ denotes $g_1 \otimes \cdots \otimes g_{k+1}$. The condition $\partial \circ \partial = 0$ is proved in [12].

When the left action agrees with the (-1) times of the right action, we get the following "natural" Leibniz cohomology:

Proposition 2.1. *Let $\mathfrak{g}$ be a Leibniz algebra and A a $\mathfrak{g}$-module with respect to the representation of $\mathfrak{g}$ on A, that is, A is endowed with a bilinear map $\mathfrak{g} \times A \to A$ such that $[g_1, g_2]a = g_1(g_2 a) - g_2(g_1 a)$. Then the operator $\partial^k : C^k(\mathfrak{g}; A) \to C^{k+1}(\mathfrak{g}; A)$ given by*

$$\partial^k c^k(g_1, \dots, g_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} g_i \left(c^k(g_1, \dots, \widehat{g_i}, \dots, g_{k+1})\right)$$
$$+ \sum_{1 \le i < j \le k+1} (-1)^i c^k(g_1, \dots, \widehat{g_i}, \dots, g_{j-1}, [g_i, g_j], g_{j+1}, \dots, g_{k+1}) \quad (2.9)$$

defines a Leibniz cohomology of $\mathfrak{g}$ with coefficients in A.

In most of the cases, we consider the Leibniz cohomology of this type, which is denoted by $HL^*(\mathfrak{g}; A)$. If $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra, we obtain the subcomplex of $(C^*(\mathfrak{g}; A), \partial)$ that consists of the skew-symmetric cochains. The cohomology of this subcomplex is just the usual cohomology $H^*_{\mathrm{Lie}}(\mathfrak{g}; A)$ of the Lie algebra $(\mathfrak{g}, [\ ,\ ])$ with coefficients in A. Thus there is a natural homomorphism

$$\iota : H^*_{\mathrm{Lie}}(\mathfrak{g}; A) \to HL^*(\mathfrak{g}; A).$$

The followings are several examples of Leibniz cohomology we have in mind.

Example 2.2. ([2, 4, 16]) Let (M, Π) be a Nambu-Poisson manifold of order p, that is, Π is a p-vector field satisfying

$$[\Pi(df_1, \dots, df_{p-1}), \Pi] = 0$$

for $f_1, \dots, f_{p-1} \in C^\infty(M)$, where $[\ ,\ ]$ denotes the Schouten bracket. It holds that $\bigwedge^{p-1} C^\infty(M)$ is a Leibniz algebra by the bracket $[\![\ ,\ ]\!]$ defined by

$$[\![f_1 \wedge \cdots \wedge f_{p-1}, g_1 \wedge \cdots \wedge g_{p-1}]\!] = \sum_{i=1}^{p-1} g_1 \wedge \cdots \wedge \Pi(df_1, \dots, df_{p-1}, dg_i) \wedge \cdots \wedge g_{p-1}$$

cohomology of the subcomplex of the skew-symmetric and $C^\infty(M)$-linear cochains.

Then the diagram

$$\begin{array}{ccc} H^*_{DR}(M) & \xrightarrow{\iota} & H^*_{GF}(\mathfrak{X}(M); C^\infty(M)) \\ \pi\circ\iota \big\downarrow & \swarrow \pi & \\ HL^*(\mathfrak{X}(M); C^\infty(M)) & & \end{array} \tag{2.13}$$

commutes where $H^*_{GF}(\mathfrak{X}(M); C^\infty(M))$ denotes the Gel'fand-Fuks cohomology. The map $\iota : H^*_{DR}(M) \to H^*_{GF}(\mathfrak{X}(M); C^\infty(M))$ is induced by the inclusion

$$\mathrm{Hom}^{\mathrm{cont}}_{C^\infty(M)}(\mathfrak{X}^k(M); C^\infty(M)) \to \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{R}}(\mathfrak{X}^k(M); C^\infty(M))$$

where $\mathfrak{X}^k(M)$ denotes the space of k-vector fields on M and $\pi : H^*_{GF}(\mathfrak{X}(M); C^\infty(M)) \to HL^*(\mathfrak{X}(M); C^\infty(M))$ is induced by the projection $\otimes^k\mathfrak{X}(M) \to \mathfrak{X}^k(M)$.

3 Leibniz algebras associated with foliations

The notion of Leibniz algebroid is introduced in [9] as a generalization of the Lie algebroid:

Definition 3.1. A Leibniz algebroid is a smooth vector bundle $\pi : A \to M$ with a Leibniz algebra structure $[\![\ ,\ ]\!]$ on $\Gamma(A)$ (the space of smooth sections of A) and a bundle map $\rho : A \to TM$, called an anchor, such that the induced map $\rho : \Gamma(A) \to$

$\mathfrak{X}(M)$ satisfies the following properties:

(1) (Leibniz algebra homomorphism)

$$\rho([\![x, y]\!]) = [\rho(x), \rho(y)]$$

(2) (derivation law)

$$[\![x, fy]\!] = (\rho(x)f)\,y + f[\![x, y]\!]$$

for all $x, y \in \Gamma(A)$ and $f \in C^\infty(M)$.

Example 3.2. If the bracket $[\![\ ,\ ]\!]$ is skew-symmetric, we recover the Lie algebroid.

Example 3.3. The bundle of $(p-1)$-forms $\bigwedge^{p-1} T^*M$ over a Nambu-Poisson manifold (M, Π) of order p is a Leibniz algebroid with the anchor map $\Pi : \bigwedge^{p-1} T^*M \to TM$ and the bracket either (2.10) ($p \geq 2$ which we assume Π is decomposable when $p = 2$) or (2.11) ($p \geq 2$).

Example 3.4. ([5]) There is a different generalization of Lie algebroid. A Filippov p-algebroid, or p-Lie algebroid, $(E, \pi, [\ ,\ldots,\ ])$ over a manifold M is a vector bundle E endowed with a p-Lie bracket $[\ ,\ldots,\ ]$ on $\Gamma(E)$, that is, the skew-symmetric bracket satisfying the Filippov (or Fundamental) identity

$$[a_1, \ldots, a_{p-1}, [b_1, \ldots, b_p]] = \sum_{i=1}^{p} [b_1, \ldots, [a_1, \ldots, a_{p-1}, b_i], \ldots, b_p]$$

for any $a_1, \dots, a_{p-1}, b_1, \dots, b_p \in \Gamma(E)$, and a bundle map $\pi : \bigwedge^{p-1} E \to TM$, called an anchor, such that the induced map $\pi : \Gamma(\bigwedge^{p-1} E) \to \mathfrak{X}(M)$ satisfies the following properties:

$$[\pi(a_1 \wedge \cdots \wedge a_{p-1}), \pi(b_1 \wedge \cdots \wedge b_{p-1})] = \sum_{i=1}^{p-1} \pi(b_1 \wedge \cdots \wedge [a_1, \dots, a_{p-1}, b_i] \wedge \cdots \wedge b_{p-1}),$$

$$[a_1, \dots, a_{p-1}, fb] = f[a_1, \dots, a_{p-1}, b] + (\pi(a_1 \wedge \cdots \wedge a_{p-1})f)b$$

for all $a_1, \dots, a_{p-1}, b_1, \dots, b_{p-1}, b \in \Gamma(E)$ and $f \in C^\infty(M)$. In this case, it is shown that $\bigwedge^{p-1} E$ is a Leibniz algebroid with the anchor π and the bracket

$$[\![a_1 \wedge \cdots \wedge a_{p-1}, b_1 \wedge \cdots \wedge b_{p-1}]\!] = \sum_{i=1}^{p-1} b_1 \wedge \cdots \wedge [a_1, \dots, a_{p-1}, b_i] \wedge \cdots \wedge b_{p-1}.$$

In the recent paper [20], it has been shown that any Nambu-Poisson manifold has an associated Filippov algebroid.

Let $\mathcal{F}$ be a transversely oriented foliation of codimension q on M. Then we deduce, by using a partition of unity, that there exists a transverse volume form ω on M such that ω is decomposable (that is, $\omega = \omega_1 \wedge \cdots \wedge \omega_q$ for some 1-forms $\omega_1, \dots, \omega_q$) and integrable ($d\omega = \gamma \wedge \omega$ for some 1-form γ). In this paper, we call a decomposable and integrable form ω on M simply an integrable form. We remark that ω needs not to be nonsingular. When ω is nonsingular, the transversely oriented foliation

$\mathcal{F}$ is recovered by $\omega_1 = \cdots = \omega_q = 0$ where $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. If ω is singular, it yields a foliation whose leaves are of codimension q where $\omega \neq 0$ and otherwise of dimension 0. Thus the equivalence class of an integrable form gives a foliation.

Now, we will prove that such a foliation given by an integrable q-form on a manifold M gives the Leibniz algebroid structure to the bundle of $(q+1)$-vectors.

Theorem 3.5. *Let M be an n-dimensional smooth manifold endowed with a decomposable and integrable q-form ω $(q < n)$. Then $\bigwedge^{q+1} TM$ becomes a Leibniz algebroid over M whose anchor is the interior product $\iota_\omega : \bigwedge^{q+1} TM \to TM$ and whose bracket is defined by*

$$[\![X, Y]\!]_\omega = [\iota_\omega X, Y] + (-1)^q \langle X | d\omega \rangle Y.$$

for any $X, Y \in \mathfrak{X}^{q+1}(M)$, where $[\ ,\ ]$ denotes the Schouten bracket, $\langle\ |\ \rangle$ the natural pairing and $\mathfrak{X}^{q+1}(M)$ the space of $(q+1)$-vector fields.

Proof. This Leibniz algebroid is essentially the same as that in Example 3.3 with the bracket (2.10) by the correspondence $\Pi = (-1)^{nq} \Phi(\omega)$ where Φ is an arbitrary co-volume field on M (that is, a dimensional multivector field). However, we will give a direct verification in the realm of multivector fields.

We abbreviate $[\![\ ,\ ]\!]_\omega$ to $[\![\ ,\ ]\!]$. It is easy to see $[\![X, fY]\!] = ((\iota_\omega X)f)Y + f[\![X, Y]\!]$.

Let us prove $\iota_\omega([\![X,Y]\!]) = [\iota_\omega X, \iota_\omega Y]$. Since ω is integrable, there is a 1-form γ such that $d\omega = \gamma \wedge \omega$. By the decomposability of ω we have $\omega(X(\omega)) = 0$. Thus

$$\iota_{X(\omega)} d\omega = (-1)^q \langle X | d\omega \rangle\, \omega. \tag{3.1}$$

Moreover,

$$\begin{aligned}(\mathcal{L}_{X(\omega)} Y)(\omega) &= \mathcal{L}_{X(\omega)}(Y(\omega)) - Y(\mathcal{L}_{X(\omega)}\omega) \\ &= [X(\omega), Y(\omega)] - (-1)^q \langle X | d\omega \rangle\, (Y(\omega)).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\iota_\omega [\![X,Y]\!] &= [X(\omega), Y](\omega) + (-1)^q \langle X | d\omega \rangle\, (Y(\omega)) \\ &= [\iota_\omega X, \iota_\omega Y]. \end{aligned} \tag{3.2}$$

Now we will see that the Leibniz identity holds. Let $X, Y, Z \in \mathfrak{X}^{q+1}(M)$. By (3.1),

$$d\iota_{X(\omega)} d\omega = \omega \wedge (d\langle X | d\omega \rangle) + (-1)^q \langle X | d\omega \rangle\, d\omega.$$

Thus we have

$$\begin{aligned}[\![X,Y]\!](d\omega) &= (\mathcal{L}_{X(\omega)} Y)(d\omega) + (-1)^q \langle X | d\omega \rangle \langle Y | d\omega \rangle \\ &= \mathcal{L}_{X(\omega)} \langle Y | d\omega \rangle - Y(\mathcal{L}_{X(\omega)} d\omega) + (-1)^q \langle X | d\omega \rangle \langle Y | d\omega \rangle \\ &= (X(\omega)) \langle Y | d\omega \rangle - Y(d\iota_{X(\omega)} d\omega) + (-1)^q \langle X | d\omega \rangle \langle Y | d\omega \rangle \\ &= (X(\omega)) \langle Y | d\omega \rangle - (Y(\omega)) \langle X | d\omega \rangle.\end{aligned}$$

Therefore, by (3.2),

$$[\![[\![X,Y]\!], Z]\!] = [[X(\omega), Y(\omega)], Z] + (-1)^q ((X(\omega)) \langle Y | d\omega \rangle - (Y(\omega)) \langle X | d\omega \rangle)\, Z.$$

Also using (3.2), we have

$$\begin{aligned}[\![X, [\![Y,Z]\!]]\!] &= [X(\omega), [\![Y,Z]\!]] + (-1)^q \langle X | d\omega \rangle\, [\![Y,Z]\!] \\ &= [X(\omega), [Y(\omega), Z] + (-1)^q \langle Y | d\omega \rangle\, Z] \\ &\quad + (-1)^q \langle X | d\omega \rangle\, ([Y(\omega), Z] + (-1)^q \langle Y | d\omega \rangle\, Z) \\ &= [X(\omega), [Y(\omega), Z]] \\ &\quad + (-1)^q ((X(\omega)) \langle Y | d\omega \rangle)\, Z + (-1)^q \langle Y | d\omega \rangle\, [X(\omega), Z] \\ &\quad + (-1)^q \langle X | d\omega \rangle\, [Y(\omega), Z] + \langle X | d\omega \rangle \langle Y | d\omega \rangle\, Z.\end{aligned}$$

In the same way, we have

$$
\begin{aligned}
[\![Y,[\![X,Z]\!]]\!] = {} & [Y(\omega),[X(\omega),Z]] \\
& + (-1)^q((Y(\omega))\langle X|d\omega\rangle)\,Z + (-1)^q\langle X|d\omega\rangle\,[Y(\omega),Z] \\
& + (-1)^q\langle Y|d\omega\rangle\,[X(\omega),Z] + \langle X|d\omega\rangle\langle Y|d\omega\rangle\,Z.
\end{aligned}
$$

Then the Leibniz identity

$$[\![X,[\![Y,Z]\!]]\!] = [\![[\![X,Y]\!],Z]\!] + [\![Y,[\![X,Z]\!]]\!]$$

is equivalent to

$$[X(\omega),[Y(\omega),Z]] = [[X(\omega),Y(\omega)],Z] + [Y(\omega),[X(\omega),Z]]$$

which is true since $[\mathcal{L}_{X(\omega)},\mathcal{L}_{Y(\omega)}] = \mathcal{L}_{[X(\omega),Y(\omega)]}$ holds. □

Corollary 3.6. (1) $(\mathfrak{X}^{q+1}(M),[\![\ ,\ ]\!])$ *is a Leibniz algebra where*

$$[\![X,Y]\!] = [\iota_\omega X, Y] + (-1)^q(X(d\omega))\,Y. \tag{3.3}$$

The interior product ι_ω is a Leibniz algebra homomorphism from $\mathfrak{X}^{q+1}(M)$ to the Lie algebra of vector fields $(\mathfrak{X}(M),[\ ,\ ])$. It also holds that $[\![\ker\iota_\omega, Y]\!] = 0$

and $[\![X,\ker\iota_\omega]\!] \in \ker\iota_\omega$ where $X,Y\in\mathfrak{X}^{q+1}(M)$.

(2) *For any non-zero function f, the multiplication by f induces an isomorphism from the Leibniz algebra $(\mathfrak{X}^{q+1}(M),[\![\ ,\ ]\!]_{f\omega})$ to $(\mathfrak{X}^{q+1}(M),[\![\ ,\ ]\!]_\omega)$. That is, the isomorphism class of Leibniz algebra structure is determined by the foliation.*

Proof. Since (1) is obvious, we will check (2). We have

$$
\begin{aligned}
[\![X,Y]\!]_{f\omega} &= f\,[X(\omega),Y] - X(\omega)\wedge Y(df) + (X(\omega\wedge df))\,Y + (-1)^q\langle X|fd\omega\rangle)\,Y \\
&= f\,[\![X,Y]\!]_\omega + (X(\omega)\wedge Y)(df).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
[\![fX,fY]\!]_\omega &= (fX(\omega\wedge df))\,Y + f\,[\![fX,Y]\!]_\omega \\
&= f^2\,[X(\omega),Y] - fX(\omega)\wedge Y(df) + (-1)^q f^2(X(d\omega))\,Y + f(X(\omega\wedge df))\,Y \\
&= f^2\,[\![X,Y]\!]_\omega + f\,(X(\omega)\wedge Y)(df).
\end{aligned}
$$

This is equal to $f\,[\![X,Y]\!]_{f\omega}$, and we obtain (2). □

In general, $(\mathfrak{X}^{q+1}(M),[\![\ ,\ ]\!])$ is not a Lie algebra unless $q=0$ or $q=n-2$.

Example 3.7. (1) The case $q=n-2$ corresponds to the Lie algebra associated with a Poisson manifold of rank 2 via the isomorphism by the volume.

(2) Consider the case $q = 0$. For any function f on M, the Lie bracket is given as

$$[X, Y]_f = f[X, Y] + (Xf)Y - (Yf)X$$

where $X, Y \in \mathfrak{X}(M)$. This corresponds to the Lie algebra associated with a Nambu-Poisson manifold coming from a volume form.

(3) Consider the case $q = n - 1$. Then the Leibniz bracket is given as

$$[\![f\Phi, g\Phi]\!]_\omega = (\, fZg - gZf + fg\langle Z|\gamma\rangle\,)\Phi$$

where Φ is a co-volume field, $f, g \in C^\infty(M)$, $d\omega = \gamma \wedge \omega$ and $Z = \Phi(\omega)$. Therefore, if ω is a closed $(n-1)$-form, $(\mathfrak{X}^n(M), [\![\ ,\ ]\!]_\omega)$ is a Lie algebra. This corresponds to $(C^\infty(M), [\ ,\ ]_Z)$ defined by an arbitrary vector field Z where

$$[f, g]_Z = fZg - gZf.$$

Sometimes, we have a Lie algebra as a Leibniz subalgebra. For example, let us consider $(\mathfrak{X}^2(\mathbb{R}^n), [\![\ ,\ ]\!]_\omega)$. By Corollary 3.6, it is a Leibniz algebra if ω is an integrable 1-form on $\mathbb{R}^n$. In the following by a constant bivector field we mean the bivector

field of the form

$$\sum_{i<j} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where $a_{ij} \in \mathbb{R}$.

Proposition 3.8. *Let f be a quadratic function on $\mathbb{R}^n$. In the Leibniz algebra $(\mathfrak{X}^2(\mathbb{R}^n), [\![\ ,\ ]\!]_{df})$, the subset of constant bivector fields $\mathfrak{X}^2_{\mathrm{const}}(\mathbb{R}^n)$ forms a Lie algebra.*

Proof. It follows from a direct computation. □

We can relate this Lie algebra to the Lie algebra of matrices; let (j, k) be the signature and l the nullity of any quadratic function f on $\mathbb{R}^n$. Denote by P_f the matrix $\mathrm{diag}(I_{j+k}, 0_l)$ where I_{j+k} is the unit matrix of size $j+k$ and 0_l is the zero matrix of size l, and by $so(j, k, l)$ the set of matrices in $gl(n)$ satisfying

$$I_{jkl}A + {}^t\!AI_{jkl} = 0$$

where $I_{jkl} = \mathrm{diag}(I_j, -I_k, I_l)$. Then,

Theorem 3.9. *$(\mathfrak{X}^2_{\mathrm{const}}(\mathbb{R}^n), [\![\ ,\ ]\!]_{df})$ is isomorphic to $(so(j, k, l), \{\ ,\ \}_{P_f})$ where $\{X, Y\}_{P_f} = XP_fY - YP_fX$ for any $X, Y \in so(j, k, l)$.*

Proof. It also follows from a direct computation. □

In case f is nondegenerate, $(\mathfrak{X}^2_{\mathrm{const}}(\mathbb{R}^n), [\![\ ,\ ]\!]_{df})$ is isomorphic to $(so(j,k), [\ ,\ ])$.

4 Central extensions of Leibniz algebras

Let us return to the Leibniz cohomology of a Leibniz algebra $\mathfrak{g}$. The condition (2.2) – (2.4) admits the case that the right action $r_g = 0$ for any $g \in \mathfrak{g}$. If this is the case, we get a different Leibniz cohomology from "natural" one given by Proposition 2.1. In this section, we assume the right action $r_g = 0$, and we use this kind of Leibniz cohomology since it is essential when we consider the extensions of Leibniz algebras.

