# Supersymmetric SO(10) grand unified theory with an intermediate scale

Joe Sato*
*Department of Physics, Kyoto University, Kyoto 606-01, Japan*
(Received 9 August 1995)

We examine a superpotential for an SO(10) GUT and show that if the parameters of the superpotential are in a certain region, the SO(10) GUT has an intermediate symmetry $SU(2)_L \otimes SU(2)_R \otimes SU(3)_C \otimes U(1)_{B-L}$ which breaks down to the group of the standard model at an intermediate scale $10^{10}$–$10^{12}$ GeV. In the model, by the breakdown of the symmetry, right-handed neutrinos acquire a mass of the intermediate scale through a renormalizable Yukawa coupling.

PACS number(s): 12.10.Dm, 12.60.Jv

## I. INTRODUCTION

When we construct a grand unified theory (GUT) based on SO(10) [1], in general, we have singlet fermions under the standard model (SM), which we call a right-handed neutrino. Under the SM, right-handed neutrinos can have Majorana masses because they are singlets. Then the scale of the right-handed neutrinos ($\equiv M_{\nu_R}$) is expected to be a scale below which the SM is realized.

It is well known that in the minimal supersymmetric standard model (MSSM), the present experimental values of gauge couplings are successfully unified at a unification scale $M_U \simeq 10^{16}$ GeV [2]. This fact implies that if we would like to consider gauge unification, it is favorable that the symmetry of the GUT breaks down to that of the SM at the unification scale. In this case the scale of the right-handed neutrinos $M_{\nu_R}$ is expected to be the unification scale $M_U$. This means also there is no intermediate scale between the supersymmetry- (SUSY-) breaking scale and the unification scale.

On the other hand, it is said that $M_{\nu_R} \sim 10^{10}$–$10^{12}$ GeV [3]. The experimental data on the deficit of the solar neutrino can be explained by the Mikheyev-Smirnov-Wolfenstein (MSW) solution [4]. According to one of the MSW solutions, the mass of the muon neutrino seems to be $m_{\nu_\mu} \simeq 10^{-3}$ eV. Such a small mass can be led by the seesaw mechanism [5]: A muon neutrino can acquire a mass of $\sim 10^{-3}$ eV if the Majorana mass of the right-handed muon neutrino is about $10^{12}$ GeV.

Then how can the right-handed neutrinos acquire mass of about $10^{12}$ GeV? It was our question in our previous paper [6], because if we take the prediction of the MSSM serious, $M_{\nu_R}$ is expected to be $M_U \simeq 10^{16}$ GeV. Our point of view was that it is more natural to consider that one energy scale corresponds to a dynamical phenomenon, for instance, symmetry breaking. That mass is given by a renormalizable coupling is also the crucial point of our view. This idea is consistent with the survival hypothesis. Thus we were led to the possibility that a certain group breaks down to the SM group at the intermediate scale at which right-handed neutrinos gain mass through a *renormalizable coupling*.

*Electronic address: joe@gauge.scphys.kyoto-u.ac.jp

In the previous work we searched possibilities to construct such a SUSY SO(10) GUT with an intermediate symmetry[1] $SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \otimes SU(3)_C$ ($\equiv G_{2231}$) which breaks down to the SM group at an intermediate scale $M_{\nu_R} \sim 10^{10}$–$10^{12}$ GeV where a right-handed neutrino gains mass.

In such a scenario, as we showed in the previous work, to make the model consistent with the gauge unification, we have to introduce several multiplets at the intermediate region between the GUT scale and the intermediate scale, in addition to ordinary matters, three generations of quarks and leptons and a pair of so-called Higgs doublets.

Although we showed a possibility to construct a SUSY SO(10) GUT with an intermediate symmetry $G_{2231}$, it is not trivial whether it is actually possible to construct such a GUT since there are many extra fields in the intermediate region. We did not show the superpotential for the theory explicitly which can realize such a scenario that we have suggested in Ref. [6].

The purpose of this paper is to show an explicit form of a superpotential for a SUSY SO(10) GUT to construct a SUSY SO(10) GUT whose symmetry breaks down to $G_{2231}$ at a GUT scale $M_U$ and $G_{2231}$ breaks down to the SM symmetry at the intermediate scale $M_{\nu_R}$.

