Control of Fusion and Cohomology of Trivial Source Modules

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Abstract

Let G be a finite group and H a subgroup. We give an algebraic proof of Mislin's theorem which states that the restriction map from G to H on mod-p cohomology is an isomorphism if and only if H controls p-fusion in G. We follow the approach of P. Symonds (Bull. London Math. Soc. 36 (2004) 623-632) and consider the cohomology of trivial source modules.

Key words: cohomology of finite groups, p-fusion, Mackey functor, trivial source modules

1 Introduction

Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. In [13], P. Symonds proved the following.

Theorem 1.1 ([13, Theorem 4.1]) *As an inflation functor, the cohomology $H^*(-, k)$ contains every simple cohomological inflation functor as a composition factor.*

Using this result, he proved the following theorem of G. Mislin [7] for finite groups.

Theorem 1.2 ([7, Theorem], [11, Theorem 1.1], [13, Theorem 1.1]) *Let H be a subgroup of G containing a Sylow p-subgroup of G. Then the following are equivalent.*

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(1) *The restriction*

$$\mathrm{res}_{G,H} : H^*(G, k) \longrightarrow H^*(H, k)$$

is an isomorphism.
(2) *If* ${}^xQ \subseteq H$ *then* $x \in HC_G(Q)$*, for any* $x \in G$ *and* p*-subgroup* $Q \leq H$*.*

The original result of Mislin is proved for compact Lie groups. The proof in [11] and Symonds' proof of Theorem 1.1 use also results from topology. The purpose of this paper is to give an algebraic proof of Theorem 1.2 using modular representation theory of finite groups.

Let G be a finite group. Then $H^n(-, k)$ is a cohomological Mackey functor for G. The simple cohomological Mackey functors are classified in [15], [16]. They are parametrized by pairs (P, V), where P is a p-subgroup of G and V is a simple $k(N_G(P)/P)$-module (up to conjugation and isomorphism). The following result is obtained from Theorem 1.1 immediately. The " only if " part comes from the fact that if $S^G_{P,V}$ appears as a compostion factor of $H^n(-, k)$ for some n, then $C_G(P)$ acts trivially on $S^G_{P,V}(P) = V$ since $C_G(P)$ acts trivially on $H^n(P, k)$.

Theorem 1.3 ([13, Corollary 4.2]) *Let* P *be a* p*-subgroup of* G *and* V *a simple* $k(N_G(P)/P)$*-module. Then the simple Mackey functor* $S^G_{P,V}$ *appears in* $H^n(-, k)$ *as a composition factor for some* $n \geq 0$ *if and only if* $C_G(P)$ *acts trivially on* V*.*

Actually we can prove Theorem 1.2 using only Theorem 1.3, see section 2. On the other hand, Theorem 1.3 is equivalent to a condition on the cohomology of a trivial source module. Let M be an indecomposable trivial source kG-module with vertex P. Then the Green correspondent of M in $N_G(P)$ is a projective cover of some simple $kN_G(P)$-module V as a $k(N_G(P)/P)$-module. We denote the trivial source module M by $M^G_{P,V}$. Then Theorem 1.3 is equivalent to the following.

Theorem 1.4 ([13, Theorem 5.3]) *Let* P *be a* p*-subgroup of* G *and* V *a simple* $k(N_G(P)/P)$*-module. Then* $H^*(G, M^G_{P,V}) \neq 0$ *if and only if* $C_G(P)$ *acts on trivially on* V*.*

We prove Theorem 1.4 in section 4. To reduce the problem to some local subgroup of G, we need a result of Benson [3], which says that every periodic module is induced from some subgroup. In [8], T. Okuyama gives an algebraic proof of Theorem 1.4 independently. Some related results are obtained in [1], [10].

