Remark on the dimension of Kohnen's spaces of half integral weight with square free level

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(Communicated by Heisuke Hironaka, M. J. A., Feb. 12, 2002)

Abstract: In this note we determine explicitly the dimension of Kohnen's spaces of half integral weight with odd square free level and arbitrary character χ, and show that it coincides with that of spaces of modular cusp forms of weight $2k$ with square free level and character χ^2.

Key words: Modular forms; modular forms of half integral weight.

Introduction. Let N and k be positive integers such that N is odd. For a character χ modulo N, we denote by $S_{k+1/2}(N,\chi)$ the Kohnen's space of half integral weight $k+1/2$ with level N and character χ. In [2], Kohnen calculated the trace of Hecke operators in $S_{k+1/2}(N,\chi)$ and showed that there exists a theory of new forms in it under the assumption that N is square free and $\chi^2 = 1$. Ueda [7] generalized those results to the case of Kohnen's space of weight $k+1/2$ with non-square free level N and character χ satisfying $\chi^2 = 1$. Kohnen [3] proved that the square of Fourier coefficients of modular forms f belonging to $S_{k+1/2}(N,\chi)$ essentially coincides with the central value of quadratic twisted L-series determined by the Shimura correspondence in the case that N is square free and $\chi = 1$. In the proof of this theorem the result in [2] plays an essential role. In [4], we extended this result [3] to the case of arbitrary N and χ under the assumption that f satisfies multiplicity one theorem of Hecke operators. It is an open problem whether there exists the theory of new forms in $S_{k+1/2}(N,\chi)$ in the case of arbitrary odd N.

The purpose of this note is to determine explicitly $\dim S_{k+1/2}(N,\chi)$ and to verify that $\dim S_{k+1/2}(N,\chi) = \dim S_{2k}(N,\chi^2)$ in the case of square free level N as a first step for the solution of above question. Using trace formula [6] and results in [2] and [7], we prove this. We remark that the above problem still remains open.

0. Notation and preliminaries. We denote by $\mathbf{Z}$ and $\mathbf{C}$ the ring of rational integers and the complex number field, respectively. For a $z \in \mathbf{C}$, we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg z^{1/2} \leqq \pi/2$ and put $z^{k/2} = (\sqrt{z})^k$ for every $k \in \mathbf{Z}$. Further we put $e[z] = \exp(2\pi i z)$ for $z \in \mathbf{C}$. For a commutative ring R with identity element, we denote by $SL(2,R)$ the special linear group of all matrices of degree 2 with coefficients in R. For a positive integer m, we put

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) \,\middle|\, c \equiv 0 (\mathrm{mod}\ m) \right\}.$$

The symbol $(\frac{*}{*})$ indicates the same as that of [5, p. 442].

1. Modular forms of half integral weight. For integers l, M and Dirichlet character ψ modulo M, we denote by $S_l(M,\psi)$ the space of modular cusp forms of weight l with level M and character ψ. Let N be an odd integer, χ a Dirichlet character modulo N such that $\chi(-1) = \epsilon$ and k a non negative integer. We denote by $S_{k+1/2}(N,\chi)$ the subspace of $S_{k+1/2}(4N,\chi_\epsilon)$ consisting of those f whose Fourier expansion has the form $f(z) = \sum_{\epsilon(-1)^k n \equiv 0,1(4)} a(n)e[nz]$, where $\chi_\epsilon = (\frac{4\epsilon}{*})\chi$ and $S_{k+1/2}(4N,\chi_\epsilon)$ is the space of cusp forms of half integral weight $k+1/2$ with level $4N$ and a character χ modulo $4N$ in the sence of Shimura [5]. By the table of [1], we derive the following theorem.

Theorem 1.1. *Let N and k be positive integers such that N is odd square free and $k \geqq 2$. Then*

(1.1) $\dim S_{k+1/2}(4N,\chi_\epsilon) = \dim S_{2k}(2N,\chi^2)$.

Proof. According to the decomposition $N = p_1 \cdots p_l$ of prime factors of N, we have the decomposition $\chi = \chi_1 \cdots \chi_p$ of χ. By Cohn-Oesterlé [1], we obtain

(1.2) $$\dim S_{k+1/2}(4N,\chi_\epsilon) = \frac{2k-1}{4}\prod_{p|N}(p+1) - 2^{l-1}\zeta$$

1991 Mathematics Subject Classification. 11F37, 11F72.

and

$$\dim S_{2k}(2N,\chi^2) = \frac{2k-1}{4}\prod_{p|N}(p+1) - 2^l + \frac{1}{4}(-1)^k \sum_{\substack{x\in\mathbf{Z}/2N\mathbf{Z}\\ x^2+1\equiv 0 \pmod{2N}}} \chi^2(x)$$

with

$$\zeta = \begin{cases} 2-\dfrac{1}{2}(-1)^k\epsilon & \text{if } p_i\equiv 1 \pmod 4 \text{ for every } i,\\ 2 & \text{otherwise.}\end{cases}$$

