ポアソン多様体と関連する多様体の諸構造の研究

平成９～１０年度科学研究費補助金（基盤研究（C)(2))

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研究成果報告書

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研究成果

水谷忠良

平成 9 年度 10 年度を通じて研究の重点は，ポアソン多様体の，シンプレクティック葉層に関連するものであった．まず，多様体 M 上に一般の 2−ベクトル場 π があったとき，それの定義する接平面場の積分可能性について調べた．すなわち，π を T^*M から TM への写像と考え, とくに π の階数が一定 ($=2k$) であると仮定した場合に π の像として定義される接平面場が積分可能であるのはスカウテンブラケット $[\pi,\pi^k]=0$ であること，および $\dim M=2k+1$ のとき $[\pi,\pi^k]\neq 0$ であることが π が M に接触構造を定義することを示した．続いて，ポアソンコホモロジーの立場から π の定義する 2 次元コサイクルがコバウンダリーになるポアソン多様体について調べ，次のような結果を得た．3 次元閉多様体で正則なポアソン多様体を考えるとコンパクトな葉を持ち得ない．また，通常考えられる例のほかに Hirsch 葉層を持つ例が構成できる．これらについては研究集会などで発表を行ったがプレプリントとしてまとめたものをこの冊子にとじてある．

このほか得られた新しい知見としては次のようなものがあった．

すなわち 2 ベクトル π が接触構造を定めるとき divergence Divπ がいわゆる Reeb ベクトル場となる接続が存在する．一方，2 ベクトル π がポアソン構造を定めるときは Divπが　一次元のポアソンコサイクルを定める．　正則 (regular) なポアソン構造については Divπ は付随する葉層のモジュラー類 (特性類の記号で h_1 と表されるもの）に $\pi : T^*M \to TM$ を通して対応していること　である．

この Divπ 自身は正則でないポアソン多様体に対してもポアソンコサイクルとしての意味を持つ点が極めて興味深い．この事実は，特異集合を持つ葉層の特性類の定義可能性とポアソン多様体の不変量の関連を示唆している．

これに従って平成 10 年度には，リー群の左不変なポアソン構造に付随する葉層構造を調べることにより．3 次元葉不変量 (h_3) を調べた．また，ポアソン多様体の一般化概念であるディラック多様体についても調べたが，これらを総合して，確定した結果を得るにはもう少し時間が必要である．

長瀬正義

当研究分担者は，主に，Spin 構造の変形物である Spin^{q}構造（Nagase: Spin^q structures, J. Math. Soc. Japan, 47(1995), 93-119 において導入）に付随する twistor 構造，断熱極限，等の研究に取り組んだ。Spin^{q}束の Spin^{c}による商束の全空間は，通常，twistor 空間と呼ばれるもの（Penrose

によるものや，Salamon達による四元数ケーラー多様体のそれ，等）と類似の構造を持つことがわかる。本研究ではこれをtwistor空間と呼んでいる。特別なケースとしてPenrose等のそれらを含むが，それらが$4n$次元多様体上の理論なのに対して，我々のtwistor空間は一般の次元で論じられ，かつ自然にSpin^c構造を持つことがわかる。特に興味深いのは奇数次元Spin^q多様体上のtwistor空間で，その空間のエータ不変量の断熱極限と底空間のエータ不変量との関係の研究は，物理学のいうグローバルアノマリーの研究に対応していると思われる。以上をNagase: Spin^q, twistor and Spin^c (Commun. Math. Phys., 189(1997),107-126) において論じている。10年度は，９年度よりの課題であった四元数スピン多様体上のSpin^q-Seiberg-Witten方程式（通常の(Spin^c-)SW方程式の類似物）の「トゥィスター空間への引き上げ理論の構成とその "理論" の断熱極限」という問題についての研究成果を，M. Nagase: Twistor space and the Seiberg-Witten equation (preprint) にまとめた。この研究は，四元数スピン多様体とその上のトゥィスター空間（ファイバーが$CP^1 = S^2$(もっとも単純な膜)であるような空間）という枠組みと最近注目を集めているＭ理論（１１次元空間の膜理論：種々の超弦理論を統一する理論と期待される）の枠組みとの類似性に着目して開始した研究であり，その出発点としてSpin^q-Seiberg-Witten理論と呼ぶべきものの "引き上げ理論" 及びそれの断熱極限（この操作によりその理論の本質的な部分が浮き彫りとなる）を考察している。

その他，関連する研究に解析的トーションの断熱極限の研究がある。このトーションは，ラプラシアン（＝ディラック作用素の二乗）の固有値より作られるゼータ関数の微分の原点における値に関係しており，本質的には熱核のトレースの（時間パラメータ$t \to 0$の場合の）漸近展開に依存している。研究対象は，そのトーションの（断熱極限パラメータ$\epsilon \to 0$の場合の）極限であり，上述漸近展開が二つのパラメータt, ϵにどう依存するかを明確にする必要がある。現在，トップ項（$t^{-1/2}$の係数）をϵの関数として書き下すことに成功しているが，それ以降の項の評価には成功していない。

江頭信二

コンパクト多様体上の横断的に区分滑らかな（piecewise-C^{1+bv} 級の）葉層 S^1-束がもつ定性的構造を明らかにした。またそれにより、この葉層の拡大度は典型的な増大度しか取らないこと、およびそれは葉のレベルと弾性葉の存在性によって分類されることを示した。

収録論文

1. Toshizumi Fukui;
   Congruence for real curves in toric surface and Newton polygons.
2. Tadayoshi Mizutani;
   On exact Poisson manifolds of dimension 3.

CONGRUENCE FOR REAL CURVES IN TORIC SURFACE AND NEWTON POLYGONS

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We consider the dividing curves in real toric surfaces. This is determined by polynomials with appropriate Newton polygon. We discuss some relation between the Euler characteristic of its positive point locus and their complex orientations under some conditions.

Introduction

Let $f(x,y) = \sum a_{ij}x^i y^j$ be a real polynomial, and $\Delta(f)$ the Newton polygon of f, that is, the convex hull of the set of points (i,j) with $a_{ij} \neq 0$. In this paper, we discuss some congruences related to zero sets of f and its Newton polygon.

Obviously, the Newton polygon $\Delta(f)$ is an integral convex polygon. Here, an integral polygon means a polygon whose vertices are integral points. A polynomial $f(x,y)$ is said to be *non-degenerate*, if the gradient of $f_\gamma(x,y) = \sum_{(i,j)\in\gamma} a_{ij}x^i y^j$ has no zeros in $(\mathbf{C}-0)^2$ for each face γ of $\Delta(f)$. If f is non-degenerate, then the zero locus of f in K^2 ($K = \mathbf{C}, \mathbf{R}$) is nonsingular except the origin.

If f is non-degenerate, the real and complex zero locus of f can be compactified in a suitable toric surface $P(K)$ ($K = \mathbf{R}, \mathbf{C}$) as nonsingular algebraic curves, and we denote the compactifications by $Z(\mathbf{R})$, $Z(\mathbf{C})$. Then, by Harnack's inequality, the number of the (connected) components of $Z(\mathbf{R})$ is at most $g+1$, where g is the genus of $Z(\mathbf{C})$. By custom, we set $M = g+1$, and call Z an $(M-i)$*-curve*, if the number of components of $Z(\mathbf{R})$ is $M-i$. In our situation, g is given by the number of integral points in the interior of $\Delta(f)$, which is equals to $1 - \frac{1}{2}\mathrm{Vol}_1(\Delta(f)) + \mathrm{Vol}_2(\Delta(f))$. (See Khovanski[12] for a proof of this fact.) Here, for a polygon Δ, $\mathrm{Vol}_2(\Delta)$ denotes the area of Δ, and $\mathrm{Vol}_1(\Delta)$ denotes the perimeter of the boundary of Δ, which coincides the number of integral points in the boundary of Δ. We understand 1 is the length of an integral segment which contains no integral points except its ends.

By the proof of Harnack's inequality appeared in the paper by G.Wilson[20], the number of connected components of $Z(\mathbf{C}) - Z(\mathbf{R})$ is at most two. We say Z is a *dividing curve*, or simply Z *divides*, if $Z(\mathbf{C}) - Z(\mathbf{R})$ is not connected. As noting ibid., an M-curve always divides, and an $(M-i)$-curve never divides, if i is odd. Assume that Z divides, and we denote $Z_{\pm}$ the closures of the connected components of $Z(\mathbf{C}) - Z(\mathbf{R})$. Obviously, $Z(\mathbf{R}) = Z_+ \cap Z_-$, and $Z(\mathbf{C}) = Z_+ \cup Z_-$. We orient $Z_{\pm}$ by their natural complex structure, and orient $Z(\mathbf{R})$ as boundaries of Z_+ or Z_-. We call them complex orientations of $Z(\mathbf{R})$.

If $\Delta(f)$ is even, that is, the twice of some integral polygon, the inequality $f(\alpha) \geq 0$ makes sense for each $\alpha \in P(\mathbf{R})$, and we denote $P^{\pm} = \{\alpha \in P(\mathbf{R}) : \pm f(\alpha) \geq 0\}$. We understand that $P^{\pm}$ is a compactification of the set $B^{\pm} := \{(x,y) \in \mathbf{R}^2 - (0,0) : \pm f(x,y) \geq 0\}$. If $\Delta(f)$ is even and f is non-degenerate, then each component of $Z(\mathbf{R})$ is an oval, that is, a connected nonsingular two-sided component of $Z(\mathbf{R})$. We say that a component of the real zero locus of f in $P(\mathbf{R})$ is said to be a 0-*oval*, if it bounds a real disc. We consider the following condition:

Condition (A). Each connected component of $Z(\mathbf{R})$ is 0-oval.

We assume that Condition (A). Then, for each connected component C of $Z(\mathbf{R})$, we have a real disc bounded by C. We consider the union of all such discs and denote it by S. Then $P(\mathbf{C}) - S$ is a component of $P(\mathbf{C}) - Z(\mathbf{R})$. Without loss of generality, we may assume that this is a component of P^-. We understand that the connected component of $P(\mathbf{R}) - C$ contained in $P(\mathbf{C}) - S$ is *outside* of the 0-oval C. Since each real disc in $P(\mathbf{R})$ can be deformed to a disc in $\mathbf{R}^2_+$, P^+ can be deformed into the first quadrant $\mathbf{R}^2_+$.

Under Condition (A), we say that an oval of $Z(\mathbf{R})$ is *even* (resp. *odd*), if it lies inside an even (resp. odd) number of other ovals of $Z(\mathbf{R})$. We denote the number of even (resp. odd) ovals by N^+ (resp. N^-). Obviously, $\chi(P^+) = N^+ - N^-$.

Now we recall the following theorems:

Theorem 0.1 (Theorem 0.3[5]) *Let f be a non-degenerate polynomial with even $\Delta(f)$. Suppose that each component of $Z(\mathbf{R})$ in some nonsingular toric surface $P(\mathbf{R})$ is a 0-oval. Then we have the following congruences.*

(i) If Z is an M-curve, then $N^+ - N^- \equiv \frac{1}{2} Vol_2(\Delta(f)) \pmod 8$.

(ii) If Z is an $(M-1)$-curve, then $N^+ - N^- \equiv \frac{1}{2} Vol_2(\Delta(f)) \pm 1 \pmod 8$.

(iii) If Z is an $(M-2)$-curve and does not divide, then
$N^+ - N^- \equiv \frac{1}{2} Vol_2(\Delta(f)) \pmod 8$, $\frac{1}{2} Vol_2\Delta(f) \pm 2 \pmod 8$.

(iv) If Z divides, then $N^+ - N^- \equiv \frac{1}{2} Vol_2(\Delta(f)) \pmod 4$.

(i) is Gudkov[8]-Rokhlin[18] type congruence. (ii) is Gudkov-Krachnov[7]-Kharlamov[11] type congruence and (iv) is Arnold[1] type congruence.

Theorem 0.2 (Theorem 0.4[5]) *Let f be a non-degenerate polynomial with even $\Delta(f)$. Then we have the following inequality.*

$$3 - e(\Delta(f)) + \frac{1}{4} Vol_1(\Delta(f)) - \frac{3}{4} Vol_2(\Delta(f))$$
$$\leq \chi(P^+) \leq 1 - \frac{1}{4} Vol_1(\Delta(f)) + \frac{3}{4} Vol_2(\Delta(f)),$$

where $e(\Delta)$ is the number of sides of a convex polygon Δ.

This is a Petrowsky[16] type inequality.

The proof in the paper [5] is basically the toric version of the proof due to A.Marin[13].

In this paper, we show more information for dividing curves. One of consequences of our discussion is the following congruence: (See Proposition 7.6, also.)

Theorem 0.3 *Assume that $\Delta(f)$ is bi-even, that is, twice of some even polygon, and each connected component of $Z(\mathbf{R})$ is a 0-oval. If Z is an M-curve and each even oval surrounds an odd number of other ovals, then*

$$N^+ - N^- \equiv -\frac{1}{2} Vol_2(\Delta(f)) \pmod{16}.$$

When $P = P^2$ (the projective plane), this was obtained by T.Fidler[4]. To see it, T.Fidler considered the congruence due to Guillou-Marin[9] for the complex projective plane. When we consider this congruence for complex toric surfaces which is reviewed in §1, we obtain similar result. We also mention some consequences for a complex orientation of $Z(\mathbf{R})$. In §3, we present Rokhlin's formula for dividing curves, and we see a fact we can expect: Roughly speaking, we assert that, for a dividing curve, some condition on its complex orientation determines the parity of $\frac{1}{4}(N^+ - N^- - \frac{1}{2}\mathrm{Vol}_2(\Delta(f)))$ under suitable suppositions. We investigate this phenomena using Guillou-Marin's congruence. We present technical details in §4-6, and some of consequences are formulated explicitly in §7.