Let H be a subgroup of G and M a kG-module. We denote by $M \downarrow_H$ the restriction of M to H. If N is a kH-module, we denote by $N \uparrow^G$ the induced

kG-module. Let M and N be kG-modules. Let

$$\widehat{\mathrm{Ext}}\,^n_{kG}(M,N) = \mathrm{Ext}^n_{kG}(M,N)$$

for $n > 0$ and

$$\widehat{\mathrm{Ext}}^0_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the set of all kG-homomorphisms which factor through projective modules. Let

$$\widehat{\mathrm{Ext}}^*_{kG}(M,N) = \bigoplus_{n\geq 0} \widehat{\mathrm{Ext}}^n_{kG}(M,N).$$

It is possible to consider $\widehat{\mathrm{Ext}}^n_{kG}(M,N)$ for $n < 0$, but we need only the non-negative part.

2 Mackey functor and fusion

In this section, we review the results of Symonds [13, section 3 and section 5]. Here we consider only Mackey functors for a finite group G though Symonds considered inflation functors and global Mackey functors. Fix a p-subgroup Q of G. Let

$$M_Q = \mathrm{Ind}^G_{N_G(Q)} \mathrm{Inf}^{N_G(Q)}_{N_G(Q)/Q} FQ_{k(N_G(Q)/QC_G(Q))}$$

where $FQ_{k(N_G(Q)/QC_G(Q))}$ is the fixed quotient functor for the $k(N_G(Q)/Q)$-module $k(N_G(Q)/QC_G(Q))$. Then M_Q is a cohomological Mackey functor, since the fixed quotient functor $FQ_{k(N_G(Q)/QC_G(Q))}$ is, and the subsequent operations used to construct M_Q preserve cohomological Mackey functors [16, (16.2) Lemma, (16.6) Corollary, (16.13) Lemma].

Theorem 2.1 *Let P be a p-subgroup of G and V a simple $k(N_G(P)/P)$-module.*
(1) *If the Mackey functor M_Q contains the simple Mackey functor $S^G_{P,V}$ as a composition factor, then $C_G(P)$ acts trivially on V.*
(2) *If $C_G(P)$ acts trivially on V then M_P contains the simple Mackey functor $S^G_{P,V}$.*

PROOF. (1) Let $N = N_G(Q)$, $\bar{N} = N/Q$ and $C = C_G(Q)$. Suppose that the Mackey functor M_Q contains $S^G_{P,V}$ as a composition factor. Let W be the projective cover of V as a $k(N_G(P)/P)$-module. Then the injective hull of $S^G_{P,V}$ as a cohomological Mackey functor is a direct summand of $FQ_{W\uparrow^G}$ by

[16, (16.12) Corollary]. Then

$$
\begin{aligned}
0 &\neq \mathrm{Hom}_{\mathrm{Mack}_k(G)}(\mathrm{Ind}_N^G \mathrm{Inf}_{\bar{N}}^N FQ_{k(N/QC)}, FQ_{W\uparrow^G}) \\
&\cong \mathrm{Hom}_{\mathrm{Mack}_k(N)}(\mathrm{Inf}_{\bar{N}}^N FQ_{k(N/QC)}, FQ_{W\uparrow^G\downarrow_N}) \\
&\cong \mathrm{Hom}_{\mathrm{Mack}_k(\bar{N})}(FQ_{k(N/QC)}, (FQ_{W\uparrow^G\downarrow_N})^-) \\
&\cong \mathrm{Hom}_{k\bar{N}}(k(N/QC), (FQ_{W\uparrow^G\downarrow_N})^-(Q/Q))
\end{aligned}
$$

by [15, (4.2) Proposition, (5.1) Proposition, (6.1) Proposition], where $^-$ is the right adjoint of the inflation functor. By the Mackey decomposition formula,