We give the scenario and the model briefly in Sec. II where we give a candidate for the matter content in the intermediate region [the spectrum (1)]. Then in Sec. III we show the most general form of the superpotential and a symmetry-breaking condition as preparation for our analysis. In Sec. IV first we calculate parameters of the theory, namely, parameters appearing in the superpotential, which produce the spectrum (1) at the intermediate region. Then we show the exact parameters which realize the MSSM below $M_{\nu_R}$. Finally, in Sec. V we give a summary and a discussion.

## II. SCENARIO AND MODEL

### A. Scenario

We construct a SUSY SO(10) GUT whose symmetry breaks down to $G_{2231}$ at a GUT scale $M_U$ and $G_{2231}$ breaks

[1] We use a notation $G_{lmn\ldots}$ to represent $SU(l) \otimes SU(m) \otimes SU(n) \cdots$. If $l=1$, it means U(1).

 

down to the SM symmetry at the intermediate scale $M_{\nu_R}$. When $G_{2231}$ breaks down to the SM symmetry, the right-handed neutrinos gain mass through a *renormalizable Yukawa coupling*.

Let us first recapitulate the content of the previous work [6]. To achieve the gauge unification in the scenario, we have to introduce a certain combination of multiplets. Because in our model right-handed neutrinos acquire mass of $O(M_{\nu_R})$ via a renormalizable Yukawa coupling by the symmetry breaking, we have to introduce at least a pair of (1,3,1,6) + H.c. multiplet under $G_{2231}$. We adopt the normalization for $\mathrm{U}(1)_{B-L}$, $T_4^{15}=\mathrm{diag}(-1,-1,-1,3)$. When we introduce only (1,3,1,6) + H.c. multiplet in addition to the ordinary matter, gauge couplings do not unify. Then we have to introduce certain matter content under $G_{2231}$.

We found very many candidates for matter content in the intermediate region between the GUT scale and the intermediate scale which lead the gauge unification. Among them we showed two candidates for the matter content as the simplest examples. Here we use another candidate which was not shown in the previous paper. In the examples appearing in [6], a (1,3,1,0) multiplet under $G_{2231}$ was not included. In constructing a GUT following the idea, however, we have to introduce a (1,3,1,0) multiplet in the intermediate region. The reason why we have to introduce a (1,3,1,0) multiplet is stated in Appendix A. Thus, we have to use another candidate for matter content.

The matter content other than quarks and leptons (including right-handed neutrinos), which we assume survive until $G_{2231}$ breaks down to the SM group at the intermediate scale, are given below:

| | | | | |
|---|---|---|---|---|
| $(1,3,1,-6)$ | 1 | $(1,3,1,6)$ | 1 | responsible for $\nu_R$ mass |
| $(2,2,1,0)$ | 2 | | | ordinary Higgs doublets |
| $(2,1,3,-1)$ | 1 | $(2,1,\overline{3},1)$ | 1 | |
| $(2,1,1,3)$ | 1 | $(2,1,1,-3)$ | 1 | |
| $(1,3,1,0)$ | 1 | | | |
| $(1,1,8,0)$ | 1 | | | (1) |

In this list, for example, (1,3,1,−6) 1 stands for that the representation of the matter under $G_{2231}$ is (1,3,1,−6) and its number is one. When we have the particle content listed here in the intermediate region, the unified coupling $\alpha_U(M_U)$ is about 1/18 if we take the intermediate scale to be $10^{12}$ GeV. As a candidate which contains (1,3,1,0), this candidate leads the smallest unified coupling.

In our scenario, at the GUT scale $M_U$ where SO(10) breaks down to $G_{2231}$, almost of all particles have mass of $O(M_U)$ while the particles listed in (1) as well as quarks and leptons are left massless. Then at the intermediate scale, where $G_{2231}$ breaks down to the SM group $G_{231}$, all extra multiplets in (1), besides a pair of Higgs doublets and right-handed neutrinos, have mass of $O(M_{\nu_R})$, that is, they decouple from the spectrum. Thus below $M_{\nu_R}$ the MSSM is realized.

