*Validity in Simple Partial Logic**

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Firstly I characterize Simple Partial Logic (=SPL) as a generalization and extension of a certain two-valued logic. Based on that characterization I present two definitions of validity in SPL. Secondly I show that given my characterization these two definitions are more appropriate than other definitions that have been prevalent, since both have some desirable semantic properties that the others lack.

1. Introduction

Partial logic is, broadly speaking, logic that allows the truth-value gap, which means that propositions may possibly be neither true nor false. Because of the gap, there occur, as it were, two conceptions of truth: being true and being not false. I call the former the strong concept of truth and the latter the weak concept of truth.

Accordingly the conceptions of validity also have similar complexity. There are strong and weak concepts of validity of a formula: being true and being not false respectively under all truth-valuations. As for the validity of an argument there can be *at least* four concepts:[1]

(a) The consequence is true whenever all the premises are true.
(b) The consequence is not false whenever all the premises are true.
(c) The consequence is true whenever no premises are false.
(d) The consequence is not false whenever no premises are false.

Logicians have adopted different concepts of validity of arguments in partial logic. Wang, for example, chose (a), which appeared most standard[2]. This concept, however, has defects despite its apparent naturalness; neither (semantic) deduction theorem nor contraposition theorem holds:

<Deduction Theorem>
$P_1, P_2, \cdots, P_n \models$ C iff $\models (P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$.

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<Contraposition Theorem>

$P_1, P_2, \cdots, P_n \models C$ iff $\neg C \models \neg(P_1 \wedge P_2 \cdots \wedge P_n)$.

Improving Wang's concept, Blamey defined validity as follows:

> The consequence is true whenever all the premises are true, *and at least one of the premises is false whenever the consequence is false*[3].

Though this modification makes contraposition theorem hold, deduction theorem still does not hold.

Below I attempt to find better conceptions of validity in partial logic. Which definitions of validity are better depends on what the logic concerned is for. So I will characterize partial logic and then present two definitions of validity both of which have their place in my characterization. At the end I prove some semantic theorems and compare my definitions with other definitions.

2. Characterization of Simple Partial Logic

In this paper I restrict my arguments to the syntax and semantics of propositional logic. Moreover I confine myself to the simplest type of partial logic which I call 'Simple Partial Logic' (=SPL).

The syntax of SPL is as follows:

(1) Atomic formulae $p, q, r, \cdots, p_1, p_2, \cdots, p_n, \cdots$ are well formed formulae (=wffs).
(2) If A is a wff, $\neg A$ and TA are wffs.
(3) If A and B are wffs, $(A \wedge B)$ is a wff (the outermost parentheses can be omitted).

In addition I introduce the following operators by definitions (I will also add a few more operators by definitions later in this section):

$$A \vee B =df \neg(\neg A \wedge \neg B)$$
$$A \rightarrow B =df \neg(A \wedge \neg B)$$
$$A \leftrightarrow B =df (A \rightarrow B) \wedge (B \rightarrow A)$$

As for the semantics there are two conditions that SPL should satisfy:

1. It is truth-functional[4].
2. It has another value (in a broader sense described below) than truth and falsity.

However, these conditions are not proper to SPL, because the systems of three-valued logic in general satisfy these conditions. So we have to find the characteristics that make SPL special among the systems of three-valued logic. The crucial point is the character of 'another value' in the second condition above. Strictly speaking, the third value of partial logic is not an independent value but *the lack* of the truth-values. I call the third value of partial logic the 'gap-value', distinguishing it from the 'truth-values', which only refer to truth and falsity. Now I attempt at formulating what conditions make the third value not an independent value but the lack of the truth-values.

Langhholm enumerates the following properties among those which may be characteristic of partial logic :[5]

(1) Determinablity
If all the atomic formulae in a wff have *truth*-values, the wff itself also has a *truth*-value.

(2) Monotonicity
If a wff that includes gap-valued (=gapped) atomic formulae has a *truth*-value, that truth-value will not be changed by giving a *truth*-value to a gapped atomic formula.