It is easy to check that

$$(1.3)\quad \sum_{\substack{x\in\mathbf{Z}/2N\mathbf{Z}\\ x^2+1\equiv 0 \pmod{2N}}} \chi^2(x) = \prod_{i=1}^{l}\left(\sum_{\substack{x_i\in\mathbf{Z}/p_i\mathbf{Z}\\ x_i^2+1\equiv 0 \pmod{p_i}}} \chi_i^2(x_i)\right).$$

Let ξ_i be a primitive root modulo p_i. If x_i is a solution of congruence $x_i^2+1\equiv 0 \pmod{p_i}$, then $p_i\equiv 1 \pmod 4$ and x_i is $\xi_i^{(p_i-1)/4}$ or $-\xi_i^{(p_i-1)/4}$. Therefore, (1.3) is equal to $\epsilon\prod_{p|N}(1+(\frac{-1}{p}))$. Combining this with (1.2), we conclude our assertion. □

2. The dimension of Kohnen's space. In this section, we shall deduce the following theorem.

Theorem 2.1. *Suppose that N and k are positive integers such that N is odd square free and $k\geqq 2$. Then*

$$(2.1)\quad \dim S_{k+1/2}(N,\chi) = \dim S_{2k}(N,\chi^2).$$

Proof. For a integer t satisfying $|t|<8$, $t\equiv 0 \pmod 4$, we put

$$(2.2)\quad B(t,1) = \left\{ A=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \,\middle|\, a,b,c,d\in\mathbf{Z},\ a+d=t,\ (a-d,b,c)=1,\ (a,b,c,d)=1,\ ad-bc=16 \text{ and } c>0\right\}.$$

Furthermore, for $A\in B(t,1)$, we put

$$(2.3)\quad D(A) = \left\{ B\in SL(2,\mathbf{Z}) \,\middle|\, 4^{-1}B^{-1}AB\in\Gamma_0(4N)\begin{pmatrix} 1 & 1/4\\ 0 & 1\end{pmatrix}\Gamma_0(4N)\right\}.$$

For $A, A'\in B(t,1)$, define an equivalence relation $A\sim A'$ by

$$(2.4)\quad A\sim A' \text{ if and only if } A'=g^{-1}Ag \text{ for some } g\in SL(2,\mathbf{Z}).$$

Then we denote by $B(t,1)/\sim$ a set of representatives of all equivalence classes of $B(t,1)$ under this relation. Moreover, $\Gamma_0(4N)$ acts on $D(A)$ by means of the multiplication from the right. We denote by $D(A)/\Gamma_0(4N)$ a set of representatives of $D(A)$ by means of this multiplication. We consider a set C determined by

$$(2.5)\quad C = \left\{\beta=\frac{1}{4}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(4N)\begin{pmatrix} 1 & 1/4\\ 0 & 1\end{pmatrix}\Gamma_0(4N) \,\middle|\, \beta \text{ is elliptic}\right\}.$$

For

$$\beta=\frac{1}{4}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in C,$$

we define $\chi(\beta)$ by

$$\chi(\beta) = \left(\frac{\operatorname{sgn}(d)}{-\operatorname{sgn}(c)}\right)\chi\left(\frac{a}{4}\right)\left(\frac{d}{b}\right)\left(\frac{\epsilon}{b}\right).$$

By [5, p. 442], we can verify the following lemma.

Lemma 2.2. *The notation being as above, the relation holds*

$$(2.6)\quad \text{(i)}\quad \chi(w\beta w) = \epsilon\chi(\beta)$$

$$\text{(ii)}\quad \chi(-w\beta w)\left(\frac{-\epsilon}{b}\right) = -\epsilon\chi(\beta)\left(\frac{\epsilon}{b}\right)$$

if $c>0$ with $w=\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}$.

Define a $\tilde{e}_0(1)$ by

$$(2.7)\quad \tilde{e}_0(1) = 2^{2k}(1+\epsilon(-1)^k\sqrt{-1}) \times \sum_{\beta=(1/2)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in C/\sim} \operatorname{sgn}(d)\chi\left(\frac{a}{4}\right)\left(\frac{d}{b}\right)p_k(t)(t+8)^{-1/2},$$

where $C/\sim$ means a set of representatives of all $\Gamma_0(4N)$-conjugacy classes $[\beta]$ containing $\beta\in C$ such that $c>0$ and

$$p_k(t) = \frac{\lambda(t)^{-2k+1}-\overline{\lambda(t)}^{-2k+1}}{\lambda(t)-\overline{\lambda(t)}} \quad \left(\lambda(t)=\frac{\sqrt{t+8}-\sqrt{t-8}}{2}\right).$$

Then, using Lemma 1.2 and the arguments in [2, p. 53] and [7, pp. 532–534], we may find