## B. Model

### *1. Matter content*

To have multiplets (1) and quarks and leptons at the intermediate region, we introduce following multiplets of SO(10):

| | | SO(10) | $G_{2231}$ |
|---|---|---|---|
| $H$ | : | **10** | $(2,2,1,0),\ldots$ |
| $A$ | : | **45** | $(1,3,1,0),(1,1,8,0),\ldots$ |
| $\Phi$ | : | **126** | $(1,3,1,-6),(2,2,1,0),\ldots$ |
| $\overline{\Phi}$ | : | $\overline{\mathbf{126}}$ | $(1,3,1,6),(2,2,1,0),\ldots$ |
| $\Delta$ | : | **210** | $(1,3,1,0),(1,1,8,0),\ldots$ |
| $\Psi_{i=1-4}$ | : | **16** | $(2,1,3,-1),(2,1,1,3)$, quarks and leptons |
| $\overline{\Psi}$ | : | $\overline{\mathbf{16}}$ | $(2,1,\overline{3},1),(2,1,1,-3),\ldots.$ |

(2)

In this list numbers in the columns of SO(10) mean SO(10) representations. In the last column we show what representation in (1) is contained in the corresponding SO(10) multiplet.

By the requirement that the right-handed neutrinos get mass through a renormalizable coupling, we introduce **126** and $\overline{\mathbf{126}}$. As a candidate of ordinary Higgs doublets 10 is introduced. There are other candidates for ordinary Higgs doublets in **126** and $\overline{\mathbf{126}}$. Then the ordinary Higgs doublets will be a mixture of these three. To break SO(10) to the SM group via $G_{2231}$, namely, to have the intermediate symmetry $G_{2231}$, we use **45** and **210**.[2] These also contain (1,3,1,0) and (1,1,8,0). 4 **16**'s and 1 $\overline{\mathbf{16}}$ represent 4 generation + 1 antigeneration. The reason why we introduce a pair of **16** and $\overline{\mathbf{16}}$ is that they contain (2,1,3,−1) + H.c. and (2,1,1,3) + H.c.

At this stage the matter content (2) is just a candidate which may realize our scenario.

As we will see, we can write down the superpotential with these matter which realize our idea.

### *2. Singlets under the SM group*

In the SO(10) multiplets (2), there are many singlets under the SM symmetry (see Appendix B for the meaning of subscripts $1,\ldots,0$):

[2]Using only **210** it is impossible to break SO(10) to $G_{231}$ through $G_{2231}$ [7]. We can break SO(10) to the SM group via $G_{2231}$ using **45**+**54**. In this case if there is no multiplet which contains (1,3,1,0) other than **45**, (3,1,1,0) is also massless. The reason is that mass terms for (1,3,1,0) and (3,1,1,0) come from the mass term of **45** and the vacuum expectation value of 54 through the coupling $\mathbf{45}^2\mathbf{54}$ and because of $D$ parity [8], they are same as each other's. Thus we can get rid of the possibility of using **45**+**54**.

| Field | : | Component | Little Group |
|---|---|---|---|
| A | : | $a_{12+34+56} \equiv \alpha$ | $G_{2231}$ |
| | : | $a_{78+90} \equiv \beta$ | $G_{241}$ |
| $\Phi$ | : | $\phi_{1-2i,3-4i,5-6i,7-8i,9-0i} \equiv \phi$ | SU(5) |
| $\bar{\Phi}$ | : | $\bar{\phi}_{1+2i,3+4i,5+6i,7+8i,9+0i} \equiv \bar{\phi}$ | SU(5) |
| $\Delta$ | : | $\delta_{7890} \equiv a$ | $G_{224}$ |
| | | $\delta_{1234+3456+5612} \equiv b$ | $G_{2231}$ |
| | | $\delta_{(12+34+56)(78+90)} \equiv c$ | $G_{2311}$ |
| $\Psi_{i=1-4}$ | : | $\psi_{i=1-4}$ | SU(5) |
| $\bar{\Psi}$ | : | $\bar{\psi}$ | SU(5) |

(3)

where $a, b, \ldots$ stand for vacuum expectation values (VEV's) of the corresponding fields. Little group means a remaining symmetry when only a corresponding component has a VEV. For example, when only $a$ gets a VEV, SO(10) breaks down to $G_{224}$.

Among them, $a, b$, and $\alpha$ are $G_{2231}$ singlets and hence their order of magnitudes is expected to be the GUT scale $M_U \sim 10^{16}$ GeV. By assumption that SO(10) breaks down to $G_{2231}$ at the GUT scale, $b$ or $\alpha$ must be of order $M_U$. Others must be of order at most $M_{\nu_R} \equiv M_U \epsilon$ by assumption because they are not $G_{2231}$ singlets. Also, $\bar{\phi}$ is required to be of order $M_{\nu_R}$,

$$\bar{\phi} \sim M_{\nu_R} (= M_U \epsilon) \tag{4}$$

because it gives masses to the right-handed neutrinos. Of course, as we will see, there are constraints among VEV's in addition to the well-known constraints: $F$-flat and $D$-flat conditions because we require that certain multiplets must have a mass of $O(M_{\nu_R})$.