Proposition 4.1. *Let $\mathfrak{g}$ be a Leibniz algebra and A a $\mathfrak{g}$-module with respect to the representation of $\mathfrak{g}$ on A, that is, A is endowed with a bilinear map $\mathfrak{g} \times A \to A$ such that $[g_1, g_2]a = g_1(g_2 a) - g_2(g_1 a)$. Then the operator $\delta^k : C^k(\mathfrak{g}; A) \to C^{k+1}(\mathfrak{g}; A)$ given by*

$$\begin{aligned}\delta^k c^k(g_1, \dots, g_{k+1}) &= \sum_{i=1}^{k} (-1)^{i-1} g_i \left(c^k(g_1, \dots, \widehat{g_i}, \dots, g_{k+1}) \right) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^i c^k(g_1, \dots, \widehat{g_i}, \dots, g_{j-1}, [g_i, g_j], g_{j+1}, \dots, g_{k+1})\end{aligned} \tag{4.1}$$

defines a Leibniz cohomology of $\mathfrak{g}$ with coefficients in A.

We denote this Leibniz cohomology by $H^*(\mathfrak{g}; A)$. Note that even though $\mathfrak{g}$ is a Lie algebra and c^k is skew-symmetric, c^{k+1} is not skew-symmetric in general.

Now, we will consider the central extensions of Leibniz algebras. A central extension $(\mathfrak{g}', [\![\ ,\ ]\!])$ of a Leibniz algebra $(\mathfrak{g}, [\ ,\ ])$ with a center A is a Leibniz algebra with a surjective homomorphism $\Pi : \mathfrak{g}' \to \mathfrak{g}$ whose kernel A is a center in the sense of $[\![A, \mathfrak{g}']\!] = 0$. This is equivalent to giving an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \mathfrak{g}' \xrightarrow{\Pi} \mathfrak{g} \longrightarrow 0$$

such that A is a center of $\mathfrak{g}'$.

The next theorem shows that an analog of the case of Lie algebra holds.

Theorem 4.2. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Leibniz algebra and A a $\mathfrak{g}$-module. Then an element of $H^2(\mathfrak{g}; A)$ determines an equivalence class of central extensions of $\mathfrak{g}$ with the center A. The action of $\mathfrak{g}$ on A is recovered by $g \cdot a = [\![s(g), a]\!]$ where $(\mathfrak{g}', [\![\ ,\ ]\!])$ is a central extension of $\mathfrak{g}$ and $s : \mathfrak{g} \to \mathfrak{g}'$ an arbitrary linear map satisfying $\Pi \circ s = \mathrm{id}_{\mathfrak{g}}$. Conversely, an equivalence class of central extensions of $\mathfrak{g}$ with the center A defines the action of $\mathfrak{g}$ on A by $g \cdot a = [\![s(g), a]\!]$ where s is as above, and determines an element of $H^2(\mathfrak{g}; A)$. That is, a central extension of a Leibniz algebra $\mathfrak{g}$ with a center A is in one-to-one correspondence to an element of $H^2(\mathfrak{g}; A)$ up to isomorphisms.*

Proof. Take an arbitrary "section" s. Then $S = s(\mathfrak{g})$ has a Leibniz bracket $[\ ,\ ]_s$ induced by s. We may write $\mathfrak{g}' = S \oplus A$. Thus it may be written $g_i' = s(g_i) + a_i$ for any $g_i' \in \mathfrak{g}'$ where $\Pi(g_i') = g_i \in \mathfrak{g}, a_i \in A$ and $i = 1, 2$. We deduce that the action of $\mathfrak{g}$ on A is independent to the choice of a section map s. It holds

$$[\![g_1', g_2']\!] = [\![s(g_1), s(g_2)]\!] + [\![s(g_1), a_2]\!],$$

and from $\Pi([\![g_1', g_2']\!]) = [s(g_1), s(g_2)]_s$ it follows

$$[\![s(g_1), s(g_2)]\!] = s[g_1, g_2] + \psi_s(g_1, g_2)$$

for some linear map $\psi_s : \mathfrak{g} \otimes \mathfrak{g} \to A$. It is shown that the Leibniz identity holds if and only if ψ_s is a 2-cocycle. Now, we will see that $[\psi_s] \in H^2(\mathfrak{g}; A)$ does not depend on the choice of s. Take a section $\tilde{s}$ and let $\tilde{a}_i = g_i' - \tilde{s}(g_i)$ $(i = 1, 2)$. Then we may define a 1-cochain $t : \mathfrak{g} \to A$ by $t(g) = \tilde{s} - s$. Since

$$s([g_1, g_2]) + [\![s(g_1), \tilde{a}_2]\!] + \psi_s(g_1, g_2) = \tilde{s}([g_1, g_2]) + [\![\tilde{s}(g_1), a_2]\!] + \psi_{\tilde{s}}(g_1, g_2),$$

we have

$$(\psi_{\tilde{s}} - \psi_s)(g_1, g_2) = g_1 \cdot t(g_2) - t([g_1, g_2]).$$

The right hand of the equation is just $\delta t(g_1, g_2)$, thus we deduce that $[\psi_s] \in H^2(\mathfrak{g}; A)$ does not depend on the choice of s. We denote this element simply by $[\psi]$.

Next we prove that by the equivalence class of extensions ψ is determined up to coboundaries. Suppose that $(\mathfrak{g}', [\![\ ,\ ]\!]), (\bar{\mathfrak{g}}', [\![\ ,\ ]\!]^-)$ are isomorphic central extensions of $\mathfrak{g}$, and $\psi_s, \bar{\psi}_{\bar{s}}$ are corresponding cocycles with respect to sections $s, \bar{s}$ respectively. We consider the commutating diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"'] \arrow[d, equal] & \mathfrak{g}' \arrow[r, "\Pi"'] \arrow[d, "f"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota^-"'] & \bar{\mathfrak{g}}' \arrow[r, "\Pi^-"'] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

where f is a Leibniz algebra isomorphism. We define 1-cochain $t : \mathfrak{g} \to A$ by $t = f \circ s - \bar{s}$. Then, from $\bar{\psi}_{f \circ s} = f \circ \psi_s$, $f|_A = 1$ and

$$f \circ s([g_1, g_2]) + [\![f \circ s(g_1), a_2]\!]^- + \bar{\psi}_{f \circ s}(g_1, g_2) = \bar{s}([g_1, g_2]) + [\![\bar{s}(g_1), a_2 + t(g_2)]\!]^- + \bar{\psi}_{\bar{s}}(g_1, g_2)$$

where $g_i' = s(g_i) + a_i$, it follows

$$(\psi_s - \bar{\psi}_{\bar{s}})(g_1, g_2) = g_1 \cdot t(g_2) - t([g_1, g_2]) = \delta t(g_1, g_2).$$

Hence we have $[\psi] = [\bar{\psi}]$. Conversely, it is not difficult to see that if corresponding cohomologies with two central extensions of $\mathfrak{g}$ are equal then they are isomorphic. $\square$

We remark that we cannot develop a Leibniz generalization of the abelian extension of a Lie algebra because $[\![[\![a_1, s(g_2)]\!], s(g_3)]\!] + [\![s(g_2), [\![a_1, s(g_3)]\!]]\!]$ does not vanish in general for $g_i' = s(g_i) + a_i$ $(i = 1, 2, 3)$.

As an example of a central extension, we have the Leibniz algebras associated with foliations. For any foliation $\mathcal{F}_\omega$ given by a q-form ω, we have shown that $(\mathfrak{X}^{q+1}(M), [\![\ ,\ ]\!]_\omega)$ is a Leibniz algebra. In fact, it follows from Corollary 3.6(1) that there is a central extension

$$0 \longrightarrow \ker \iota_\omega \xrightarrow{\ \iota\ } \mathfrak{X}^{q+1}(M) \xrightarrow{\ \iota_\omega\ } \mathfrak{X}_\omega(M) \longrightarrow 0.$$

where $\mathfrak{X}_\omega(M)$ denotes the image of ι_ω, which yields the foliation $\mathcal{F}_\omega$. We will calculate the 2-cocycle of this extension. When ω is nonsingular, that is, the given foliation is regular, we may take a section s by $s(X) = Z \wedge X$ where Z is an arbitrary q-vector field satisfying $\omega(Z) = 1$, and then ψ_s is given by

$$\psi_s(X, Y) = \mathcal{L}_X Z \wedge Y + \langle X|\gamma\rangle(Z \wedge Y)$$

where $d\omega = \gamma \wedge \omega$. Therefore, if a foliation is given by $\omega_i = 0$ for non-zero 1-forms

$\omega_1, \dots, \omega_q$,

$$\psi_s(X, Y) = \mathcal{L}_X(Z_1 \wedge \cdots \wedge Z_q) \wedge Y + \langle X| \sum_{i=1}^{q} \gamma_{ii}\rangle(Y \wedge Z_1 \wedge \cdots \wedge Z_q)$$

where $d\omega_i = \sum_{k=1}^{q} \gamma_{ik} \wedge \omega_k$ and $\omega_i(Z_j) = \delta_{ij}$. For $\omega' = f\omega$ where f is an arbitrary function, using a metric g we may take a section

$$s(X) = \frac{1}{|\omega'|^2} Z \wedge X$$

where $g(Z) = \omega'$, which is well-defined since both Z and an element of $\mathfrak{X}_{\omega'}(M)$ is divisible by $|\omega'|$. Using the metric g satisfying $|\omega|_g = 1$, the corresponding cocycle with s is given by

$$\psi_s(X, Y) = \mathcal{L}_X Z' \wedge Y' + \langle X|\gamma\rangle(Z' \wedge Y')$$

where Z' and X' denote $|f|^{-1}Z$ and $|f|^{-1}X$, respectively.

Conversely, by the theorem above, an arbitrary element of $H^2(\mathfrak{X}_\omega(M); \ker \iota_\omega)$ determines a Leibniz algebra structure on $\mathfrak{X}^{q+1}(M)$.

The following consideration gives us homomorphisms between Leibniz algebras.

Proposition 4.3. *Suppose that a p-form α and a q-form β which are both integrable are given, and that $\alpha \wedge \beta \neq 0$. Then $\alpha \wedge \beta$ is also integrable, and we get the exact* *sequence*

$$0 \longrightarrow \ker \iota_{\alpha\wedge\beta} \xrightarrow{\iota} \mathfrak{X}^{p+q+1}(M) \xrightarrow{\iota_{\alpha\wedge\beta}} \mathfrak{X}_{\alpha\wedge\beta}(M) \longrightarrow 0. \tag{4.2}$$

The following diagram of Leibniz algebra commutes where $\iota'_\beta = (-1)^{pq}\iota_\beta$ *and* $\mathfrak{X}^{q+1}_\alpha(M) \subset \mathfrak{X}^{q+1}(M)$ *is the image of the interior product* $\iota_\alpha : \mathfrak{X}^{p+q+1}(M) \to \mathfrak{X}^{q+1}(M)$.

$$\begin{tikzcd}
 & (\mathfrak{X}^{p+q+1}(M), [\![\ ,\ ]\!]_{\alpha\wedge\beta}) \arrow[dl, "\iota_\alpha"'] \arrow[dr, "\iota'_\beta"] \arrow[dd, "\iota_{\alpha\wedge\beta}"] & \\
(\mathfrak{X}^{q+1}_\alpha(M), [\![\ ,\ ]\!]_\alpha) \arrow[dr, "\iota_\beta"'] & & (\mathfrak{X}^{p+1}_\beta(M), [\![\ ,\ ]\!]_\beta) \arrow[dl, "\iota_\alpha"] \\
 & (\mathfrak{X}_{\alpha\wedge\beta}(M), [\ ,\ ]) &
\end{tikzcd}$$

Proof. It is easy to see that $\alpha\wedge\beta$ is an integrable $(p+q)$-form. Let us show the above diagram commutes. All the maps are well-defined since $\iota_\beta(\mathfrak{X}^{q+1}_\alpha(M)), \iota_\alpha(\mathfrak{X}^{p+1}_\beta(M)) \subset \mathfrak{X}_{\alpha\wedge\beta}(M)$. For any $X, Y \in \mathfrak{X}^{p+q+1}(M)$, we calculate

$$\begin{aligned}
[\iota_{\alpha\wedge\beta}X, Y](\alpha) &= \mathcal{L}_{X(\alpha\wedge\beta)}(Y(\alpha)) - Y\left(\mathcal{L}_{X(\alpha\wedge\beta)}\alpha\right) \\
&= [(\iota_\alpha X)(\beta), \iota_\alpha Y] - (-1)^{p+q}\langle X|d\alpha\wedge\beta\rangle\,(Y(\alpha)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iota_\alpha([\![X,Y]\!]_{\alpha\wedge\beta}) &= \left([\iota_{\alpha\wedge\beta}X, Y] + (-1)^{p+q}\langle X|d(\alpha\wedge\beta)\rangle\, Y\right)(\alpha)\\
&= [(\iota_\alpha X)(\beta), \iota_\alpha Y] + (-1)^q\langle X|\alpha\wedge d\beta\rangle\,(Y(\alpha))\\
&= [(\iota_\alpha X)(\beta), \iota_\alpha Y] + (-1)^q\langle \iota_\alpha X|d\beta\rangle\,(Y(\alpha))\\
&= [\![\iota_\alpha X, \iota_\alpha Y]\!]_\beta
\end{aligned}$$

and thus we conclude that $\iota_\alpha : \mathfrak{X}^{p+q+1}(M) \to \mathfrak{X}^{q+1}(M)$ is a Leibniz homomorphism; similarly for $\iota'_\beta : \mathfrak{X}^{p+q+1}(M) \to \mathfrak{X}^{p+1}(M)$. □

Example 4.4. We consider the Lie algebra $sl(2,\mathbb{R})$ with the basis

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then it holds that

$$[e_1, e_2] = e_2, \qquad [e_2, e_3] = 2e_1, \qquad [e_1, e_3] = -e_3.$$

Let us take the dual e_1^*, e_2^*, e_3^* of e_1, e_2, e_3. From $de_2^* = -e_1^* \wedge e_2^*$, it follows that $(\wedge^2 sl(2,\mathbb{R}), [\![\ ,\ ]\!]_{e_2^*})$ is a Leibniz algebra which is a central extension of the Lie algebra

$\mathfrak{g}$ where

$$\mathfrak{g} = \mathrm{span}(e_1, e_3), \quad [e_3, e_1] = e_3.$$

As we mentioned before, we may take a section $s(X) = e_2 \wedge X$. Then the corresponding cocycle $\psi \in H^2(\mathfrak{g}; \mathfrak{g}\wedge\mathfrak{g})$ is given by

$$\psi(X,Y) = [X, e_2] \wedge Y - \langle X|e_1^*\rangle\, e_2 \wedge Y$$

for any $X, Y \in \mathfrak{g}$. Since it follows that

$$\psi(e_1,e_1) = \psi(e_1,e_3) = \psi(e_3,e_1) = 0,\ \psi(e_3,e_3) = 2a$$

where $a = e_3 \wedge e_1 \in \mathfrak{g}\wedge\mathfrak{g}$, we may write $\psi = 2a\, e_3^* \otimes e_3^*$.

When we replace e_2^* with ce_2^* where c is a non-zero constant, which preserves the foliation, then by

$$[\![g_1, g_2]\!]_{ce_2^*} = c\,[\![g_1, g_2]\!]_{e_2^*}$$

we deduce that the cocycle ψ is replaced with $c^{-1}\psi$.

Now, let us elucidate all the central extension $(\wedge^2 sl(2,\mathbb{R}), [\![\ ,\ ]\!])$ of $\mathfrak{g}$. The action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$ is given by

$$e_1 \cdot a = -2a, \quad e_3 \cdot a = 0, \tag{4.3}$$

that is, $g \cdot a = 2\mathcal{L}_g a$, and any 1-cochain t is generated by

$$t_1 = a\, e_1^*, \quad t_3 = a\, e_3^*.$$

Since $\delta t(g_1, g_2) = g_1 \cdot t(g_2) - t([g_1, g_2])$, we have

$$\delta t_1(e_1, e_1) = -2a, \quad \delta t_1(e_1, e_3) = \delta t_1(e_3, e_1) = \delta t_1(e_3, e_3) = 0, \tag{4.4}$$

$$\delta t_3(e_1, e_3) = \delta t_3(e_3, e_1) = -a, \quad \delta t_1(e_1, e_3) = \delta t_1(e_3, e_3) = 0, \tag{4.5}$$

thus we may write $\delta t_1 = 2\kappa_{11}$ and $\delta t_3 = \kappa_{13} + \kappa_{31}$ where κ_{ij} denotes $-a\, e_i^* \otimes e_j^*$. A direct computation shows $\delta\kappa_{13}(e_1, e_3, e_3) \neq 0$, that is, κ_{13} is not a cocycle, thus we deduce that $H^2(\mathfrak{g}; \mathfrak{g} \wedge \mathfrak{g})$ is 1-dimensional and a cocycle $c\kappa_{33}$ where c is a constant determines a element in $H^2(\mathfrak{g}; \mathfrak{g} \wedge \mathfrak{g})$. The corresponding Leibniz algebra structure on $\wedge^2 sl(2, \mathbb{R})$ is then given by

$$\begin{aligned} [\![e_2 \wedge e_1, e_2 \wedge e_1]\!] &= 0, & [\![e_2 \wedge e_1, e_2 \wedge e_3]\!] &= -e_2 \wedge e_3, \\ [\![e_2 \wedge e_3, e_2 \wedge e_1]\!] &= e_2 \wedge e_3, & [\![e_2 \wedge e_3, e_2 \wedge e_3]\!] &= -c\, e_3 \wedge e_1 \end{aligned}$$

(the rest is given by (4.3) and $[\![\mathfrak{g} \wedge \mathfrak{g}, \wedge^2 sl(2, \mathbb{R})]\!] = 0$). Thus we have shown that, on any central extension of $\mathfrak{g}$ with the center $\mathfrak{g} \wedge \mathfrak{g}$, the Leibniz algebra structure is necessarily of this type.

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VARIATIONAL PROBLEMS OF NORMAL CURVATURE TENSOR AND CONCIRCULAR SCALAR FIELDS

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ABSTRACT. We consider the integral of (the square of) the length of the normal curvature tensor for immersions of manifolds into real space forms, especially into spheres. The first variation formula is given and the Euler-Lagrange equation is expressed in terms of the isothermal coordinates when the submanifold is 2-dimensional. The relations between the critical surfaces and Willmore surfaces are discussed. We also give formulas concerning the residue of logarithmic singularities of S-Willmore points or estimate it by a conformal invariant.

We show that if a compact critical surface satisfies certain conditions and the immersion is not totally umbilical, then the Gauss curvature is a non-negative constant and the immersion is minimal. To prove this result, We study 2-dimensional Riemannian manifolds admitting concircular scalar fields whose characteristic functions are polynomials of degree 2. Moreover, the case that the characteristic functions are polynomials of degree 3 is studied.

0. INTRODUCTION

In the 1960's, T. J. Willmore proposed to study the functional

$$\mathcal{L}[\phi] = \int_M (\eta^2 - K)\, dv$$

on the space of immersions $\phi : M \to \mathbb{R}^3$ of a compact orientable surface M into a Euclidean space $\mathbb{R}^3$, where η is the mean curvature of ϕ, K the Gauss curvature of the induced metric and dv the volume element. The functional $\mathcal{L}[\phi]$ is called Willmore functional and the critical surface is called a *Willmore surface.*

R. Bryant [4] studied Willmore surfaces in S^3 and contributed to the subject. He defined a conformal Gauss map of a surface M in S^3 into the de Sitter space of all oriented small spheres of S^3 and showed that M is a Willmore surface if and only if the conformal Gauss map is harmonic. Furthermore, he obtained a duality theorem for Willmore surfaces in S^3. N. Ejiri [10] introduced *S-Willmore surface* and generalized the Bryant's duality theorem to S-Willmore surfaces in S^n. He also proved that Willmore surfaces of genus 0 in $S^4(1)$ is a S-Willmore surface and classified them. Recently, F. Hélein [12] constructed a Weierstrass type representation of all Willmore immersions in terms of closed one-forms. In the studies mentioned above, the most important fact about Willmore surfaces is that $\mathcal{L}[\phi]$ is invariant under conformal transformations of the ambient space. The Willmore functional is

1991 *Mathematics Subject Classification.* 53c42.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 10640063), Ministry of Education, Science and Culture, Japan.

generalized to submanifolds in a Euclidean space or a sphere. One is Pinkall's conformal invariant ([17]) and the other is given in [18]. The generalized Willmore functional dealt with in this paper coincides with the latter. For a general presentation of the problem, see [23].