In the following I modify these two properties to fit in with my characterization of SPL.

First I adopt the following usage of terms :

Total Valuation : the valuation which assigns a *truth*-value to every atomic formula.
Total Logic : the logic which contains no gapped wffs.

Partial valuation and partial logic are their negations.

(1′) Reliable Determinability (=Reliability)

First of all I characterize SPL as a generalization of a certain total (two-valued) logic. In other words, SPL presupposes a particular total logic. I call the presupposed total logic the 'basic logic' of the SPL concerned and the operators included in the basic logic the 'basic operators'.

By 'generalization' I mean, in the first place, the partialization of valuation, namely the allowance of the truth-value gap in valuation. In total logic every formula must have a truth-value. In contrast a formula in partial logic does not necessarily have a truth-value. The truth-tables of partial logic do not only determine the conditions as to when a formula is true (or false), but also the conditions as to *when the formula has a truth-value*. Secondly the truth-tables that define the meanings of the basic operators in SPL have to coincide with those of its basic logic under total valuation. In other words, the truth-value of the formula that includes only the basic operators have the same truth-value with that of basic logic, if all the atomic formulae included have truth-values. This is a restriction of determinability defined above. I call such property 'Reliable Determinability' or 'Reliability' in short.

(2′) Weak Monotonicity

Blamey restricted the operators of partial logic to monotonic operators(6). I agree with him on the basic operators. The basic operators should have the property of monotonicity as well as reliability. However, I do not think that SPL should exclude all the non-monotonic operators, because there are useful and significant ones for partial logic among them. For example, the following Truth-operator that Woodruff adopted for his System Q, which can be taken as a kind of SPL, is useful for expressing significant concepts which are related to partiality :(7)

A	TA
t	t
-	f
f	f

Using this Truth-operator we can also define the following operators :(8)

$FA \quad =df \quad T\neg A$ (A is false.)
$LA \quad =df \quad TA \vee FA$ (A has a truth-value.)
$A \Rightarrow B \quad =df \quad LA \rightarrow TB$ (A presupposes B.)

These operators are common in that they always bring about a *truth*-value under partial valuation. In my view, they express a kind of (extensional) modality which results from partial valuation. I call the operators which always bring about a truth-value under partial valuation the 'modal operators' (In the following I call the formulae that do not include modal operators 'basic formulae'). Adding these operators to the basic operators makes SPL an extension of some total two-valued logic as well as its generalization.

Though the modal operators destroy the monotonicity of partial logic, I do not think that they make the 'spirit' of partial logic totally lost, for the monotonicity still remains at the basic level and it is no wonder that the propositions which describe the modal facts in terms of partiality itself are non-monotonic. Partiality implies that there are cases where a gapped proposition gets a truth-value by adding some information. Therefore, for example, the proposition '$\neg LA$' (A has no truth-value.), which includes a modal operator, may well have opportunities to change its truth-value from truth to falsity.

Moreover the modal operators keep a weakened monotonicity in the following sense:

> If a formula that includes gapped atomic formulae has a *truth*-value, it does not make its truth-value *lost* (namely, be changed to gap-value) by altering the gap-value to a truth-value.

I call such a monotonicity 'weak monotonicity'.

To summarize, SPL is defined as a three-valued logic (in a broader sense) that includes the basic operators and the modal operators; the former are monotonic and coincidental with a certain total two-valued logic under total valuation, while the latter are weakly monotonic and always bring about a truth-value under partial valuation. By this definition, SPL is characterized both as a generalization and extension of a total two-valued logic; SPL is its generalization in the sense that SPL introduces the partiality of truth-valuation and is its extension in the sense that SPL includes modal operators that describe the modality related with partial valuation.

3. Definitions of Validity in SPL

To define validity in SPL, I first define two concepts of satisfiability in SPL:

(1) Strong Satisfiability
The set of formulae Σ is strongly satisfiable. $=df$ There is a partial valuation which makes all the formulae in Σ true.
(2) Weak Satisfiability
The set of formulae Σ is weakly satisfiable. $=df$ There is a partial valuation which makes no formulae in Σ false.