$$W\uparrow^G\downarrow_N \cong \bigoplus_{g\in N_G(P)\backslash G/N} U(g)$$

where $U(g) = {}^gW\downarrow_{N_G({}^gP)\cap N}\uparrow^N$. If $Q \not\subseteq {}^gP$, then $(FQ_{U(g)})^-(Q/Q) = 0$. On the other hand, if $Q \subseteq {}^gP$ then $(FQ_{U(g)})^-(Q/Q) = U(g)$. So there exists $g \in G$ such that $Q \subseteq {}^gP$ and

$$\mathrm{Hom}_{kN}(k(N/QC), U(g)) \neq 0.$$

Then

$$\mathrm{Hom}_{k(N_G({}^gP)\cap QC)}(k, {}^gW\downarrow_{(N_G({}^gP)\cap QC)}) \neq 0$$

and

$$\mathrm{Hom}_{kN_G(P)}(k(N_G(P)/C_G(P)), W) \cong \mathrm{Hom}_{kC_G(P)}(k, W\downarrow_{C_G(P)}) \neq 0$$

since $C_G({}^gP) \subseteq C$. Hence $C_G(P)$ acts trivially on V.
(2) Let $N = N_G(P)$ and $C = C_G(P)$. Since C acts trivally on V, there exists a non-zero $k(N/P)$-morphism from $k(N/PC)$ to V. By [15, (6.1) Proposition], this gives a non-zero morphism of N/P-Mackey functors

$$FQ_{k(N/PC)} \longrightarrow FP_V$$

whose image contains the socle $S_{1,V}^{N/P}$ of FP_V. Now applying the exact functor $\mathrm{Ind}_N^G\mathrm{Inf}_{N/P}^N$ shows that $S_{P,V}^G = \mathrm{Ind}_N^G\mathrm{Inf}_{N/P}^N S_{1,V}^{N/P}$ appears as a composition factor of M_P.

Let $T_G(Q,H) = \{x \in G \mid {}^xQ \subseteq H\}$ for $H \leq G$. Then the following result is clear from the definition of M_Q and [15, (4.3) Proposition].

Proposition 2.2 *Let Q be a p-subgroup of G and $Q \leq H \leq G$. Then*

$$\dim M_Q(H) = 1$$

if and only if

$$T_G(Q,H) = HC_G(Q).$$

Now let explain how we can obtain Theorem 1.2 from Theorem 1.3 following [13]. Assume that $H \leq G$ and H contains a Sylow p-subgroup of G. Then

$$\mathrm{res}_{G,H} : H^*(G,k) \longrightarrow H^*(H,k)$$

is an isomorphism if and only if

$$r^G_H : S^G_{P,V}(G) \longrightarrow S^G_{P,V}(H)$$

is an isomorphism for any p-subgroup P and simple $k(N_G(P)/PC_G(P))$-module V by Theorem 1.3. On the other hand, these maps are isomorphisms if and only if $r^G_H : M_Q(G) \longrightarrow M_Q(H)$ is an isomorphism for any p-subgroup Q by Theorem 2.1. By Proposition 2.2, these maps are isomorphisms if and only if $T_G(Q,H) = HC_G(Q)$ for any p-subgroup Q of H.

Next we consider the cohomology of some trivial source modules. Let P be a p-subgroup of G and V a simple $k(N_G(P)/P)$-module. Let P_V be the projective cover of V as a $k(N_G(P)/P)$-module. Let $M^G_{P,V}$ be the Green correspondent of P_V with respect to $(G, N_G(P), P)$. Then by [13, section 5], the simple Mackey functor $S^G_{P,V}$ is a composition factor of $H^n(-,k)$ if and only if $H^n(G, M^G_{P,V^*}) \neq 0$ where $V^* = \mathrm{Hom}_k(V,k)$. Hence Theorem 1.3 is equivalent to Theorem 1.4.