It is well known that, for a submanifold M^m in a space form, the normal curvature tensor $R^\perp \in C^\infty(\wedge^2 T^*M \otimes (T^\perp M)^* \otimes T^\perp M)$ is invariant under conformal transformations of the ambient space. Therefore the functional

$$\mathcal{R}^\perp[\phi] = \int_M \|R^\perp\|^q \, dv$$

on the space of immersions $\phi : M^m \to \tilde{M}(c)$ is also a conformal invariant if $q = m/2$. However, for the most part, we shall deal with the functional $\mathcal{R}^\perp[\phi]$ in the case that $q = 2$, because it is Yang-Mills integral of the normal bundle. We shall also deal with the case that $q = 1$ when M is a surface. We here note that the geometric meaning of $\mathcal{L}[\phi]$ and $\mathcal{R}^\perp[\phi]$ for surfaces is as follows: The integrand of $\mathcal{L}[\phi]$ is equal, up to a constant factor, to the sum of the square of lengths of major and minor axes of the curvature ellipse in the normal space at each point. On the other hand, the integrand $\|R^\perp\|^2$ of $\mathcal{R}^\perp[\phi]$ is equal to the square of the area encircled by the curvature ellipse up to a constant factor.

I. V. Guadalupe and L. Rodriguez [11] studied the integral of the normal curvature and obtained some inequalities relating the area of the surface and the integral of the square of the length of the mean curvature vector with topological invariants. Their integral of the normal curvature is different from ours. We should note that $\mathcal{R}^\perp[\phi]$ is the integral of the absolute value (or the square of the length) of the normal curvaure.

In §1, we give the fundamental formulas in the theory of submanifolds in a real space form. We also rewrite the corresponding formulas in terms of isothermal coordinates when the submanifold is two dimensional.

In §2, we obtain the first variation formulas of $\mathcal{L}[\phi]$ and $\mathcal{R}^\perp[\phi]$. The Euler-Lagrange equation of $\mathcal{L}[\phi]$ has already known as mentioned above. However the computation in this paper seems to be more brief than that of [19]. The Euler-Lagrange equation of $\mathcal{R}^\perp[\phi]$ is given in Theorem 2.7. The functional $\mathcal{R}^\perp[\phi]$ where $q = 2$ is a conformal invariant if the submanifold is of dimension 4 and is Yang-Mills integral. We shall prove in Theorem 2.8 that if $\phi : M^4 \to \tilde{M}(c)$ is an immersion of a 4-dimensional compact oriented manifold M^4 into an n-dimensional space form $\tilde{M}(c)$ and the normal connection is self-dual or anti-self-dual, then ϕ is a critical immersion of $\mathcal{R}^\perp[\phi]$. We should note that since $\mathcal{R}^\perp[\phi]$ is a functional defined on a space of immersions, the normal bundle and the induced metric vary with ϕ.

In §3, we reduce the Euler-Lagrange equation of $\mathcal{R}^\perp[\phi]$ to the situation that the submanifold is a surface. The result is given in Theorem 3.1.

In §4, we shall study critical surfaces of $\mathcal{L}[\phi]$ and $\mathcal{R}^\perp[\phi]$. We give formulas relating the sum of residues of logarithmic singularities of S-Willmore points in a compact oriented Willmore surface with conformal invariants. In particular, the conformal invariant appeared in the formula (4.8) is the Willmore integral. We conclude this section by showing Theorem 4.9 that is stated as follows:

Let $\phi : M^2 \to S^n(c)$ be an immersion of compact surface M^2 into $S^n(c)$. If ϕ is a critical immersion of $\mathcal{R}_2^\perp[\phi]$, the mean curvature vector is parallel and the curvature ellipses are circles, then the Gauss curvature is constant and the immersion is a standard minimal immersion of a sphere, a minimal immersion of a flat torus or a totally umbilical immersion.

To complete the proof of Theorem 4.9, we need §6 where we study concircular scalar fields.

In §5, we shall consider the equation satisfied by concircular scalar fields on a 2-dimensional manifold M as Euler-Lagrange equation of the functional

$$\mathcal{F}_J[g] = \int_M J(K) \, dv_g,$$

where J is a function on $\mathbb{R}$. Moreover we shall introduce Tashiro's work concerning concircular scalar fields.

In §6, by making use of elliptic functions, we classify complete 2-dimensional manifolds admitting concircular scalar fields whose characteristic functions are polynomials of the scalar field and of degree 2 or 3. The classification is given in Theorem 6.3. The proof of Theorem 4.9 is completed by using Theorem 6.4.

The author would like to express his hearty thanks to Prof. M. Okumura who taught him the results by Tashiro [21] explained in §5.

1. Submanifolds in a space form

Let $\phi : M \to \tilde{M}$ be an immersion of an m-dimensional C^∞-manifold M into an n-dimensional Riemannian manifold $\tilde{M}$. We shall denote the Riemannian metric on $\tilde{M}$ by $\tilde{g}$ and the induced metric M by g. Indices i, j, k, ℓ run over the range $\{1, \dots, m\}$, $\lambda, \mu, \nu, \kappa$ the range $\{1, \dots, n\}$ and u, v the range $\{m+1, \dots, n\}$. The differential $d\phi$ of the map ϕ can be regarded as a C^∞-section of the bundle $T^*M \otimes \phi^*T\tilde{M}$, namely $d\phi \in C^\infty(T^*M \otimes \phi^*T\tilde{M})$ and, in terms of local coordinates $\{x^1, \dots, x^m\}$ (resp. $\{y^1, \dots, y^n\}$) in M (resp. in $\tilde{M}$), it is represented as

$$d\phi = \frac{\partial \phi^\lambda}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\lambda}, \qquad (y^\lambda = \phi^\lambda(x^1, \dots, x^m)), \tag{1.1}$$

where we have used the so-called Einstein summantion convention. The induced metric g is given by

$$g(X, Y) = \tilde{g}((d\phi(X), d\phi(Y)) \tag{1.2}$$

for any vector fields X and Y tangent to M. Let $N : T^\perp M \to \phi^*T\tilde{M}$ be the inclusion map of the normal bundle $T^\perp M$ into $\phi^*T\tilde{M}$. Then it is regarded as a C^∞-section of $Hom(T^\perp M, \phi^*T\tilde{M})$. The connection on $\phi^*T\tilde{M}$ induced from the Levi-Civita connection on $\tilde{M}$ and the normal connection on $T^\perp M$ induce a connection ∇ on $Hom(T^\perp M, \phi^*T\tilde{M})$. Then Weingarten equation for ϕ becomes

$$\nabla N = -d\phi \circ A, \tag{1.3}$$

where $A \in C^\infty(T^*M \otimes TM \otimes (T^\perp M)^*)$ and, for a normal vector field ξ, $A_\xi \in C^\infty(T^*M \otimes TM)$ is the shape operator correspnding to ξ. The relation between A_ξ and the second fundamental form $h \in C^\infty(S^2T^*M \otimes T^\perp M)$ is

$$g(A_\xi X,\, Y) = \tilde{g}(h(X,\, Y),\, \xi) \tag{1.4}$$

for vector fields X and Y tangent to M. We shall put $H = N \circ h$, which belongs to $C^\infty(S^2T^*M,\, \phi^*T\tilde{M})$. Gauss equation is given by

$$\nabla d\phi = H, \tag{1.5}$$

∇ being the induced connection on the bundle $T^*M \otimes \phi^*T\tilde{M}$. Let $\tilde{M}$ be a space form $\tilde{M}(c)$ of constant sectional curvature c. Then the structure equations of Gauss, Codazzi, Ricci are given by

(1.6)

$$\begin{aligned} g(R(X,\, Y)Z,\, W) = {} & c\{g(X,\, W)g(Y,\, Z) - g(X,\, Z)g(Y,\, W)\} \\ & + \tilde{g}(H(X,\, W),\, H(Y,\, Z) - \tilde{g}(H(X,\, Z),\, H(Y,\, W)), \end{aligned}$$

$$(\nabla h)(X,\, Y,\, Z) = (\nabla h)(Y,\, X,\, Z), \tag{1.7}$$

$$R^\perp(X,\, Y) = h(X,\, A_\xi Y) - h(Y,\, A_\xi X) \tag{1.8}$$

for $X, Y, Z, W \in TM$ and $\xi \in T^\perp M$ (cf. [8]), where $R^\perp \in C^\infty(\wedge^2 T^*M \otimes (T^\perp M)^* \otimes T^\perp M)$ is the normal curvature tensor. We note that ∇ in (1.7) is the induced connection on $(T^*M)^2 \otimes T^\perp M$. In the sequel, we shall use the same notation ∇ for each connection induced on various vector bundle composed of $TM, T^\perp M$ and $\phi^*T\tilde{M}$ except for the 2-dimensional case and shall not state to which vector bundle various tensors belong. From (1.6), we have formulas for Ricci tensor Ric and scalar curvature ρ:

$$\begin{aligned} \mathrm{Ric}(X,\, Y) = {} & c(m-1)g(X,\, Y) + m\tilde{g}(h(X,\, Y),\, \eta) \\ & - \sum_i \tilde{g}(h(X,\, X_i),\, h(Y,\, X_i)) \end{aligned} \tag{1.9}$$

$$\rho = cm(m-1) + m^2\|\eta\|^2 - \|H\|^2, \tag{1.10}$$

where η is the mean curvature vector field defined by $\eta = (\sum_i h(X_i,\, X_i))/m$, $\{X_1, \ldots, X_m\}$ being an orthonormal frame tangent to M.

Next, we deal with oriented C^∞-surfaces differentiably immresed in a sphere $S^n(c) = \{p \in \mathbb{R}^{n+1} \mid \|p\| = 1/\sqrt{c}\}$. Using isothermal coordinates $z = x + iy$, we write the induced metric g as

$$g = 2F(z,\, \bar{z})\, |dz|^2, \tag{1.11}$$

where F is a positive valued C^∞-function. We note that F becomes real analytic if the immersion ϕ is minimal, has parallel mean curvature vector or make M to be a Willmore surface (cf. [10]). The area element is given by

$$\begin{aligned} dv &= 2F\, dx \wedge dy \\ &= iF\, dz \wedge d\bar{z}. \end{aligned} \tag{1.12}$$

For integers p and q, let $E^{p,q}$ be the complex line bundle over M whose elements are equivalence classes of $(U,\, z,\, P,\, w)$, where

(a) U is an open domain in M and $P \in U$,
(b) z is a local isothermal parameter defined in U and $w \in \mathbb{C}$,
(c) $(U,\, z,\, P,\, w) \sim (U',\, z',\, P',\, w')$ if and only if

(i) $P = P' \in \dot{U} \cap U'$
(ii) $w' = w\left(\frac{\partial z}{\partial z'}(P)\right)^p \overline{\left(\frac{\partial z}{\partial z'}(P)\right)}^q$

see [7] for details. We shall sometimes use the complex conjugation $E^{p,q} \to E^{q,p}$ in our computation. If $\alpha = \bar{\alpha}$ for $\alpha \in E^{p,q}$, then it is said to be real. For instance, $F(= F(z,\, \bar{z})dz \otimes d\bar{z})$ is in $E^{1,1}$ and real. The Gauss curvature K of g is given by

$$\begin{aligned} K &= -\frac{1}{F}\partial\bar{\partial}\log F \\ &= \frac{1}{2}\Delta \log F, \end{aligned} \tag{1.13}$$

where $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$ and $\Delta = -2F^{-1}\partial\bar{\partial}$. The metric g induces Levi-Civita connection ∇ on the bigraded algebra $E = \sum_{p,q} E^{p,q}$ with tensor product. The covariant differential operator ∇ splits into ∇' and ∇'', where ∇' (resp. ∇'') is a differential operator of bidegree (1,0) (resp. (0,1)). The operators ∇' and ∇'' are defined by

$$\begin{aligned} \nabla'\alpha &= (\partial\alpha(z,\, \bar{z}) - p\,\partial\log F \cdot \alpha(z,\, \bar{z}))(dz)^{p+1} \otimes (d\bar{z})^q, \\ \nabla''\alpha &= (\bar{\partial}\alpha(z,\, \bar{z}) - q\,\bar{\partial}\log F \cdot \alpha(z,\, \bar{z}))(dz)^p \otimes (d\bar{z})^{q+1} \end{aligned} \tag{1.14}$$

for $\alpha = \alpha(z,\, \bar{z})(dz)^p \otimes (d\bar{z})^q \in C^\infty(E^{p,q})$. In particular, we have $\nabla' F = 0 = \nabla'' F$. For the Ricci identity, we have

$$[\nabla',\, \nabla'']\,\alpha = (q-p)KF \otimes \alpha. \tag{1.15}$$

All higher order derivatives of ϕ will be considered as functions with values in $\mathbb{C}^{n+1} = \mathbb{R}^{n+1} \otimes_{\mathbb{R}} \mathbb{C}$. Let the symmetric product $\mathbf{a} = (a_1, \ldots, a_{n+1})$ and $\mathbf{b} = (b_1, \ldots, b_{n+1})$ in $\mathbb{C}^{n+1}$ be defined by

$$\langle \mathbf{a},\, \mathbf{b} \rangle = \sum_{h=1}^{n+1} a_h b_h. \tag{1.16}$$

Then the Hermitian product on $\mathbb{C}^{n+1}$ is given by $\langle \mathbf{a},\, \bar{\mathbf{b}} \rangle$. The norm of $\alpha \in E^{p,q} \otimes \mathbb{C}^{n+1}$ is defined as

$$|\alpha|^2 = F^{-(p+q)}\langle \alpha,\, \bar{\alpha} \rangle. \tag{1.17}$$

We immediately have

$$\langle \nabla'\phi,\, \nabla'\phi \rangle = 0, \quad \langle \nabla''\phi,\, \nabla''\phi \rangle = 0, \quad \langle \nabla'\phi,\, \nabla''\phi \rangle = F. \tag{1.18}$$

Let $x^1 = x$ and $x^2 = y$. We put $H_{ij} = H(\partial/\partial x^i,\, \partial/\partial x^j)$, where H is the second fundamental form of the immersion $\phi : M \to S^n(c)$. If we consider H_{ij} as a vector in $\mathbb{C}^{n+1}$, then we see that the Gauss equation (1.5) becomes

$$\nabla'^2\phi = \frac{1}{4}(H_{11} - H_{22} - 2iH_{12})(dz)^2. \tag{1.19}$$

The right hand side is a vector normal to M in $S^n(c)$, which we shall denote by γ. The mean curvature vector field η satisfies

$$\eta = c\phi - \frac{1}{2}\Delta\phi, \tag{1.20}$$

because $\Delta\phi = -(2/F)\nabla'\nabla''\phi$. Taking a local orthonormal cross sections $\{N_3, \ldots, N_n\}$ in $T^\perp M$ and regarding them as $\mathbb{C}^{n+1}$-valued functions, we have

$$(1.21) \qquad (\partial N_u + \eta^u \partial\phi + \frac{1}{F}\gamma^u \bar{\partial}\phi)dz = \omega_u^v N_v,$$

where $\eta = \eta^v N_v$, $\gamma = \gamma^v N_v$ and ω_u^v is the components of the normal connection extended to the complexification $\mathbb{C}T^\perp M$ of the normal bundle. For $\xi \in C^\infty(E^{p,q} \otimes \mathbb{C}T^\perp M)$, we may define the covariant differentiation of ξ by

$$(1.22) \qquad {}'\nabla^\perp \xi = ({}'\nabla \xi^u + \omega_v^u \xi^v) N_u, \qquad {}''\nabla^\perp \xi = ({}''\nabla \xi^u + \bar{\omega}_v^u \xi^v) N_u\,,$$

where $\xi = \xi^u N_u = \xi^u(z, \bar{z})(dz)^p \otimes (d\bar{z})^q N_u$. Then the Weingarten equation (1.3) becomes

$$(1.23) \qquad \nabla'\xi - {}'\nabla^\perp \xi = -\langle \xi, \eta \rangle \nabla'\phi - \frac{1}{F}\langle \xi, \gamma \rangle \nabla''\phi,$$

in virtue of (1.21). The structure equation (1.6)$\sim$ (1.8) of Gauss, Codazzi and Ricci are the following:

$$(1.24) \qquad K = c + \|\eta\|^2 - |\gamma|^2,$$

$$(1.25) \qquad {}''\nabla^\perp \gamma = F\, {}'\nabla^\perp \eta,$$

$$(1.26) \qquad \mathfrak{R}^\perp \xi = \frac{1}{F}\{\langle \xi, \bar{\gamma} \rangle \gamma - \langle \xi, \gamma \rangle \bar{\gamma}\}\,,$$

for $\xi \in C^\infty(E^{p,q} \otimes \mathbb{C}T^\perp M)$, where $\mathfrak{R}^\perp = \nabla'\bar{\omega} - \nabla''\omega + [\omega, \bar{\omega}] \in C^\infty(E^{1,1} \otimes \mathrm{Hom}(\mathbb{C}T^\perp M, \mathbb{C}T^\perp M))$. We note that the components of $\mathfrak{R}^\perp$ are given by

$$(1.27) \qquad \mathfrak{R}^\perp{}_u{}^v = \frac{i}{2} R^\perp{}_{12u}{}^v dz \otimes d\bar{z},$$

and hence it is a pure imaginary. We finally note that Ricci identity for $\xi \in C^\infty(E^{p,q} \otimes \mathbb{C}T^\perp M)$ is

$$(1.28) \qquad \left[{}'\nabla^\perp, {}''\nabla^\perp\right] \xi = (q-p)KF \otimes \xi + \mathfrak{R}^\perp \xi.$$

2. Variation of the length of normal curvature tensor

Let ϕ be an immersion of an oriented m-dimensional manifold M into an n-dimensional Riemannian manifold $\tilde{M}$.

Definition. By a compactly supported variation of ϕ, we mean a C^∞-map $\Phi : (-\epsilon, \epsilon) \times M \to \tilde{M}$, where ϵ is a positive real number, such that

(a) each map $\phi_t = \Phi(t, \cdot) : M \to \tilde{M}$ is an immersion and $\phi = \phi_0$,
(b) the closure of the set $\{p \in M \mid \phi_t(p) \neq \phi(p)$ for some $t \in (-\epsilon, \epsilon)\}$ is compact.

The variation vector V of Φ is a vector field along ϕ which is defined by $V = d\Phi(\partial/\partial t)\,|_{t=0}$. Thus, if we put $W = d\Phi(\partial/\partial t)$, then $W(0, p) = V(p)$ for every $p \in M$. We decompose W into tangential and normal parts:

$$(2.1) \qquad W = d\phi_t(T) + N_t\zeta,$$

where N_t is the inclusion map of the normal bundle $(T^\perp M)_t$ into $\phi_t^* T\tilde{M}$ with respect to the immersion ϕ_t. We note that T and ζ depend on t, but we do not put t on them. On $(-\epsilon, \epsilon) \times M$, we define an operator δ_t by

$$(2.2) \qquad \delta_t f = \frac{\partial f}{\partial t}, \qquad \delta_t \left(\frac{\partial}{\partial y^\lambda}\right)_\Phi = W^\mu \tilde{\Gamma}_\mu{}^\nu{}_\lambda \left(\frac{\partial}{\partial y^\nu}\right)_\Phi, \qquad \delta_t \frac{\partial}{\partial x^i} = 0$$

for every λ and i, where f is a C^∞-function on $(-\epsilon, \epsilon) \times M$, $\tilde{\Gamma}_\mu{}^\nu{}_\lambda$ is the Christoffel's symbols of the Levi-Civita connection of $\tilde{M}$ and $(\partial/\partial y^\lambda)_\Phi$ is the natural local frame in $\Phi^* T\tilde{M}$. We extend δ_t as a derivation to the tensor bundle $\sum T_s^r(M) \otimes \Phi^* T_q^p(\tilde{M})$.