1 Toric surface

In this section, we briefly recall the definition and some properties of toric surface. See a survey paper by V.I.Danilov[3], and books by W.Fulton[6], T.Oda[15], for detailed discussion. Set $K = \mathbf{C}$, or $\mathbf{R}$.

1.1 *Definition*

Let $v_0, v_1, ..., v_d = v_0$ be a sequence of lattice points in $\mathbf{Z}^2$ in counterclockwise order such that the successive pairs generate the lattice $\mathbf{Z}^2$. For convenience, we set $v_{d+1} = v_1$. Then we have $v_{i-1} + v_{i+1} + c_i v_i = 0, 1 \leq i \leq d$ for some integer c_i. Let $\mathbf{C}_i^2$ $(i = 1, ..., d)$ be copies of $\mathbf{C}^2$ with a complex coordinate system (z_i, w_i). Then, we obtain a compact nonsingular toric surface $P(\mathbf{C})$ gluing $\mathbf{C}_i^2$'s by $z_{i+1} = z_i^{-c_i} w_i$, $w_{i+1} = z_i^{-1}$ $(i = 1, ..., d)$. We denote $P(\mathbf{R})$ the real part of $P(\mathbf{C})$.
Example 1: Set $v_0 = \binom{1}{0}, v_1 = \binom{0}{1}, v_2 = \binom{-1}{-1}, v_3 = v_0$. P is the projective plane P^2. And each nonsingular compact toric surface with $d = 3$ is isomorphic to P^2.
Example 2: Let a be a non-negative integer. Consider the toric surface obtained by setting $v_0 = \binom{1}{0}, v_1 = \binom{0}{1}, v_2 = \binom{-1}{-a}, v_3 = \binom{0}{-1}, v_4 = v_0$. This surface is called by *Hirzebruch surface*, and is denoted by F_a. We remark that each nonsingular compact toric surface with $d = 4$ is isomorphic to F_a for some a. In particular, $F_0 = P^1 \times P^1$.
Lemma 1.1 *The Euler characteristic of $P(\mathbf{C})$ is d, and the signature of $P(\mathbf{C})$ equals $4 - d$. The Euler characteristic of $P(\mathbf{R})$ is $4 - d$.*

1.2 *Divisors*

Let $D_i(K)$ be the divisor of $P(K)$ defined by $w_{i+1} = z_i = 0$ for $i = 0, 1, ..., d$. We understand $D_0 = D_d$. Then, we have

$$D_i(\mathbf{C}).D_j(\mathbf{C}) = \begin{cases} c_i & \text{if } i = j, \\ 1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The last two equalities are trivial by definition. To see the first equality, we construct a small perturbation of $D_i(\mathbf{C})$. Let k_i be the maximal integer with $2k_i \leq |c_i|$ and $\alpha_1, \alpha_2, ..., \alpha_{k_i}$ positive numbers with $\alpha_1 < \alpha_2 < \cdots < \alpha_{k_i}$. Let $\tilde{D}_i$ denote the closure of the set $\{w_i = F_{\varepsilon,\delta}(z_i)\}$ in $P(\mathbf{C})$ where

$$F_{\varepsilon,\delta}(z) = \begin{cases} \varepsilon f_\delta(z) & \text{if } c_i \geq 0, \\ \varepsilon f_\delta(z)^{-1} & \text{if } c_i < 0 \text{ and } |f_\delta(z)| \geq \sqrt{|\varepsilon|}, \\ \varepsilon \overline{f_\delta(z)} & \text{if } c_i < 0 \text{ and } |f_\delta(z)| < \sqrt{|\varepsilon|}, \end{cases}$$

and

$$f_\delta(z) = \begin{matrix} \sqrt{-1}\prod_{j=1}^{k_i}(z^2 + \alpha_j) & \text{if } |c_i| = 2k_i, \\ \sqrt{-1}\prod_{j=1}^{k_i}(z^2 + \alpha_j)(z - 1 - \delta\sqrt{-1}) & \text{if } |c_i| = 2k_i + 1. \end{matrix}$$

Here δ is a small positive number and ε is a real number close to 0. By construction, we see $D_i(\mathbf{C}).\tilde{D}_i = c_i$.

We remark that $H_2(P(K); \mathbf{Z})$ is generated by $D_i(K)$'s $(i = 1, ..., d)$, and the homology class represented by $\sum_{i=1}^d v_{i,1} D_i(K), \sum_{i=1}^d v_{i,2} D_i(K)$ is zero in $H_2(P(K); \mathbf{Z})$, where we write $v_i = \binom{v_{i,1}}{v_{i,2}}$ for $i = 1, ..., d$. We set

$$D_i^{\pm} = \{w_i = 0, \pm\mathrm{Im} z_i \geq 0\} \cup \{z_{i+1} = 0, \mp\mathrm{Im} w_{i+1} \geq 0\}, \text{ and}$$
$$D_I(K) = \bigcup_{i \in I} D_i(K) \quad (K = \mathbf{R}, \mathbf{C}), \quad \text{for } I \subset \{1, ..., d\}.$$

Then, $D_I(\mathbf{C}).D_I(\mathbf{C}) = \sum_{i \in I} c_i + 2\#I_D$, where $\#I_D$ is the number of double points of $D_I(\mathbf{C})$. Note that $-D_{\{1,...,d\}}$ is the canonical divisor of P.

1.3 *Sections of line bundles*

Set $D = \sum a_i D_i$. Let $\mathcal{O}(D)$ be the sheaf of algebraic sections of the line bundle $[D]$ defined by the divisor D. Then, we have

$$\Gamma(P(K), \mathcal{O}(D)) = \bigoplus_{m \in \mathbf{Z}^2 \cap \Delta_D} K x^m,$$

$$\text{where} \quad \Delta_D = \{u \in \mathbf{R}^2 : \langle u, v_i \rangle \geq -a_i, \forall i\}. \tag{1}$$

If $f(x, y)$ is a K-coefficient polynomial with $\Delta_D = \Delta(f)$, we can consider f as a section of the line bundle $[D]$, using the isomorphism above, and the zero locus of f defines an algebraic curve $Z(K)$ in the toric surface $P(K)$.

If $\Delta(f)$ is even, that is, twice of some integral polygon, then the inequality $f \geq 0$ make sense on the toric surface $P(\mathbf{R})$. In this case, we set

$$P^{\pm} = \{\alpha \in P(\mathbf{R}) : \pm f(\alpha) \geq 0\}.$$

The self-intersection number $D(\mathbf{C})^2$ is equal to $2\mathrm{Vol}_2(\Delta_D)$. See the articles [3,15,6] for its proof.

Let $f(x, y)$ be a non-degenerate real polynomial, and V be the set of primitive vectors supporting edges of $\Delta(f)$. We choose $v_1, ..., v_d$ in §1.1 so that $\{v_1, ..., v_d\} \supset V$. Then we can find a divisor $D = \sum a_i D_i$ with $\Delta_D = \Delta(f)$, by (1). Let E_i denote the side of $\Delta(f)$ supported by v_i, for $i = 1, ..., d$. If $\Delta(f)$ is even, then $D = 2\overline{D}$ for some divisor $\overline{D} = \sum \overline{a}_i D_i$.

Throughout this paper, we assume that $f(x, y)$ is a non-degenerate real polynomial, and use the notation above.

2 Lemmas for self-intersection numbers

We prepare two lemmas.

Lemma 2.1 *Let A and B be oriented surfaces with boundaries in $P(\mathbf{R})$, and v a tangent vector field of A with finite zeros in the interior of B. Assume the boundaries ∂A and ∂B intersect transversely, that v does not tangent to ∂A and ∂B, and that v looks outward (resp. inward) along both ∂A and ∂B. Let $\tilde{A}$ be a small perturbation of A by the vector $\sqrt{-1}v$.*

(i) If the orientation of A agrees with that of B for each point of $A \cap B$, then the intersection number of $\tilde{A}$ and the interior of B is equal to $-\chi(A \cap B)$.

(ii) If the orientation of A disagrees with that of B for each point of $A \cap B$, then the intersection number of $\tilde{A}$ and the interior of B is equal to $\chi(A \cap B)$.

Proof We first assume that the orientation of A agrees with that of B for each point of $A \cap B$, Then, the desired intersection number is the sum of indices of $\sqrt{-1}v$, which is equal to $-\chi(A \cap B)$. Changing the orientation of B, we obtain (ii) from (i). □

See the paper by C.C.Pugh[17] for general treatment.

The self-intersection number of a surface embedded in an oriented 4-manifold is its normal Euler number.

Lemma 2.2 *Let S be an immersed surface in an oriented 4-manifold M. Assume that there are a point $P \in M$, a coordinate neighborhood U near P, and an orientation preserving complex coordinate system $z = (z_1, z_2) : U \to \mathbf{C}^2$ so that $S \cap U = z^{-1}\{z_1 z_2 = 0\}$ and that $z(P) = 0$. Here we understand that $\mathbf{C}^2$ is oriented by its natural complex structure. Let r_1, r_2 be positive numbers satisfying $r_1 < r_2$ and $\{|z| \le r_2\} \subset z(U)$. Let $\rho_{r_1 r_2}(r)$ be a C^∞-function on $[0, \infty)$ such that $\rho_{r_1 r_2}(r) = 1$ for $0 \le r \le r_1$, and that $\rho_{r_1 r_2}(r) = 0$ for $r \ge r_2$. We set*

$$S_+ = (S - U) \cup z^{-1}\{z_1 z_2 = \varepsilon\rho(|z|)\}$$
$$S_- = (S - U) \cup z^{-1}\{z_1 \overline{z_2} = \varepsilon\rho(|z|)\}.$$

Here ε is a small positive number. Then the normal Euler number of S_+ (resp. S_-) is equal to the sum of the normal Euler number of S in M and 2 (resp. -2).

I learned this fact from O.Saeki. His proof is based on computation of linking number. Same proof of Lemma 2.2 can be found in the §5 of the paper by Y.Yamada[21].

We call S_+ (resp. S_-) a *positive* (resp. *negative*) *smoothing* of S at double point P.

3 Rokhlin's formula

Assume that Z is a dividing curve and Condition (A). Let ℓ be the number of connected components of $Z(\mathbf{R})$. We consider the orientation of $Z(\mathbf{R})$ as boundary of Z_+ (or Z_-). We call an injective pair of ovals of Z, i.e. a pair of ovals one of which lies inside of the other, *positive* if the orientations of the ovals induce an orientation of annulus bounded by them in $P(\mathbf{R})$, and *negative* in the opposite case, and we denote the number of positive pairs by Π^+ and the number of negative pairs by Π^-. An odd oval of Z is called *disoriented* if it forms a negative pair with the innermost of the ovals outside of it. The number of disoriented ovals is denoted by d^*, the number of positive pairs with disoriented outer ovals by D^+, and the number of negative pairs with disoriented outer ovals by D^-. We then have that $\Pi^+ - \Pi^- = N^- - 2(d^* + D^- - D^+)$. Denote B_C the disc bounded by the oval C in $P(\mathbf{R})$. Attaching small perturbations of discs B_C's in $P(\mathbf{C})$ to $Z_\pm$, we obtain surfaces $X_\pm$, which represent $\mathbf{Z}$-homology classes in $H_2(P(\mathbf{C}), \mathbf{Z})$. Then the self-intersection numbers of $X_\pm$ is equal to $\frac{1}{2}Z(\mathbf{C}).Z(\mathbf{C}) - \ell + 2(\Pi^+ - \Pi^-)$, because of Lemma 2.1. On the other hand, since $X_\pm$ represent the integral homology classes of $\overline{D}(\mathbf{C})$, we have the self-intersection numbers of $X_\pm$ are that of $\overline{D}(\mathbf{C})$ which are equal to $\frac{1}{2}\mathrm{Vol}_2(\Delta(f))$. Therefore, we obtain

$$N^- - 2(d^* + D^- - D^+) = \Pi^+ - \Pi^- = \frac{1}{2}(\ell - \frac{1}{2}\mathrm{Vol}_2(\Delta(f))) = \overline{g} - i/2, \quad (2)$$

where $\overline{g}$ is the virtual genus of $\overline{D}(\mathbf{C})$. This formula was originally formulated by Rokhlin[19] for dividing plane curves with even degree.

This gives some restriction about topology of dividing curve. It is an interesting problem to construct dividing curves with prescribed topology. Sometime this is a delicate problem.