## III. PREPARATION

### A. Superpotential

With the multiplets (2), the most general form of the superpotential $W$ is written as

$$W = W_{\rm mass} + W_{\rm int} + W_{\Psi}. \tag{5}$$

$W_{\rm mass}$ consists of the most general bilinear terms:

$$W_{\rm mass} = \frac{1}{2} M_H H^2 + M_\Phi \bar{\Phi}\Phi + \frac{1}{2} M_\Delta \Delta^2 + \frac{1}{2} M_A A^2 + M_\Psi \bar{\Psi}\Psi_4. \tag{6}$$

We define only $\Psi_4$ has a mass term with $\bar{\Psi}$, because by a redefinition of $\Psi_4$, namely, by a rotation among $\Psi_{i=1-4}$, it is possible that only the new $\Psi_4$ has a mass term with $\bar{\Psi}$.

We require all mass parameters are $O(M_U)$ because $M_U$ is the natural order for them.

$W_{\rm int}$ has the most general interaction terms without **16** and $\overline{\mathbf{16}}$:

$$W_{\rm int} = Y_{H\Phi\Delta} H\Phi\Delta + Y_{H\bar{\Phi}\Delta} H\bar{\Phi}\Delta + \frac{1}{3!} Y_\Delta \Delta^3 + Y_{\Phi\Delta} \bar{\Phi}\Delta\Phi + Y_{\Phi A} \bar{\Phi} A \Phi + \frac{1}{2} Y_{\Delta A^2} A^2 \Delta + \frac{1}{2} Y_{\Delta^2 A} A \Delta^2. \tag{7}$$

We require all Yukawa couplings are at most of order 1. More exactly, as an expansion parameter for the perturbation, we require they are at most of order 1. As an expansion parameter for the perturbation, they appear multiplied by a certain overall factor. The overall factors for each couplings are given in Appendix B 3.

Finally, $W_\Psi$ represents the most general interaction terms with **16** and $\overline{\mathbf{16}}$:

$$W_\Psi = \sum_{i=3}^{4} Y_{\Psi\Delta i} \bar{\Psi}\Delta\Psi_i + \sum_{i=2}^{4} Y_{\Psi A i} \bar{\Psi} A \Psi_i + \sum_{ij} y_{ij} \Psi_i \Psi_j \bar{\Phi} + y' \overline{\Psi\Psi}\Phi + \sum_{ij} \tilde{y}_{ij} \Psi_i \Psi_j H + \tilde{y}' \overline{\Psi\Psi} H. \tag{8}$$

By the same reason that only $\Psi_4$ has a mass term with $\bar{\Psi}$, only $\Psi_{3,4}$ have couplings with $\Delta$ and only $\Psi_{2,3,4}$ have couplings with $A$.

To see in which direction the gauge group SO(10) can break down, we examine the $D$-term and the $F$-term conditions.

### B. *D*-flat condition

To keep the SUSY, all $D$-terms must be zero up to SUSY-breaking scale:

$$\Phi^\dagger T^a_\Phi \Phi + \bar{\Phi}^\dagger T^a + \sum_i \Psi_i^\dagger T^a_\Psi \Psi_i + \bar{\Psi}^\dagger T^a_{\bar{\Psi}} \bar{\Psi} + \Delta^\dagger T^a_\Delta \Delta + A^\dagger T^a_A A = 0.$$

Since the $D$ term for real representations automatically vanishes [9,10],

$$2(|\phi|^2 - |\bar{\phi}|^2) + \left( \sum_{i=1}^{4} |\psi_i|^2 - |\bar{\psi}|^2 \right) = 0 \tag{9}$$

must be satisfied. The factor 2 reflects the difference of U(1) charge which corresponds to a broken generator.

Later we put $\psi_i$'s and $\bar{\psi}$ zeros. In this case

$$|\phi|^2 - |\bar{\phi}|^2 = 0. \tag{10}$$

### C. *F*-flat condition

First we examine the $F$-flat condition for **16** and $\overline{\mathbf{16}}$ with a mass term for $(1,2,1,-3)+$H.c. component because the singlet components of **16** and $\overline{\mathbf{16}}$ are contained in it and therefore there is a relation between the mass term and the $F$-flat condition. By such an examination we see that both $\psi_i$ and $\bar{\psi}$ should be zeros though it is not a strict reason for it.