Lemma 2.1. *Let g_t be the induced metric $\phi_t^*\tilde{g}$ on $\{t\} \times M$ for each $t \in (-\epsilon, \epsilon)$, $\mathfrak{L}_T$ denote the Lie derivative with respect to T and $(H_t)_\zeta$ be defined by $(H_t)_\zeta(X, Y) = \tilde{g}(H_t(X, Y), N_t\zeta)$ for $X, Y \in TM$, where H_t is the second fundamental form of the immersion ϕ_t. Then we have*

$$(2.3) \qquad \delta_t g_t = \mathfrak{L}_T g_t - 2\,(H_t)_\zeta\,.$$

Proof. We first note that

$$(2.4) \qquad \delta_t \tilde{g} = 0,$$

since

$$\begin{aligned}
\delta_t \left(\tilde{g}_{\lambda\mu} dy^\lambda \otimes dy^\mu\right) &= W^\nu \frac{\partial}{\partial y^\nu} \tilde{g}_{\lambda\mu} dy^\lambda \otimes dy^\mu \\
&\quad - \tilde{g}_{\lambda\mu} W^\kappa \tilde{\Gamma}_\kappa{}^\lambda{}_\nu dy^\nu \otimes dy^\mu - \tilde{g}_{\lambda\mu} W^\kappa \tilde{\Gamma}_\kappa{}^\mu{}_\nu dy^\lambda \otimes dy^\nu \\
&= W^\nu \nabla_\nu \tilde{g}_{\lambda\mu} dy^\lambda \otimes dy^\mu = 0.
\end{aligned}$$

It follows from (2.4) that

$$\begin{aligned}
(\delta_t g_t)(X, Y) &= \delta_t\left(g_t(X, Y)\right) = \delta_t\left(\tilde{g}\left(d\phi_t(X), d\phi_t(Y)\right)\right) \\
&= \tilde{g}\left((\delta_t d\phi_t)(X), d\phi_t(Y)\right) + \tilde{g}\left(d\phi_t(X), (\delta_t d\phi_t)(Y)\right)
\end{aligned}$$

for every $X, Y \in TM$. Since

$$\begin{aligned}
\delta_t d\phi_t &= \delta_t \left(\frac{\partial \Phi^\lambda}{\partial x^i} dx^i \otimes \left(\frac{\partial}{\partial y^\lambda}\right)_\Phi\right) \\
&= \left(\frac{\partial W^\lambda}{\partial x^i} + W^\nu \tilde{\Gamma}_\nu{}^\lambda{}_\mu \frac{\partial \Phi^\mu}{\partial x^i}\right) dx^i \otimes \left(\frac{\partial}{\partial y^\lambda}\right)_\Phi \\
&= \nabla W,
\end{aligned}$$

we have, from (1.5),

$$\begin{aligned}
(\delta_t g_t)(X, Y) &= \tilde{g}\left(\nabla_X W, d\phi_t(Y)\right) + \tilde{g}\left(d\phi_t(X), \nabla_Y W\right) \\
&= X \cdot \tilde{g}\left(W, d\phi_t(Y)\right) - \tilde{g}\left(W, \left(\nabla_X^t d\phi_t\right)(Y)\right) - \tilde{g}\left(W, d\phi_t\left(\nabla_X^t Y\right)\right) \\
&\quad + Y \cdot \tilde{g}\left(d\phi_t(X), W\right) - \tilde{g}\left(\left(\nabla_Y^t d\phi_t\right)(X), W\right) - \tilde{g}\left(d\phi_t\left(\nabla_Y^t X\right), W\right) \\
&= X \cdot g_t(T, Y) - g_t\left(T, \nabla_X^t Y\right) + Y \cdot g_t(X, T) - g_t\left(\nabla_Y^t X, T\right) \\
&\quad - \tilde{g}\left(N_t\zeta, H_t(X, Y)\right) - \tilde{g}\left(H_t(Y, X), N_t\zeta\right) \\
&= \left(\mathfrak{L}_T g_t\right)(X, Y) - 2\,(H_t)_\zeta\,(X, Y),
\end{aligned}$$

where ∇^t is the induced connection on the bundle $T^*M \otimes \phi_t^*\tilde{M}$ over $\{t\} \times M$. $\square$

Lemma 2.2. *Let g_t^{-1} be the inverse matrix of the metric g_t and dv_t be the volume form on M with respect to g_t. Then we have*

$$\delta_t g_t^{-1} = -g_t^{-1}\left\{\mathfrak{L}_T g_t - 2\left(H_t\right)_\zeta\right\} g_t^{-1}, \tag{2.5}$$

$$\delta_t dv_t = \{\mathrm{div}T - m\tilde{g}(\eta, \zeta)\}\, dv_t, \tag{2.6}$$

$\mathrm{div}T$ denoting the divergence of the vector field T.

Proof. Since

$$0 = \delta_t\left(g_t g_t^{-1}\right) = \left(\delta_t g_t\right) g_t^{-1} + g_t\left(\delta_t g_t^{-1}\right),$$

substituting (2.3) into the first term, we obtain (2.5). We denote the determinant of g_t by $\mathfrak{g}_t$ and (i, k)-cofactor by $\triangle_{ik}$. Then $\partial\sqrt{\mathfrak{g}_t}/\partial t = (\partial\mathfrak{g}_t/\partial t)/2\sqrt{\mathfrak{g}_t}$ and

$$\begin{aligned}
\partial\mathfrak{g}_t/\partial t &= \sum_{k=1}^{m}\left\{\frac{\partial(g_t)_{1k}}{\partial t}\triangle_{1k} + \frac{\partial(g_t)_{2k}}{\partial t}\triangle_{2k} + \cdots + \frac{\partial(g_t)_{mk}}{\partial t}\triangle_{mk}\right\} \\
&= \sum_k\sum_i \left(\delta_t g_t\right)_{ik}\left(g_t\right)^{ik}\mathfrak{g}_t \\
&= \mathfrak{g}_t\left(\mathfrak{L}_T g_t - 2\left(H_t\right)_\zeta\right)_{ik}\left(g_t\right)^{ik} \\
&= 2\left\{\mathrm{div}T - m\tilde{g}(\eta, \zeta)\right\}\mathfrak{g}_t.
\end{aligned}$$

Thus we have (2.6). $\square$

Lemma 2.3. *The variation $\delta_t H_t$ of the second fundamental form H_t is given by*

$$\begin{aligned}
\left(\delta_t H_t\right)(X, Y) = \left(\nabla^t\nabla W\right)(X, Y) &+ \tilde{R}\left(W, d\phi_t(X)\right) d\phi_t(Y) \\
&- d\phi_t\left(\left(\delta_t\Gamma\right)(X, Y)\right),
\end{aligned} \tag{2.7}$$

for every $X, Y \in TM$, where $\delta_t\Gamma = (\partial\Gamma_j{}^i{}_k(t))/\partial t)dx^j \otimes dx^k \otimes \partial/\partial x^i$, $\Gamma_j{}^i{}_k(t)$ being the Christoffel's symbols of the Levi-Civita connection of g_t.

Proof. From (1.5), we have

$$\begin{aligned}
\left(\delta_t H_t\right)(X, Y) &= \left(\delta_t\nabla^t d\phi_t\right)(X, Y) \\
&= \left(\nabla^t\delta_t d\phi_t\right)(X, Y) + \tilde{R}\left(V, d\phi_t(X)\right) d\phi_t(Y) \\
&\quad - d\phi_t\left(\left(\delta_t\Gamma\right)(X, Y)\right),
\end{aligned}$$

for $X, Y \in TM$, where we note that

$$\begin{aligned}
\nabla^t d\phi_t = &\left(\frac{\partial^2}{\partial x^i\partial x^j}\Phi^\lambda + \frac{\partial\Phi^\nu}{\partial x^i}\tilde{\Gamma}_\nu{}^\lambda{}_\mu\frac{\partial\Phi^\mu}{\partial x^j} - \Gamma_i{}^k{}_j(t)\frac{\partial\Phi^\lambda}{\partial x^k}\right) \\
&dx^i \otimes dx^j \otimes \left(\partial/\partial y^\lambda\right)_\Phi
\end{aligned}$$

and hence we need the last term $-d\phi_t((\phi_t\Gamma)(X, Y))$ in the Ricci formula for $[\delta_t, \nabla^t]d\phi_t$. $\square$

Remark. In later computation, we shall take an inner product of $\delta_t H_t$ with normal vectors and so we need not compute $\delta_t\Gamma$. Here, we only note that $\delta_t\Gamma$ is a tensor field on M.

We next compute the first and second terms of the right hand side of (2.7). Hereafter we assume that $\tilde{M}$ is a Riemannian manifold $\tilde{M}(c)$ of constant sectional curvature c.

Lemma 2.4. *We have*

$$\tilde{R}\left(W, d\phi_t(X)\right) d\phi_t(Y) \equiv c g_t(X, Y) N_t\zeta \qquad \mathrm{mod}\ d\phi_t(TM), \tag{2.8}$$

$$\begin{aligned}
&(\nabla^t\nabla W)(X, Y) \\
&\equiv N_t\{(\nabla^t h_t)(X, Y, T) + h_t(\nabla_X^t T, Y) \\
&\qquad + h_t(X, \nabla_Y^t T) + (\nabla^t\nabla^t\zeta)(X, Y) \\
&\qquad - h_t(X, A_\zeta^t Y)\} \qquad \mathrm{mod}\ d\phi_t(TM),
\end{aligned} \tag{2.9}$$

for every $X, Y \in TM$.

Proof. Since $\tilde{M} = \tilde{M}(c)$, we have

$$\begin{aligned}
\tilde{R}(W, d\phi_t(X))d\phi_t(Y) &= c\tilde{g}(d\phi_t(X), d\phi_t(Y))W - \tilde{g}(W, d\phi_t(Y))d\phi_t(X) \\
&\equiv c g_t(X, Y)N_t\zeta \qquad \mathrm{mod}\ d\phi_t(TM).
\end{aligned}$$

Equation (2.9) is proved as the following:

$$\begin{aligned}
\nabla_Y W &= \nabla_Y^t(d\phi_t(T) + N_t\zeta) \\
&= H_t(Y, T) + d\phi_t(\nabla_Y^t T) - d\phi_t(A_\zeta^t Y) + N_t\nabla_Y^t\zeta.
\end{aligned}$$

Therefore, if $X \in T_pM$ and Y is a vector field on M such that $\nabla_X Y = 0$ at p, then

$$\begin{aligned}
(\nabla^t\nabla W)(X, Y) &= \nabla_X^t\nabla_Y W \\
&= \nabla_X^t\{N_t h_t(Y, T) + d\phi_t(\nabla_Y^t T) - d\phi_t(A_\zeta^t Y) + N_t\nabla_Y^t\zeta\} \\
&\equiv N_t(\nabla^t h_t)(X, Y, T) + N_t h_t(Y, \nabla_X^t T) \\
&\quad + H_t(X, \nabla_Y^t T) - H_t(X, A_\zeta^t Y) + N_t\nabla_X^t\nabla_Y^t\zeta \\
&= N_t\{(\nabla^t h_t)(X, Y, T) + h_t(X, \nabla_Y^t T) + h_t(Y, \nabla_X^t T) \\
&\qquad - h_t(X, A_\zeta^t Y) + (\nabla^t\nabla^t\zeta)(X, Y)\} \qquad \mathrm{mod}\ d\phi_t(T_pM),
\end{aligned}$$

where A_ζ^t is the shape operator of ϕ_t with respect to ζ. $\square$

It follows from (2.7) $\sim$ (2.9) that

$$\begin{aligned}
&(\delta H)(X, Y) \\
&\equiv N\{(\nabla\nabla\zeta)(X, Y) + c g(X, Y)\zeta + (\nabla h)(T, X, Y) \\
&\qquad - h(X, A_\zeta Y) + h(\nabla_X T, Y) + h(X, \nabla_Y T)\} \\
&\qquad\qquad \mathrm{mod}\ d\phi(TM),
\end{aligned} \tag{2.10}$$

where we have put $\delta = \delta_t|_{t=0}$ and so on. For the mean curvature vector η, we have

$$\delta(N\eta) \equiv N\left\{\frac{1}{m}(-\Delta\zeta + S^\perp\zeta) + c\zeta + \nabla_T\eta\right\} \qquad \mathrm{mod}\ d\phi(TM), \tag{2.11}$$

$S^\perp$ being the symmetric transformation $T^\perp M \to T^\perp M$ defined by $\tilde{g}(S^\perp\xi, \xi') = \mathrm{trace}(A_\xi A_{\xi'})$. Here we take an orthonormal local frame field $\{N_u\}$ in $T^\perp M$. The equation (2.11) is proved as follows:

$$\begin{aligned}
\delta(N\eta) &= \frac{1}{m}\delta\left\{g^{ij}H_{ij}{}^\lambda\left(\frac{\partial}{\partial y^\lambda}\right)_\phi\right\}\\
&= \frac{1}{m}\left\{(\delta g^{-1})^{ij}H_{ij}{}^\lambda\left(\frac{\partial}{\partial y^\lambda}\right)_\phi + g^{ij}(\delta H)_{ij}{}^\lambda\left(\frac{\partial}{\partial y^\lambda}\right)_\phi\right\}\\
&\equiv \frac{1}{m}\{-(\nabla^i T^j + \nabla^j T^i) + 2h^{ij}{}_u\zeta^u\}H_{ij}{}^\lambda\left(\frac{\partial}{\partial y^\lambda}\right)_\phi\\
&\quad + \frac{1}{m}N(-\Delta\zeta - S^\perp\zeta + cm\zeta + m\nabla_T\eta)\\
&\quad + \frac{2}{m}(\nabla^i T^j)H_{ij}{}^\lambda\left(\frac{\partial}{\partial y^\lambda}\right)_\phi\\
&= \frac{1}{m}N(-\Delta\zeta + S^\perp\zeta) + cN\zeta + N\nabla_T\eta \qquad \mathrm{mod}\ d\phi(TM),
\end{aligned}$$

because of (2.5) and (2.10). Let $\{X_i\}_{i=1,\dots,m}$ be an orthonormal base in T_pM.

Lemma 2.5. *Let S be the symmetric transformation of TM defined by $g(SX, Y) = \sum_i \tilde{g}(H(X, X_i), H(Y, X_i))$. Then the variation of the length of the second fundamental form and the mean curvature vector are given by*

$$(2.12)\quad \begin{aligned}\delta\|H\|^2 &= 2\sum_{i,j}\tilde{g}(H(X_i, X_j), (\nabla\nabla\zeta)(X_i, X_j)) + 2\sum_i H_\zeta(SX_i, X_i)\\ &\quad + 2mc\tilde{g}(\eta, \zeta) + d\|H\|^2(T),\end{aligned}$$

$$(2.13)\quad \delta\|\eta\|^2 = -\frac{2}{m}\tilde{g}(\Delta\zeta, \eta) + \frac{2}{m}\tilde{g}(S^\perp\zeta, \eta) + 2c\tilde{g}(\zeta, \eta) + d\|\eta\|^2(T),$$

respectively.

Proof. Since

$$\|H\|^2 = H_{ij}{}^\lambda H_{k\ell}{}^\mu g^{ik}g^{j\ell}\tilde{g}_{\lambda\mu} \quad \text{and} \quad \|\eta\|^2 = \tilde{g}(N\eta, N\eta),$$

equations (2.12) and (2.13) are derived from (2.4), (2.5), (2.10) and (2.11) by a routine calculation. □

Next, we shall compute the variation of the length of the tensor field L, which we define by $L = h - \eta g$, and the normal curvature tensor $R^\perp$. We note that NL and $R^\perp$ are conformally invariant, that is, $N^*L^* = NL$ and $(R^\perp)^* = R^\perp$ under the change $\tilde{g}^* = e^f\tilde{g}$. Since

$$(2.14)\quad \|L\|^2 = \|H\|^2 - m\|\eta\|^2,$$

equations (2.12) and (2.13) imply that

(2.15)

$$\begin{aligned}\delta\|L\|^2 &= 2\sum_{i,j}\tilde{g}(L(X_i, X_j), (\nabla\nabla\zeta)(X_i, X_j)) + 2\sum_i\tilde{g}(h(SX_i, X_i), \zeta)\\ &\quad - 2\tilde{g}(S^\perp\eta, \zeta) + d\|L\|^2(T).\end{aligned}$$

Therefore we have the following result which was obtained in [18] and [19].

Theorem 2.6 ([18, 19, 23]). *The Euler-Lagrange equation of the conformally invariant functional*

$$\mathcal{L}[\phi] = \int_M \|L\|^m dv$$

is

(2.16)

$$\Box(\|L\|^{m-2}L) - \|L\|^{m-2}\Big\{(m-1)Q^\perp\eta - \sum_{i,j}\mathrm{Ric}(X_i, X_j)L(X_i, X_j)\Big\} = 0,$$

where $Q^\perp : T^\perp M \to T^\perp M$ is a symmetric transformation defined by $\tilde{g}(Q^\perp\xi, \xi') = \sum\tilde{g}(L(X_i, X_j), \xi)\tilde{g}(L(X_i, X_j), \xi')$ and $\Box B = -(\nabla^i\nabla^j B_{ij}{}^u)N_u$ for any section B of $T^*M \otimes T^*M \otimes T^\perp M$.

Proof. We see from (2.6) and (2.15) that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}[\phi_t]|_{t=0} &= \int \delta\|L\|^m dv\\
&= \int\Big\{\frac{m}{2}\|L\|^{m-2}\delta\|L\|^2 + \|L\|^m(\mathrm{div}T - m\tilde{g}(\eta, \zeta))\Big\}dv\\
&= m\int\Big\{-\tilde{g}(\Box(\|L\|^{m-2}L), \zeta)\\
&\quad + \|L\|^{m-2}\tilde{g}\Big(\sum h(SX_i, X_i) - S^\perp\eta - \|L\|^2\eta, \zeta\Big)\Big\}dv.
\end{aligned}$$

Since $Q^\perp\xi = S^\perp\xi - m\tilde{g}(\eta, \xi)\eta$ and the second term in the integrand is equal to

$$\|L\|^{m-2}\tilde{g}\Big(-\sum\mathrm{Ric}(X_i, X_j)L(X_i, X_j) + (m-1)Q^\perp\eta, \zeta\Big),$$

we get (2.16). □

Thus if the Ricci tensor is proportional to the metric tensor, then (2.16) reduces to

$$\Box(\|L\|^{m-2}L) - (m-1)\|L\|^{m-2}Q^\perp\eta = 0.$$

In particular, we have the following result obtained in [18, 22].

Corollary. *If $m = 2$, then* (2.16) *reduces to*

$$(2.17)\quad \Delta\eta - Q^\perp\eta = 0.$$

Proof. We have only to show $\Box L = (m-1)\Delta\eta$. We can easily show that by using Codazzi equation (1.7). □

Definition. Willmore surface is a surface satisfying (2.17) immersed in a space form.

Let us consider a variational problem for another conformal invariant $R^\perp$. We shall compute the Euler-Lagrange equation for the functional

$$\mathcal{R}^\perp[\phi] = \int \|R^\perp\|^q dv.$$

We note that if $q = m/2$, then $\mathcal{R}^{\perp}[\phi]$ is a conformal invariant. However we are also interested in the case $q = 2$ for any dimension m, because the right hand side of the definition of $\mathcal{R}^{\perp}[\phi]$ is a Yang-Mills integral. Here we explain the geometric meaning of $\|R^{\perp}\|$ in the case that $q = 1$ and $m = 2$ (cf. [11]). For arbitrarily fixed point $p \in M$, the curvature ellipse E_p at p is defined as the set $\{h(X, X) | X \in T_pM, \|X\| = 1\}$. This is an ellipse lying on the plane Π_p which pass through η and is spanned by normal vectors $\mathbf{a} = (h_{11} - h_{22})/4F$ and $\mathbf{b} = h_{12}/2F$ in the normal space $T_p{}^{\perp}M$. We easily see that $4|\gamma|^2 (= 2\|L\|^2)$ is equal to $4(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)$ and hence equal to the sum of the square of lengths of major and minor axes. The square of the area surrounded by E_p in Π_p is equal to $\pi^2(\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \langle \mathbf{a}, \mathbf{b}\rangle^2)$. It follows that it is equal to $\pi^2(|\gamma|^4 - |\langle\gamma, \gamma\rangle|^2)/4$ at p. Since $|\mathfrak{R}^{\perp}|^2 = F^{-2}\sum_{u,v} \mathfrak{R}^{\perp}{}_u{}^v \overline{\mathfrak{R}^{\perp}{}_u{}^v}$, we see from (1.26) that $(\text{area})^2 = \pi^2|\mathfrak{R}^{\perp}|^2/8 = \pi^2\|R^{\perp}\|^2/16$ at p.