Based on these definitions, we can define the following definitions of unsatisfiability (Notice that the order of strength reverses):

(1a) Weak Unsatisfiability
The set of formulae Σ is weakly unsatisfiable. $=df$ There are no partial valuations which make all the formulae in Σ true.
(2a) Strong Unsatisfiability
The set of formulae Σ is strongly satisfiable. $=df$ There are no partial valuations

which make no formulae in Σ false.

Finally we can define the following definitions of validity :[9]

(1b) Weak Validity
The argument that has the set of premises Σ and the consequence C is w-valid ($\Sigma \models w\ C$). $=df$ The set of formulae $\Sigma \cup \{\neg C\}$ is weakly unsatisfiable.
(When Σ is empty, we say that the formula C is w-valid and call C 'Weak Tautology'.)
(2b) Strong Validity
The argument that has the set of premises Σ and the consequence C is s-valid ($\Sigma \models s\ C$). $=df$ The set of formulae $\Sigma \cup \{\neg C\}$ is strongly unsatisfiable.
(When Σ is empty, we say that the formula C is s-valid and call C 'Strong Tautology'.)

Here we are required to define the truth-condition of the negation operator. If we choose classical logic as the basic logic, there can be only one monotonic negation operator :

A	$\neg A$
t	f
–	–
f	t

Adopting this truth-table, we can paraphrase the above definitions of validity in the following way :

(1c) $\Sigma \models w\ C$ iff whenever all the formulae in Σ are true, C is true or gapped (= C is not false).
(2c) $\Sigma \models s\ C$ iff whenever all the formulae in Σ are true or gapped (=whenever no formulae in Σ are false), C is true.

Compared with the naturalness of (1a), (2a) and (1b), (2b), they may seem somewhat arbitrary because of the asymmetry between the premises and the consequence in terms of being gapped. Indeed I believe that it is the reason why these definitions have not attracted much attention from logicians. However, if we take the essential features of partial logic into consideration, I think that these definitions should be accepted.

In the semantics of total (two-valued) logic there is only one axis of valuation : which truth-value a formula has, namely truth or falsity. In the semantics of partial logic, on the other hand, there is another axis of valuation presupposed: whether a formula has a truth-value or not. I give other definitions here:

Weak Formula: the formula which does not always have a truth-value under partial valuation
Strong Formula: the formula which always has a truth-value under partial valuation

We have to take both of these axes into consideration when we define the concept of validity. For example, we should distinguish between the following concepts of validity of a formula (these are respectively the explanatory meanings of 'C is true or gapped' and 'C is true' in the above definitions (1c) and (2c).):

Weak Tautology: the Formula which is always true *if it has a truth-value*
Strong Tautology : the Formula which *always has a truth- value* and is always true

Considering these points, the explanatory meaning of weak and strong validity of arguments are as follows ((b) is logically equivalent to (a)):

<Weak Validity>
(a) Whenever all the premises have a truth-value and are true, the consequence is, if it has a truth-value, true.
(b) If all the formulae in the argument have a truth-value, whenever all the premises are true, the consequence is true.

<Strong Validity>
(a) Whenever all the premises that have a truth-value are true, the consequence has a truth-value and is true.
(b) At least one formula in the argument always has a truth-value. Among the formulae that have a truth-value, whenever all the premises are true, the consequence is true.

Weak validity means that if we assume that all the formulae in Σ and the negation of C have truth-values, they cannot be true at the same time. To say that the negation of C is true, assuming that it has a truth-value, is the same as saying that the negation of C has the value of truth (strong concept of truth). Accordingly 'the negation of C cannot be true' here means that 'the negation of C cannot have the value of truth', namely, 'the negation of C must be gapped or has the value of falsity'. This is equivalent to 'C must be gapped or have the value of truth', namely, 'if it has a truth-value, C must be true (weak concept of truth)'.