The $F$-flat conditions for **16** and $\overline{\mathbf{16}}$ are as follows [see Appendix B to know how to calculate the Clebsch-Gordan (CG) coefficient]:

$$\frac{\partial W}{\partial \psi_1} = 2 \sum_{j=1}^{4} y_{1j} \psi_j \bar{\phi} = 0, \tag{11}$$

$$\frac{\partial W}{\partial \psi_2} = 2 \sum_{j=1}^{4} y_{2j} \psi_j \bar{\phi} - Y_{\Psi A 2} (\sqrt{6} i \alpha + 2 i \beta) \bar{\psi} = 0, \tag{12}$$

$$\frac{\partial W}{\partial \psi_3}=2\sum_{j=1}^{4} y_{3j}\psi_j\bar{\phi}-Y_{\Psi A3}(\sqrt{6}i\alpha+2i\beta)\bar{\psi}$$
$$-Y_{\Psi\Delta 3}(2\sqrt{6}a+6\sqrt{2}b+12c)=0, \tag{13}$$

$$\frac{\partial W}{\partial \psi_4}=2\sum_{j=1}^{4} y_{4j}\psi_j\bar{\phi}-Y_{\Psi A4}(\sqrt{6}i\alpha+2i\beta)\bar{\psi}$$
$$-Y_{\Psi\Delta 4}(2\sqrt{6}a+6\sqrt{2}b+12c)+M_\Psi$$
$$=0, \tag{14}$$

$$\frac{\partial W}{\partial}=2y'\bar{\psi}\phi+\sum_{i=2}^{4}-Y_{\Psi Ai}(\sqrt{6}i\alpha+2i\beta)\psi_i$$
$$-\sum_{j=3}^{4}Y_{\Psi\Delta i}(2\sqrt{6}a+6\sqrt{2}b+12c)\psi_i+M_\Psi\psi_4$$
$$=0. \tag{15}$$

By the way, in the intermediate region where $G_{2231}$ is realized, $\beta=c=0$ and the mass term for $(1,2,1,-3)+\mathrm{H.c.}$ is given by

$$\frac{\partial^2 W}{\partial\psi_i\partial\psi}=\begin{pmatrix}0\\ -\sqrt{6}iY_{\Psi A2}\alpha\\ -\sqrt{6}iY_{\Psi A3}\alpha-2\sqrt{6}Y_{\Psi\Delta 3}(a+\sqrt{3}b)\\ -\sqrt{6}iY_{\Psi A4}\alpha-2\sqrt{6}Y_{\Psi\Delta 4}(a+\sqrt{3}b)+M_\Psi\end{pmatrix}. \tag{16}$$

If $\phi,\bar{\phi},\psi_i,\bar{\psi}=O(\epsilon)$, using $F$-flat conditions [Eqs. (12)–(14)], all elements of the mass term for (1,2,1,-3)+H.c., (16), are calculated to be of order $M_{\nu_R}$. This, however, contradicts with the mass spectrum (1). Though we may be able to make some elements of the mass term $O(M_U)$, for example, by making $\bar{\psi}$ $O(\epsilon^2)$ [with an appropriate value of $\psi_i,\bar{\phi}=O(\epsilon)$], we put both $\psi_i$ and $\bar{\psi}$ zeros since what we try to do is to show a possibility of SUSY SO(10) GUT with an intermediate scale and to take $\psi_i=\bar{\psi}=0$ as the solution of the $F$-flat conditions for **16** and $\overline{\mathbf{16}}$ is the easiest way of doing it.