Theorem 2.7. *Let $C \in C^{\infty}(T^*M \otimes T^*M \otimes T^{\perp}M)$ be defined by*

$$C(X, Y) = \sum_{i=1}^{m} R^{\perp}(Y, X_i)h(X, X_i).$$

Then the Euler-Lagrange equation of the functional

$$\mathcal{R}^{\perp}[\phi] = \int \|R^{\perp}\|^q dv$$

is given by

(2.18)
$$\square(\|R^{\perp}\|^{q-2}C) - \frac{1}{2}\|R^{\perp}\|^{q-2}\Big\{\sum_{i,j} P(X_i, X_j)h(X_i, X_j) - \frac{m}{2q}\|R^{\perp}\|^2\eta\Big\} = 0,$$

where P is defined by $P(X, Y) = -\sum \operatorname{trace}(R^{\perp}(X, X_i)R^{\perp}(Y, X_i))$. In particular, if $q = 2$, then (2.18) *becomes*

$$\square C - \frac{1}{2}\Big\{\sum_{i,j} P(X_i, X_j)h(X_i, X_j) - \frac{m}{4}\|R^{\perp}\|^2\eta\Big\} = 0. \tag{2.19}$$

Proof. Since

$$\begin{aligned}
&\frac{d}{dt}\mathcal{R}^{\perp}[\phi_t]|_{t=0} \\
&= \int \|R^{\perp}\|^{q-2}\left\{\frac{q}{2}\delta\|R^{\perp}\|^2 + \|R^{\perp}\|^2(\operatorname{div}T - m\tilde{g}(\eta, \zeta))\right\} dv,
\end{aligned}$$

we need to compute $\delta\|R^{\perp}\|^2$. Define D_t by

$$D_t = (H_t)_{ik}{}^{\mu}(H_t)_{j\ell}{}^{\lambda}(g_t)^{k\ell}dx^i \otimes dx^j \otimes \left(\frac{\partial}{\partial y^{\lambda}}\right)_{\Phi} \otimes \left(\frac{\partial}{\partial y^{\mu}}\right)_{\Phi}.$$

Then, from (1.8), we have

$$\|R_t{}^{\perp}\|^2 = \{(D_t)_{ij}{}^{\lambda\mu} - (D_t)_{ji}{}^{\lambda\mu}\}\{(D_t)_{k\ell}{}^{\nu\kappa} - (D_t)_{\ell k}{}^{\nu\kappa}\}(g_t)^{ik}(g_t)^{j\ell}\tilde{g}_{\lambda\nu}\tilde{g}_{\mu\kappa}.$$

Therefore

$$\begin{aligned}
\frac{1}{2}\delta\|R^{\perp}\|^2 &= \{(\delta D)_{ij}{}^{\lambda\mu} - (\delta D)_{ji}{}^{\lambda\mu}\}\{D_{k\ell}{}^{\nu\kappa} - D_{\ell k}{}^{\nu\kappa}\}g^{ik}g^{j\ell}\tilde{g}_{\lambda\nu}\tilde{g}_{\mu\kappa} \\
&\quad + (D_{ij}{}^{\lambda\mu} - D_{ji}{}^{\lambda\mu})(D_{k\ell}{}^{\nu\kappa} - D_{\ell k}{}^{\nu\kappa})(\delta g^{-1})^{ik}g^{j\ell}\tilde{g}_{\lambda\nu}\tilde{g}_{\mu\kappa}.
\end{aligned} \tag{2.20}$$

Next we compute δD:

$$\begin{aligned}
\delta D &= \delta\left(H_{ik}{}^{\mu}H_{j\ell}{}^{\lambda}g^{k\ell}dx^i \otimes dx^j \otimes \left(\frac{\partial}{\partial y^{\lambda}}\right)_{\phi} \otimes \left(\frac{\partial}{\partial y^{\mu}}\right)_{\phi}\right) \\
&= \{(\delta H)_{ik}{}^{\mu}H_{j\ell}{}^{\lambda}g^{k\ell} + H_{ik}{}^{\mu}(\delta H)_{j\ell}{}^{\lambda}g^{k\ell} + H_{ik}{}^{\mu}H_{j\ell}{}^{\lambda}(\delta g^{-1})^{k\ell}\} \\
&\qquad dx^i \otimes dx^j \otimes \left(\frac{\partial}{\partial y^{\lambda}}\right)_{\phi} \otimes \left(\frac{\partial}{\partial y^{\mu}}\right)_{\phi}.
\end{aligned}$$

Using (2.5) and (2.10), we obtain

$$\begin{aligned}
(\delta D)_{ij}{}^{\lambda\mu} \equiv\ & \nabla_i\nabla_k\zeta^v N_v{}^{\mu}H_j{}^{k\lambda} + \nabla_j\nabla_k\zeta^v N_v{}^{\lambda}H_i{}^{k\mu} \\
& + c(H_{ij}{}^{\lambda}N_v{}^{\mu}\zeta^v + H_{ij}{}^{\mu}N_v{}^{\lambda}\zeta^v) \\
& + T^{\ell}(\nabla_{\ell}h_{ik}{}^v N_v{}^{\mu}h_j{}^{k\lambda} + \nabla_{\ell}h_{jk}{}^v N_v{}^{\lambda}h_i{}^{k\mu}) \\
& + \nabla_i T^k D_{jk}{}^{\mu\lambda} + \nabla_j T^k D_{ik}{}^{\lambda\mu} \qquad \bmod (d\phi^{\lambda}, d\phi^{\mu}).
\end{aligned}$$

Substituting this result into the first term of the right hand side of (2.20) and putting $D_{ij}{}^{uv} = h_{ik}{}^v h_j{}^{ku}$, we have

$$\begin{aligned}
&\frac{1}{2}\delta\|R^{\perp}\|^2 \\
&= 4\nabla_i\nabla_k\zeta^v h_j{}^{ku}R^{\perp ij}{}_{uv} + 2P_{ij}h^{ij}{}_u\zeta^u \\
&\quad + 2(2T^k\nabla_i h_{k\ell}{}^v h_j{}^{\ell u} + 2\nabla_j T^k D_{ik}{}^{uv} - \nabla_i T^k R^{\perp}{}_{kj}{}^{uv})R^{\perp ij}{}_{uv}.
\end{aligned} \tag{2.21}$$

The third term of the right hand side of (2.21) is equal to

$$2T^k R^{\perp}{}_{iju}{}^v \nabla^i R^{\perp j}{}_{kv}{}^u.$$

The second Bianchi identity for $R^{\perp}$ implies that this is equal to $(\nabla_T\|R^{\perp}\|^2)/2$. It follows that

$$\delta\|R^{\perp}\|^2 = 8\nabla_i\nabla_k\zeta^v h_j{}^{ku}R^{\perp ij}{}_{uv} + 4P_{ij}h^{ij}{}_u\zeta^u + T^i\nabla_i\|R^{\perp}\|^2,$$

and hence

$$\begin{aligned}
\frac{d}{dt}\mathcal{R}^{\perp}[\phi_t]|_{t=0} = \int & \{4q\nabla_k\nabla_i(\|R^{\perp}\|^{q-2}h_j{}^{ku}R^{\perp ij}{}_{uv}) \\
& + 2q\|R^{\perp}\|^{q-2}P_{ij}h^{ij}{}_v - m\|R^{\perp}\|^q\eta_v\}\zeta^v dv
\end{aligned}$$

in virtue of Green's theorem. $\square$

When $m = 4$ and $q = 2$, $\mathcal{R}^{\perp}[\phi]$ is a conformal invariant and Yang-Mills integral in the vector bundle $T^{\perp}M$.

Theorem 2.8. *Let $\phi : M^4 \to \tilde{M}(c)$ be an immersion of a 4-dimensional compact oriented manifold M^4 into an n-dimensional space form $\tilde{M}(c)$. If the normal connection is self-dual or anti-self-dual, then ϕ is critical for the functional $\mathcal{R}^{\perp}[\phi]$.*

It follows that

$$\nabla^k C_{1k}{}^v = \frac{1}{F}\Big\{ -\frac{1}{F}\,{}'\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\bar{\gamma}^u) + \frac{1}{F}\,{}''\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\gamma^u) - {}'\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\eta^u) + {}''\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\eta^u)\Big\}, \tag{3.8}$$

$$\nabla^k C_{2k}{}^v = \frac{i}{F}\Big\{ \frac{1}{F}\,{}'\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\bar{\gamma}^u) + \frac{1}{F}\,{}''\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\gamma^u) - {}'\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\eta^u) - {}''\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\eta^u)\Big\}.$$

Moreover, we compute $\nabla^j\nabla^k C_{jk}{}^v$. Since

$$\begin{aligned}\nabla^j\nabla^k C_{jk}{}^v &= g^{j\ell}\nabla_\ell\nabla^k C_{jk}{}^v \\ &= g^{j\ell}(\partial_\ell\nabla^k C_{jk}{}^v - \Gamma_\ell{}^i{}_j\nabla^k C_{ik}{}^v + \Gamma^v_{\ell u}\nabla^k C_{jk}{}^u)\end{aligned}$$

and from (3.8)

$$\nabla^k C_{1k}{}^v + i\,\nabla^k C_{2k}{}^v = \frac{2}{F}\Big\{-\frac{1}{F}\,{}'\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\bar{\gamma}^u) + {}''\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\eta^u)\Big\},$$

we have

$$\begin{aligned}&\nabla^j\nabla^k C_{jk}{}^v \\ &= \frac{1}{2F}\{\partial(\nabla^k C_{1k}{}^v + i\,\nabla^k C_{2k}{}^v) + \bar{\partial}(\nabla^k C_{1k}{}^v - i\,\nabla^k C_{2k}{}^v) \\ &\qquad + \ell^v_u(\nabla^k C_{1k}{}^u + i\,\nabla^k C_{2k}{}^u) + \bar{\ell}^v_u(\nabla^k C_{1k}{}^u - i\,\nabla^k C_{2k}{}^u)\} \\ &= \frac{1}{F^3}\{{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp{}_u{}^v\gamma^u) + {}'\nabla^\perp\,{}'\nabla^\perp(\bar{\mathfrak{R}}^\perp{}_u{}^v\bar{\gamma}^u)\} + \frac{1}{F^2}\mathfrak{R}^\perp{}_w{}^v\mathfrak{R}^\perp{}_u{}^w\eta^u.\end{aligned}$$

We thus obtain

$$-\Box C = \frac{2}{F^3}\Re[{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma)] + \frac{1}{F^2}(\mathfrak{R}^\perp)^2\eta, \tag{3.9}$$

where $\Re[\ \ ]$ means the real part of $[\ \ ]$. The second term of (2.19) is computed as follows:

$$\begin{aligned}&\sum_{i,j} P(X_i, X_j)h(X_i, X_j) \\ &= R^\perp{}_{ikuv}R^\perp{}_j{}^{kuv}h^{ijw}N_w \\ &= \frac{1}{8F^3}(4F^2|\mathfrak{R}^\perp|^2h_{11}{}^w + 4F^2|\mathfrak{R}^\perp|^2h_{22}{}^w)N_w \\ &= 2|\mathfrak{R}^\perp|^2\eta.\end{aligned}$$

Here we note that $|\mathfrak{R}^\perp|^2 = F^{-2}\langle\mathfrak{R}^\perp, \bar{\mathfrak{R}}^\perp\rangle = \|R^\perp\|^2/2$. Therefore we have obtained

Theorem 3.1. *The Euler-Lagrange equation of the functional*

$$\mathcal{R}^\perp_2[\phi] = \int \|R^\perp\|^2 dv$$

for immersions $\phi : M^2 \to S^n(c)$ of an oriented surface M^2 is given by

$$\frac{2}{F^3}\Re[{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma)] + \frac{1}{F^2}(\mathfrak{R}^\perp)^2\eta + \frac{1}{2}|\mathfrak{R}^\perp|^2\eta = 0. \tag{3.10}$$

For the conformally invariant functional

$$\mathcal{R}^\perp_1[\phi] = \int \|R^\perp\| dv,$$

the Euler-Lagrange equation is given by

$$\Box(\|R^\perp\|^{-1}C) = 0. \tag{3.11}$$

Corollary. *If the normal curvature tensor is parallel, then the immersion ϕ is critical for the functional $\mathcal{R}^\perp_1[\phi]$. Moreover if ϕ is minimal, then ϕ is critical for the functional $\mathcal{R}^\perp_2[\phi]$.*

Proof. We immediately have

$$\begin{aligned}\nabla_1 R^\perp{}_{12u}{}^v &= -2i\,({}'\nabla^\perp\mathfrak{R}^\perp{}_u{}^v + {}''\nabla^\perp\mathfrak{R}^\perp{}_u{}^v), \\ \nabla_2 R^\perp{}_{21u}{}^v &= -2\ \ ({}'\nabla^\perp\mathfrak{R}^\perp{}_u{}^v - {}''\nabla^\perp\mathfrak{R}^\perp{}_u{}^v).\end{aligned}$$

It follows that the normal curvature tensor parallel if and only if ${}''\nabla^\perp\mathfrak{R}^\perp = 0$. From the assumption, we see that ${}''\nabla^\perp\mathfrak{R}^\perp = 0$ and so $|\mathfrak{R}^\perp|$ is constant. If $\mathfrak{R}^\perp = 0$, then it is trivial that ϕ is critical. Assume that $\mathfrak{R}^\perp \neq 0$. Then we see from (3.9) that (3.11) is equivalent to

$$\frac{2}{F}\Re[{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma)] + (\mathfrak{R}^\perp)^2\eta = 0. \tag{3.12}$$

Since

$$\begin{aligned}&{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma) + {}'\nabla^\perp\,{}'\nabla^\perp(\bar{\mathfrak{R}}^\perp\bar{\gamma}) \\ &= \mathfrak{R}^\perp({}''\nabla^\perp\,{}''\nabla^\perp\gamma - {}'\nabla^\perp\,{}'\nabla^\perp\bar{\gamma}) \\ &= F\mathfrak{R}^\perp({}''\nabla^\perp\,{}'\nabla^\perp\eta - {}'\nabla^\perp\,{}''\nabla^\perp\eta) \\ &= -F\mathfrak{R}^\perp\mathfrak{R}^\perp\eta,\end{aligned}$$

we have (3.12). $\Box$

The following proposition shows that (3.12) is equivalent to the defining equation (3.7) of Willmore surfaces under certain assumption.

Proposition 3.2. *We assume that $\|R^\perp\|$ is a non-zero constant and the curvature ellipse is a circle at every point. Then (3.7) is equivalent to (3.12).*

Proof. We first note that the curvature ellipse is a circle if and only if $\langle\gamma, \gamma\rangle = 0$. Thus from the assumption, we have $\mathfrak{R}^\perp\gamma = F|\gamma|^2\gamma$ and

$$\begin{aligned}|\mathfrak{R}^\perp|^2 &= F^{-2}\langle\mathfrak{R}^\perp, \bar{\mathfrak{R}}^\perp\rangle \\ &= F^{-4}\sum_{u,v}(\bar{\gamma}^u\gamma^v - \gamma^u\bar{\gamma}^v)(\gamma^u\bar{\gamma}^v - \bar{\gamma}^u\gamma^v) \\ &= 2(|\gamma|^4 - |\langle\gamma, \gamma\rangle|^2) \\ &= 2|\gamma|^4\end{aligned}$$

which is a non-zero constant. Assume that (3.7) holds. Then

$$\begin{aligned}{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma) &= F|\gamma|^2\,{}''\nabla^\perp\,{}''\nabla^\perp\gamma \\ &= -F|\gamma|^2\langle\eta, \bar{\gamma}\rangle\gamma.\end{aligned}$$

Therefore we have

$$\begin{aligned}&{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma)+{}'\nabla^\perp\,{}'\nabla^\perp(\bar{\mathfrak{R}}^\perp\bar\gamma)\\&=-F|\gamma|^2\{\langle\eta,\,\bar\gamma\rangle\gamma+\langle\eta,\,\gamma\rangle\bar\gamma\}.\end{aligned}$$

Since

$$\begin{aligned}(\mathfrak{R}^\perp)^2\eta&=F^{-1}\mathfrak{R}^\perp\{\langle\eta,\,\bar\gamma\rangle\gamma-\langle\eta,\,\gamma\rangle\bar\gamma\}\\&=\langle\eta,\,\bar\gamma\rangle|\gamma|^2\gamma+\langle\eta,\,\gamma\rangle|\gamma|^2\bar\gamma,\end{aligned}$$

we have (3.12). Conversely, assume that (3.12) holds. Then

$$\begin{aligned}0&=F^{-1}\{{}''\nabla^\perp\,{}''\nabla^\perp(\mathfrak{R}^\perp\gamma)+{}'\nabla^\perp\,{}'\nabla^\perp(\bar{\mathfrak{R}}^\perp\bar\gamma)\}+(\mathfrak{R}^\perp)^2\eta\\&=|\gamma|^2({}''\nabla^\perp\,{}''\nabla^\perp\gamma+{}'\nabla^\perp\,{}'\nabla^\perp\bar\gamma)+|\gamma|^2\{\langle\eta,\,\bar\gamma\rangle\gamma+\langle\eta,\,\gamma\rangle\bar\gamma\}.\end{aligned}$$

Since

$$\begin{aligned}{}''\nabla^\perp\,{}''\nabla^\perp\gamma-{}'\nabla^\perp\,{}'\nabla^\perp\bar\gamma&=F({}''\nabla^\perp\,{}'\nabla^\perp\eta-{}'\nabla^\perp\,{}''\nabla^\perp\eta)\\&=-F\mathfrak{R}^\perp\eta\\&=\langle\eta,\,\gamma\rangle\bar\gamma-\langle\eta,\,\bar\gamma\rangle\gamma,\end{aligned}$$

we obtain (3.7). □

4. Critical surfaces

Firstly, we shall study Willmore surfaces. Let $\phi: M\to S^n(c)$ be an isometric immersion of a compact oriented surface M into $S^n(c)$. Define $\Psi\in C^\infty(E^{3,0}\otimes\wedge^2\mathbb{C}T^\perp M)$ by $\Psi=\gamma\wedge{}'\nabla^\perp\eta$. The immersion $\phi: M\to S^n(c)$ is called a *S-Willmore surface* if $\gamma\wedge\bar\gamma\neq 0$ and $\Psi=0$ everywhere on M. It is known that S-Willmore surfaces are Willmore surfaces and there exist Willmore surfaces which are not S-Willmore surfaces ([10, 9]). In the following, we shall obtain an integral formula for the sum of residues of logarithmic singularities of $\log|\Psi|^2$. We note that the Willmore surface equation (3.6) and Codazzi equation (1.25) imply that Ψ is a holomorphic section of $E^{3,0}\otimes\wedge^2\mathbb{C}T^\perp M$, *i.e.*, ${}''\nabla^\perp\Psi=0$ and hence either Ψ is identically zero, or else the zeros of Ψ can be at most isolated. Define the symmetric product of two p-vectors $\xi=\xi_1\wedge\cdots\wedge\xi_p$ and $\zeta=\zeta_1\wedge\cdots\wedge\zeta_p$ in $\wedge\mathbb{C}T^\perp M$ by

$$\langle\xi,\,\zeta\rangle=\frac{1}{p!}\det\bigl(\langle\xi_A,\,\zeta_B\rangle\bigr)_{A,B=1,\dots,p}.\tag{4.1}$$

Then we have

Lemma 4.1. *Let $\phi: M\to S^n(c)$ be a compact oriented Willmore surface such that $\Psi\neq 0$ identically. Let Σ denote the set $\{p\in M|\,\Psi(p)=0\}$ and $2j_p$ the real analytic order of the zero of $|\Psi|^2$ at $p\in\Sigma$. Set $\mathcal{N}=\sum_{p\in\Sigma}j_p$. Then we have*

$$-2\pi\mathcal{N}=6\pi\chi(M)+\int\left\{\frac{2}{|\Psi|^2}(|\Psi\gamma|^2-|\Psi\bar\gamma|^2)+\frac{A}{|\Psi|^4}\right\}dv,\tag{4.2}$$

where $\Psi\gamma=\{\langle{}'\nabla^\perp\eta,\,\gamma\rangle\gamma-\langle\gamma,\,\gamma\rangle{}'\nabla^\perp\eta\}/2$, $\Psi\bar\gamma=\{\langle{}'\nabla^\perp\eta,\,\bar\gamma\rangle\gamma-\langle\gamma,\,\bar\gamma\rangle{}'\nabla^\perp\eta\}/2$ and $A=|{}'\nabla^\perp\Psi|^2|\Psi|^2-|\langle{}'\nabla^\perp\Psi,\,\bar\Psi\rangle|^2$.