3 Periodic modules

In this section we state some results on the cohomology and the variety of kG-modules. If $\zeta \in H^n(G,k)$, then ζ is considered as a kG-homomorphism $\Omega^n(k) \longrightarrow k$. Let L_ζ be the kernel of this map if $\zeta \neq 0$. Set $L_\zeta = \Omega^n(k) \oplus \Omega(k)$ if $\zeta = 0$. Then there exists a short exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n(k) \oplus (\text{projective}) \longrightarrow k \longrightarrow 0.$$

Let $V_G(k)$ be the maximal ideal spectrum of $H^*(G,k)$. Let M be a finitely generated kG-module. Let $V_G(M)$ be the homogeneous closed subset of $V_G(k)$ defined by the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$ in $H^*(G,k)$. This closed subset is called the variety of M. Let $\zeta_1, \ldots, \zeta_m$ be homogeneous elements in $H^*(G,k)$. We denote by $V_G(\zeta_1, \ldots, \zeta_m)$ the closed subset defined by the ideal $(\zeta_1, \ldots, \zeta_m)$. It is known that $V_G(\otimes^m_{i=1} L_{\zeta_i}) = V_G(\zeta_1, \ldots, \zeta_m)$. For details, see [2, Chapter 5]. We need the following result in the next section. Note that the proof of this Proposition in [3] uses Rickard's idempotent module [9] which is not finitely generated.

Proposition 3.1 ([3, Corollary 3.2]) *Let E be a subgroup of order p in G and $H = N_G(E)$. Let $\tilde{l} = \mathrm{res}^*_{G,E}(V_E(k))$ and $l = \mathrm{res}^*_{H,E}(V_E(k))$. Let M be* *a kG-module with $V_G(M) = \tilde{l}$. If $M \downarrow_H = X \oplus Y$ where $V_H(X) = l$ and $V_H(Y) \cap l = \{0\}$, then*

$$X \uparrow^G \cong M \oplus Z$$

for some projective module Z.

4 Cohomology of trivial source modules

In this section we prove Theorem 1.4. We prove the " if " part since the " only if " part follows from the fact that $C_G(P)$ acts trivially on $H^n(P,k)$.

Lemma 4.1 *Let P be a p-subgroup of G and V a simple $k(N_G(P)/PC_G(P))$-module. If $G \geq H \geq P$, then there exists a simple $k(N_H(P)/PC_H(P))$-module V' such that $M^H_{P,V'}$ is a direct summand of $M^G_{P,V}$ as a kH-module.*

PROOF. Let P_V be a projective cover of V as a $k(N_G(P)/P)$-module. There exists a composition factor V' of $V \downarrow_{N_H(P)}$ such that

$$\mathrm{Hom}_{N_H(P)}((P_V) \downarrow_{N_H(P)}, V') \neq 0.$$

Then V' is a $k(N_H(P)/PC_H(P))$-module and $P_{V'}$ is a direct summand of $P_V \downarrow_{N_H(P)}$, where $P_{V'}$ is a projective cover of V' as a $k(N_H(P)/P)$-module. So, $P_{V'}$ is a direct summand of $U \downarrow_{N_H(P)}$ for some indecomposable direct summand U of $M^G_{P,V} \downarrow_H$. Then $U \cong M^H_{P,V'}$ by Burry-Carlson-Puig theorem [14, Exercise (20.5)].

Next we prove some results on the module L_ζ.

Lemma 4.2 *Let X and Y be kG-modules. Let $\zeta \in H^n(G,k)$.*
(1) *If $\widehat{\mathrm{Ext}}^*_{kG}(X,Y) = 0$, then $\widehat{\mathrm{Ext}}^*_{kG}(X \otimes L_\zeta, Y) = 0$.*
(2) *Let I be a prime ideal of $H^*(G,k)$ and $\zeta \in I$. If* ann $\widehat{\mathrm{Ext}}^*_{kG}(X,Y)$, *the annihilator of $\widehat{\mathrm{Ext}}^*_{kG}(X,Y)$ in $H^*(G,k)$, is contained in I, then*

$$\mathrm{ann}\ \widehat{\mathrm{Ext}}^*_{kG}(X \otimes L_\zeta, Y) \subseteq I.$$