Then, other $F$-term conditions are

$$\frac{\partial W}{\partial a}=24\sqrt{2}iY_{\Delta^2A}\alpha b-\frac{Y_{\Delta A^2}\beta^2}{2\sqrt{6}}+\frac{Y_\Delta c^2}{12\sqrt{6}}+M_\Delta a+\frac{Y_{\Phi\Delta}\bar{\phi}\phi}{10\sqrt{6}}$$
$$=0, \tag{17}$$

$$\frac{\partial W}{\partial b}=24\sqrt{2}iY_{\Delta^2A}a\alpha-\frac{Y_{\Delta A^2}\alpha^2}{3\sqrt{2}}+\frac{Y_\Delta b^2}{18\sqrt{2}}+24\sqrt{2}iY_{\Delta^2A}\beta c$$
$$+\frac{Y_\Delta c^2}{18\sqrt{2}}+M_\Delta b+\frac{Y_{\Phi\Delta}\phi\bar{\phi}}{10\sqrt{2}}$$
$$=0, \tag{18}$$

$$\frac{\partial W}{\partial c}=-\frac{Y_{\Delta A^2}\alpha\beta}{\sqrt{6}}+24\sqrt{2}iY_{\Delta^2A}b\beta+\frac{Y_\Delta ac}{6\sqrt{6}}$$
$$+16\sqrt{6}iY_{\Delta^2A}\alpha c+\frac{Y_\Delta bc}{9\sqrt{2}}+M_\Delta c+\frac{Y_{\Phi\Delta}\phi\bar{\phi}}{10}$$
$$=0, \tag{19}$$

$$\frac{\partial W}{\partial \alpha}=24\sqrt{2}iY_{\Delta^2A}ab-\frac{\sqrt{2}Y_{\Delta A^2}\alpha b}{3}-\frac{Y_{\Delta A^2}\beta c}{\sqrt{6}}$$
$$+8\sqrt{6}iY_{\Delta^2A}c^2+M_A\alpha+\frac{\sqrt{6}Y_{\Phi A}\phi\bar{\phi}}{10}$$
$$=0, \tag{20}$$

$$\frac{\partial W}{\partial \beta}=-\frac{Y_{\Delta A^2}a\beta}{\sqrt{6}}-\frac{Y_{\Delta A^2}\alpha c}{\sqrt{6}}+24\sqrt{2}iY_{\Delta^2A}bc+M_A\beta$$
$$+\frac{Y_{\Phi A}\phi\bar{\phi}}{5}$$
$$=0, \tag{21}$$

$$\frac{\partial W}{\partial \phi}=\left[Y_{\Phi A}\left(\frac{\sqrt{6}\alpha}{10}+\frac{\beta}{5}\right)+Y_{\Phi\Delta}\left(\frac{a}{10\sqrt{6}}+\frac{b}{10\sqrt{2}}+\frac{c}{10}\right)\right.$$
$$\left.+M_\phi\right]\bar{\phi}$$
$$=0. \tag{22}$$

## IV. ANALYSIS

The purpose of this paper is to give the input parameters appearing in the superpotential (5). Though VEV's listed in (3) are functions of the input parameters, we will express them in the terms of the VEV's since we know the desirable values of the VEV's.

### A. First step

First, we check whether it is possible to break SO(10) down to $G_{2231}$ consistently with the requirement that the spectrum (1) remains massless up to $O(\epsilon)=O(M_{\nu_R}/M_U)$.

#### *1. Multiplets under $G_{2231}$*

First, we show what multiplets exist in the SO(10) multiplets (2).