Proof. On $M\backslash\Sigma$, we have

$$\begin{aligned}&F^{-1}\,\nabla'\,\nabla''\log|\Psi|^2\\&=\frac{F^{-4}}{|\Psi|^4}\{(\langle{}'\nabla^\perp\Psi,\,{}''\nabla^\perp\bar\Psi\rangle+\langle\Psi,\,{}'\nabla^\perp\,{}''\nabla^\perp\bar\Psi\rangle)|\Psi|^2\\&\qquad-F^{-3}\langle\Psi,\,{}''\nabla^\perp\bar\Psi\rangle\langle{}'\nabla^\perp\Psi,\,\bar\Psi\rangle\}\\&=\frac{1}{|\Psi|^2}F^{-4}\langle\Psi,\,{}'\nabla^\perp\,{}''\nabla^\perp\bar\Psi\rangle+\frac{A}{|\Psi|^4}.\end{aligned}$$

Since

$${}'\nabla^\perp\,{}''\nabla^\perp\bar\Psi=3FK\bar\Psi+\mathfrak{R}^\perp\bar\Psi-\bar\Psi\mathfrak{R}^\perp,$$

we obtain

$$\begin{aligned}&F^{-4}\langle\Psi,\,{}'\nabla^\perp\,{}''\nabla^\perp\bar\Psi\rangle\\&=3K|\Psi|^2+2F^{-4}\sum\mathfrak{R}^\perp{}_w{}^u\bar\Psi^{wv}\Psi^{uv}\\&=3K|\Psi|^2+2F^{-5}\sum(\bar\gamma^w\gamma^u-\gamma^w\bar\gamma^u)\bar\Psi^{wv}\Psi^{uv}\\&=3K|\Psi|^2+2(|\Psi\gamma|^2-|\Psi\bar\gamma|^2).\end{aligned}\tag{4.3}$$

The residue of the logarithmic singularities of $\log|\Psi|^2$ is given by

$$-2\pi\mathcal{N}=\lim_{\epsilon\to 0}\int_{\Sigma_\epsilon}\bigl(F^{-1}\,\nabla'\,\nabla''\log|\Psi|^2\bigr)\,dv,$$

where Σ_ϵ denotes the complement in M of an ϵ-neighborhood of all points of Σ. In virtue of the Gauss-Bonnet formula :

$$\int K\,dv=2\pi\chi(M),$$

we obtain (4.2). □

Theorem 4.2. *Let $\phi: M\to S^n(c)$ be a compact oriented Willmore surface. Assume that $\Psi\neq 0$ identically. Then we have*

$$4\pi(3\chi(M)+\mathcal{N})\leq\int\frac{|\mathfrak{R}^\perp|^2}{|\gamma|^2}dv.$$

The equality holds if and only if $\langle\Psi\gamma,\,\bar\gamma\rangle=0$ and ${}'\nabla^\perp\Psi$ is proportional to Ψ.

Proof. We compute the first term of the integrand of (4.2) on $M\backslash\Sigma$. Since

$$|\Psi\gamma\wedge\gamma|^2=\frac{1}{4}|\langle\gamma,\,\gamma\rangle|^2|\Psi|^2,$$

we have

$$|\Psi\gamma|^2|\gamma|^2=\frac{1}{2}|\langle\gamma,\,\gamma\rangle|^2|\Psi|^2+|\langle\Psi\gamma,\,\bar\gamma\rangle|^2.$$

We also have

$$
\begin{aligned}
|\Psi\bar\gamma|^2 &= \frac{1}{4}F^{-5}\langle \langle {}'\nabla^\perp\eta,\, \bar\gamma\rangle\gamma - \langle\gamma,\, \bar\gamma\rangle\, {}'\nabla^\perp\eta,\, \langle {}''\nabla^\perp\eta,\, \gamma\rangle\bar\gamma - \langle\gamma,\, \bar\gamma\rangle\, {}''\nabla^\perp\eta\rangle \\
&= \frac{1}{4}\{|\gamma|^4|{}'\nabla^\perp\eta|^2 - |\langle {}'\nabla^\perp\eta,\, \bar\gamma\rangle|^2|\gamma|^2\} \\
&= \frac{1}{2}|\gamma|^2|\Psi|^2.
\end{aligned}
$$

It follows that

$$
\frac{1}{|\Psi|^2}(|\Psi\gamma|^2 - |\Psi\bar\gamma|^2) = -\frac{|\mathfrak{R}^\perp|^2}{4|\gamma|^2} + \frac{|\langle\Psi\gamma,\, \bar\gamma\rangle|^2}{|\Psi|^2|\gamma|^2}, \tag{4.4}
$$

where we have used

$$
|\mathfrak{R}^\perp|^2 = 2(|\gamma|^4 - |\langle\gamma,\, \gamma\rangle|^2).
$$

Therefore we see from (4.4) and the non-negativity of A that

$$
\begin{aligned}
&F^{-1}\, \nabla'\, \nabla'' \log|\Psi|^2 \\
&= 3K - \frac{|\mathfrak{R}^\perp|^2}{2|\gamma|^2} + 2\frac{|\langle\Psi\gamma,\, \bar\gamma\rangle|^2}{|\Psi|^2|\gamma|^2} + \frac{A}{|\Psi|^4} \\
&\geq 3K - \frac{|\mathfrak{R}^\perp|^2}{2|\gamma|^2}.
\end{aligned} \tag{4.5}
$$

Integrating (4.5), we have the desired inequality. □

Surfaces with isotropic γ in $S^4(c)$ are S-Willmore surfaces, because γ and $\bar\gamma$ form an orthogonal basis of $\mathbb{C}T^\perp M$ and

$$
\begin{aligned}
F\langle {}'\nabla^\perp\eta,\, \gamma\rangle &= \langle {}''\nabla^\perp\gamma,\, \gamma\rangle \\
&= 0.
\end{aligned}
$$

This fact was proved in [10]. We also have

Theorem 4.3. *Let $\phi : M \to S^5(c)$ be a Willmore surface whose curvature ellipse is a circle everywhere. Then it is a S-Willmore surface.*

This is immediately derived from the following lemma.

Lemma 4.4. *Let $\phi : M \to S^6(c)$ be a Willmore surface whose curvature ellipse is a circle everywhere. If $\Psi \neq 0$ identically, then γ, $\bar\gamma$, ${}'\nabla^\perp\eta$ and ${}''\nabla^\perp\eta$ form an orthogonal basis of $\mathbb{C}T^\perp M$ on $M\backslash\Sigma$.*

Proof. Since $\langle\gamma,\, \gamma\rangle = 0$, we get

$$
\langle {}'\nabla^\perp,\, \gamma\rangle = 0, \qquad \langle {}'\nabla^\perp\gamma,\, \gamma\rangle = 0, \tag{4.6}
$$

and so, using (3.6),

$$
\begin{aligned}
0 &= \nabla''\langle {}'\nabla^\perp\eta,\, \gamma\rangle \\
&= \langle {}''\nabla^\perp\, {}'\nabla^\perp\eta,\, \gamma\rangle + \langle {}'\nabla^\perp\eta,\, F\, {}'\nabla^\perp\eta\rangle \\
&= F\langle {}'\nabla^\perp\eta,\, {}'\nabla^\perp\eta\rangle.
\end{aligned}
$$

Therefore we see that the subspace spanned by γ and ${}'\nabla^\perp\eta$ in $\mathbb{C}T^\perp M$ is isotropic. Thus γ, $\bar\gamma$, ${}'\nabla^\perp\eta$ and ${}''\nabla^\perp\eta$ form an orthogonal basis of $\mathbb{C}T^\perp M$ on $M\backslash\Sigma$. □

Theorem 4.5. *Let $\phi : M \to S^6(1)$ be a compact oriented Willmore surface. Assume that $\Psi \neq 0$ identically and the curvature ellipse is a circle everywhere. Then we have*

$$
-2\pi\mathcal{N} = 6\pi\chi(M) - \int|\gamma|^2 dv + \int\frac{|\alpha|^2}{|\Psi|^2}dv, \tag{4.7}
$$

where $\alpha = \langle {}'\nabla^\perp\, {}'\nabla^\perp\eta,\, \gamma\rangle = -\langle {}'\nabla^\perp\eta,\, {}'\nabla^\perp\gamma\rangle$. In particular, if M is a topological sphere, then we have

$$
\int|\gamma|^2 dv = 12\pi + 2\pi\mathcal{N}. \tag{4.8}
$$

Proof. By Lemma 4.4, we can set

$$
\begin{aligned}
{}'\nabla^\perp\, {}'\nabla^\perp\eta &= a\gamma + b\bar\gamma + c\, {}'\nabla^\perp\eta + d\, {}''\nabla^\perp\eta, \\
{}'\nabla^\perp\gamma &= a'\gamma + b'\bar\gamma + c'\, {}'\nabla^\perp\eta + d'\, {}''\nabla^\perp\eta.
\end{aligned}
$$

Taking the symmetric product of the both hand sides of the above equation and $\gamma({}'\nabla^\perp\eta)$, we obtain

$$
\begin{aligned}
F^2 b &= \frac{\alpha|{}'\nabla^\perp\eta|^2}{2|\Psi|^2}, & F^3 d &= \frac{-\alpha\langle {}'\nabla^\perp\eta,\, \bar\gamma\rangle}{2|\Psi|^2}, \\
F^3 b' &= \frac{\alpha\langle {}''\nabla^\perp\eta,\, \gamma\rangle}{2|\Psi|^2}, & F^3 d' &= \frac{-\alpha|\gamma|^2}{2|\Psi|^2}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
{}'\nabla^\perp\, {}'\nabla^\perp\eta\wedge\Psi &= \frac{\alpha F^{-3}}{2|\Psi|^2}\{F|{}'\nabla^\perp\eta|^2\bar\gamma\wedge\Psi - \langle {}'\nabla^\perp\eta,\, \bar\gamma\rangle\, {}''\nabla^\perp\eta\wedge\Psi\}, \\
{}'\nabla^\perp\gamma\wedge\Psi &= \frac{\alpha F^{-3}}{2|\Psi|^2}\{\langle {}''\nabla^\perp\eta,\, \gamma\rangle\bar\gamma\wedge\Psi - F^2|\gamma|^2\, {}''\nabla^\perp\eta\wedge\Psi\}.
\end{aligned} \tag{4.9}
$$

The following general formula for decomposable 2-vectors are easily proved:

$$
\begin{aligned}
&\langle p\wedge s_1,\, \bar q\wedge\bar t_1\rangle\langle p\wedge s_2,\, \bar q\wedge\bar t_2\rangle - \langle p\wedge s_1,\, \bar q\wedge\bar t_2\rangle\langle p\wedge s_2,\, \bar q\wedge\bar t_1\rangle \\
&= \frac{3}{2}\langle p,\, \bar q\rangle\langle p\wedge s_1\wedge s_2,\, \bar q\wedge\bar t_1\wedge\bar t_2\rangle.
\end{aligned} \tag{4.10}
$$

Using this formula, we compute A defined in Lemma 4.1. We have

$$
{}'\nabla^\perp\Psi = \theta\wedge\delta + \gamma\wedge\omega,
$$

where we have put $\theta = {}'\nabla^\perp\gamma$, $\delta = {}'\nabla^\perp\eta$ and $\omega = {}'\nabla^\perp\, {}'\nabla^\perp\eta$. Then A is computed as following:

$$
\begin{aligned}
A &= |\theta\wedge\delta + \gamma\wedge\omega|^2|\gamma\wedge\delta|^2 - |\langle\theta\wedge\delta + \gamma\wedge\omega,\, \bar\gamma\wedge\bar\delta\rangle|^2 \\
&= |\theta\wedge\delta|^2|\gamma\wedge\delta|^2 - |\langle\theta\wedge\delta,\, \bar\gamma\wedge\bar\delta\rangle|^2 \\
&\quad + |\gamma\wedge\omega|^2|\gamma\wedge\delta|^2 - |\langle\gamma\wedge\omega,\, \bar\gamma\wedge\bar\delta\rangle|^2 \\
&\quad + F^{-4}\langle\theta\wedge\delta,\, \bar\gamma\wedge\bar\omega\rangle|\gamma\wedge\delta|^2 - F^{-7}\langle\theta\wedge\delta,\, \bar\gamma\wedge\bar\delta\rangle\langle\bar\gamma\wedge\bar\omega,\, \gamma\wedge\delta\rangle \\
&\quad + F^{-4}\langle\gamma\wedge\omega,\, \bar\theta\wedge\bar\delta\rangle|\gamma\wedge\delta|^2 - F^{-7}\langle\gamma\wedge\omega,\, \bar\gamma\wedge\bar\delta\rangle\langle\bar\theta\wedge\bar\delta,\, \gamma\wedge\delta\rangle \\
&= \frac{3}{2}\{|\delta|^2|\delta\wedge\theta\wedge\gamma|^2 + |\gamma|^2|\gamma\wedge\omega\wedge\delta|^2 \\
&\qquad - 2\Re(F^{-7}\langle\bar\delta,\, \gamma\rangle\langle\omega\wedge\gamma\wedge\delta,\, \bar\theta\wedge\bar\gamma\wedge\bar\delta\rangle)\}.
\end{aligned}
$$

Thus we have

$$A = \frac{3}{2}\{|\gamma|^2|'\nabla^\perp\, '\nabla^\perp\eta\wedge\Psi|^2 + |'\nabla^\perp\eta|^2|'\nabla^\perp\gamma\wedge\Psi|^2 - 2\Re(F^{-7}\langle\gamma,\, ''\nabla^\perp\eta\rangle\langle'\nabla^\perp\, '\nabla^\perp\eta\wedge\Psi,\, ''\nabla^\perp\bar\gamma\wedge\bar\Psi\rangle)\}. \tag{4.11}$$

Substituting (4.9) into (4.11), we get $A = |\alpha|^2|\Psi|^2$. Since $\Psi\gamma = 0$, (4.2) reduces to (4.7).

Furthermore if M is a topological sphere, α vanishes. To prove this result, we have only to show that α is a holomorphic differential of degree 4. By (3.6), we obtain

$$\begin{aligned}
&\nabla''\langle'\nabla^\perp\eta,\, '\nabla^\perp\gamma\rangle\\
&= \langle''\nabla^\perp\, '\nabla^\perp\eta,\, '\nabla^\perp\gamma\rangle + \langle'\nabla^\perp\eta,\, ''\nabla^\perp\, '\nabla^\perp\gamma\rangle\\
&= -F^{-1}\langle\bar\gamma,\, \eta\rangle\langle\gamma,\, '\nabla^\perp\gamma\rangle + \langle'\nabla^\perp\eta,\, F\, '\nabla^\perp\, '\nabla^\perp\eta + 2KF\gamma - \Re^\perp\gamma\rangle\\
&= \frac{1}{2}F\nabla'\langle'\nabla^\perp\eta,\, '\nabla^\perp\eta\rangle\\
&= 0.
\end{aligned}$$

□

Secondly, we study surfaces satisfying (3.10). If the normal connection is flat, then (3.10) trivially holds. By the same proof as that of Proposition 3.2, we obtain

Lemma 4.6. *Under the assumption that the curvature ellipses are circles of constant radius on M, (3.10) is equivalent to*

$$''\nabla^\perp\, '\nabla^\perp\eta + F^{-1}\langle\eta,\, \bar\gamma\rangle\gamma + \frac{F}{2}|\gamma|^2\eta = 0. \tag{4.12}$$

An isometric immersion $\phi : M \to \tilde M$ is said to be *constant isotropic* if $\|H(X,\, X)\|^2$ is constant on the unit tangent bundle of M. In the case that M is a surface, we easily see that ϕ is constant isotropic if and only if it is pseudo-umbilical ($\langle\gamma,\, \eta\rangle = 0$), the curvature ellipses are circles ($\langle\gamma,\, \gamma\rangle = 0$) and $\|\eta\|^2 + |\gamma|^2/2$ is constant. In [20], we determined constant isotropic surfaces in $S^5(c)$. All of them were of constant Gauss curvature. In connection with this result, we state

Theorem 4.7. *Let $\phi : M \to S^n(c)$ be a pseudo-umbilical immersion of a surface M. If the curvature ellipses are circles of constant radius on M and ϕ satisfies (3.10), then K is of constant Gauss curvature.*

Proof. Since $\langle'\nabla^\perp\eta,\, \gamma\rangle = 0$, we have, from (4.12),

$$\begin{aligned}
0 &= \nabla''\langle'\nabla^\perp\eta,\, \gamma\rangle\\
&= \langle''\nabla^\perp\, '\nabla^\perp\eta,\, \gamma\rangle + \langle'\nabla^\perp\eta,\, F\, '\nabla^\perp\eta\rangle\\
&= -\frac{F}{2}|\gamma|^2\langle\eta,\, \gamma\rangle + F\langle'\nabla^\perp\eta,\, '\nabla^\perp\eta\rangle\\
&= F\langle'\nabla^\perp\eta,\, '\nabla^\perp\eta\rangle.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
0 &= \nabla''\langle'\nabla^\perp\eta,\, '\nabla^\perp\eta\rangle\\
&= -|\gamma|^2\langle\eta,\, '\nabla^\perp\eta\rangle\\
&= -\frac{1}{2}|\gamma|^2\nabla'\|\eta\|^2.
\end{aligned}$$

If $\gamma = 0$, then ϕ is totally umbilical. If $\|\eta\|^2$ is constant, then K is constant because of the Gauss equation (1.24). □

Remark. If $\phi \to S^n(c)$ is a minimal immersion of a surface M, then the assumption that ϕ is pseudo-umbilical and satisfies (3.10) is trivially satisfied. Minimal surfaces of constant Gauss curvature in $S^n(c)$ were determined in [5].

Lemma 4.8. *Assume that the mean curvature vector is parallel and (3.10) holds. If the normal curvature $\Re^\perp$ does not vanish identically, then the immersion ϕ is minimal.*

Proof. Take the symmetric product of the both hand sides of (3.10) and η. From the assumption, we have

$$2\Re(F^{-3}\nabla''\nabla''\langle\Re^\perp\gamma,\, \eta\rangle) + F^{-2}\langle(\Re^\perp)^2\eta,\, \eta\rangle + \frac{1}{2}|\Re^\perp|^2\|\eta\|^2 = 0.$$

Since

$$\begin{aligned}
\Re^\perp\eta &= '\nabla^\perp\, ''\nabla^\perp\eta - ''\nabla^\perp\, '\nabla^\perp\eta\\
&= 0,
\end{aligned}$$

we see that $\eta = 0$ on the open dense set $\Sigma' = \{p \in M | \Re^\perp(p) \neq 0\}$ and hence $\eta = 0$ on M. □

Theorem 4.9. *Let $\phi : M \to S^n(c)$ be an immersion of compact surface M. If ϕ satisfies (3.10), the mean curvature vector is parallel and the curvature ellipses are circles everywhere, then the Gauss curvature of M is constant and the immersion is a standard minimal immersion of a sphere, a minimal immersion of a flat torus (cf. [5, 15]) or a totally umbilical immersion.*

Proof. If $\Re^\perp \equiv 0$, then $\gamma = 0$ and so ϕ is totally umbilical. Assume that $\Re^\perp$ does not vanish identically. Then, by Lemma 4.8, we see that ϕ is minimal. Equation (3.10) reduces to

$$\Re[''\nabla^\perp\, ''\nabla^\perp(\Re^\perp\gamma)] = 0.$$

It follows that

$$(\nabla''\nabla''|\gamma|^2)\gamma + (\nabla'\nabla'|\gamma|^2)\bar\gamma = 0.$$

Vectors γ and $\bar\gamma$ are linearly independent on Σ'. Thus $\nabla'\nabla'|\gamma|^2 = 0$ on Σ' and hence on M, which implies that $\nabla'\nabla'K = 0$ on M in virtue of the Gauss equation (1.24). In the subsequent sections, we study a 2-dimensional Riemannian manifold $(M,\, g)$ which admits a function satisfying $\nabla\nabla f = \tau g$. In the section 6, we prove that if the Gauss curvature K of a compact 2-dimensional Riemannian manifold satisfies $\nabla'\nabla'K = 0$, then K is constant. From this and the result of [5], we have the assertion. □

Remark. For any standard minimal immersion of a sphere, the curvature ellipses are circles. On the other hand, there are minimal immersions of flat tori such that curvature ellipses are not circles.