(2.8)

$$\tilde{e}_0(1) = 2^{2k}(1+\epsilon(-1)^k\sqrt{-1}) \times p_k(-4)4^{-1/2} \sum_{[A]\in B(-4,1)/\sim} \sum_{\substack{B\in D(A)/\Gamma_0(4N)\\ B^{-1}AB=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}} \chi\left(\frac{a}{4}\right).$$

For $x \in (\mathbf{Z}/N\mathbf{Z})^\times$ and $A \in B(-4,1)$, put

$$(2.9)\quad V(x,A) = \left\{ B \in SL(2,\mathbf{Z}) \,\middle|\, B^{-1}AB \equiv \begin{pmatrix} 4x+4N\nu & * \\ 0 & * \end{pmatrix} \pmod{16N} \right\}.$$

Then, we may check the following decomposition.

$$(2.10)\quad D(A)/\Gamma_0(4N) = \bigcup_{x\in(\mathbf{Z}/N\mathbf{Z})^\times} V(x,A)/\Gamma_0(4N)$$

(a disjoint union).

By [2, p. 53] and [7, p. 533], we see that

$$(2.11)\quad \sharp(B(-4,1)/\sim) = 2 \quad\text{and}\quad \sharp(V(x,A)/\Gamma_0(4N)) = \begin{cases} 1 & \text{if } x^2+x+1\equiv 0 \pmod N, \\ 0 & \text{otherwise.}\end{cases}$$

This implies that

$$(2.12)\quad \tilde{e}_0(1) = (1+\epsilon(-1)^k\sqrt{-1})\tilde{p}_k \sum_{\substack{x\in\mathbf{Z}/N\mathbf{Z}\\ x^2+x+1\equiv 0 \pmod N}} \chi(x),$$

where

$$\tilde{p}_k = \begin{cases} 1 & \text{if } k\equiv 0 \pmod 3, \\ -1 & \text{if } k\equiv 1 \pmod 3, \\ 0 & \text{otherwise.}\end{cases}$$

Define $\tilde{p}_0(1)$ by

$$(2.13)\quad \tilde{p}_0(1) = \frac{(1+\epsilon(-1)^k\sqrt{-1})}{2} \times\left(-2^l + \epsilon(-1)^k \prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right)\right).$$

Then, using Kohnen[2, pp. 47–58] and Ueda [7, pp. 528–538], we may deduce that

$$(2.14)\quad \dim S_{k+1/2}(N,\chi) = \frac{1}{3}\tilde{p}_k \sum_{\substack{x\in\mathbf{Z}/N\mathbf{Z}\\ x^2+x+1\equiv 0 \pmod N}} \chi(x) + \frac{1}{6}\left(-2^l + \epsilon(-1)^k \prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right)\right) + \frac{1}{3}\dim S_{k+1/2}(4N,\chi_\epsilon).$$

Therefore, by Theorem 1.1 and [1], we may confirm the following

$$(2.15)\quad \dim S_{k+1/2}(N,\chi) - \dim S_{2k}(N,\chi^2) = \frac{1}{3}\tilde{p}_k\left(\sum_{\substack{x\in\mathbf{Z}/N\mathbf{Z}\\ x^2+x+1\equiv 0 \pmod N}} \chi(x) - \sum_{\substack{x\in\mathbf{Z}/N\mathbf{Z}\\ x^2+x+1\equiv 0 \pmod N}} \chi^2(x)\right).$$

The solution x_i of the congruence $x_i^2+x_i+1\equiv 0 \pmod{p_i}$ is given by

$$(2.16)\quad x_i = \begin{cases} \xi_i^{(p_i-1)/3} \text{ or } (\xi_i^{(p_i-1)/3})^{-1} & \text{if } \left(\frac{-3}{p_i}\right) = 1, \\ 1 & \text{if } p_i = 3. \end{cases}$$

Assume that $p_i = 3$ or $p_i \equiv 1 \pmod 3$ for every i. Then

$$(2.17)\quad \sum_{\substack{x\in\mathbf{Z}/N\mathbf{Z}\\ x^2+x+1\equiv 0 \pmod N}} \chi(x) = \prod_{i=1,p_i\neq 3}^{l} (\chi_i(\xi_i^{(p_i-1)/3}) + \overline{\chi}_i(\xi_i^{(p_i-1)/3}))$$

and

$$\sum_{\substack{x\in\mathbf{Z}/N\mathbf{Z}\\ x^2+x+1\equiv 0 \pmod N}} \chi^2(x) = \prod_{i=1,p_i\neq 3}^{l} (\chi_i^2(\xi_i^{(p_i-1)/3}) + \overline{\chi}_i^2(\xi_i^{(p_i-1)/3})).$$

Since $(\chi_i(\xi_i^{(p_i-1)/3}))^3 = 1$, we conclude our assertion. $\square$

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