| Multiplet under $G_{2231}$ | under SO(10), contained in | NG1 | NG2 |
|---|---|---|---|
| (2,2,1,0) | $\mathbf{10}, \mathbf{126}, \overline{\mathbf{126}}$ | | |
| (1,1,3,2)+H.c. | $\mathbf{10}, \mathbf{126}, \overline{\mathbf{126}}$ | | |
| (3,1,1,0) | $\mathbf{45}, \mathbf{210}$ | | |
| (1,3,1,0) | $\mathbf{45}, \mathbf{210}$ | | $\tilde{z}$ |
| (1,1,3,−4)+H.c. | $\mathbf{45}, \mathbf{210}$ | $x$ | $\tilde{x}$ |
| (1,1,8,0) | $\mathbf{45}, \mathbf{210}$ | | |
| (2,2,3,2)+H.c. | $\mathbf{45}, \mathbf{210}$ | $y$ | $\tilde{y}$ |
| (3,1,1,6)+H.c. | $\mathbf{126}+\overline{\mathbf{126}}$ | | |
| (3,1,3,2)+H.c. | $\mathbf{126}+\overline{\mathbf{126}}$ | | |
| (3,1,6,−2)+H.c. | $\mathbf{126}+\overline{\mathbf{126}}$ | | |
| (1,3,1,−6)+H.c. | $\mathbf{126}+\overline{\mathbf{126}}$ | | $\tilde{z}$ |
| $(1,3,\bar{3},-2)$+H.c. | $\mathbf{126}+\overline{\mathbf{126}}$ | | $\tilde{x}$ |
| $(1,3,\bar{6},2)$+H.c. | $\mathbf{126}+\overline{\mathbf{126}}$ | | |
| (2,2,3,−4)+H.c. | $\mathbf{126}, \overline{\mathbf{126}}$ | | $\tilde{y}$ |
| (2,2,8,0)+H.c. | $\mathbf{126}, \overline{\mathbf{126}}$ | | |
| (3,1,3,−4)+H.c. | $\mathbf{210}$ | | |
| (1,3,3,−4)+H.c. | $\mathbf{210}$ | | $\tilde{x}$ |
| (3,1,8,0)+H.c. | $\mathbf{210}$ | | |
| (1,3,8,0)+H.c. | $\mathbf{210}$ | | |
| (2,2,1,6)+H.c. | $\mathbf{210}$ | | |
| (2,1,3,−1)+H.c. | $\mathbf{16}+\overline{\mathbf{16}}$ | | $\tilde{y}$ |
| $(1,2,\bar{3},1)$+H.c. | $\mathbf{16}+\overline{\mathbf{16}}$ | | $\tilde{x}$ |
| (2,1,1,3)+H.c. | $\mathbf{16}+\overline{\mathbf{16}}$ | | |
| (1,2,1,−3)+H.c. | $\mathbf{16}+\overline{\mathbf{16}}$ | | $\tilde{z}$ |

(23)

In this table NG1 means a Nambu-Goldstone (NG) mode associated with the breakdown of SO(10) to $G_{2231}$. An NG mode associated with the SO(10) breaking down to the SM group $G_{231}$ is contained in a multiplet with $\tilde{x}, \tilde{y}$, and $\tilde{z}$ in the column NG2. Under $G_{231}$, certain components of the multiplets with $\tilde{x}$ $(\tilde{y},\tilde{z})$ have the same quantum numbers and mix with each other. One of the combinations of $\tilde{x}$ $(\tilde{y},\tilde{z})$ is massless which is swallowed by a gauge boson.

There are also singlets of $G_{2231}$ which we denote $a$, $b$, and $\alpha$.

### 2. F-flat condition

In the intermediate region $c,\beta,\phi=0$. And hence, the $F$-term conditions [Eqs. (17)–(22)] are reduced to

$$\frac{\partial W}{\partial a}=24\, i\sqrt{2}Y_{\Delta^2 A}\alpha b+M_\Delta a=0, \tag{24}$$

$$\frac{\partial W}{\partial b}=24\, i\sqrt{2}aY_{\Delta^2 A}\alpha-\frac{Y_{\Delta A^2}\alpha^2}{3\sqrt{2}}+\frac{Y_\Delta b^2}{18\sqrt{2}}+M_\Delta b =0, \tag{25}$$

$$\frac{\partial W}{\partial \alpha}=24\, i\sqrt{2}Y_{\Delta^2 A}ab-\frac{\sqrt{2}Y_{\Delta A^2}\alpha b}{3}+M_A\alpha=0. \tag{26}$$

### 3. Tuning of parameters

From now on, as we stated at the top of this section, we express the input parameters in the terms of the VEV's.

Using the $F$-flat conditions [Eqs. (24) and (26)], $M_\Delta$ and $M_A$ are expressed as

$$M_\Delta=M_\Delta(Y_{\Delta^2 A},a,b,\alpha)=\frac{-24\sqrt{2}iY_{\Delta^2 A}\alpha b}{a}, \tag{27}$$

$$M_A=M_A(Y_{\Delta^2 A},Y_{\Delta A^2},a,b,\alpha) =\frac{-72\sqrt{2}iY_{\Delta^2 A}ab+\sqrt{2}Y_{\Delta A^2}\alpha b}{3\alpha}. \tag{28}$$

There is an additional constraint which is obtained by eliminating $M_\Delta$ from Eqs. (24) and (25):

$$-24\sqrt{2}iY_{\Delta^2 A}a^2\alpha+\frac{Y_{\Delta A^2}a\alpha^2}{3\sqrt{2}}-\frac{Y_\Delta ab^2}{18\sqrt{2}}+24\sqrt{2}iY_{\Delta^2 A}\alpha b^2 =0. \tag{29}$$