5. Equation $\nabla\nabla f = \tau g$

In the proof of Theorem 4.9, we used the result that a compact surface whose Gauss curvature satisfies $\nabla'\nabla' K = 0$ is only of constant curvature. The equation $\nabla'\nabla' K = 0$ can be rewritten as a tensor equation $\nabla\nabla K = \tau g$, τ being a C^∞ function on M. In the present and next sections, we shall study a complete two-dimensional Riemannian manifold M which admits a function f satisfying $\nabla\nabla f = \tau g$ (cf. [3, 14, 16, 21]).

Let M be a two-dimensional C^∞ manifold. We assume that M is compact and orientable. Let $\mathcal{M}$ denote the set of Riemannian metrics on M. Furthermore let $\mathcal{M}_1$ denote the subset $\{g \in \mathcal{M} | \int_M dv_g = 1\}$ and $\mathcal{M}_2$ the subset $\{g \in \mathcal{M}_1 | dv_g = \mu\}$, where μ is a positive density on M with total volume $\int_M \mu = 1$ (cf. [2]). In the compact open C^∞ topology, $\mathcal{M}$ is an open convex cone in the set $C^\infty(S^2T^*M)$ of all C^∞ sections of S^2T^*M. We consider the following functional $\mathcal{F}_J$ from $\mathcal{M}$ to $\mathbb{R}$:

$$\mathcal{F}_J[g] = \int_M J(K)\, dv_g,$$

where $J = J(x)$ is a function defined on $\mathbb{R}$, $J(K)$ the composition $J \circ K$ and dv_g the area element of $g \in \mathcal{M}$. The Euler-Lagrange equation is given by

$$\nabla\nabla \dot{J}(K) + \left\{\Delta \dot{J}(K) - K\dot{J}(K) + J(K)\right\} g = 0 \tag{5.1}$$

for a critical point $g \in \mathcal{M}$, where ∇ denotes the covariant derivative with respect to g, the Laplace operator is defined by $\Delta = -g^{ij}\nabla_i\nabla_j$ and $\dot{J}(K)$ the composite $\dot{J} \circ K$. The equation (5.1) for the case $J(x) = x^2$ is well-known (cf. [2], Chapter 4). However, to complete this paper, we give the proof in the following.

Let $g(t)$ be a smooth curve $(-\epsilon, \epsilon) \to \mathcal{M}$ such that $g(0) = g$. We compute $\mathcal{F}'_J[g] := (d/dt\mathcal{F}_J[g(t)])(0)$. Since

$$\mathcal{F}'_J[g] = \int_M \dot{J}(K)\frac{\partial K}{\partial t}(0)\, dv_g + \int_M J(K)\,(dv_{g(t)})'(0),$$

we have to compute $\frac{\partial K}{\partial t}(0)$ and $(dv_{g(t)})'(0)$. Let $k \in T_g\mathcal{M}$ be defined by $k = g'(0)$. Then it is easy to show that

$$\left(\frac{\partial}{\partial t} g^{ij}\right)(0) = -k^{ij}$$

and hence

$$\mathfrak{g}'(0) = (\mathrm{tr}\, k)\mathfrak{g},$$

where $\mathfrak{g}(t) = \det(g_{ij}(t))$, $\mathfrak{g} = \mathfrak{g}(0)$ and $\mathrm{tr}\, k = k_{ij}g^{ij}$. Therefore we have

$$\begin{aligned}(dv_{g(t)})'(0) &= \frac{\mathfrak{g}'(0)}{2\sqrt{\mathfrak{g}}}\, dx^1 \wedge dx^2 \\ &= \frac{1}{2}\mathrm{tr}\, k\, dv_g.\end{aligned}$$

The derivative $(\frac{\partial}{\partial t}\Gamma_j{}^i{}_k)(0)$ of the coefficients of Riemannian connection ∇ is given by

$$\left(\frac{\partial}{\partial t}\Gamma_j{}^i{}_k\right)(0) = \frac{1}{2}g^{ip}\left(\nabla_j k_{pk} + \nabla_k k_{jp} - \nabla_p k_{jk}\right).$$

Using this equation in the derivation of the Rimannian curvature tensor $R_{ijk}{}^l$:

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{K\left(g_{jk}\delta^l_i - g_{ik}\delta^l_j\right)\right\} \\ &= \left(\frac{\partial}{\partial t}R_{ijk}{}^l\right)(0) \\ &= \left\{\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i}\Gamma_j{}^l{}_k - \frac{\partial}{\partial x^j}\Gamma_i{}^l{}_k + \Gamma_i{}^l{}_p\Gamma_j{}^p{}_k - \Gamma_j{}^l{}_p\Gamma_i{}^p{}_k\right)\right\}(0),\end{aligned}$$

we get

$$2\left(\frac{\partial}{\partial t}K\right)(0) = \nabla_i\nabla_j k^{ij} + \Delta \mathrm{tr}\, k - K\mathrm{tr}\, k.$$

By integration by parts, we have

$$\begin{aligned}2\mathcal{F}'_J[g] &= \int_M \left\{\dot{J}(K)(\nabla_i\nabla_j k^{ij} + \Delta\mathrm{tr}\, k - K\mathrm{tr}\, k) + J(K)\mathrm{tr}\, k\right\} dv_g \\ &= \int_M \left[\nabla_i\nabla_j\dot{J}(K) + \{\Delta\dot{J}(K) - K\dot{J}(K) + J(K)\}g_{ij}\right] k^{ij}\, dv_g.\end{aligned} \tag{5.2}$$

The equation (5.1) is the necessary and sufficient condition for that $\mathcal{F}'_J[g] = 0$ for arbitrary $k \in T_g\mathcal{M}$. The equation (5.1) implies that

$$\Delta\dot{J}(K) = 2\left\{K\dot{J}(K) - J(K)\right\}$$

and hence is rewritten as

$$\nabla\nabla\dot{J}(K) = \left\{J(K) - K\dot{J}(K)\right\} g. \tag{5.3}$$

We now introduce C^∞-functions on M

$$f = \dot{J}(K), \qquad \tau = J(K) - K\dot{J}(K).$$

Then (5.3) becomes

$$\nabla\nabla f = \tau g, \tag{5.4}$$

which shows that f is a *concircular scalar field* on M (cf. [21]). Recall that $T_g\mathcal{M}_1 = \{k \in T_g\mathcal{M} | \int \mathrm{tr} k\, dv_g = 0\}$ and $T_g\mathcal{M}_2 = \{k \in T_g\mathcal{M} | \mathrm{tr} k = 0\}$. Thus g is a critical point of $\mathcal{F}_J|_{\mathcal{M}_1}$ (resp. $\mathcal{F}_J|_{\mathcal{M}_2}$) if and only if the orthogonal projection of the left hand side of (5.1) onto $T_g\mathcal{M}_1$ (resp. $T_g\mathcal{M}_2$) is zero. Thus if g is a critical point for the functionals $\mathcal{F}_J, \mathcal{F}_J|_{\mathcal{M}_1}$ or $\mathcal{F}_J|_{\mathcal{M}_2}$, then we have a concircular scalar field on M. The function τ, called the *characteristic function* of f, can be considered as a function $\tau(f)$ of f if $\dot{J}$ is strictly monotone, i.e., $\ddot{J} \neq 0$ anywhere on $\mathbb{R}$. In fact, we have $\tau = -\Delta f/2$ and covariantly differentiating the both hand sides of (5.4) and using the Ricci identity, we obtain

$$\nabla\Delta f = 2K\nabla f.$$

We finally start from the assumption that K satisfies (5.8). Substituting (5.8) into the Ricci identity :

$$\nabla_h\nabla_i\nabla_j K = \nabla_i\nabla_h\nabla_j K - K(\delta^p_h g_{ij} - \delta^p_i g_{hj})\nabla_p K,$$

we easily obtain

$$\begin{aligned}&\dot\varphi(K)\nabla_h K\, g_{ij} + \dot\psi(K)\nabla_h K\,\nabla_i K\,\nabla_j K + \psi(K)\nabla_i K\,\nabla_h\nabla_j K\\ &= \dot\varphi(K)\nabla_i K\, g_{hj} + \dot\psi(K)\nabla_i K\,\nabla_h K\,\nabla_j K + \psi(K)\nabla_h K\,\nabla_i\nabla_j K\\ &\quad - K(\delta^p_h g_{ij} - \delta^p_i g_{hj})\nabla_p K,\end{aligned}$$

where $\dot{}$ denotes the differentiation with respect to K. Transvecting with g^{hj} and using (5.8) again, we have

$$\{\dot\varphi(K) + K + \psi(K)\Delta K + \varphi(K)\psi(K) + \psi^2(K)\|\nabla K\|^2\}\,\nabla K = 0.$$

Substituting

$$\Delta K = -2\varphi(K) - \psi(K)\|\nabla K\|^2$$

into this, we obtain

$$\dot\varphi(K) + K - \varphi(K)\psi(K) = 0 \tag{5.9}$$

on the set of non-critical points of K. Assume that $\varphi(K) \neq 0$ at an arbitrary critical point of K. Then the critical points are isolated and (5.9) holds on M. Consider a nontrivial function $v = v(K)$ satisfying

$$\frac{d^2 v}{dK^2} + \psi(K)\frac{dv}{dK} = 0.$$

Define $J(K)$ by

$$J(K) = Kv(K) + \dot v(K)\varphi(K) \tag{5.10}$$

Then, by virtue of (5.9),

$$\begin{aligned}\dot J(K) &= v(K) + \{K - \varphi(K)\psi(K) + \dot\varphi(K)\}\,\dot v(K)\\ &= v(K).\end{aligned}$$

Therefore if we set

$$v(K) = C\int e^{-\Psi(K)}\,dK, \qquad \Psi(K) = \int \psi(K)\,dK \tag{5.11}$$

with some non-zero constant C, then $J(K)$ defined by (5.10) satisfies $\dot J(K) = v(K)$ and hence

$$\begin{aligned}\nabla\nabla \dot J(K) &= \ddot v(K)\nabla K \otimes \nabla K + \dot v(K)\nabla\nabla K\\ &= \{J(K) - K\dot J(K)\}g\end{aligned}$$

because of (5.8) and (5.10).

Theorem 5.3. *If the Gauss curvature K of a compact, orientable Riemannian 2-manifold M satisfies (5.8) and $\varphi(K) \neq 0$ at every critical point of K, then the metric of M is a critical point of the functional $\mathcal{F}_J$ where $\dot J = v$ and v is defined by (5.11).*

6. Surfaces admitting a concircular scalar field

All facts in this section about elliptic functions are well-known; for instance, see [1, 6, 13].

If the equation $\nabla\nabla f = \tau g$ is restricted to a geodesic, then it reduces to an ordinary differential equation $f'' = \tau(f)$. When τ is a linear function of f, then the Riemannian manifold which admits the concircular scalar field f was determined in [16, 21]. So we study the cases that τ is a polynomial of degree 2 or 3 under the assumption that M is a complete Riemannian 2-manifold, although the results are easily generalized to the case that the dimension is not restricted.

We consider the real solutions of the following differential equations with constant real coefficients:

$$f'' = 6f^2 - \frac{1}{2}g_2, \tag{6.1}$$

$$f'' = 2f^3 + 6a_2 f + 2a_3. \tag{6.2}$$

Since, making use of solutions of (6.1), we can obtain those of (6.2), we first deal with (6.1). We have from (6.1)

$$(f')^2 = 4f^3 - g_2 f - g_3, \tag{6.3}$$

where g_3 is a constant real number. The roots of the polynomial

$$p(x) = 4x^3 - g_2 x - g_3$$

will be denoted by e_1, e_2 and e_3. The discriminant D is given by

$$\begin{aligned}D &= 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2\\ &= {g_2}^3 - 27{g_3}^2.\end{aligned}$$

We have the relations:

$$e_1 + e_2 + e_3 = 0, \qquad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2, \qquad e_1e_2e_3 = \frac{1}{4}g_3. \tag{6.4}$$

Since f is a function defined on M, f restricted to a geodesic is defined on $\mathbb{R}$. So we have to exclude solutions which diverge at a finite number $t_0 \in \mathbb{R}$ from nontrivial solutions of (6.1).

In the case that $D > 0$, the roots are real numbers. Fistly, we assume that $D > 0$ and $e_3 < e_2 < e_1$. Clearly, the real solution f satifies $e_3 \leq f \leq e_2$ or $e_1 \leq f$. In the case that $e_3 \leq f \leq e_2$, the solution with initial conditions $f(0) = e_3$ and $f'(0) = 0$ is given by

$$f(t) = \wp(t + \omega'), \tag{6.5}$$

where $\wp$ is the Weierstrass elliptic function with periods 2ω and $2\omega'$:

$$\omega = \int_{e_1}^{\infty}\frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \qquad \omega' = i\int_{-e_3}^{\infty}\frac{dx}{\sqrt{4x^3 - g_2x - g_3}}. \tag{6.6}$$

We note that $\wp(\omega) = e_1$, $\wp(\omega + \omega') = e_2$ and $\wp(\omega') = e_3$. Using the Jacobi elliptic functions, (6.5) becomes

$$f(t) = e_3 + (e_2 - e_3)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\,t), \tag{6.7}$$

where the modulus κ is given by $\kappa^2 = (e_2 - e_3)/(e_1 - e_3)$.

In the case that $e_1 \le f$, the solution with initial conditions $f(0) = e_1$ and $f'(0) = 0$ is given by

$$f(t) = \wp(t + \omega). \tag{6.8}$$

In this case, we have $\lim_{t\to-\omega} f(t) = \infty$.

Secondly, we consider the case that $D = 0$. Assume that $e_3 < e_2 = e_1$. Then the real solution f satisfies $e_3 \le f < e_1$ or $f \ge e_1$. In the case that $e_3 \le f < e_1$, the solution with initial condition $f(0) = e_3$ and $f'(0) = 0$ is given by

$$f(t) = e_3 + (e_1 - e_3)\tanh^2(\sqrt{e_1 - e_3}\, t), \tag{6.9}$$

which is the limit solution of (6.7) as $\kappa^2 \to 1$. We note that $\lim_{t\to\infty} f(t) = e_1$. If $f \ge e_1$, then the solution with $\lim_{t\to\infty} f(t) = e_1$ and $\lim_{t\to\infty} f'(t) = 0$, $(\omega = \infty)$, is given by

$$f(t) = e_1 + (e_1 - e_3)\frac{1}{\sinh^2(\sqrt{e_1 - e_3}\, t)}, \tag{6.10}$$

which shows that $\lim_{t\to 0} f(t) = \infty$. Next assume that $e_3 = e_2 < e_1$. In this case, $\kappa = 0$ and $\sqrt{e_1 - e_3}\,\omega = \pi/2$. The real solution f satisfies $f \ge e_1$. The solution with initial condition $f(0) = e_1$ and $f'(0) = 0$ is given by

$$f(t) = e_1 + (e_1 - e_3)\tan^2(\sqrt{e_1 - e_3}\, t). \tag{6.11}$$

Thus $\lim_{t\to t_0} f(t) = \infty$, where $t_0 = \pi/(2\sqrt{e_1 - e_3})$. Assume that $e_1 = e_2 = e_3$. From (6.4), we see that $e_1 = e_2 = e_3 = 0$. Therefore the solution is

$$f(t) = \frac{1}{(t + c)^2}, \tag{6.12}$$

where c is a constant. We also have $\lim_{t\to -c} f(t) = \infty$.

Thirdly, let us assume that $D < 0$. One of roots, say e_2, is real and the others are conjugate complex numbers. We also see that the periods 2ω and $2\omega'$ are conjugate complex numbers and so $\omega + \omega'$ is real, which is given by

$$\omega + \omega' = -\int_{e_2}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} \tag{6.13}$$

(cf. [1]). In this case, the real solution with $f(0) = e_2$ and $f'(0) = 0$ is given by

$$f(t) = \wp(t + \omega + \omega'). \tag{6.14}$$

Thus we have $\lim_{t\to-(\omega+\omega')} f(t) = \infty$. After all, we have

Lemma 6.1. *Among the nonconstant solutions of* (6.1), *the solutions which are defined on the whole line* $\mathbb{R}$ *are* (6.7) *in the case* $(D > 0, e_3 < e_2 < e_1)$ *and* (6.9) *in the case* $(D = 0, e_3 < e_2 = e_1)$, *up to the change of variable* $: t \to t + a$.

Let us turn to the differential equation (6.2). We assume that the polynomial

$$q(x) = x^4 + 6a_2x^2 + 4a_3x + a_4$$

has at least a real root. We denote the minimum of the real roots by x_4. The nontrivial solutions of (6.2) satisfy

$$(f')^2 = q(f). \tag{6.15}$$

If x_4 is a triple root, then the solution of (6.15) is given by

$$f(t) = \frac{x_1 - 4x_4{}^3(t + c)^2}{1 - 4x_4{}^2(t + c)^2},$$

and hence $\lim_{t\to t_0} |f(t)| = \infty$, where $t_0 + c = 1/2x_4$. If x_4 is a quadruple root, then

$$f(t) = x_4 + \frac{1}{c - t},$$

and hence $\lim_{t\to c} |f(t)| = \infty$. Thus we first assume that x_4 is a simple root. Let $x_1, \dots, x_4$ be the roots of $q(x)$. Put $x = x_4 + 1/z$. Then we have

$$(z')^2 = -\alpha(z - \alpha_1)(z - \alpha_2)(z - \alpha_3),$$

where $\alpha = (x_1 - x_4)(x_2 - x_4)(x_3 - x_4) > 0$ and $\alpha_i = (x_i - x_4)^{-1}$ $(i = 1, 2, 3)$. Furthermore we put $z = Ay + B$. Then

$$(y')^2 = -\alpha A\left(y + \frac{B - \alpha_1}{A}\right)\left(y + \frac{B - \alpha_2}{A}\right)\left(y + \frac{B - \alpha_3}{A}\right). \tag{6.16}$$

So if we define A and B by $A = -4/\alpha$ and $B = (\alpha_1 + \alpha_2 + \alpha_3)/3$ respectively, then we can rewrite (6.16) as

$$(y')^2 = 4(y - e_1)(y - e_2)(y - e_3), \tag{6.17}$$

where $e_i = (\alpha_i - B)/A$ $(i = 1, 2, 3)$. This means that a part of real solutions of (6.15) can be obtained by setting $f = x_4 + 1/(Ay + B)$ for a real solutions y of (6.17). We note that $A, B \in \mathbb{R}$ and

$$e_1 = \frac{1}{4}(2a_2 - x_1x_4 - x_2x_3), \qquad e_2 = \frac{1}{4}(2a_2 - x_2x_4 - x_1x_3), \tag{6.18}$$
$$e_3 = \frac{1}{4}(2a_2 - x_3x_4 - x_1x_2).$$

Let us consider the case that $x_4 < x_3 < x_2 < x_1$ (real). By (6.18) we see that $e_3 < e_2 < e_1$. The solution of (6.15) corresponding to (6.7) is

$$f(t) = \frac{x_3(x_2 - x_4) - x_4(x_2 - x_3)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t)}{(x_2 - x_4) - (x_2 - x_3)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t)}. \tag{6.19}$$

This solution attains the minimum x_3 at $t = 0$, the maximum x_2 at $t = \omega$ and is a periodic function with period 2ω. The solution corresponding to (6.8) is

$$f(t) = \frac{x_1(x_2 - x_4) - x_2(x_1 - x_4)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t)}{(x_2 - x_4) - (x_1 - x_4)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t)}, \tag{6.20}$$

which attains the minimum x_1 at $t = 0$ and diverges at $t = t_0$ such that $\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t_0) = (x_2 - x_4)/(x_1 - x_4)$. The function $\wp(t)$ is certainly a real solution of (6.3) which coincides with (6.8) up to the change of variable $: t \to t + \omega$. The solution of (6.15) corresponding to $\wp(t)$ is

$$f(t) = \frac{x_4(x_1 - x_3) - x_3(x_1 - x_4)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t)}{(x_1 - x_3) - (x_1 - x_4)\,\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t)}. \tag{6.21}$$

This attains the maximum x_4 at $t = 0$ and $\lim_{t\to t_0} f(t) = -\infty$, where $\mathrm{sn}^2(\sqrt{e_1 - e_3}\, t_0) = (x_1 - x_3)/(x_1 - x_4)$.