To the contrary, strong validity means that even if we do not assume that all the formulae in Σ and the negation of C have truth-values, they cannot be true at the same time. To say that the negation of C is true, without assuming that it has a truth-value, is the same as saying that the negation of C is gapped or has the value of truth (weak concept of truth). Accordingly 'the negation of C cannot be true' here means that 'the negation of C can neither be gapped nor have the value of truth', namely, 'the negation of C must have the value of falsity'. This is equivalent to 'C must have the value of truth', namely, 'C must have a truth-value and be true (strong concept of truth)'.

Some readers may still be irritated by the remaining asymmetry in our two definitions. If so, I want to point out that at least these definitions result from two natural notions of unsatisfiability shown above. In my view, the relation of logical consequence or validity in a truth-functional logic cannot but be unsatisfiability.

I am not sure as to whether these two definitions can capture the features of our ordinary inferences better than others. My principal concern is the generalization of total logic. I want to show that *total logic is*, as it were, *a specialized partial logic*. The more these two logics share properties, the better for me. Therefore if the apparent naturalness of other definitions is the only reason to adopt them, I would prefer the definitions described in this paper, as they retain some advantageous semantic properties which the others lack. In the next section I will explore those properties. To say in advance, the holding of semantic deduction theorem is one of them. Thanks to deduction theorem we can also grasp w-valid and s-valid arguments as the arguments which make the corresponding conditional formulae a weak and strong tautology respectively.

Before moving to the next section, it should be noticed that the above justification of the two definitions of validity crucially depends on my characterization of partial logic. For my justification presupposes that in SPL there is only one negation operator, which is reliable and monotonic. If we admit another negation operator such as external negation, which is non-monotonic, we can equally justify other definitions of validity.(10)

4. Semantic Theorems

We can easily prove the following semantic theorems:

<Th.1>

$\Sigma \models_s C \;\rightarrow\; \Sigma \models_w C$

[proof] It results from the definitions of validity. ■

<*Df*>

$\Sigma \models C \quad =df$ The argument which derives C from Σ is valid in the basic logic.

(This implies that the argument includes only the basic formulae.)

<Th.2>
If an argument includes only the basic formulae,

$\Sigma \models w\ C \ \rightarrow\ \Sigma \models\ C$

[proof] It results from reliable determinability. ■

Now I prove deduction theorem and contraposition theorem. For that we have to choose the basic logic and define the truth-conditions for other basic operators in addition to the negation operator above. Here I adopt classical logic as the basic logic and the truth-tables which Kleene defined for his strong three-valued logic.(11)

$A \wedge B$	A\B	t	−	f
	t	t	−	f
	−	−	−	f
	f	f	f	f

$A \vee B$	A\B	t	−	f
	t	t	t	t
	−	t	−	−
	f	t	−	f

$A \rightarrow B$	A\B	t	−	f
	t	t	−	f
	−	t	−	−
	f	t	t	t

These operators have another desirable property for partial logic that is called 'strength'; an operator is stronger than another under partial valuation iff the former gives a truth-value to the formula which it bounds whenever the latter gives one to the same formula. Kleene's operators above are the strongest definitions that are both reliable and monotonic relative to classical logic.(12)

<Th.3w> Weak Deduction Theorem (=WDT)

$P_1, P_2, \cdots, P_n \models w\ C$ iff $\models w\ (P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$

<Th.3s> Strong Deduction Theorem(=SDT)

$P_1, P_2, \cdots, P_n \models s\ C$ iff $\models s\ (P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$

<Th.4w> Weak Contrapostition Theorem (=WCT)

$P_1, P_2, \cdots, P_n \models w\ C$ iff $\neg C \models w\ \neg(P_1 \wedge P_2 \cdots \wedge P_n)$

<Th.4s> Strong Contrapostition Theorem (=SCT)

$P_1, P_2, \cdots, P_n \models s\ C$ iff $\neg C \models s\ \neg(P_1 \wedge P_2 \cdots \wedge P_n)$

[proof]

(1) When n=1, the theorems above are expressed in the following way:

T1 $P \models w\ C$ iff $\models w\ P \rightarrow C$

T2 $P \models s\ C$ iff $\models s\ P \rightarrow C$

T3 $P \models w\ C$ iff $\neg C \models w\ \neg P$

T4 $P \models s\ C$ iff $\neg C \models s\ \neg P$

We can confirm them by the truth-table below:

P	C	$P\to C$	$\neg C$	$\neg P$
t	t	t	f	f
t	—	—	—	f
t	f	f	t	f
—	t	t	f	—
—	—	—	—	—
—	f	—	t	—
f	t	t	f	t
f	—	t	—	t
f	f	t	t	t

(2) We can prove that when $n \geqq 2$, both WDT and SDT hold in the following way:

(i) In the case that $(P_1 \wedge P_2 \cdots \wedge P_{n-1})$ is true,
All the premises $P_1, P_2, \cdots, P_n$ are true iff P_n is true. So

$P_1, P_2, \cdots, P_n \models w\ C$ iff $P_n \models w\ C$.

Moreover the value of '$(P_1 \wedge P_2 \cdots \wedge P_n) \to C$' coincides with that of '$P_n \to C$'. So

$\models w\ (P_1 \wedge P_2 \cdots \wedge P_n) \to C$ iff $\models w\ P_n \to C$

By $T1$, $P_n \models w\ C$ iff $\models w\ P_n \to C$.
Therefore, $P_1, P_2, \cdots, P_n \models w\ C$ iff $\models w\ (P_1 \wedge P_2 \cdots \wedge P_n) \to C$.
This is WDT. Similarly for SDT.

(ii) In the case that $(P_1 \wedge P_2 \cdots \wedge P_{n-1})$ is false, $(P_1 \wedge P_2 \cdots \wedge P_n)$ is also false. So both '$P_1, P_2, \cdots, P_n \models w\ C$' and '$\models w\ (P_1 \wedge P_2 \cdots \wedge P_n) \to C$' trivially hold. So WDT holds. Similarly for SDT.

(iii) In the case that $(P_1 \wedge P_2 \cdots \wedge P_{n-1})$is gapped, the value of $(P_1 \wedge P_2 \cdots \wedge P_n)$ coincides with that of P_n. If P_n is true or false, it is the same as (i) and (ii) respectively. If it is gapped, both of '$P_1, P_2, \cdots, P_n \models w\ C$' and '$\models w\ (P_1 \wedge P_2 \cdots \wedge P_n) \to C$' trivially hold. So WDT holds. As for '$P_1, P_2, \cdots, P_n \models s\ C$' and '$\models s\ (P_1 \wedge P_2 \cdots \wedge P_n) \to C$', both of them hold iff C is true. So SDT holds.
By(i), (ii) and (iii), WDT and SDT hold in any case when $n \geqq 2$.
By (1) and (2), WDT and SDT hold for any natural number n. ■

We can also prove contraposition theorems in the similar way. However I omit part-(2) of their proof for brevity.

Using deduction theorems we can prove the following theorems :

<Th.5>
If an argument includes only the basic formulae,

$\Sigma \models C \quad \rightarrow \quad \Sigma \models w \ C$

[proof] (by Reductio ad Absurdum)

Assume that $\Sigma = \{P_1, P_2, \cdots, P_n\}$. By WDT, $\Sigma \models w \ C$ iff $\models w \ (P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$. If <Th.5> does not hold, it means that there are cases where '$\Sigma \mid = C$' holds but '$\Sigma \models w$ C' does not. Because of reliable determinability it is possible only if there is a valuation which makes '$(P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$' false. But this contradicts the monotonicity of the basic operators, since under total valuation the formula is true. So <Th.5> holds. ■

By <Th.2> and <Th.5> we can conclude that the validity of classical logic coincides with the weak validity of SPL restricted to the basic operators. We can say that it is another way to hold classical validity allowing the truth-value gap than supervaluational logic.(13)

<Th.6>

There are no s-valid arguments which include only the basic formulae.(14)