Example: Let Δ be a convex hull of the three points $(0,0)$, $(6,0)$, $(0,4)$, and f be a non-degenerate polynomial with $\Delta(f) = \Delta$. Using Theorems 0.1, 0.2, it is not difficult to see that the isotopy class of $Z(\mathbf{R})$ is one the following table.

$$\begin{array}{cccccccccccc}
 & & \frac{5}{1}2 & & & & & & & & \frac{1}{1}6 & \\
 & \frac{5}{1}1 & & \frac{4}{1}2 & & & & & & \frac{1}{1}5 & & 7 \\
\frac{5}{1} & & \frac{4}{1}1 & & \frac{3}{1}2 & & \frac{2}{1}3 & & \frac{1}{1}4 & & 6 & \\
 & \frac{4}{1} & & \frac{3}{1}1 & & \frac{2}{1}2 & & \frac{1}{1}3 & & 5 & & \\
 & & \frac{3}{1} & & \frac{2}{1}1 & & \frac{1}{1}2 & & 4 & & & \\
 & & & \frac{2}{1} & & \frac{1}{1}1 & & 3 & & & & \\
 & & & & \frac{1}{1} & & 2 & & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & \emptyset & & & & & &
\end{array}$$

Here we use the standard notation appeared in the articles[8,10]. If Z divides, then (iv) of Theorem 0.1 says some restriction about the isotopy type of $Z(\mathbf{R})$, and this is one of the following:

$$\frac{5}{1}2,\ \frac{4}{1}1,\ \frac{3}{1},\ \frac{2}{1}3,\ \frac{1}{1}2,\ 2,\ \frac{1}{1}6,\ 6.$$

This Rokhlin's formula says that the isotopy type 2 is not appeared. Moreover, the formula says that the isotopy type with complex orientation is one of the following:

$$\frac{3^+,2^-}{1^+}2,\ \frac{2^+,2^-}{1^+}1,\ \frac{1^+,2^-}{1^+},\ \frac{1^+,1^-}{1^+}3,\ \frac{1^-}{1^+}2,\ \frac{1^+}{1^+}6,\ 6.$$

Here we follow the notation used by A.Marin[13]. In this case, we can observe that the parity of the number of disoriented ovals (or negative injective pairs) is the parity of $\frac{1}{4}(\chi(P^+) - \frac{1}{2}\mathrm{Vol}_2(\Delta(f)))$. This observation will be generalized in §7. Since M-curves with prescribed topology above exist, and M-curves are dividing curves, we can claim the existence of the dividing curves whose isotopy types are $\frac{5}{1}2$ and $\frac{1}{1}6$. The existence of dividing curves with other isotopy types is not clear, and seems to be open.

4 Characteristic surfaces of $P(\mathbf{C})$

Let F be an immersed surface in a closed oriented 4-manifold M. F is called a *characteristic surface* if the $\mathbf{Z}/2$-homology class of F is dual to the second Stiefel-Whitney class $w_2(M)$. This is equivalent to that the $\mathbf{Z}/2$-valued intersection number $F.x$ is equal to the $\mathbf{Z}/2$-valued self-intersection number of x for each $x \in H_2(M; \mathbf{Z}/2)$.

Lemma 4.1 *$P(\mathbf{R})$ is a characteristic surface of $P(\mathbf{C})$.*

Proof It is enough to show that for each i the number of intersection points of $P(\mathbf{R})$ and a generic perturbation of $D_i(\mathbf{C})$ is congruent with c_i modulo 2. The intersection of $\tilde{D}_i$ with $P(\mathbf{R})$ is empty, if c_i is even; and one point defined by $(z_i, w_i) = (1, \varepsilon\delta)$, if c_i is odd. Since f_δ has no multiple zeros, this completes the proof. □

O.Saeki showed Lemma 4.1 in the following way. Let M be a branched double covering of the 4-sphere S^4 and R its ramification locus in M. Then he showed that the $\mathbf{Z}/2$-homology class of R is the Poincaré dual of $w_2(M)$. Since the quotient space of $P(\mathbf{C})$ by the natural complex conjugation is homeomorphic to S^4 (see Lemma 2.8[5]), we obtain Lemma 4.1.

Lemma 4.2 *Let J be the subset of $\{1, ..., d\}$ so that $i \in J$ is equivalent to that both $v_{i,1}, v_{i,2}$ are odd. Then $D_J(\mathbf{C})$ is a characteristic surface of $P(\mathbf{C})$.*

Proof It is enough to see $D_i(\mathbf{C}).D_J(\mathbf{C}) \equiv c_i \pmod 2$ for each i. This is direct computation. (Remark that successive numbers do not belong to J.) □

Suppose that Z divides. Let $F^\pm$ be smoothings of $Z_+ \cup P^\pm$ in $P(\mathbf{C})$. As stating in T.Fidler[4], the self-intersection number of $F^\pm$ in $P(\mathbf{C})$ is obtained by the following:

$$F^\pm.F^\pm = \frac{1}{2}D(\mathbf{C}).D(\mathbf{C}) - \chi(P^\pm). \tag{3}$$

Here, we remark that $\frac{1}{2}D(\mathbf{C}).D(\mathbf{C}) = \mathrm{Vol}_2(\Delta(f))$.

We set $\varepsilon_i^\pm$ is 1, if c_i is odd and $\pm f(t^{v_{i,1}}, t^{v_{i,2}})$ is positive for sufficiently large positive t; 0, otherwise.

We remark that the orientation of the first quadrant induced by the coordinate system (x, y) is agree with that induced by the restriction of the coordinate system (z_i, w_i) to the real part, since $x = z_i^{v_{i-1,1}} w_i^{v_{i,1}}$ and $y = z_i^{v_{i-1,2}} w_i^{v_{i,2}}$. Assume that ε is a small positive number. Set

$$I_{int}^\pm = \{i \in I^\pm : \varepsilon_i^\pm = 1\}.$$

Note that $P^\pm \cap \tilde{D}_i$ is a point, if $\varepsilon_i^\pm = 1$; empty, if $\varepsilon_i^\pm = 0$. Let C_i be a small circle centered at $P_i = P(\mathbf{R}) \cap \tilde{D}_i$. Here P_i is expressed by $(x, y) = (t^{v_{i,1}}, t^{v_{i,2}})$ for some sufficiently large positive t for $i \in I_{int}^\pm$. We may assume that C_i $(i \in I^\pm)$ is in $\tilde{F}_{I^\pm}^\pm$. Set $\tilde{Z}_{I^\pm}$ a surface with boundary obtained by taking positive smoothings at all double points of $Z_+ \cup D_{I^\pm}$. Note that $\tilde{Z}_{I^\pm}$ has an orientation induced by the natural complex structure of Z_+ and $\tilde{D}_i$ with $i \in I^\pm$. This orientation of $\tilde{Z}_{I^\pm}$ induces an orientation of C_i with $i \in I_{int}^\pm$. This orientation agrees with that induced by the coordinate system (x, y). In fact, since P_i is very close to the point Q_i defined by $(z_i, w_i) = (1, 0)$, it is enough to see the same assertion at Q_i. Near Q_i, $w = \prod_{i=1}^k (z_i^2 + a_i)(z_i - 1)$ and w_i give a complex coordinate system of $P(\mathbf{C})$ defined over real. If $c_i \geq 0$, then the image of the embedding defined by $\mathbf{C} \ni w \mapsto (w, w_i) = (w, \varepsilon\sqrt{-1}w) \in \mathbf{C}^2$ is $\tilde{D}_i$, near Q_i. If $c_i < 0$, then the image of the embedding defined by $\mathbf{C} \ni w \mapsto (w, w_i) = (w, \varepsilon\sqrt{-1}\overline{w}) \in \mathbf{C}^2$ is $\tilde{D}_i$, near Q_i. In any case, by elementary computation, we have

$$\begin{aligned}&(\text{orientation of } \tilde{D}_i) + (\text{orientation of } \mathbf{R}^2 \text{ induced by } (x, y))\\ &\qquad = \text{orientation of } \mathbf{C}^2 \quad \text{near } Q_i \quad (i \in I^\pm).\end{aligned}$$

This shows the following:

Lemma 4.3 $P^\pm.\tilde{D}_i = \varepsilon_i^\pm$.

Suppose that the $\mathbf{Z}/2$-homology class of $F^\pm$ is that of $\sum_{i=1}^d b_i^\pm D_i(\mathbf{C})$, where $b_i^\pm \in \{0, 1\}$.

Lemma 4.4 *The numbers $b_1^{\pm}, ..., b_d^{\pm}$ are obtained by solving the following equations:*

$$b_{i-1}^{\pm} + b_i^{\pm} c_i + b_{i+1}^{\pm} \equiv \varepsilon_i^{\pm} + \frac{1}{2} Vol_1(E_i) \pmod 2, \quad \text{for } i = 1, ..., d.$$

By Poincaré duality, these equations determine $b_1^{\pm}, ..., b_d^{\pm}$. In particular, Condition (A) implies that the $\mathbf{Z}/2$-homology class of F^+ is that of $\overline{D}(\mathbf{C})$.

Proof By §1.2, $F^{\pm}.\tilde{D}_i = b_{i-1}^{\pm} + b_i^{\pm} c_i + b_{i+1}^{\pm}$ for $i = 1, ..., d$. On the other hand, since $F^{\pm} = P^{\pm} \cup Z_+$, $F^{\pm}.\tilde{D}_i = \varepsilon_i^{\pm} + \frac{1}{2}\mathrm{Vol}_1(E_i)$ for $i = 1, ..., d$. Then, we have the first assertion. The remaining assertions are trivial. □

Lemma 4.5 *Let I^+, I^- be subsets of $\{1, ...d\}$ so that $F_{I^+}^{+} := F^+ \cup \tilde{D}_{I^+}$ and $F_{I^-}^{-} := F^- \cup \tilde{D}_{I^-}$ are characteristic surfaces. Here $\tilde{D}_I = \bigcup_{i \in I} \tilde{D}_i$, for $I \subset \{1, ..., d\}$. Then the $\mathbf{Z}/2$-homology class of $D_{I^+} + D_{I^-}$ is dual to $w_2(P(\mathbf{C}))$.*

Proof Easy computation. □

At each double point of $F_{I^\pm}^{\pm}$, we consider the orientations defined above. Taking a positive smoothing of $F_{I^+}^{+}$ (resp. $F_{I^-}^{-}$) at each double points, we obtain a nonsingular characteristic surface of $P(\mathbf{C})$, and we denote it by $\tilde{F}_{I^+}^{+}$ (resp. $\tilde{F}_{I^-}^{-}$).

Lemma 4.6

$$\tilde{F}_{I^\pm}^{\pm}.\tilde{F}_{I^\pm}^{\pm} = Vol_2(\Delta(f)) + \sum_{i \in I^\pm} (c_i + Vol_1(E_i)) + 2\#I_D^{\pm} + 2\#I_{int}^{\pm} - \chi(P^{\pm}),$$

where $\#I_D^{\pm}$ is the number of double points of $D_{I^\pm}$, and $\#I_{int}^{\pm}$ is the number of elements of $I_{int}^{\pm}$. Here E_i is the side of $\Delta(f)$ supported by v_i as defined in §1.

Proof Consequence of §1.2, Lemma 2.2 and (3). □

5 Guillou-Marin's Congruences

We first review the Brown invariant.

5.1 Brown invariant

Let V be a finite-dimensional vector space over $\mathbf{Z}/2$ with non-degenerate bilinear form $. : V \times V \to \mathbf{Z}/2$. A $\mathbf{Z}/4$-*quadratic* is a map $q : V \to \mathbf{Z}/4$ with $q(u + v) = q(u) + q(v) + 2u.v$.

Example: Here are examples of $\mathbf{Z}/4$-quadratics.

$$P_{\pm} = (V = \mathbf{Z}/2(v); v.v = 1; q(v) = \pm 1),$$

$$T_0 = (V = \mathbf{Z}/2(u) \oplus \mathbf{Z}/2(v); u.u = v.v = 0, u.v = 1; q(u) = q(v) = 0),$$
$$T_4 = (V = \mathbf{Z}/2(u) \oplus \mathbf{Z}/2(v); u.u = v.v = 0, u.v = 1; q(u) = q(v) = 2),$$

and their direct sum.

Since any indecomposable $\mathbf{Z}/4$-quadratic is isomorphic to one of $P_\pm, T_0, T_4$, a $\mathbf{Z}/4$-quadratic is isomorphic to some direct sum of $P_\pm, T_0, T_4$'s. If q is isomorphic to $aP_+ \oplus bP_- \oplus cT_0 \oplus dT_4$, we define $\beta(q)$ by $a - b + 4d$ mod 8 and call it by the *Brown invariant* of q. We have that

$$\exp \frac{\beta(q)\pi\sqrt{-1}}{4} = 2^{-\frac{\dim V}{2}} \sum_{v \in V} \exp \frac{q(v)\pi\sqrt{-1}}{2}. \tag{4}$$

Lemma 5.1 *(i) If there is a subspace H with $q(H) = 0$, $\dim H = \frac{1}{2}\dim V$, then $\beta(q) = 0$.*

(ii) Let $q_i : V \to \mathbf{Z}/4$ $(i = 1, 2)$ be two $\mathbf{Z}/4$-quadratics with respect to the same non-degenerate bilinear form .. Then $q_2(u) = q_1(u) + 2u.x$ for some $x \in V$, and $\beta(q_2) = \beta(q_1) - 2q_1(x)$.

See E.H.Brown Jr.[2] for proofs of (4), Lemma 5.1 and details on Brown invariant.