Next we consider the case $x_4 < x_3 < x_2 = x_1$. We have $e_3 < e_2 = e_1$. The solution corresponding to (6.9) (i.e., the limit of (6.19) as $\kappa^2 \to 1$) is

$$f(t) = \frac{x_3(x_1 - x_4) - x_4(x_1 - x_3)\tanh^2(\sqrt{e_1 - e_3}\, t)}{(x_1 - x_4) - (x_1 - x_3)\tanh^2(\sqrt{e_1 - e_3}\, t)}. \tag{6.22}$$

This function satisfies $f(0) = x_3$, which is the minimum, and $\lim_{t\to\pm\infty} f(t) = x_1$. For (6.10), we can show that there exists t_0 such that $Ay(t_0)+B = 0$. In fact, the range of the function (6.10) is $[e_1, \infty)$ and $-B/A = (a_2 + x_4{}^2)/2 > (a_2 + x_1{}^2)/2 = e_1$. Thus we have $\lim_{t\to t_0} |f(t)| = \infty$ for (6.10). Since $0 < (x_1 - x_3)/(x_1 - x_4) < 1$, we also see that there exists t_0 such that $\lim_{t\to t_0} |f(t)| = \infty$ for the limit solution of (6.21) as $\kappa^2 \to 1$.

Consider the case that $x_4 < x_3 = x_2 < x_1$. Then we have $e_3 = e_2 < e_1$. Thus the solution corresponding to (6.11) is

$$f(t) = \frac{x_1(x_2 - x_4) - x_2(x_1 - x_4)\sin^2(\sqrt{e_1 - e_3}\, t)}{(x_2 - x_4) - (x_1 - x_4)\sin^2(\sqrt{e_1 - e_3}\, t)}. \tag{6.23}$$

Since $0 < (x_2 - x_4)/(x_1 - x_4) < 1$, there exists t_0 such that $\lim_{t\to t_0} |f(t)| = \infty$. For f of (6.23), the function $f(t + \pi/2)$ is the real solution with initial condition $f(0) = x_4$. We also have t_0 such that $\lim_{t\to t_0} |f(t)| = \infty$.

In the case that $x_4 < x_3 = x_2 = x_1$, we have $e_1 = e_2 = e_3$ and hence the solution corresponding to (6.12) is

$$f(t) = \frac{x_4 - 4x_1{}^3(t + c)^2}{1 - 4x_1{}^2(t + c)^2}. \tag{6.24}$$

Clearly we have t_0 such that $\lim_{t\to t_0} |f(t)| = \infty$.

In the case that x_1 and x_3 are conjugate complex numbers, we see that e_2 is real and $e_3 (= \bar{e}_1)$ is not real. The real solution with $f(0) = x_2$ corresponds to (6.14) and that with $f(0) = x_4$ to $y(t) = \wp(t)$. The range of (6.14) is $[e_2, \infty)$ and $-B/A > e_2$. Thus we see that $f(t) = x_4 + 1/(Ay(t) + B)$ diverses at some t_0.

Next assume that x_4 is a double root ($x_4 = x_3$). Consider the case that $x_4 = x_3 < x_2 < x_1$. Set $\tilde{f}(t) = -f(t)$. Then $\tilde{f}$ satisfies

$$(\tilde{f}')^2 = \tilde{f}^4 + 6a_2\tilde{f}^2 - 4a_3\tilde{f} + a_4.$$

The roots of the polynomial : $\tilde{q}(x) = x^4 + 6a_2x^2 - 4a_3x + a_4$ are $\tilde{x}_4 = -x_1$, $\tilde{x}_3 = -x_2$, $\tilde{x}_2 = \tilde{x}_1 = -x_4$. We note that e_i $(i = 1, 2, 3)$ does not change. Therefore the real solution with initial condition $f(0) = x_2$ can be obtained by making use of (6.22):

$$f(t) = \frac{x_2(x_1 - x_4) - x_1(x_2 - x_4)\tanh^2(\sqrt{e_1 - e_3}\, t)}{(x_1 - x_4) - (x_2 - x_4)\tanh^2(\sqrt{e_1 - e_3}\, t)}. \tag{6.25}$$

We note that f attains the maximum x_2 at $t = 0$ and $\lim_{t\to\pm\infty} f(t) = x_4$. The other real solution in this case with initial condition $f(0) = x_1$ (or $f(0) = x_4$) satisfies $\lim_{t\to t_0} |f(t)| = \infty$ for some $t_0 \in \mathbb{R}$.

In the case that $x_4 = x_3 < x_2 = x_1$, we directly solve (6.15). Using the relation $x_1 + x_4 = 0$, we have

$$f(t) = \pm x_1 \tanh(x_1(t + c)) \tag{6.26}$$

or

$$f(t) = \pm x_1 \coth(x_1(t + c)), \tag{6.27}$$

where c is a constant. The solution (6.27) diverges at $t = -c$.

We consider the case that $x_4 = x_3$ and x_2 $(= \bar{x}_1)$ is not real. We put $f = x_4 + 2/(y + \alpha_1 + \alpha_2)$, where $\alpha_i = 1/(x_i - x_4)$ $(i = 1, 2)$. We note that $\alpha_2 = \bar{\alpha}_1$ and so $\alpha_1 + \alpha_2$ is real. Then we have

$$(y')^2 = \alpha\{y^2 - (\alpha_2 - \alpha_1)^2\}, \qquad (\alpha = (x_1 - x_4)(x_2 - x_4) > 0). \tag{6.28}$$

It is easy to solve (6.28). If we put $\alpha_2 - \alpha_1 = ib$, then the solution is $y = \pm b\sinh(t + c)$. Thus we have

$$f(t) = x_4 + \frac{2}{\pm b\sinh(t + c) + \alpha_1 + \alpha_2}, \tag{6.29}$$

and hence there exists $t_0 \in \mathbb{R}$ such that $f(t)$ diverges as $t \to t_0$.

The remainder case is that the polynomial q has not a real root. If $x_4 = \bar{x}_3 = \bar{x}_2 = x_1$, then $(f')^2 = (f - x_1)^2(f - \bar{x}_1)^2$ and hence $f(t) = \Re x_1 \pm (\Im x_1)\tan\{(\Im x_1)t + c\}$, which shows that there exists t_0 such that $f(t)$ diverges as $t \to t_0$.

Finally, we deal with the case that $x_4 = \bar{x}_3$, $\bar{x}_2 = x_1$ and $x_1 \neq x_4$. We reduce

$$\int \frac{df}{\sqrt{|f - x_4|^2|f - x_1|^2}} = \pm(t + c) \tag{6.30}$$

to a Jacobi normal form (cf. pp. 106 $\sim$ 109, [1]). If $\Re x_1 = \Re x_4$, then $\Re x_1 = 0$, so that (6.30) becomes

$$\int \frac{df}{\sqrt{(f^2 + b_1{}^2)(f^2 + b_4{}^2)}} = \pm(t + c), \tag{6.31}$$

where $b_i = \Im x_i$ $(i = 1, 4)$. Suppose that $\Re x_1 \neq \Re x_4$. We put $c_i = \Re x_i$ $(i = 1, 4)$. Let $f = (py + q)/(y + 1)$, where p and q $(p > q)$ are roots of the equation:

$$X^2 + \frac{1}{2c_1}(b_4{}^2 - b_1{}^2)X - \frac{1}{2}(2c_1{}^2 + b_1{}^2 + b_4{}^2) = 0.$$

Then the integral of (6.30) becomes

$$\frac{p - q}{|p - x_1||p - x_4|}\int \frac{dy}{\sqrt{(y^2 + \alpha^2)(y^2 + \beta^2)}} \tag{6.32}$$

$$\left(\alpha = \left|\frac{q - x_1}{p - x_1}\right|,\ \beta = \left|\frac{q - x_4}{p - x_4}\right|\right).$$

Since the integrals of (6.31) and (6.32) reduce to the normal form:

$$\int \frac{du}{\sqrt{(1 - u^2)(1 - \kappa^2u^2)}} \qquad (\kappa^2 = (\alpha^2 - \beta^2)/\alpha^2)$$

by putting $y^2 = \beta^2u^2/(1 - u^2)$, the straightforward computation shows that

$$f(t) = \frac{q + p\beta\,\mathrm{tn}\{\gamma(t + c)\}}{1 + \beta\,\mathrm{tn}\{\gamma(t + c)\}} \qquad \left(\gamma^2 = \frac{1}{4}(|x_4 - x_1| + |x_4 - x_2|)^2\right). \tag{6.33}$$

In particular, there exists t_0 such that $f(t)$ diverges as $t \to t_0$.

Summing up, we obtain

Lemma 6.2. *Among the nonconstant solutions of* (6.2), *the solutions which are defined on the whole line* $\mathbb{R}$ *are, up to the change of the variable* : $t \to t+a$, (6.19) *in the case* $(x_4 < x_3 < x_2 < x_1)$, (6.22) *in the case* $(x_4 < x_3 < x_2 = x_1)$, (6.25) *in the case* $(x_4 = x_3 < x_2 < x_1)$ *and* (6.26) *in the case* $(x_4 = x_3 < x_2 = x_1)$.

Let us return to the study of the manifold M admitting a concircular scaler field f. We consider the case that the characteristic function τ is a polynomial of f whose degree is less than 3. Tashiro [21] determined M when the polynomial is of degree ≤ 1. We may assume that the polynomial $\tau(f)$ is of the form of the right hand side of (6.1) or (6.2).

Theorem 6.3. *Let* M *be a complete* 2*-dimensional Riemannian manifold and suppose that it admits a concircular scaler field* f *whose characteristic function is a polynomial of* f. *If the degree is* 2 *or* 3, *then* M *is one of the following:*

[I] (deg $= 2$). *M is diffeomorphic to $\mathbb{R}^2$ and the metric ds^2 is given by*

$$ds^2 = du^2 + \frac{1}{e_1 - e_3}\tanh^2(\sqrt{e_1 - e_3}\,u)\,\mathrm{sech}^4(\sqrt{e_1-e_3}\,u)d\theta^2$$

in terms of the geodesic polar coordinates $\{u, \theta\}$ in $\mathbb{R}^2$ where $e_1 > e_3$ and $2e_1 + e_3 = 0$. It is isometric to the surface of revolution which is obtained by rotating the unit speed curve:

$$x(u) = \frac{1}{\sqrt{e_1 - e_3}}\tanh(\sqrt{e_1-e_3}\,u)\,\mathrm{sech}^2(\sqrt{e_1-e_3}\,u),$$
$$z(u) = \frac{1}{6{e_1}^2}\int_{e_3}^{f(u)}\frac{1}{e_1 - \xi}\sqrt{(2e_1 - \xi)(2{e_1}^2+\xi^2)}\,d\xi$$

in the x-z plane around the z-axis in $\mathbb{R}^3$, where $f(u) = e_3 + (e_1 - e_3)\tanh^2(\sqrt{e_1-e_3}\,u)$.

[II] (deg $= 3$).

(1) *M is isometric to $\mathbb{R}\times Z$ with warped product metric:*

$$ds^2 = du^2 + {x_1}^4\,\mathrm{sech}^4(x_1 u)d\theta^2,$$

where θ is a local coordinate in a complete 1-dimensional manifold Z and x_1 a positive constant.

(2) *M is diffeomorphic to $\mathbb{R}^2$ and the metric is given by*

$$ds^2 = du^2 + a(u)^2 d\theta^2,$$

where $\{u, \theta\}$ is the geodesic polar coordinates in $\mathbb{R}^2$,

$$a(u) = \frac{2f'(u)}{(x_3 - x_1)^2(x_3 - x_4)},$$
$$f(u) = \frac{x_3(x_1 - x_4) - x_4(x_1 - x_3)\tanh^2(\sqrt{e_1-e_3}\,t)}{(x_1-x_4) - (x_1 - x_3)\tanh^2(\sqrt{e_1-e_3}\,t)}$$

and x_1, x_3, x_4 are constants satisfying $x_4 < x_3 < x_1$ and $2x_1 + x_3 + x_4 = 0$.

(3) *M is isometric to S^2 with metric*

$$ds^2 = du^2 + a(u)^2 d\theta^2,$$

in terms of the geodesic polar coordinates $\{u, \theta\}$ whose center is a critical point of f, where

$$a(u) = \frac{f'(u)}{x_2({x_1}^2 - {x_2}^2)},$$
$$f(u) = \frac{-x_2\{x_1 + x_2 - 2x_1\,\mathrm{sn}^2((x_1+x_2)u/2)\}}{(x_1+x_2) - 2x_2\,\mathrm{sn}^2((x_1+x_2)u/2)}$$

and $0 < x_2 < x_1$.

Proof. Recall the results proved by Tashiro ((1) $\sim$ (3) in section 5). The integral curves of grad f are geodesics. When the concircular scalar field f is restricted to a geodesic, it satisfies (6.1) (resp. (6.2)) if the degree of the polynomial $\tau(f)$ is 2 (resp. 3). Thus f restricted to a geodesic is one of solutions given in Lemmas 6.1 and 6.2. Suppose that the degree of $\tau(f)$ is equal to 2. Then we see from Lemma 6.1 that the number m of the critical points of the concircular scalar field f is 1 or 2. If $m = 1$, then f restricted to the geodesic γ which coincides with the integral curve of grad$f$ is given by (6.9). If $m = 2$, then $f|_\gamma$ is given by (6.7). Since $ds^2 = du^2 + a(u)^2 d\theta^2$ (see (3) in section 5) and the metric $ds^2 (= g)$ is smooth at the critical point $P \in W$, we require the function a to satisfy

$$a(0) = 0, \quad a'(0) = 1, \quad a^{(2k)}(0) = 0 \quad (k = 1, 2, \ldots).$$

If $f|_\gamma$ is given by (6.9), then $a = cf'$ is an odd function and satisfies the condition $a'(0) = 1$ by setting $c = 1/\{2(e_1 - e_3)^2\}$. Let $f|_\gamma$ be given by (6.7). Since we have from (6.4)

$$\begin{aligned} a'(0) &= c\wp''(\omega') \\ &= c\left(6\wp(\omega')^2 - \frac{1}{2}g_2\right) \\ &= 2c(e_1 - e_3)(e_2 - e_3), \end{aligned}$$

we have to put $c = 1/\{2(e_1 - e_3)(e_2 - e_3)\}$. Furthermore since ds^2 gives the smooth metric g at another critical point $Q \in W$, we also require a to satisfy

$$a(\omega) = 0, \quad a'(\omega) = -1, \quad a^{(2k)}(\omega) = 0 \quad (k = 1, 2, \ldots).$$

However we have

$$\wp''(\omega + \omega') = 2(e_2 - e_3)(e_2 - e_1)$$

and hence

$$\begin{aligned} a'(\omega) &= cf''(\omega) \\ &= -\frac{e_1 - e_2}{e_1 - e_3} \\ &= \kappa^2 - 1 > -1. \end{aligned}$$

We conclude that the case $m = 2$ does not occur if the degree is equal to 2. It is easy to see that $\mathbb{R}^2$ with metric given in [I] is isometric to a surface of revolution.

Suppose that the degree of $\tau(f)$ is equal to 3. If $m = 0$, then $f|_\gamma$ is the function given in (6.26) and M is isometric to $\mathbb{R} \times Z$ with warped product metric given in [II](1). If $m = 1$, then $f|_\gamma$ is the function given in (6.22) which is essentially the same as (6.25). Therefore, in the case that $m = 1$, we set $c = 2/\{(x_3 - x_1)^2(x_3 - x_4)\}$ so that $a'(0) = cf''(0) = 1$, and get the case [II](2). Next assume that $m = 2$. The function $f|_\gamma$ is that given in (6.19). In order that a satisfies $a'(0) = 1$, the constant c must be equal to $2/\{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)\}$. At another critical point (i.e., $t = \omega$), we require $a'(\omega) = -1$. Noting that $x_1 + x_2 + x_3 + x_4 = 0$ and

$$\begin{aligned} a'(\omega) &= cf''(\omega) \\ &= -\frac{(x_2 - x_1)(x_2 - x_4)}{(x_3 - x_1)(x_3 - x_4)}, \end{aligned}$$

we have $x_3 = -x_2$. Thus we see that M is diffeomorphic to S^2 and

$$a(u) = \frac{f'(u)}{x_2(x_1 + x_2)(x_1 - x_2)},$$

$$f(u) = \frac{x_2\{x_1 + x_2 - 2x_1 \operatorname{sn}^2((x_1 + x_2)u/2)\}}{(x_1 + x_2) - 2x_2 \operatorname{sn}^2((x_1 + x_2)u/2)},$$

where we note that $\sqrt{e_1 - e_3} = (x_1 + x_2)/2$. We have got the case [II](3). $\square$

Corollary. *Let M be a complete 2-dimensional Riemannian manifold. If the Gauss curvature K satisfies $\nabla\nabla K = \tau g$, then K is constant or M is isometric to $\mathbb{R}^2$ with the metric whose curvature is given by $K = -x''/x$, x being the function given in* [I] *of Theorem* 6.3.

Proof. Since $\nabla\nabla K = \tau g$, we have $-\Delta K = 2\tau$ and $\nabla \Delta K = 2K\nabla K = \nabla K^2$. It follows that $-2\tau = K^2 - \lambda$, where λ is a constant. Thus we have

$$\nabla\nabla K = -\frac{1}{2}(K^2 - \lambda). \tag{6.34}$$

We put $f = -K/12$ and $g_2 = \lambda/12$. Then (6.34) becomes

$$\nabla\nabla f = \left(6f^2 - \frac{1}{2}g_2\right) g. \tag{6.35}$$

The characteristic function $\tau(f)$ is a polynomial of f of degree 2. Thus the assertion is obtained from Theorem 6.3. $\square$

Now we consider the functional on $\mathcal{M}_1$:

$$\mathcal{F}_2[g] = c \int_M (\delta - K)^2 \, dv_g \qquad (c \neq 0), \tag{6.36}$$

δ being a constant. Suppose that g is a critical point of $\mathcal{F}_2$. Then $f = \dot{J}(K) = 2c(K - \delta)$ is a concircular scalar field if K is not constant. Thus, from the above Corollary, we conclude the following:

Theorem 6.4. *Let M be a compact orientable 2-dimensional Riemannian manifold. If the metric g of M is critical with respect to $\mathcal{F}_2$, then the Gauss curvature is constant.*

Next for a given positive number δ, we consider the functional on $\mathcal{M}_{K,\delta} = \{g \in M | K \leq \delta \text{ on } M\}$:

$$\mathcal{F}_{3/2}[g] = c \int_n (\delta - K)^{3/2} \, dv_g \qquad (c \neq 0). \tag{6.37}$$

If the metric g is critical for the functional $\mathcal{F}_{3/2}$, then $f = \dot{J}(K)$ satisfies (5.3) and hence

$$\nabla\nabla f = \left(\frac{4}{27c^2} f^3 - \delta f\right) g. \tag{6.38}$$

Thus it is convenient to put $c = \sqrt{2}/(3\sqrt{3})$. If the metric g is critical for the functional $\mathcal{F}_{3/2}|_{\mathcal{M}_1}$, then

$$\nabla\nabla f = \left(2f^3 - \delta f + 2a_3\right) g, \tag{6.39}$$

where the constant a_3 is chosen in such a way that $\int_M (2f^3 - \delta f + 2a_3) \, dv_g = 0$. Since $J(K) = c(\delta - K)^{3/2}$, (5.8) becomes

$$\nabla\nabla K = \frac{2}{3}(\delta - K)(K + 2\delta)g - \frac{1}{2(\delta - K)} \nabla K \otimes \nabla K. \tag{6.40}$$

In the case (3)[II] of Theorem 6.3, we have $\delta = x_1{}^2 + x_2{}^2$ and $6f^2 = \delta - K$. Therefore the maximum of K is equal to δ. We see that the metric given in (3)[II] of Theorem 6.3 is on the boundary of $\mathcal{M}_{K,\delta} \cap \mathcal{M}_1$.

Theorem 6.5. *Let M be a manifold diffeomorphic to S^2. If a metric g on M is critical with respect to $\mathcal{F}_{3/2}|_{\mathcal{M}_1}$, then the Gauss curvature of g is a positive constant or g is the metric given in* (3)[II] *of theorem* 6.3 *and on the boundary of $\mathcal{M}_{K,\delta} \cap \mathcal{M}_1$.*

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