[proof] (by Reductio ad Absurdum)

Assume that '$P_1, P_2, \cdots, P_n \models s \ C$' is such an argument. By SDT, $P_1, P_2, \cdots, P_n \models s \ C$ iff $\models s \ (P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$. It implies that there is a partial valuation which makes '$(P_1 \wedge P_2 \cdots \wedge P_n) \rightarrow C$' true, while all the atomic formulae in it are gapped. But no monotonic operators make such a valuation possible. Therefore <Th.6> holds. ■

If we adopt the modal operators given in §2 together with Kleene's, the following are among s-valid arguments:

$TA \models s \ A$ (cf. $A \not\models s \ A$, $A \not\models s$ TA)

$LA \models s \ A \vee \neg A$ (cf. $\not\models s \ A \vee \neg A$)

$\models s \ TA \vee \neg TA$

$(A \Rightarrow B) \wedge LA \models s \ TB$

We can take these s-valid arguments as showing the validity which is proper to SPL, in contrast with w-valid arguments.

5. Comparison with other Definitions of Validity

Weak, strong and Wang's validity are three of the four conceptions of validity I enumerated in the introduction. The final one is the following:

(d) The consequence is not false whenever no premises are false.

Blamey's validity is the conjunction of Wang's and (d). Wang's validity and (d) cannot be ordered in terms of strength. Accordingly we can make two order sequences of strength :

Weak Validity < Wang's Validity < Blamey's Validity < Strong Validity
Weak Validity < (d) < Blamey's Validity < Strong Validity

Based on these orders, I rename here the intermediate concepts of validity in the following way:

Blamey's Validity :	Medium Validity,	being m-valid,	$\models m$
Wang's Validity :	Low1 Validity,	being l_1-valid,	$\models l_1$
(d) :	Low2 Validity,	being l_2-valid,	$\models l_2$

Among them contraposition theorem holds only in weak, medium and strong validity. Deduction theorem holds only in weak and strong validity.

As for medium validity, if we adopt Lukasiewicz's conditional instead of Kleene's, deduction theorem holds. However, as Lukasiewicz's conditional is non-monotonic, Blamey cannot adopt it, because he insists that the operators of partial logic should be monotonic.(15)

Blamey gives two theorems which make his definition preferable:

<*Df*>
$A \equiv B$ $=df$ A and B always have the same value under partial valuation.
<Th.m1> $A \equiv B$ iff $A \models m\ B$ and $B \models m\ A$
<Th.m2> $A \equiv (A \wedge B)$ iff $B \equiv (A \vee B)$ iff $A \models m$ B

Neither of them holds in weak nor strong validity. However, the following theorems which correspond to <Th.m1> hold in both:

<Th.m1w> $\models w\ \ A \leftrightarrow B$ iff $A \models w\ B$ and $B \models w\ A$
<Th.m1s> $\models s\ \ A \leftrightarrow B$ iff $A \models s\ B$ and $B \models s\ A$

In my view, this formulation is more appropriate for partial logic than Blamey's. '$A \equiv B$' is the so called three-valued logical equivalence, which requires the coincidence of gap-values besides truth-values. This implies that the gap-value is treated as an equally qualified value as the truth-values. In fact if we express it using an operator in the object language it has to be non-monotonic, which is against Blamey's characterization of partial logic. That is why he used the equivalence operator only in the meta-lenguage. On the contrary, our formulation requires only the coincidence of *truth*-values and the equivalence can be expressed with a basic operator in the object language.

As for <Th.m2>, the following <Th.m2w>, which is modified in the same way with <Th.m1>, holds, while <Th.m2s> does not:

<Th.m2w> $\models w \quad A \leftrightarrow (A \wedge B)$ iff $\models w \quad B \leftrightarrow (A \vee B)$ iff $A \models w \; B$

<Th.m2s> $\models s \quad A \leftrightarrow (A \wedge B)$ iff $\models s \quad B \leftrightarrow (A \vee B)$ iff $A \models s \; B$

However, if we scrutinize the way in which <Th.m2s> fails, we can confirm that it is not against the point of the theorem, which requires the coincidence of the truth-values of A and $A \wedge B$ and the truth-values of B and $A \vee B$, iff $A \models s \; B$:

The value assignment which makes the theorem fail is the following:

<1>When A is gapped and B is true, $A \leftrightarrow (A \wedge B)$ is gapped.