5.2 Rokhlin form

Let F be a nonsingular surface in an oriented 4-dimensional manifold M. Suppose that the natural map $H_1(F; \mathbf{Z}/2) \to H_1(M; \mathbf{Z}/2)$ is zero. Then any curve C embedded in F bounds a membrane $\mathcal{M}$ in M. Here, a *membrane* is a surface $\mathcal{M}$ in M, which bounded by C and normal to F along C, and nowhere tangent to F. Let $n(\mathcal{M})$ be the integer obtained by evaluating the obstruction class to extend the normal bundle of the embedding $C \subset F$ to a subline bundle in the normal bundle of the immersion $\mathcal{M} \subset M$ by the fundamental class of $(\mathcal{M}, C)$. Suppose that F is a characteristic surface. Then, $q(C) = n(\mathcal{M}) + 2\mathcal{M}.F \pmod 4$ is determined by the $\mathbf{Z}/2$-homology class of C, and the induced map $q : H_1(F; \mathbf{Z}/2) \to \mathbf{Z}/4$ is a $\mathbf{Z}/4$-quadratic with respect to the $\mathbf{Z}/2$-valued intersection form of F. The map q is called the *Rokhlin form* of F.

Theorem 5.2 (Guillou-Marin)

$$\textit{Signature of } M \equiv F.F + 2\beta(q) \pmod{16},$$

where $F.F$ is the self-intersection number of F in M.

See L.Guillou-A.Marin[9], and Y.Matsumoto[14], for proof of Theorem 5.2 and detailed discussion on the Rokhlin form.

Lemma 5.3 *Let F be a characteristic surface of M, and U an open set in M which is homeomorphic to a 4-ball. Let $z = (z_1, z_2) : U \to \mathbf{C}^2$ be a complex coordinate system whose image is a 4-ball centered at the origin in $\mathbf{C}^2$. Suppose that $F \cap U = z^{-1}\{z_1 z_2 = \varepsilon^2\}$ for some positive number ε. Set $C = z^{-1}\{z_1 z_2 = \varepsilon^2, z_1 = \overline{z_2}\}$. Then, $q(C) = 2$.*

Proof Set $D^2 = \{z \in \mathbf{C} : |z| \leq 1\}$. Let ι denote a map of D^2 to $\mathbf{C}^2$ defined by $\iota(z) = (\varepsilon z, \varepsilon \bar{z})$. Then the boundary of the image of ι is $z(C)$, and the image of ι is a membrane of $z(C)$. Consider the vector field $v := \mathrm{Re}(z(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_2}}))$. By elementary computation, v is tangent to $\{z_1 z_2 = \varepsilon^2\}$ and normal to the membrane. Since v has only non-degenerate zero at the origin, we have $q(C) = 2$. □

Remark that the 1-cycle $\{z_1 z_2 = \varepsilon^2, z_1 = \overline{z_2}\}$ tend to the origin in $\mathbf{C}^2$ and finally vanishes when ε tends to 0.

Proposition 5.4 *Let $q_{P(\mathbf{R})}$ be the Rokhlin form of $P(\mathbf{R})$ in $P(\mathbf{C})$. We have $q_{P(\mathbf{R})}(D_I(\mathbf{R})) = D_I(\mathbf{C})^2 \pmod 4$, and $\beta(q_{P(\mathbf{R})}) = 4 - d \pmod 8$.*

Proof By Lemma 4.1, $P(\mathbf{R})$ is a characteristic surface. The first assertion followed by the following: $q_{P(\mathbf{R})}(D_i(\mathbf{R})) = D_i(\mathbf{C})^2 = c_i \pmod 4$. This will be proved a discussion similar to our proof of Lemma 6.3. We next remark that the self-intersection number of $P(\mathbf{R})$ in $P(\mathbf{C})$ is $-\chi(P(\mathbf{R}))$, by Lemma 2.1. By §1.1, Lemma 4.1 and Lemma 5.2, we complete the proof. □

Proposition 5.5 *Let $q_{D_J(\mathbf{C})}$ be the Rokhlin form of $D_J(\mathbf{C})$ in $P(\mathbf{C})$. We have $q_{D_J(\mathbf{C})} = 0 \pmod 4$, and $\sum_{j \in J} c_j \equiv 4 - d \pmod{16}$.*

Proof By Lemma 4.2, $D_J(\mathbf{C})$ is a characteristic surface. Since D_J has no double points, we obtain that $H_1(D_J(\mathbf{C}); \mathbf{Z}/2) = 0$. Thus $q_{D_J(\mathbf{C})} = 0$, and the proposition holds, because of Lemma 4.3 and Theorem 5.2. □

6 Computation of Rokhlin form

Let $q^{\pm}$ be the Rokhlin form of $\tilde{F}^{\pm}_{I^{\pm}}$. Then, by Lemma 4.6 and Theorem 5.2 we have the following congruence:

$$\chi(P^{\pm}) \equiv \mathrm{Vol}_2(\Delta(f)) + d - 4 + \sum_{i \in I^{\pm}} (c_i + \mathrm{Vol}_1(E_i)) + 2(\# I^{\pm}_{D} + \# I^{\pm}_{int} + \beta(q^{\pm})) \pmod{16}. \quad (5)$$

Thus, if we compute the Brown invariant $\beta(q^{\pm})$, we obtain a congruence for $\chi(P^{\pm})$ modulo 16. In this section, we compute the Rokhlin form $q^{\pm}$ and their Brown invariant $\beta(q^{\pm})$ under some conditions. We first introduce notation. Hereafter, we understand that Z_{+} and D_i ($i \in I^{\pm}$) are oriented

by their natural complex structures. Set $\tilde{Z}^{\pm}(\mathbf{R}) = Z(\mathbf{R}) \cup \bigcup_{i \in I^{\pm}} C_i$, and $\tilde{P}^{\pm} = P^{\pm} - \bigcup_{i \in I^{\pm}} B_i$ where B_i is the disc bounded by C_i in $P(\mathbf{R})$.

Let $L^{\pm}$ be the subspace of $H_1(F^{\pm}; \mathbf{Z}/2)$ generated by components of $Z(\mathbf{R})$, and $V^{\pm}$ the image of the natural map $H_1(F^{\pm}; \mathbf{Z}/2) \to H_1(F, Z(\mathbf{R}); \mathbf{Z}/2)$. Obviously, we have $H_1(F; \mathbf{Z}/2) \simeq L^{\pm} \oplus V^{\pm}$. We set

$$U^{\pm} = \mathrm{Im}\{H_1(\tilde{P}^{\pm}; \mathbf{Z}/2) \to H_1(\tilde{P}^{\pm}, \tilde{Z}^{\pm}(\mathbf{R}); \mathbf{Z}/2)\}, \text{ and}$$
$$W^{\pm} = \mathrm{Im}\{H_1(\tilde{Z}_{I^{\pm}}; \mathbf{Z}/2) \to H_1(\tilde{Z}_{I^{\pm}}, \tilde{Z}^{\pm}(\mathbf{R}); \mathbf{Z}/2)\}.$$

By the isomorphism

$$H_1(\tilde{F}^{\pm}, \tilde{Z}^{\pm}(\mathbf{R}); \mathbf{Z}/2) \simeq H_1(\tilde{P}^{\pm}, \tilde{Z}^{\pm}(\mathbf{R}); \mathbf{Z}/2) \oplus H_1(\tilde{Z}_{I^{\pm}}, \tilde{Z}^{\pm}(\mathbf{R}); \mathbf{Z}/2),$$

we consider $U^{\pm} \oplus 0$ and $0 \oplus W^{\pm}$ are subspaces of $V^{\pm}$. By abuse of language, we denote them by $U^{\pm}$, $W^{\pm}$, respectively. We set

$$\tilde{L}^{\pm} = \{u \in H_1(\tilde{F}^{\pm}_{I^{\pm}}; \mathbf{Z}/2) : u.v = 0, \forall v \in U^{\pm} \oplus W^{\pm}\}.$$

If Z is an $(M - i)$-curve, then

$$\dim W^{\pm} = i + \frac{1}{2} \sum_{i \in I^{\pm}} \mathrm{Vol}_1(E_i) + \# I^{\pm}_{D} - \# I^{\pm}_{int},$$
$$\dim L^{\pm} = M - i + \# I^{\pm}_{int} - h_0(\tilde{P}^{\pm}), \text{ and}$$
$$\dim \tilde{L}^{\pm} = 2 \dim L^{\pm},$$

where $h_0(\tilde{P}^{\pm})$ is the number of components of $\tilde{P}^{\pm}$. Moreover, the restrictions of the $\mathbf{Z}/2$-valued intersection form . to $U^{\pm}$, $W^{\pm}$, and $\tilde{L}^{\pm}$ are non-degenerate. Computing the Euler characteristic of $\tilde{F}^{\pm}_{I^{\pm}}$, we have

$$H_1(\tilde{F}^{\pm}_{I^{\pm}}; \mathbf{Z}/2) \simeq U^{\pm} \oplus \tilde{L}^{\pm} \oplus W^{\pm}.$$

Thus, if we set $\beta^{\pm}_{U} = \beta(q^{\pm}|U^{\pm}), \beta^{\pm}_{W} = \beta(q^{\pm}|W^{\pm}), \beta^{\pm}_{L} = \beta(q^{\pm}|\tilde{L}^{\pm})$, we have the following:

Lemma 6.1 $\beta(q^{\pm}) = \beta^{\pm}_{U} + \beta^{\pm}_{W} + \beta^{\pm}_{L}$.

Lemma 6.2 $\beta^{\pm}_{W} = 0, 4$. *If Z is an M-curve and $I^{\pm} = \emptyset$, then $\beta^{\pm}_{W} = 0$.*

Proof Since $\tilde{Z}^{+}_{I^{\pm}}$ is orientable, the classification of $\mathbf{Z}/4$-quadratics gives the lemma. □

We next compute the Rokhlin form on $L^{\pm} \oplus U^{\pm}$.

Lemma 6.3 *If a closed curve C in $P^{\pm}$ is boundary of a surface $\mathcal{M}$ in $P(\mathbf{R})$, then $q^{\pm}(C) = 2\chi(\mathcal{M} \cap P^{\mp}) \pmod 4$.*

Proof By III Remarque 3 in the paper by Guillou-Marin[9], we use here the method of vector fields. For a vector field v on $\mathcal{M}$ normal to C, consider an extension $\tilde{v}$ of $\sqrt{-1}v$ to some neighborhood of $\mathcal{M}$. Since $\tilde{v}|C$ is a normal vector field of the embedding $C \subset F$, the obstruction number to extend $\tilde{v}|C$ to a normal vector field of the embedding $\mathcal{M} \subset P_\Delta(\mathbf{C})$ is the sum of its indices, which is equal to $-\chi(\mathcal{M})$. Thus, $n(\mathcal{M}) = 2(-\chi(\mathcal{M}))$. By Lemma 2.1, $\mathcal{M}.F = \chi(\mathcal{M} \cap F)$. Therefore, $q^{\pm}(C) = 2(-\chi(\mathcal{M}) + \chi(\mathcal{M} \cap P^{\pm})) = -2\chi(\mathcal{M} \cap P^{\mp})$.
□

Under Condition (A), by Lemma 6.3, we have the following: If C is an odd oval surrounds an odd (resp. even) number of other ovals, then $q^+(C) = 0$ (resp. 2). If C is an even oval surrounds an odd (resp. even) number of other ovals, then $q^-(C) = 0$ (resp. 2). These conditions were first formulated by T.Fidler[4].

Lemma 6.4 *Assume that $D_i(\mathbf{R})$ is in $P^{\pm}$. Then D_i^+ is a membrane of $D_i(\mathbf{R})$, and we have $n(D_i(\mathbf{R})) = c_i$. Thus,*

$$q^{\pm}(D_i(\mathbf{R})) = c_i + 2Z_+.D_i^+ + 2\tilde{D}_{I^\pm}.D_i^+ \pmod 4.$$

Proof In order to compute $n(D_i(\mathbf{R}))$, we present here a discussion on counting "the number of half twists of the normal bundle of C in F in the restriction to C of a trivialization of the normal bundle of the membrane in the ambient 4-manifold," due to Y.Matsumoto[14]. See III Remarque 2 of the paper Guillou-Marin[9]. To avoid the term 2Self(C) in the definition of the Rokhlin form at the bottom of page 132 of the paper by Y.Matsumoto[14], we take the orientations of fibers of the normal bundle of the membrane opposite to that defined ibid.. Considering that the coordinate system (z_{i+1}, w_{i+1}) gives a trivialization of the normal bundle of D_i^+ in $P_\Delta(\mathbf{C})$, we obtain the number of half twists of the normal bundle of C in $P(\mathbf{C})$ is equal to c_i. Since $D_i^+.\tilde{F}_{I^\pm}^{\pm} = Z_+.D_i^+ + \tilde{D}_{I^\pm}.D_i^+$, we complete the proof. □

Lemma 6.5 *If $\frac{1}{2}Vol_1(E_i)$ is even, then $Z_+.D_i^+ \equiv \frac{1}{4}Vol_1(E_i) \pmod 2$.*

Proof Replacing D_i^+ by D_i^- in the proof above, we obtain that $q^{\pm}(D_i(\mathbf{R})) = c_i + 2Z_+.D_i^- + 2\tilde{D}_{I^\pm}.D_i^- \pmod 4$. Thus,

$$Z_+.D_i^+ + \tilde{D}_{I^\pm}.D_i^+ \equiv Z_+.D_i^- + \tilde{D}_{I^\pm}.D_i^- \pmod 2, \tag{6}$$

and $\frac{1}{2}\mathrm{Vol}_1(E_i) + \tilde{D}_{I^\pm}.D_i(\mathbf{C}) = (Z_+ + \tilde{D}_{I^\pm}).D_i(\mathbf{C})$ is even. Since $\frac{1}{2}\mathrm{Vol}_1(E_i)$ is even, so is $\tilde{D}_{I^\pm}.D_i(\mathbf{C})$, and $Z_+.D_i^+ \equiv \frac{1}{4}\mathrm{Vol}_1(E_i) \pmod 2$, because of (6).
□