PROOF. (1) By the long exact sequence of cohomology, we have the following exact sequence,

$$\widehat{\mathrm{Ext}}^i_{kG}(\Omega^n(X), Y) \longrightarrow \widehat{\mathrm{Ext}}^i_{kG}(X \otimes L_\zeta, Y) \longrightarrow \widehat{\mathrm{Ext}}^{i+1}_{kG}(X,Y)$$

for any $i \geq 0$. Then the result follows since $\widehat{\mathrm{Ext}}^i_{kG}(\Omega^n(X), Y) \cong \widehat{\mathrm{Ext}}^{i+n}_{kG}(X,Y)$.
(2) Let $A = H^*(G,k)$ and $B = \widehat{\mathrm{Ext}}^*_{kG}(X,Y)$. Since there exists an exact

sequence,

$$\widehat{\operatorname{Ext}}^i_{kG}(X,Y) \longrightarrow \widehat{\operatorname{Ext}}^{i+n}_{kG}(X,Y) \longrightarrow \widehat{\operatorname{Ext}}^i_{kG}(X\otimes L_\zeta,Y)$$

we have $\operatorname{ann}\widehat{\operatorname{Ext}}^*_{kG}(X\otimes L_\zeta,Y)\subseteq\sqrt{\operatorname{ann}_A B/\zeta B}$. If $\operatorname{ann}_A B/\zeta B$ is not contained in I, then $B_I=(\zeta B)_I$ and $B_I=IB_I$ since $\zeta\in I$, where B_I is the localization of B with respect to I. Hence $B_I=0$ but this contradicts the assumption that $\operatorname{ann}_A B\subseteq I$.

Let A be a k-algebra and B an A-module. We say an element $a\in A$ is B-regular if $B\neq 0$ and $ab\neq 0$ for any nonzero element $b\in B$.

Lemma 4.3 *Let E be a subgroup of order p in the center of G and $\zeta\in H^m(G,k)$. Let M be a $k(G/E)$-module. Suppose that $H^*(G,M)\neq 0$. If $\operatorname{res}_{G,E}(\zeta)$ is not nilpotent, then ζ is $H^*(G,M)$-regular.*

PROOF. This lemma is proved using the argument in [6, Proof of Theorem 10.3.1]. It is also proved in [5, Proposition 4] for $M=k$. We include the proof using the method in [5, Proposition 4].

Let $v\in H^n(G,M)$. Assume that $v\neq 0$ and $\zeta v=0$. Let

$$\mu_1^*: H^*(G,k)\longrightarrow H^*(E,k)\otimes H^*(G,k)$$

$$\mu_2^*: H^*(G,M)\longrightarrow H^*(E,k)\otimes H^*(G,M)$$

be the homomorphisms induced by the multiplication $\mu: E\times G\longrightarrow G$ in G. Then

$$\mu_1^*(\zeta)=\operatorname{res}_{G,E}(\zeta)\otimes 1+\tilde{\zeta}$$
$$\mu_2^*(v)=\sum v_j$$

where $\tilde{\zeta}\in\sum_{i=1}^m H^{m-i}(E,k)\otimes H^i(G,k)$ and $v_j\in H^{n-j}(E,k)\otimes H^j(G,M)$. Take a minimal k such that $v_k\neq 0$. Then

$$0=\mu_2^*(\zeta v)=(\operatorname{res}_{G,E}(\zeta)\otimes 1)v_k+w$$

with $w\in H^*(E,k)\otimes(\oplus_{j>k}H^j(G,M))$. Hence $\operatorname{res}_{G,E}(\zeta)$ is a divisor of 0 in $H^*(E,k)$. This is a contradiction since E is a cyclic group.