<2>When A is false and B is gapped, $A \leftrightarrow (A \vee B)$ is gapped.

However, in these cases the following hold:

<1a> When A is gapped, $A \wedge B$ is also gapped.

<2a> When B is gapped, $A \vee B$ is also gapped

So these cases do not contradict the point of the theorem at all, even if it required the coincidence of the gap-value, too.

6. Conclusion

SPL can be characterized as a generalization and extension of a total two-valued logic to deal with partiality of truth-valuation. It consists of basic operators, which are monotonic and reliable relative to its basic logic, and modal operators, which are weakly monotonic and always bring about truth-values under partial valuation.

Given such a characterization, weak and strong validity as defined above are more appropriate than other definitions of validity in partial logic. If we adopt Kleene's definitions of basic operators, which have an additional desirable property of strength, (semantic) deduction theorem and contraposition theorem hold. In this case weak validity coincides with classical validity if its operators are restricted to the basic operators. So SPL can be viewed as an alternative to supervaluational logic to retain classical validity allowing the truth-value gap. By contrast no valid arguments in classical logic are strongly valid. Strong validity shows the validity of arguments which are proper to SPL.

NOTES

(1) Besides these we can define other concepts of validity by making a conjunction or a disjunction out of them. Blamey's definition shown below is an example of the conjunct definitions. Langholm combined and generalized these definitions into a single figure and showed the way to choose one of them. [Langhholm, T. 1996] [Fenstad, J.E. 1997]

(2) [Wang, H. 1961]

(3) [Blamey, S. 1986] p. 5f. Here I described his definition in a more generalized way.

(4) Langholm calls this property 'Compositionality'. [Langhholm, T. 1988] p. 17f. Below I take after most of his terminology for partial logic, except for 'Monotonicity', which he calls 'Persistence'.

(5) [Langhholm, T. 1988] pp. 3-6.

(6) [Blamey, S. 1986] p. 9.

(7) [Woodruff, P. 1970]

(8) [Woodruff, P. 1970] For '$\neg$','$\vee$' and '$\rightarrow$', Woodruff adopted the definitions of Kleene's strong three-valued logic shown in §4 below.

(9) Fenstad also used the symbol '$\models w$' and '$\models s$'. However, he meant Wang's validity by the latter. [Fenstad, J.E. 1997] p. 670.

(10) External negation is the one defined by the following table: [Busch, D. 1996] p. 58.

A	$\sim A$
t	f
—	t
f	t

(11) [Kleene, S.C. 1952] The truth tables of '$A \vee B$' and '$A \rightarrow B$' result from their definitions.

(12) [Langhholm, T. 1996] p. 11f. Blamey and Wang also adopted Kleene's operators.

(13) It has merits in that it keeps extensionality and does not include the quantification over valuations in the valuation of a formula. [Woodruff, P. 1984] [Langhholm, T. 1988] p. 8.

(14) Strictly speaking, this theorem presupposes that the basic operators do not include the truth-constant operator, which always brings about truth under any partial valuation, though it is also monotonic.

(15) Lukasiewicz's implication is the one defined by the following table:

$A \sim B$

$A \setminus B$	t	—	f
t	t	—	f
—	t	$\underline{t}$	—
f	t	t	t

It differs from Kleene's strong implication only at the underlined value, which makes this implication non-monotonic.

If we adopt Schmitt's implication, deduction theorem holds for Low1-validity (= Wang's validity). Schmitt's implication is the one defined using external negation and Kleene's strong disjunction as follows:

$$A \supset B \quad =df \quad \sim A \vee B$$

However, this implication is also non-monotonic. [Busch, D. 1996] p. 58f.

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