Remark 6.6 *By definition of $\tilde{D}_{I^\pm}$, it is easy to see $\tilde{D}_{I^\pm}.D_i^+ = k_i + \chi_{I^\pm}(i-1) + \chi_{I^\pm}(i+1)$. Here, $\chi_I(j) = 1$, if $j \in I$; 0, otherwise, for $I \subset \{1, ..., d\}$.*

Lemma 6.7 *Condition (A) implies $\beta_U^+ = 0$. If $\Delta(f)$ is bi-even, that is, twice of some even polygon, then Condition(A) implies $I^- = \emptyset$ and $\beta_U^- = 4 - d - \frac{1}{4}Vol_2(\Delta(f))$.*

Proof The first sentence is trivial, since $U^+ = 0$. We assume that $\Delta(f)$ is bi-even. Then, $D = 4\overline{\overline{D}}$, where $\overline{\overline{D}} = \sum \overline{\overline{a}}_i D_i$. Remark that $\overline{\overline{D}}(\mathbf{C})^2 = (\frac{1}{4})^2 D(\mathbf{C})^2 = \frac{1}{8}\mathrm{Vol}_2(\Delta(f))$. By Lemma 4.4, the $\mathbf{Z}/2$-homology class of F^+ is zero. By Lemma 4.1, F^- is characteristic, and $I^- = \emptyset$. Let x be the $\mathbf{Z}/2$-homology class of $\overline{\overline{D}}(\mathbf{R})$ in $U^- \subset H_1(F^-; \mathbf{Z}/2)$, and $q_R : U^- \to \mathbf{Z}/4$ the $\mathbf{Z}/4$-quadratic defined by $q_R(D_I(\mathbf{R})) = D_I(\mathbf{C})^2 \pmod 4$ for $I \subset \{1, ..., d\}$ with $D_I(\mathbf{R}) \subset P^-$. Since $x.D_i(\mathbf{R}) = \frac{1}{4}\mathrm{Vol}_1(E_i)$ for $D_i(\mathbf{R}) \subset P^-$, we obtain $q|U^-(u) = q_R(u) + 2x.u$ for $u \in U^-$, and thus $\beta(q|U^-) = \beta(q_R) - q_R(\overline{\overline{D}}(\mathbf{R})) = 4 - d - \frac{1}{4}\mathrm{Vol}_2(\Delta(f)) \pmod 8$, by (ii) of Lemma 5.1 and Lemma 5.4. □

A similar argument shows the following

Lemma 6.8 *Let C be a closed curve in $P^{\pm}$, and $\mathcal{M}_0$ is the closure of the union of some components of $P(\mathbf{R}) - \bigcup_{i=1}^d D_i(\mathbf{R}) - C$ whose boundary contains C. Let I be a subset of $\{1, ..., d\}$ so that $C \cup \bigcup_{i \in I} D_i(\mathbf{R})$ is the boundary of $\mathcal{M}_0$. Then $q^{\pm}(C) = 2\chi(\mathcal{M}_0 \cap P^{\mp}) + \sum_{i \in I^\pm}(c_i + 2Z_+.D_i^+ + 2\tilde{D}_{I^\pm}.D_i^+) + 2\#I_D^{\pm}$. Here $\#I_D^{\pm}$ is the number of double points of $D_{I^\pm}$.*

We omit the detailed proof, since we do not use it later.

Thirdly we compute the Brown invariant $\beta_L^{\pm}$. To do this we need some definitions and suppositions. Remark that $D_J(\mathbf{R})$ is a subset of $P(\mathbf{R})$ so that $P(\mathbf{R}) - D_J(\mathbf{R})$ is orientable. We suppose the following:

Condition (B). There is a deformation D' of $D_J(\mathbf{R})$ in $P(\mathbf{R})$ so that $Z(\mathbf{R}) \cap D' = \emptyset$.

Remark that Condition (A) implies Condition (B). Because $L^{\pm}.L^{\pm} = 0$, we remark that $q(L^{\pm}) \subset \{0, 2\}$. Let $L_2^{\pm}$ be the set of components C of $Z(\mathbf{R})$ with $q^{\pm}(C) = 2$. Since Z_+ is an oriented surface, $Z(\mathbf{R})$ has an orientation as boundary of Z_+. This orientation is called a *complex orientation* of $Z(\mathbf{R})$. Next we consider an orientation of $P(\mathbf{R}) - D'$. This induces an orientation of $P^{\pm} - D'$, and thus induces an orientation of $Z(\mathbf{R})$ as boundary of $P^{\pm} - D'$. We call this orientation a *real orientation* of $Z(\mathbf{R})$.

We say that $L_2^{\pm}$ is *even* (resp. *odd*) *oriented*, if the number of components in $L_2^{\pm}$ of which the real and complex orientations are disagree are even (resp. odd).

Lemma 6.9 *For $\beta_L^{\pm}$, the followings hold.*

(i) If $q^{\pm}(L^{\pm}) = 0$, then $\beta_L^{\pm} = 0$.

(ii) If $L_2^{\pm}$ is even (resp. odd) oriented, then $\beta_L^{\pm} = 0, 4$ (resp. $2, -2$).

Proof (i) is a consequence of (i) of Lemma 5.1. (ii) is a consequence of a combinatorial discussion. The key facts are Lemma 5.1(i) and the following: Let V be a 2-dimensional $\mathbf{Z}/2$-vector space generated by u and v, i.e. $V = \mathbf{Z}/2(u) \oplus \mathbf{Z}/2(v)$, . a nondegenerate bilinear form with $u.u = 0, v.v = u.v = 1$, and $q : V \to \mathbf{Z}/4$ a $\mathbf{Z}/4$-quadratic. If $q(u) = 2$, then the Brown invariant of q is ± 2. □

Lemma 6.10 *(i) If $\Delta(f)$ is bi-even, then the number of elements of $L_2^{\pm}$ is even.*

(ii) If Condition (A) holds, then the number of elements of L_2^{+} is even.

Proof (i): Replacing Z_+ by Z_- in the discussion above, we obtain a congruence similar to (5). Trivially each terms in the right side in (5) does not change except $\beta(q^{\pm})$. By Lemma 6.2, $\beta_W^{\pm}$ (mod 4) do not change. By Lemmas 6.4–6.8, $\beta_U^{\pm}$ do not change either. Note that the complex orientation of $Z(\mathbf{R})$ induced from Z_- is opposite to that from Z_+. If the number of elements of $L_2^{\pm}$ is odd, then $L_2^{\pm}$ must change. This is a contradiction.

(ii): Since $Z(\mathbf{R})$ is boundary of $\tilde{Z}^{+}_{I^+}$, (ii) is trivial. □

7 Consequences

We discuss some consequences come from the discussion above. Throughout this section, we assume that Z is a dividing curve.

7.1 *Projective plane*

First we consider the case $P = P^2$ (the projective plane). Let Δ be the convex hull of the three points $(0,0), (2k, 0), (0, 2k)$, and $v_1 = \binom{0}{1}, v_2 = \binom{-1}{-1}, v_3 = \binom{1}{0}$. Let f be a non-degenerate real polynomial with $\Delta(f) = \Delta$. Then, $D = 2\overline{D}$ with $\overline{D} = kD_2$. Suppose that Condition (A) holds.

Proposition 7.1 *In the case k is even, we have the followings.*

(i) If Z is an M-curve and $L_2^- = \emptyset$, then $N^+ - N^- \equiv -k^2 \pmod{16}$.

(ii) If L_2^+ is even (resp. odd) oriented, then

$N^+ - N^- \equiv k^2$ *(resp.* $k^2 + 4$) (mod 8).

(iii) If L_2^- is even (resp. odd) oriented, then

$N^+ - N^- \equiv k^2$ *(resp.* $k^2 + 4$) (mod 8).

Proof In this case, we have $I^+ = \{2\}, I^- = \emptyset$. By (5), Lemmas 6.1, 6.4 and 6.5, we obtain $\chi(P^+) = 2k^2 + 2k + 2\beta_W^+ + 2\beta_L^+$, and $\chi(P^-) = 2k^2 - 1 + 2\beta_W^- + 2\beta_L^- + 2\beta_U^-$. By §1.1, Lemmas 6.2, 6.7, and 6.9, these complete the proof. □

Proposition 7.2 *In the case k is odd, the followings hold.*

(i) If Z is an M-curve and $L_2^+ = \emptyset$, then $N^+ - N^- \equiv 1 \pmod{16}$.

(ii) If L_2^+ is even (resp. odd) oriented, then $N^+ - N^- \equiv 1$ *(resp.* 5) (mod 8).

(iii) If L_2^- is even oriented and $Z_+.D_2^+$ is even (resp. odd), then

$N^+ - N^- \equiv -1 - 2k$ *(resp.* $3 - 2k$) (mod 8).

(iv) If L_2^- is odd oriented and $Z_+.D_2^+$ is even (resp. odd), then

$N^+ - N^- \equiv 3 - 2k$ *(resp.* $-1 - 2k$) (mod 8).

Proof In this case, we have $I^+ = \emptyset, I^- = \{2\}$. By (5), Lemmas 6.1, 6.4 and 6.5, we obtain $\chi(P^+) = 2k^2 - 1 + 2\beta_W^+ + 2\beta_L^+$, and $\chi(P^-) = 2k^2 + 2k + 2 + 2\beta_W^- + 2\beta_L^- + 2\beta_U^-$. §1.1, Lemmas 6.2, 6.7 and 6.9 complete the proof. □

T. Fidler[4] showed the congruences (i) of Proposition 7.1 and (i) of Proposition 7.2 above.

7.2 *Hirzebruch surfaces*

We next consider the case $P = F_a$ (the Hirzebruch surface). Let Δ be the convex hull of the four points $(0,0), (2k_1 + 2ak_2, 0), (2k_1, 2k_2), (0, 2k_2)$, and $v_1 = \binom{0}{1}, v_2 = \binom{-1}{-a}, v_3 = \binom{0}{-1}, v_4 = \binom{1}{0}$. Let f be a non-degenerate real polynomial with $\Delta(f) = \Delta$. Then, $D = 2\overline{D}$ with $\overline{D} = (k_1 + ak_2)D_2 + k_2 D_3$, and, the $\mathbf{Z}/2$-homology class of $F^{\pm}$ is that of $(a\varepsilon_2^{\pm} + k_1 + ak_2)D_2(\mathbf{C}) + (\varepsilon_2^{\pm} + k_2)D_3(\mathbf{C})$.

Proposition 7.3 *Assume that* $a\varepsilon_2^{\pm} + k_1 + ak_2 \equiv \varepsilon_2^{\pm} + k_2 \equiv 0 \pmod 2$. *Then, the followings hold.*

(i) If Z is an M-curve and $L_2^{\pm} = \emptyset$, then

$\chi(P^{\pm}) \equiv 2k_2(2k_1 + ak_2) + 2\beta_U^{\pm} \pmod{16}$.

(ii) If $L_2^{\pm}$ is even oriented, then $\chi(P^{\pm}) \equiv 2k_2(2k_1 + ak_2) + 2\beta_U^{\pm} \pmod 8$.

(iii) If $L_2^{\pm}$ is odd oriented, then $\chi(P^{\pm}) \equiv 2k_2(2k_1 + ak_2) + 2\beta_U^{\pm} + 4 \pmod 8$.

If $k_1 \equiv k_2 \equiv 0 \pmod 2$, *then Condition (A) implies* $\beta_U^+ = 0$, *and* $\beta_U^- = (2k_1 + ak_2)k_2/2$.

Proof By assumption, $I^{\pm} = \emptyset$. By (5), Lemmas 6.1, 6.2, 6.7, and 6.9, we obtain (i) -(iii). The last statement is a consequence of Lemma 6.7. □

7.3 Other toric surfaces

It is possible to obtain some consequence for other toric surface. We first present this by an example. Let Δ be the convex hull of the three points $(0,0), (6k,0), (0,4k)$, and $v_1 = \binom{0}{1}, v_2 = \binom{-1}{-1}, v_3 = \binom{-2}{-3}, v_4 = \binom{-1}{-2}, v_5 = \binom{0}{-1}, v_6 = \binom{1}{0}$. Let f be a non-degenerate real polynomial with $\Delta(f) = \Delta$. Then, $D = 2\overline{D}$ with $\overline{D} = k(3D_2 + 6D_3 + 4D_4 + 2D_5)$. Note that $D_2(\mathbf{C})$ is a characteristic surface. Assume that Condition (A) holds.

Proposition 7.4 *In the case k is even, we have the following:*

(i) *If Z is an M-curve and $L_2^- = \emptyset$, then* $N^+ - N^- \equiv -6k^2 \pmod{16}$.

(ii) *If L_2^+ is even (resp. odd) oriented, then* $N^+ - N^- \equiv 0$ (*resp.* 4) $\pmod 8$.

(iii) *If L_2^- is even (resp. odd) oriented, then* $N^+ - N^- \equiv 0$ (*resp.* 4) $\pmod 8$.