Corollary 4.4 *Let P be a p-subgroup of G and E a subgroup of order p in the center of P. Let M be a P-projective kG-module. Suppose that E acts on M trivially and $H^*(G,M)\neq 0$. Let $\zeta\in H^*(G,k)$. If $\operatorname{res}_{G,E}(\zeta)$ is not nilpotent, then ζ is $H^*(G,M)$-regular. In particular,* $\operatorname{ann} H^*(G,M)\subseteq\sqrt{\operatorname{Ker}\operatorname{res}_{G,E}}$.

PROOF. Since M is P-projective, the restriction

$$\mathrm{res}_{G,E} : H^*(G,M) \longrightarrow H^*(P,M)$$

is injective. Hence the result follows from Lemma 4.3.

Now we prove Theorem 1.4 by induction on the order of G. Let P be a p-subgroup of G and V a simple $k(N_G(P)/PC_G(P))$-module. Note that if $P = 1$, then $M^G_{P,V}$ is the projective cover of the simple kG-module V. Then $H^*(G, M^G_{P,V}) = H^0(G, M^G_{P,V})$. This is non-zero if and only if V is the trivial module. In the rest of this section, we assume that $P \neq 1$. Let E be a subgroup of order p in the center of P. First, we consider the case $G = C_G(E)$.

Proposition 4.5 *If $G = C_G(E)$, then $H^*(G, M^G_{P,V}) \neq 0$.*

PROOF. The following argument is based on the idea of [12]. Let $M = M^G_{P,V}$. Then $M \cong M^{G/E}_{P/E,V}$ as $k(G/E)$-modules since E is central in G. By induction, we may assume that $H^*(G/E, M) \neq 0$. Take minimal $n \geq 0$ such that $H^n(G/E, M) \neq 0$. Consider the spectral sequence,

$$E_2^{pq} = H^p(G/E, H^q(E,M)) \cong H^p(G/E, M) \otimes H^q(E,k) \Rightarrow H^{p+q}(G,M).$$

If $p < n$ then $E_2^{pq} = 0$ for any $q \geq 0$ since $H^p(G/E, M) = 0$. Hence $E_\infty^{n0} \cong E_2^{n0} = H^n(G/E, M) \neq 0$.

Let $H = N_G(E)$ and $C = C_G(E)$. Then H/C is a cyclic group of order prime to p and H/C acts on $H^*(C,k)$. We have the following result which is analogous to [4, Lemma 6.7].

Lemma 4.6 *There exists $\zeta \in H^*(C,k)$ such that $\mathrm{res}_{C,E}(\zeta)$ is not nilpotent and the one dimensional subspace $k\zeta$ affords a faithful representation of H/C.*

PROOF. Let $\xi(\neq 0) \in H^2(E,k)$. We can construct, using Evens norm map, a homogeneous element $\zeta_0 \in H^m(C,k)$ such that $\mathrm{res}_{C,E}(\zeta_0) = \xi^{p^a}$ for some $a \geq 0$ ([2, Lemma 5.6.2]). Let $J_m = \mathrm{Ker}\ \mathrm{res}_{C,E} \cap H^m(C,k)$. Then H/C acts on $H^m(C,k)/J_m = k(\zeta_0 + J_m)$ faithfully. The short exact sequence

$$0 \longrightarrow J_m \longrightarrow H^m(C,k) \longrightarrow k(\zeta_0 + J_m) \longrightarrow 0$$

splits as a sequence of $k(H/C)$-modules. So there exists $\zeta \in H^m(C,k)$ such that $\zeta \notin J_m$ and $k\zeta$ affords a faithful representation of (H/C).

