プロジェクト名：**Ga-作用を有しないログ単線織的な３次元アフィン代数多様体の研究 (On log uniruled affine algebraic threefolds which do not possess effective Ga-actions)**

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<u>Report on the project entitled “</u>On log uniruled affine algebraic threefolds which do not possess effective Ga-actions “

In what follows, we shall summarize a development about the research which were partially supported by the project of Saitama University. First of all, we mention the background and motivation of our research, then we shall state the results that are obtained under the financial support.

<u>Background:</u> It is easy to confirm that a log uniruled affine algebraic curve is automatically isomorphic to the affine line **C**, which is obviously equipped with an effective Ga-action. (For economize the space, we do not mention the definition of being log uniruled.) Meanwhile, it is well known that there exist infinitely many smooth affine algebraic surfaces, which are log uniruled. Thus, it seems not to be reasonable to classify in an explicit fashion all of such affine surfaces. However, there is fortunately a very useful invariant to characterize them, so-called log Kodaira dimension, which has been introduced in the 1970’s by Prof. Iitaka. In fact, in terms of log Kodaira dimension, it follows that a smooth affine algebraic surface is log uniruled if and only if its log Kodaira dimension is equal to -∞, furthermore, if and only if it admits a fibration by affine lines. In the theory of affine surfaces, this result mentioned just above plays very important and essential roles. Whereas let us consider whether or not the similar statement holds true for the higher dimensional case also. Namely, the following question:

Question: Let **X** be a smooth affine algebraic variety defined over the filed of complex numbers **C**. Then is it true that the following three conditions (i), (ii) and (iii) are equivalent to each other ?

(i) X admits a fibration by the affine lines.
(ii) X is log uniruled.
(iii) The log Kodaira dimension of X equals -∞.

The implications (i) => (ii) and (i) => (iii) are easy to ascertain, notwithstanding, the question is quite difficult to seetle out in general situation even in the case of dimension three. For instance, it is unknown so far whether or not the implication (ii) => (i) holds in general.

<u>Results:</u> In the present research project, we devoted ourselves mainly to the construction of a counter-example with respect to the implication (ii) => (i) and (iii) => (i) in dimension three in an explicit fashion. More precisely to say, we are able to recognize the following result, which yields the first counter-example clarifying that quite interesting, simultaneously, chaotic situations occur in the case of higher dimension.

Theorem 1. *Let S be a smooth cubic hypersurface in P^3, and let us denote by X its complement. Then we have the following:*

(1) X is log uniruled, and the log Kodaira dimension of X is equal to -∞, whereas:
(2) X admits neither a fibration by affine lines nor an effective action by any unipotent algebraic group.

For the proof of the above mentioned theorem, we are obliged to make use of the result about the non-rationality on smooth cubic threefolds. The article about this result has been already written in the joint work with Adrien Dubouloz (Universite de Bourgogne), which is submitted to the international journal.

Other relevant works: Our above mentioned result can be translated into the non-existence of an effective action of the additive group of dimension one **Ga** on X. In affine algebraic geometry, the additive group actions play very important roles, in fact, it is one of the main streams. For instance, such an action can be explained in terms of purely algebraic concept, so-called locally nilpotent derivation on the coordinate rings. Hence, depending on the situations, we are free of the choice about the method to investigate existence of group actions of the additive group **Ga**. Nevertheless, it is usually difficult to find locally nilpotent derivation or to deny the possibility of the existence of locally nilpotent derivations. Thus it is helpful for us to obtain a criterion in order to judge whether or not an effective action of **Ga** exists on a given affine algebraic variety. This is of course an ambiguous problem in some sense. Then, we restrict the consideration to the case of affine algebraic varieties, which are realized as (generalized) affine cones over projective varieties. One of the advantages of this circumstance lies in the fact that we are able to apply projective (birational) geometry. Indeed, in the joint work with Yuri Prokhorov and Mikhail Zaidenberg, we obtain the necessary and sufficient condition concerning effective action of **Ga** in terms of cylinders of the base projective varieties. As an application of this criterion, we can further ascertain that any affine cones over (pluri) anti-canonically embedded smooth del Pezzo surfaces of degree 1 or 2 does not admit such an action. This is quite remarkable because effective Ga-actions do exist in case of affine cones over (pluri) anti-canonically embedded del Pezzo surfaces of degree greater than 3 as shown in our previous works. These results are also submitted to the international journals.

Development: In the works with Prokhorov and Zaidenberg, we observe furthermore affine cone over smooth Fano threefolds, which is one of the output of minimal model program. In this direction, we are willing to generalize to the other type Mori fiber spaces, so-called del Pezzo fibrations. This generalization is never straightforward, but it seems that the techniques of maximal singularities in birational geometry can be successfully applied.