Proof In this case, we have $I^+ = \{2\}, I^- = \emptyset$. By (5), Lemmas 6.1, and 6.7 we obtain $\chi(P^+) = 12k^2 - 1 + 2\beta_W^+ + 2\beta_L^+ + 2\beta_U^+$, and $\chi(P^-) = 12k^2 + 2 + 2\beta_W^- + 2\beta_L^- + 2\beta_U^-$. §1.1, Lemmas 6.2, 6.7 and 6.9 complete the proof. □

Proposition 7.5 *In the case k is odd, we have the following:*

(i) *If Z is an M-curve and $L_2^+ = \emptyset$, then* $N^+ - N^- \equiv 6 \pmod{16}$.

(ii) *If L_2^+ is even (resp. odd) oriented, then* $N^+ - N^- \equiv 6$ (*resp.* 2) $\pmod 8$.

(iii) *If L_2^- is even (resp. odd) oriented, then* $N^+ - N^- \equiv 2$ (*resp.* 6) $\pmod 8$.

Proof In this case, we have $I^+ = \emptyset, I^- = \{2\}$. By (5), Lemmas 6.1, and 6.7, we obtain $\chi(P^+) = 12k^2 + 2 + 2\beta_W^+ + 2\beta_L^+ + 2\beta_U^+$, and $\chi(P^-) = 12k^2 + 2\beta_W^- + 2\beta_L^- + 2\beta_U^-$. §1.1, Lemmas 6.2, 6.7 and 6.9 complete the proof. □

For general case, we can state the following

Proposition 7.6 *Assume that $\Delta(f)$ is bi-even and Condition (A) holds. Then we have*

(i) *If Z is an M-curve and $L_2^- = \emptyset$, then*
$N^+ - N^- \equiv -\frac{1}{2} Vol_2(\Delta(f)) \pmod{16}$.

(ii) *If L_2^- is even oriented, then* $N^+ - N^- \equiv -\frac{1}{2} Vol_2(\Delta(f)) \pmod 8$.

(iii) *If L_2^- is odd oriented, then* $N^+ - N^- \equiv -\frac{1}{2} Vol_2(\Delta(f)) + 4 \pmod 8$.

Proof In this case, we have $I^- = \emptyset$. By (5), Lemmas 6.1, 6.2, 6.7, 6.9, and §1.1, we complete the proof. □

Proposition 7.7 *Assume that $\Delta(f)$ is bi-characteristic, that is, $\overline{D}(\mathbf{C})$ represents a class which is Poincaré dual to $w_2(P(\mathbf{C}))$, and Condition (A) holds. Then we have*

(i) *If Z is an M-curve and $L_2^+ = \emptyset$, then*
$N^+ - N^- \equiv Vol_2(\Delta(f)) + d - 4 \pmod{16}$.

(ii) *If L_2^+ is even oriented, then* $N^+ - N^- \equiv Vol_2(\Delta(f)) + d - 4 \pmod 8$.

(iii) *If L_2^+ is odd oriented, then* $N^+ - N^- \equiv Vol_2(\Delta(f)) + d \pmod 8$.

Proof In this case, we have $I^+ = \emptyset$. By (5), Lemmas 6.1, 6.2, 6.7, 6.9, and §1.1, we complete the proof. □

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On Exact Poisson Manifolds of Dimension 3

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Abstract

We investigate topological properties of the foliation which is associated with an exact Poisson manifold and shows that there are many examples of exact Poisson structures on closed manifolds.

1 Introduction

A Poisson manifold is a pair (M, Π) of a C^∞-manifold and a 2-vector field on it, which satisfies

$$[\Pi, \Pi] = 0,$$

where [,] denotes the Schouten bracket.

We call the condition $[\Pi, \Pi] = 0$ for a 2-vector field Π the *Poisson condition* and Π the *Poisson bi-vector* . The Poisson bracket $\{f, g\}$ of $f, g \in C^\infty(M)$ is defined by

$$\{f, g\} = \Pi(df, dg).$$

It satisfies the following well- known property.

(1) $(f, g) \mapsto \{f, g\} \quad f, g \in C^\infty(M)$ gives a Lie algebra structure (over $\boldsymbol{R}$) of $C^\infty(M)$, that is, the pairing $\{f, g\}$ is skew-symmetric bilinear on both components and it satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

(2) $\{f, gh\} = \{f, g\}h + g\{f, h\}$ holds for $f, g, h \in C^\infty(M)$.

For any 2-vector field Π on M, we define a homomorphism of bundles

$$I = I_\Pi : T^*M \to TM$$

which, at each point $x \in M$, is given by

$$I_x(\alpha_x) = \Pi_x(\alpha_x, \cdot) = i_{\alpha_x}\Pi, \qquad \alpha_x \in T^*_x M.$$

Here, we used the notation of interior product to express a contraction of tensors.

The rank of the linear map I_x is called the *rank of* Π *at* x and it is denoted by rank Π_x. If the rank Π_x is constant on the whole manifold, (M, Π) is called *regular*. In this paper, we are mainly concerned with regular Poisson manifolds.

One of the geometric aspects of a Poisson manifold (M, Π) is the fact that the distribution (plane field) given by

$$Image(I_x) \subset T_x M$$

is integrable ([9]), and hence it defines a smooth foliation, at least when (M, Π) is regular. We denote this foliation (integrable distribution) by $\mathcal{F} = \mathcal{F}_\Pi$. It is called the *characteristic foliation* of (M, Π), and its leaves are called *symplectic leaves*, since Π restricted to each leaf naturally defines a symplectic structure on it.

In section2, we review some basic facts about the Schouten bracket, especially its relationship with the generalized divergence of a multi-vector field. We also give a necessary and sufficient condition for a plane field defined by a regular 2-vector field to be integrable, in terms of the Schouten bracket and the generalized divergence. In sections 4, 5, we consider exact Poisson manifolds. A Poisson manifold (M, Π) is called an *exact Poisson manifold* ([9]), if there exists a vector field Z such that $[Z, \Pi] = -\Pi$ ([1]). We ask ourselves which codimension one foliation of a closed 3-manifold has such exact Poisson manifold structure. In section 5 , we will give an explicit construction of exact Poisson structure whose characteristic foliation has exceptional leaves.

All the manifolds in this paper are assumed C^∞. $\Lambda^p TM$ denotes the p-th exterior space bundle of the tangent bundle of M and $\Gamma(\Lambda^p TM)$ denotes the set of smooth section of it, that is the space of p-vector fields.

2 Generalized Divergence and the Schouten Bracket

Let M be a smooth manifold and P a p-vector field, that is, $P \in \Gamma(\Lambda^p(TM))$, $(p \geq 0)$. Let

$$c : \Gamma((T^*M)) \otimes \Gamma(\Lambda^p(TM)) \to \Gamma(\Lambda^{(p-1)}(TM))$$

denote the contraction.

DEFINITION 1 *Let ∇ be a connection (covariant differenciation) on M and P a p-vector filed. Then the $(p-1)$-vector field $Div_\nabla P$ given by*

$$Div_\nabla P = c(\nabla P)$$

is called a generalized divergence of P associated with the connection ∇.

It is shown that if ∇ is the Levi-Civita connection of a Riemannian metric and $P = X$ is a vector field, $Div_\nabla X$ coincides with the usual divergence $divX$ with respect to the Riemannian volume Ω, that is, $L_X \Omega = (divX)\Omega$ holds (L_X denotes the Lie derivation).

Although the generalized divergence of a p -vector field depends on the choice of the connection ∇, we often omit ∇ from the notation and write $DivP$ for $Div_\nabla P$. It is not

always true that $Div^2 = Div \circ Div = 0$. It is proved, however, if one chooses a connection which preserves a volume form that $Div^2 = 0$ holds. In fact, if ∇ preserves a volume form Ω, one can see the following relation of Div and d(= exterior differential) holds.

$$d(\Omega(P)) = (-1)^p \Omega(DivP).$$

One of the definition of the Schouten bracket $[P, Q]$ is the following ([6]).

DEFINITION 2 *Let Div be a generalized divergence associated with a torsion free connection of a manifold M. Let $P \in \Gamma(\Lambda^p(TM)), Q \in \Gamma(\Lambda^p(TM))$ be p-vector field and q-vector field on M, respectively. The $(p+q-1)$-vector field $[P, Q]$ defined by*

$$[P, Q] = Div(P \wedge Q) - (Div(P) \wedge Q + (-1)^p P \wedge Div(Q))$$

is called the Schouten bracket of P and Q.

It is proved that $[P, Q]$ is well-defined, namely, it is independent of the choice of torsion free connection involved.

The following is a list of some basic properties of the Schouten bracket ([9]). Here, f, g are smooth functions and P, Q, R are a p-vector field, a q-vector field and an r-vector field, respectively. Also, we use the interior product notation for the contraction.

1. For $f, g \in C^\infty(M)$, $[f, g] = 0$,
2. $[f, Q] = i_{df} Q$, more generally, $[fP, Q] = (-1)^p P \wedge (i_{df} Q) + f[P, Q]$,
3. $[P, Q] = (-1)^{pq}[Q, P]$,
4. Let $P = X$ be a vector field, then $[X, Q] = L_X Q$ (the Lie derivative),
5. $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]$,
6. $Div[P, Q] = -[DivP, Q] - (-1)^p [P, DivQ]$, when $Div^2 = 0$,
7. $[P, [Q, R]] = (-1)^{p-1}[[P, Q], R] + (-1)^{(p-1)(q-1)}[Q, [P, R]]$ (generalized Jacobi identity).

Integrability of a plane field

Let Π be a 2-vector field on M. Assume the rank of Π is equal to $2l$ $(0 \leq 2l \leq \dim M)$ everywhere on M. Recall that Π defines a distribution $\mathcal{F}_\Pi$ which gives the following subspace of $T_x M$;

$$\mathcal{F}_{\Pi, x} = \{\Pi_x(\alpha_x, \cdot) \in T_x M | \alpha_x \in T^* M\}.$$

In this subsection, we prove the following

THEOREM 1 *The distribution $\mathcal{F}_\Pi$ defined by a regular 2-vector field Π whose rank is $2l$, is integrable if and only if $[\Pi, \Pi^l] = [\Pi, \Pi \wedge \cdots \wedge \Pi] = 0$ holds.*

We first prove following formula.

LEMMA 2 *If* rank $\Pi = 2l$, *then*

$$[\Pi, \Pi^l] = -2Div\Pi \wedge \Pi^l,$$

where $Div\Pi$ *is defined by choosing any torsion free connection on* TM.

PROOF Since $\Pi \wedge \Pi^l = 0$, we have

$$[\Pi, \Pi^l] = -Div\Pi \wedge \Pi^l - \Pi \wedge Div\Pi^l. \tag{1}$$

Plugging the following

$$Div\Pi^l = Div\Pi \wedge \Pi^{l-1} + \Pi \wedge Div\Pi^{l-1} + [\Pi, \Pi^{l-1}] \tag{2}$$

into the above (1), we have

$$[\Pi, \Pi^l] = -2Div\Pi \wedge \Pi^l - \Pi^2 \wedge Div\Pi^{l-1} - \Pi \wedge [\Pi, \Pi^{l-1}]. \tag{3}$$

Again plugging (2) for $l-1$ into (3) above we obtain

$$-3Div\Pi \wedge \Pi^l - \Pi^3 \wedge Div\Pi^{l-2} - \Pi^2 \wedge [\Pi, \Pi^{l-2}] - \Pi \wedge [\Pi, \Pi^{l-1}]. \tag{4}$$

Repeating this we have

$$[\Pi, \Pi^l] = -lDiv\Pi \wedge \Pi^l - \sum_{i=1}^{l-1} \Pi^i \wedge [\Pi, \Pi^{l-i}] \tag{5}$$

Using $[\Pi, \Pi^k] = k\,[\Pi, \Pi] \wedge \Pi^{k-1}$ *for* $k \geq 1$, we get $\frac{l^2}{2}[\Pi, \Pi] \wedge \Pi^{l-1} = -l\, Div\Pi \wedge \Pi^l$. From this, we obtain

$$[\Pi, \Pi^l] = l\,[\Pi, \Pi] \wedge \Pi^{l-1} = -2Div\Pi \wedge \Pi^l.$$

□

Proof of Theorem 1 Let the distribution $\mathcal{F}_\Pi$ be of codimension q and defined by a local equation

$$\alpha_1 = \cdots = \alpha_q = 0.$$

Then we have

$$\Pi(\alpha_j, \cdot) = 0, \quad j = 1, \ldots, q.$$

Taking the covariant derivative, we have

$$(\nabla\Pi)(\alpha_j, \cdot) + \Pi(\nabla\alpha_j, \cdot) = 0.$$

Then by a contraction, we have

$$Div\Pi(\alpha_j) + \Pi(d\alpha_j) = 0. \tag{6}$$

Thus, if $\{\alpha_1 \ldots \alpha_q\}$ satisfies the Frobenius integrability condition, $\Pi(d\alpha_j) = 0$ and we have

$$Div\Pi(\alpha_j) = 0.$$

This shows that $Div\Pi$ is a vector field tangent to $\mathcal{F}_\Pi$ and $Div\Pi \wedge \Pi^l = 0$, since rank $\Pi = 2l$.

Conversely, if $Div\Pi \wedge \Pi^l = 0$, taking a contraction $\langle \alpha_j, Div\Pi \wedge \Pi^l \rangle$, we can see $Div\Pi(\alpha_j) = 0$ and hence by (6), we get $\Pi(d\alpha_j) = 0$ for each j. This means that each $d\alpha_j$ should be of the form

$$\sum_{k=1}^{q} \alpha_k \wedge \beta_{k,j}$$

for some 1-forms $\{\beta_{k,j}\}$. This shows $\{\alpha_1, \ldots, \alpha_q\}$ satisfies the Frobenius condition. □

We get the following well-known fact.