Proposition 4.7 *If $G = N_G(E)$, then $H^*(G, M^G_{P,V}) \neq 0$.*

PROOF. Let $M = M^G_{P,V}$ and $C = C_G(E)$. By Lemma 4.1, there exists a simple $k(N_C(P)/PC_C(P))$-module V' such that $M^C_{P,V'}$ is a direct summand of $M \downarrow_C$. By Proposition 4.5, we have $H^*(C,M) \neq 0$. There exists $v (\neq 0) \in H^*(C,M)$ such that the one dimensional subspace kv is G/C invariant. Let $\zeta \in H^*(C,k)$ as in Lemma 4.6. Then ζ is $H^*(C,M)$-regular by Lemma 4.3 and $\zeta^b v$ is G/C invariant for some $b \geq 0$. Hence $H^*(G,M) \cong H^*(C,M)^{G/C} \neq 0$.

Finally we consider the general case. Let $H = N_G(E)$ and $C = C_G(E)$. Let $\zeta_1, \ldots, \zeta_m$ be a homogeneous generating set of $\sqrt{\text{Ker res}_{G,E}}$. Let $\tilde{l} = \text{res}^*_{G,E}(V_E(k))$ and $l = \text{res}^*_{H,E}(V_E(k))$. Consider the tensor product $L = \otimes^m_{i=1} L_{\zeta_i}$. Then $V_G(L) = \tilde{l}$. We can decompose $L \downarrow_H$ as $L \downarrow_H \cong X \oplus Y$ where $V_H(X) = l$ and $V_H(Y) \cap l = \{0\}$. Then by Proposition 3.1,

$$L \oplus (\text{projective}) \cong X \uparrow^G .$$

Let $\eta_i = \text{res}_{G,H}(\zeta_i)$. There exists $\eta_0 \in H^*(H,k)$ such that $V_H(\eta_0, \eta_1, \ldots, \eta_m) = l$ and $V_H(Y) \cap V_H(\eta_0) = \{0\}$. Then $Y \otimes L_{\eta_0}$ is projective since

$$V_H(Y \otimes L_{\eta_0}) = V_H(Y) \cap V_H(L_{\eta_0}) = \{0\}$$

(see [2, Theorem 5.1.1]). We may assume that $\eta_0 \text{Ext}^*_{kH}(X,X) = 0$ since some power of η_0 annihilates $\text{Ext}^*_{kH}(X,X)$. Then, modulo projectives,

$$\bigotimes_{i \geq 0} L_{\eta_i} \cong (X \oplus Y) \otimes L_{\eta_0} \cong X \otimes L_{\eta_0} \cong \Omega(X) \oplus \Omega^n(X)$$

for some $n \geq 0$ by [2, Proposition 5.9.5].

Proposition 4.8 $H^*(G, M^G_{P,V}) \neq 0$.

PROOF. By Lemma 4.1, there exists a simple $k(N_H(P)/PC_H(P))$-module V' such that $M^H_{P,V'}$ is a direct summand of $M^G_{P,V} \downarrow_H$. Let $M = M^G_{P,V}$ and $M' = M^H_{P,V'}$. Then $H^*(H,M') \neq 0$ by Proposition 4.7. Let $I = \sqrt{\text{Ker res}_{H,E}} = \sqrt{(\eta_0, \eta_1, \ldots, \eta_m)}$. Then $\text{ann}_{H^*(H,k)} H^*(H,M') \subseteq I$ by Corollary 4.4. So

$$\text{ann}_{H^*(H,k)} \widehat{\text{Ext}}^*_{kH}(X, M') \subseteq \text{ann}_{H^*(H,k)} \widehat{\text{Ext}}^*_{kH}(\bigotimes_{i \geq 0} L_{\eta_i}, M') \subseteq I$$

by Lemma 4.2 (2). In particular $\widehat{\text{Ext}}^*_{kH}(X, M') \neq 0$ thus

$$\widehat{\text{Ext}}^*_{kG}(L, M) \cong \widehat{\text{Ext}}^*_{kG}(X \uparrow^G, M) \cong \widehat{\text{Ext}}^*_{kH}(X, M) \neq 0.$$

Then $H^*(G,M) \neq 0$ by repeated application of Lemma 4.2 (1) for $Y = M$, starting with $X = k$, since $L = L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_m}$. This completes the proof.

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