Corollary 3 *Let* (M, Π) *be a regular Poisson manifold. Then the characteristic distribution is integrable.*

PROOF Let rank $\Pi = l$. Since $[\Pi, \Pi] = 0$, $[\Pi, \Pi^l] = 2l[\Pi, \Pi] \wedge \Pi^{l-1} = 0$. Thus, the result follows. □

3 Exact Poisson Manifolds of Special Kind

If a Poisson manifold (M, Π) has a vector field Z satisfies $L_Z\Pi = [Z, \Pi] = -\Pi$, it is called an *exact* Poisson manifold and Z is called a *homothetic vector vector field* of (M, Π). From the view point of the Poisson cohomology, such a manifold is a Poisson manifold whose Poisson bi-vector field Π represents 0 in $H^2_{LP}(M)$. Recall that the Poisson cohomology is a cohomology whose p-th cochain group is $\Gamma(\Lambda^p TM)$ and the coboundary operator $\sigma : \Gamma(\Lambda^p TM) \to \Gamma(\Lambda^{p+1} TM)$ is given by $\sigma(P) = -[\Pi, P]$ ([9]).

It is not difficult to give examples of exact Poisson manifolds which are non-compact. The following two are standard ones.

Example 1 (Cotangent Bundle)

Let $(T^*M, d\lambda)$ be the standard symplectic structure of the cotangent bundle of a manifold M, where λ is the Liouville form. Let Π be the 2-vector field on T^*M such that $d\lambda(\Pi) = \Pi$, considering $d\lambda$ as an isomorphim $T^*M \to TM$.

Note that this means $I_\Pi(d\lambda) = \Pi$. Let $Z = \Pi(\lambda, \cdot)$ be the vector field on T^*M. Then

$$L_Z\Pi = [\Pi(\lambda, \cdot), \Pi] = -\sigma(\Pi(\lambda, \cdot)) = -\sigma(I_\Pi(\lambda)) = I_\Pi(-d\lambda) = -\Pi.$$

Thus, $(T^*M, \Pi, Z = \Pi(\lambda, \cdot))$ is an exact Poisson manifold. Here, we used the formula; $\sigma(I(\alpha)) = -I(d\alpha)$, for any form α ([9]).

Example 2 (Lie Poisson structure.)

Let $(\mathfrak{g}^*, \Pi)$ be the Lie Poisson structure on the dual space of a Lie algebra $\mathfrak{g}$. The Poisson bi-vector field Π is defined as follows. We have an identification $T^*\mathfrak{g}^* \approx \mathfrak{g}^* \times \mathfrak{g}^{**}$ by translations. If $T^*\mathfrak{g}^* \ni p, q$ are represented under this identification as $p = (\alpha, x), q = (\alpha, y)$, then Π is given by

$$\Pi_\alpha(p, q) = \alpha([x, y]).$$

Let Z be the radial vector field on $\mathfrak{g}^*$. Namely, Z_α is a tangent vector which corresponds to the curve $e^t \cdot \alpha$ in $\mathfrak{g}^*$. We regard p, q as the constant 1-forms. In other words, we consider p as a 1-form on $\mathfrak{g}^*$, given by $\alpha \mapsto (\alpha, x)$, where $x \in \mathfrak{g} = \mathfrak{g}^{**}$ is constant.

Then we have

$$L_Z(\Pi(p, q)) = L_Z(\alpha([x, y])) = \frac{d}{dt}\Big|_{t=0}(e^t\alpha([x, y])) = \alpha([x, y]) = \Pi(p, q).$$

On the other hand,

$$\begin{aligned} L_Z(\Pi(p, q)) &= (L_Z\Pi)(p, q) + \Pi(L_Z p, q) + \Pi(p, L_Z q) \\ &= (L_Z\Pi)(p, q), \end{aligned}$$

since p and q are constant vector fields.

From these we have

$$(L_Z\Pi)(p, q) = \Pi(p, q).$$

This shows $(\mathfrak{g}^*, \Pi, -Z)$ is an exact Poisson structure.

Of course, the homotheic vector field in the above is not unique. In fact if Z, Z' are both homothetic vector field for Π, then clearly the Lie derivative $L_{Z-Z'}\Pi$ vanishes. Thus the set of homothetic vector field of a Poisson structure forms an affine subspace of the vector space of all the vector fields, whose associated vector space is the space of vector fields which preserve the Poisson bi-vector field. Recall that any Hamitonian vector field $I(df)$, $f \in C^\infty(M)$ preserves Π.

Now, we are interested in the following problem:

Problem : What kind of codimension one foliation does appear as an underlying foliation of an exact Poisson manifold which is compact ?

We will consider this problem in the case when M is a closed 3-dimensional manifold.

First we note that every orientable foliation of dimension 2 is an underlying foliation of a Poisson structure. In fact, let $(M, \mathcal{F})$ be a foliation whose leaves are 2-dimensional.

and let $\Pi \in \Gamma(\Lambda^2\mathcal{F})$ be a non-zero cross section. Then it is naturally considered as a 2-vector field on M.

It is easily checked that the image of I_Π coincides with $\mathcal{F}$. The Poisson condition on Π is satisfied since in this dimension, it is equivalent to the integrability of $\mathcal{F}$ (see Section 2).

Exact Poisson Structure of Special Kind

In this subsection, we give two examples of exact Poisson manifolds which we call 'special'. The underlying manifolds are closed ones and quotient manifolds of 3-dimensional Lie groups.

Example 3 Let X_1, X_2, X_3 be the right invariant vector field of $G =$ SL $(2, \boldsymbol{R})$ corresponding to $\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. They satisfy the following bracket relations;

$$[X_1, X_2] = -X_2, \quad [X_1, X_3] = X_3, \quad [X_2, X_3] = 2X_1.$$

The 2-vector field $\Pi = X_1 \wedge X_2$ satisfies the Poisson condition $[\Pi, \Pi] = 0$ and it defines a Poisson structure. If we choose a uniform discrete subgroup Γ of $G =$ SL $(2, \boldsymbol{R})$, we obtain an induced Poisson structure on $M = G/\Gamma = SL(2, \boldsymbol{R})/\Gamma$ which is a closed manifold. The underlying foliation $\mathcal{F}_\Pi$ is known as an Anosov foliation spanned by X_1 and X_2. It is known that each leaf of this foliation is dense in M.

Let $Z = X_1 + aX_2$, (a is a constant) then

$$\begin{aligned} L_Z\Pi = [Z, \Pi] &= \frac{1}{2}[X_1, X_1 \wedge X_2] \\ &= \frac{1}{2}X_1 \wedge [X_1, X_2] = -X_1 \wedge X_2 = -\Pi. \end{aligned}$$

Thus (M, Π, Z) is a closed exact Poisson manifold.

Similarly, we have the following second example.

Example 4 Let G be a simply connected 3-dimensional solvable Lie group whose Lie algebra is generated by X_1, X_2, X_3 with the relations

$$[X_1, X_2] = -X_2, \quad [X_1, X_3] = X_3, \quad [X_2, X_3] = 0.$$

Like as in the case of Example 3, let $\Pi = X_1 \wedge X_2$, $Z = X_1 + aX_2$ be the right invariant fields on G. By the same computation, we see that Π defines an exact Poisson structure and Z is a homothetic vector field. Also, if we choose a uniform discrete subgroup Γ, we obtain an exact Poisson structure on a closed 3-dimensional manifold. In this case, $M = G/\Gamma$ is a T^2-bundle over S^1 and the foliation is a suspension of a dense linear foliation of T^2, hence the leaves of $\mathcal{F}_\Pi$ are all dense again.

Note that, in both of the above examples, the symplectic leaves of the characteristic foliations are generated by the vector fields X_1, X_2 with the relation $[X_1, X_2] = -X_2$, which generate the Lie algebra of 2-dimensional affine group GA. From this, we can see that the leaves are the orbits of a locally free actions of GA.

We call the Poisson manifold which is obtained as in the above examples *special* exact Poisson manifold.

One of the property of the homothetic vector field of a special Poisson manifold is that its divergence (with respect to the canonical volume form) vanishes everywhere. In fact, let Ω be the volume form on M, such that $\Omega(X_1 \wedge X_2 \wedge X_3) \equiv 1$. Then by an easy computation we can see $L_Z\Omega = 0$ which is, by definition, equal to $(div Z)\Omega$ and $(div Z) = 0$.

In the following , we prove that this property characterizes the special exact Poisson manifolds.

THEOREM 4 *Let (M, Π, Z) be an exact (regular) Poisson manifold, where M is a closed 3-dimensional. Suppose that the homothetic vector field Z is divergence free with respect to some volume form Ω on M and tangent to $\mathcal{F}_\Pi$. Then (M, Π, Z) is diffeomorphic to a special exact Poisson manifold.*

PROOF Choose a Riemannian metric on Mwhose associated volume form is equal to Ω. We will use the generalized divergence with respect to the Riemannian connection of this metric. By assumption, $Z \wedge \Pi = 0, Div Z = div Z = 0$, hence, we have

$$-\Pi = [Z, \Pi] = Div(Z \wedge \Pi) - Div Z \wedge \Pi + Z \wedge Div\Pi = Z \wedge Div\Pi.$$

This shows that Z and $Div\Pi$ are two vector fields tangent to the leaves of $\mathcal{F}_\Pi$, which are linearly independent at each point of M. Taking Div of both sides of $[Z, \Pi] = -\Pi$ (see Section 2), we have

$$[Z, Div\Pi] = -Div\Pi.$$

This shows that there exists an locally free action on M of the 2-dimensional affine group GA. Since $div Z = 0$, by the assumption and $div(Div\Pi) = Div^2\Pi = 0$ holds, we have $L_Z\Omega = L_{Div\Pi}\Omega = 0$. Thus the action of GA preserves the volume Ω. Now a theorem of Ghys ([2]) concerning the rigidity of the action of GA on 3-manifolds says that this action is smoothly conjugate to one of the standard ones. That is, it is equivalent to a natural action of GA on one of the quotient manifolds G/Γ in the examples of this section. This means that there is a diffeomorphism $\varphi : M \to G/\Gamma$ sending Z to X_1 and $Div\Pi$ to X_2. □

Remark In the next section, we prove that the homothetic vector field Z will always tangent to the foliation in the case of codimesion one Poisson structure. So, in the above theorem, in fact, one can drop the assumption Z *is tangent to* $\mathcal{F}_\Pi$.

4 Exact Poisson Structure on Closed 3-manifolds

In this section, we consider regular Poisson structures and find some topological conditions of an exact Poisson manifold .

We start with the following

LEMMA 5 *Let (M, Π, Z) be an exact Poisson manifold. Then the homothetic vector field Z preserves the foliation $\mathcal{F}_\Pi$.*

PROOF Let the foliation $\mathcal{F}_\Pi$ be defined locally by Pfaffian forms

$$\alpha_1, \dots, \alpha_q$$

which span $KerI_\Pi$. The relation $\Pi(\alpha_i, \cdot) = 0$ leads to the following equation

$$(L_Z\Pi)(\alpha_i) + \Pi(L_Z\alpha_i, \cdot) = 0.$$

Since $(L_Z\Pi)(\alpha_i) = -\Pi(\alpha_i, \cdot) = 0$, we have

$$\Pi(L_Z\alpha_i, \cdot) = 0, \quad (i = 1, \dots, q).$$

Thus each $L_Z\alpha_i$ is a functional linear combination of $\alpha_1, \dots, \alpha_q$. Now let X be a local vector field which is tangent to the leaves. Then we have

$$\alpha_i(L_ZX) = L_Z(\alpha_i(X)) - (L_Z\alpha_i)(X) = 0, \quad (i = 1, \dots q).$$

Thus L_ZX is also tangent to the leaves. This means Z preserves the foliation $\mathcal{F}_\Pi$. □

By the above lemma, the subset of M, where Z is transverse to $\mathcal{F}_\Pi$, is an open *saturated subset* (the subset which is a union of leaves) of M.

LEMMA 6 *Let (M, Π, Z) be a codimension one exact Poisson manifold. That is, M is an exact Poisson manifold such that $\mathfrak{F}_\Pi$ is a codimension one foliation. Let U be an open saturated subset of M, where Z is transverse to the foliation $\mathcal{F}_\Pi$. Then the foliation $\mathcal{F}_\Pi|_U$ restricted on U is defined by a closed 1-form.*

PROOF Take a 1-form α which satisfies $I_\Pi(\alpha) = 0.$ and $\alpha(Z) \equiv 1$ on U. It is easy to see that

$$d\alpha = L_Z\alpha \wedge \alpha.$$

By Lemma 5, $L_Z\alpha$ is a functional multiple of α, hence we have $d\alpha = 0$ on U. □

LEMMA 7 *Let (M, Π, Z) be an exact Poisson manifold. If L is a leaf of $\mathcal{F}_\Pi$ such that Z is tangent to L, then L is a non-compact.*

PROOF Take a leafwise symplectic 2-form ω on M such that $\langle\omega, I_\Pi(\omega) = \Pi$. If rank $\Pi = 2k$

$$\omega^k = \omega \wedge \cdots \wedge \omega, \ (k\text{-times})$$

restricts to a volume form on each leaf. Since the pairing $\langle\omega^k, \Pi^k\rangle$ is a non-zero constant and $L_Z\Pi^k = -k\Pi^k$, $\langle L_Z\omega^k, \Pi^k\rangle$ is also a non-zero constant. Thus the restriction $(L_Z\omega^k)|_L$ of $L_Z\omega^k$ to L is a non-zero multiple of $(\omega^k|_L)$. Since Z is tangent to L, this shows $(divZ)|_{\omega^k|L} = C$(the divergence with respect to the volume $\omega^k|_L$.) for some non-zero constant C. This is impossible when L is compact. □

LEMMA 8 *Let (M,Π,Z) be a codimesion one exact Poisson manifold, where M is a closed manifold. Then the subset of M, which is the union of leaves where the homothetic vector field Z is tangent to each leaf is a non-empty closed saturated set.*

PROOF Closedness of the set is clear. If it is empty, Z is transverse to $\mathcal{F}_\Pi$ everywhere on M and $Z \wedge \Pi^k$ is nowhere zero($2k$ is the rank of Π). Let Ω be a volume form on M dual to $Z \wedge \Pi^k$ (i.e. Ω satisfies $\Omega(Z \wedge \Pi) \equiv 1$). Then it is easily seen that $div_\Omega Z \equiv -1$ which is impossible on a closed manifold M. □

Let (M,Π,Z) be an exact Poisson manifold of a closed manifold, which is of codimesion one. We put rank $\Pi = 2k$. As we have seen in Lemma 7, the homothetic vector field Z is not tangent to a compact leaf L. Assume that Z is transverse to a compact leaf L, since the 1-parameter subgroup ϕ_t generated by Z preserves the foliation $\mathcal{F}_\Pi$, the union $\cup_{t\in\boldsymbol{R}}\phi_t(L)$ consists of compact leaves which are diffeomorphic to L. If $\cup_{t\in\boldsymbol{R}}\phi_t(L)$ is not whole M, there exists a leaf which is the limit leaf of a subset of $\cup_{t\in\boldsymbol{R}}\phi_t(L)$, which itself should be compact and $Z$is transverse to it. This implies that it has to be contained in $\cup_{t\in\boldsymbol{R}}\phi_t(L)$.

Thus, we can conclude that (M,Π,Z) has no compact leaves. Therefore, for example, there is no exact Poisson structures on S^3 since every codimension one foliation of S^3 has a compact leaf doffeomorphic to T^2 ([7]). Also, we saw Z is not everywhere transverse to $\mathcal{F}_\Pi$. Moreover in the case of special exact Poisson manifold, the homothetic vector field is everywhere tangent to leaves on the whole manifold. Hence it is natural to ask the following question:

Question : Are there any examples of (M,Π,Z) on which Z is tangent to the leaves of $\mathcal{F}_\Pi$ on one part and transverse to them on the other part?

The following theorem shows there is no such example on a closed manifold provided (M,Π,Z) is codimension one.

THEOREM 9 *Let (M,Π,Z) be an exact Poisson structure of a closed manifold, which we assume regular and codimension one. Then the homothetic vector field Z is tangent to the foliation $\mathcal{F}_\Pi$ everywhere on M.*

PROOF To the contrary, we assume that there exits an open subset of M, where Z is transverse to $\mathcal{F}_\Pi$. Let $M = U \cup U'$ the partition into two part; on U, Z is transverse to $\mathcal{F}_\Pi$ and on U', Z is tangent to $\mathcal{F}_\Pi$. By Lemma 5, both U and U' are saturated sets. By Lemma 8, U' is non-empty. Let F be a leaf contained in U'. Since $\mathcal{F}_\Pi$ has no compact leaves or dense leaves, $\bar{F}$ contains an exceptional minimal set. Let E denote it. By a theorem of Sacksteder([8]), if a codimension one foliation is of class C^2, exceptional minimal set contains a leaf which has a contracting holonomy. Take such a leaf L contained in E. Choose a point $x \in L$ and then take a transverse small arc $I \approx (-1,1)$ through x, where 0 corresponds to x. The contracting holonomy gives a germ of a map $\psi : (-\epsilon,\epsilon) \to (-1,1)$ at 0. The intersection $I \cap E$ is a Cantor set and $I \cap U$ is a union of open intervals. Let $\{\varphi_t\}$ be the 1-parameter subgroup of Z. Since φ_t maps a leaf into a leaf, we can choose and fix a small t_0 so that φ_{t_0} induces a local diffeomorphism $\tilde{\varphi}$ of I at x. (consider I as the set of plaques near x).

Clearly, $\tilde{\varphi}$ fixes $I \cap E$ and preserves the open intervals of $I \cap U$. It follows easily that the germs of $\tilde{\varphi}$ and ψ at commute each other. Since $\tilde{\varphi}$ has fixed points accumulating to x, by a lemma of Kopell ([5]), ψ can not be of class C^2, contradicting our assumption that the foliation is C^∞ . This proves the theorem. □

In the above theorem, we have in fact proved the following

THEOREM 10 *Let $(M, \mathcal{F}$ be a codimesion one smooth foliation of a closed manifold without compact leaves. If Z is a vector field on M, whose 1-parameter group preserves $\mathcal{F}$, then Z is everywhere tangent to leaves of $\mathcal{F}$.*

5 A Construction of Exact Poisson Manifolds

In this section, we will give an explicit example of an exact Poisson structure which is different previous ones. The manifold we will constuct is a closed 3-dimensional manifold and the underlying foliation of the Poisson structure is so-called *Hirsch foliation*([4]).

We begin with describing such a type of codimension one foliations.

Let Σ_0 be an orientable 2-dimensional compact manifold whose boundary is a circle. Make a product $S^1 \times \Sigma_0$ and choose an embedding $j : S^1 \to S^1 \times \Sigma_0$ whose image intersects each $\{t\} \times \Sigma_0$, $(t \in S^1)$ exactly at 2-points. Thus the composition

$$p \circ j : S^1 \to S^1,$$

where $p : S^1 \times \Sigma_0 \to S^1$ is the projection to the first factor, is a double covering. We choose j so that this double covering is the natural one and Image j is in the interior of $S^1 \times \Sigma_0$. Delete a small open tubular neighbourhood on Image j from $S^1 \times \Sigma_0$. Let N denote the resulting manifold. It is worthwhile to note that N is also obtained as a mapping torus of a diffeomorphism of a 3-times punctured surface Σ_1 and is a fiber bundle over S^1. There is a codimension one foliation on N defined by the fibers of this bundle.

Let $\partial_{in}N$ denote the *'interior boundary'* of N. We will fix the trivialization of the bundle $\partial_{in}N \to S^1$ as the boundary of the tubular neighbourhood of Image j. Similarly, let $\partial_{ex}N$ denote the *'exterior boundary'* and we will fix a trivialization $\partial_{ex}N \to S^1$ as the boundary of $S^1 \times \Sigma_0$.

Let ϕ be an element of DiffS^1. Then we have a diffeomorphism

$$\tilde{\phi} = \phi \times id : S^1 \times S^1 \to S^1 \times S^1$$

which gives a diffeomorphism

$$f : \partial_{in}N \to \partial_{ex}N.$$

Identifying the boundary tori of N through f, we obtain a closed 3- manifold M. M has a naturally defined C^∞ codimension one foliation $\mathcal{F}_\phi$ which is induced from that of N. The foliation $\mathcal{F}_\phi$ obtained in this way, is called a *Hirsch foliation.*

It is easy to see that the leaves of $\mathcal{F}_\phi$ are all non-compact. If $\phi = id_{S^1}$, for example, all the leaves of $\mathcal{F}_\phi$ are dense in M. One can also choose ϕ so that $\mathcal{F}_\phi$ has exceptional leaves ([4]).

In order to construct an exact Poisson structure on M, we note the following simple lemma.

LEMMA 11 *Let Ω be a volume form on Σ_1 and η a 1-form satisfying $d\eta = \Omega$. Let Z be the vector field which is determined by*

$$\Omega(Z, \cdot) = \eta, \qquad (i_Z\Omega = \eta).$$

Then

$$L_Z\Omega = \Omega.$$

PROOF By a well-known formula, we have $L_Z\Omega = di_Z\Omega + i_Z d\Omega = d\eta = \Omega$. □

In te next lemma, we use the following notations. Let U_0 $(resp. U_1)$ be an 'exterior '(resp. 'interior') collar neighbourhood of the unit circle in the Euclidean plane. (r, θ) is the standard polar coordinate on $\boldsymbol{R}^2 - 0$. Let $\partial\Sigma_1 = C_0 \cup C_1 \cup C_2$ denote the union of circles where C_0 is the fiber of the exterioir boundary of N while C_1 and C_2 are those of the interior boundary of N.

LEMMA 12 *On Σ_1, we have a 1-form η which satisfies the following;*

(1) $d\eta$ is a volume form of Σ_1,

(2) On the neighbourhood of C_0, η is diffeomorphich to $(1/2)r^2 d\theta|_{U_0}$ and on some neighbourhood of C_1, C_2, η is diffeomorphic to $(1/2)r^2 d\theta|_{U_1}$.

PROOF

We choose on Σ_1 a volume form Ω which is described as follows. First, around the boundary C_0, we consider a collar neighbourhood which is diffeomorphic to U_0. And similarly, around C_1 and C_2, we consider collar neighbourhoods diffeomorphic to U_1.

Then, we introduce the Euclidean volume form on these collar neighbourhoods by the above identification. We extend thsese forms to a volume form Ω on the whole Σ_1 in such a way that they are unchanged in smaller neighbourhoods of the boundary; i.e. $\Omega = rdr \wedge \theta$ near the boundary. This Ω is an orientation we consider on Σ_1. If it is needed, we multiply Ω by a suitable positive function then may assume

$$\int_{\Sigma_1} \Omega = \pi.$$

(Ω again should be unchanged in a small neighbourhood of the bounddary). On the other hand, let η' be any 1-form on Σ_1 which is equivalent to $(1/2)r^2 d\theta|_{U_0}$ near C_0 and to $(1/2)r^2 d\theta|_{U_1}$ near C_1 and C_2. We have

$$\begin{aligned}
\int_{\Sigma_1} d\eta' &= \int_{\partial\Sigma_1} \eta' \\
&= \int_{C_0} \eta' + \int_{C_1} \eta' + \int_{C_2} \eta' \\
&= -(1/2)\int_{S^1} d\theta + \int_{S^1} d\theta = \pi.
\end{aligned}$$

(Note that the orientation of C_i is determined by taking the interior product $i_X\Omega$ by an outward normal X.)

Then the difference $\Omega - d\eta'$ is a closed 2-form whose support is contained in the in the interior of Σ_1. By the above calculation it represents zero in $H^2_{compact}(Int\Sigma_1)$. Namely, there exists a 1-form η'' whose support is in $Int\Sigma_1$ which satisfies $\Omega - \eta' = d\eta.''$

Put

$$\eta = \eta' + \eta''.$$

Then η satisfies the required conditions (1) and (2). □

Now, we are going to construct an exact Poisson structure on M.

Let Π_0 be the 2-vector field on Σ_1 such that $\langle\Omega, \Pi_0\rangle = 1$ and Z_0 be the vector field such that

$$i_{Z_0}\Omega = \eta.$$

Then, using Lemma 11, we have

$$\begin{aligned}
0 &= L_{Z_0}\langle\Omega, \Pi_0\rangle = \langle L_{Z_0}\Omega, \Pi_0\rangle + \langle\Omega, L_{Z_0}\Pi_0\rangle \\
&= \langle di_{Z_0}\Omega, \Pi_0\rangle + \langle\Omega, L_{Z_0}\Pi_0\rangle = 1 + \langle\Omega, L_{Z_0}\Pi_0\rangle.
\end{aligned}$$

From this we have

$$L_{Z_0}\Pi_0 = -\Pi_0.$$

Now it is not difficult to get a 2-vector field Π and homothetic vector field Z on M. To see this notice that N is obtained from

$$[0, 1] \times \Sigma_1$$

by pasting $\{0\} \times \Sigma_1$ and $\{1\} \times \Sigma_1$ by diffeomorphism $k : \Sigma_1 \to \Sigma_1$, which is an involution.

Taking $\frac{1}{2}(\eta + k^*\eta)$ instead of η if necessary, we can assume everything is k-invariant. Consider the obvious liftings of Π_0 and Z_0 onto the $[0,1] \times \Sigma_1$. Then the fields we are considering on the top and the bottom of the product manifold fit togeter under the diffeomorphism k. This gives N a well-defined 2vector filed and vector fields. Finally, pasting the boundary of by a diffeomorphism

$$f : \partial_{in} N \to \partial_{ex} N,$$

we obtain a 2-vector field Π and the homothetic vector field Z on $(M, \mathcal{F})$. By our construction, (Π, Z) clearly satisfies the relation $L_Z\Pi = -\Pi$.

This finishes our construction of an exact Poisson structure on M whose undelying foliation is a Hirsch type foliation.

Remark It seems an interesting question if a similar construction is possible in higher dimensions. That is: Is it possible to construct an exact Poisson manifold starting from a higher dimensional symplectic manifold with boundary in stead of Σ_0, and proceed similarly to the above construction?

Of course the following procedure is possible. Let $(M_1, \Pi_1, Z_1), (M_2, \Pi_2, Z_2)$ be two exact Poisson manifolds. Let us denote the liftings Π_1, Π_2, Z_1 and Z_2 to the product manifold by the same letters. Then we obtain an exact Poisson manifold $(M_1 \times M_2, \Pi_1 + \Pi_2, Z_1 + Z_2)$.

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