We can interpret that this constraint with (27) and (28) is equivalent with that determinant of the mass matrix for (1,1,3,−4) [$\equiv M(1,1,3,-4)$, an explicit form is given in Appendix C] vanishes because (1,1,3,−4) is an NG mode and hence when we substitute VEV's into the mass matrix for it, there must be one massless mode which mean the determinant vanishes:

$$\det M(1,1,3,-4)=M_A M_\Delta+\frac{Y_\Delta M_A b}{18\sqrt{2}}-\frac{Y_{\Delta A^2}M_\Delta b}{3\sqrt{2}} +1152\, Y_{\Delta^2 A}^2 a^2+16\, iY_{\Delta A^2}Y_{\Delta^2 A}a\alpha -\frac{Y_{\Delta A^2}^2\alpha^2}{18}-\frac{Y_\Delta Y_{\Delta A^2}b^2}{108} =0. \tag{30}$$

Now, we required that one (1,1,8,0) mode be massless and therefore determinant of the mass matrix for it [$\equiv M(1,1,8,0)$] should vanish:

$$\det M(1,1,8,0)=M_A M_\Delta-\frac{Y_\Delta M_A b}{18\sqrt{2}}+\frac{Y_{\Delta A^2}M_\Delta b}{3\sqrt{2}} +1152\, Y_{\Delta^2 A}^2 a^2+16\, iY_{\Delta A^2}Y_{\Delta^2 A}a\alpha -\frac{Y_{\Delta A^2}^2\alpha^2}{18}-\frac{Y_\Delta Y_{\Delta A^2}b^2}{108} =0. \tag{31}$$

Substituting (27) and (28) into (30) and (31), we find

$$\frac{-8\, i}{3}Y_\Delta Y_{\Delta^2 A}a^2+\frac{Y_\Delta Y_{\Delta A^2}a\alpha}{27}+16\, iY_{\Delta A^2}Y_{\Delta^2 A}\alpha^2=0 \tag{32}$$

$$\{[(30)-(31)]a\alpha/b^2\}$$

and

$$2304\, Y_{\Delta^2 A}^2 a^3 + 32\, i Y_{\Delta A^2} Y_{\Delta^2 A} a\alpha - \frac{Y_{\Delta A^2}^2 a\alpha^2}{9} - \frac{Y_\Delta Y_{\Delta A^2} a b^2}{54} - 2304\, Y_{\Delta^2 A}^2 a b^2 - 32\, i Y_{\Delta A^2} Y_{\Delta^2 A} \alpha b^2 = 0 \tag{33}$$

$$\{[(30)+(31)]*a\}.$$

Solving simultaneously Eqs. (32) and (33), we get forms of $Y_\Delta$ and $Y_{\Delta A^2}$ as functions of $Y_{\Delta^2 A}, a, b, \alpha$. Then, by substituting these expressions into (27) and (28), we find the following three sets of solutions for $M_\Delta$, $M_A$, $Y_\Delta$, and $Y_{\Delta A^2}$ as functions of $Y_{\Delta^2 A}, a, b, \alpha$.

Solution 1:

$$M_\Delta = \frac{-24\sqrt{2} i Y_{\Delta^2 A} \alpha b}{a},$$

$$M_A = \frac{24\sqrt{2} i Y_{\Delta^2 A} a b}{\alpha},$$

$$Y_\Delta = \frac{-864 i Y_{\Delta^2 A} \alpha}{a},$$

$$Y_{\Delta A^2} = \frac{144 i Y_{\Delta^2 A} a}{\alpha}. \tag{34}$$

Solution 2:

$$M_\Delta = \frac{-24\sqrt{2} i Y_{\Delta^2 A} \alpha b}{a},$$

$$M_A = \frac{-24 i Y_{\Delta^2 A} b}{\sqrt{2} a\alpha}(-a^2+3b^2-\sqrt{a^4-10a^2b^2+9b^4}),$$

$$Y_\Delta = \frac{-432 i Y_{\Delta^2 A} \alpha(-3a^2+3b^2-\sqrt{a^4-10a^2b^2+9b^4})}{(-a^3+3ab^2-a\sqrt{a^4-10a^2b^2-9b^4})},$$

$$Y_{\Delta A^2} = \frac{-36 i Y_{\Delta^2 A}}{a\alpha}(-3a^2+3b^2-\sqrt{a^4-10a^2b^2+9b^4}). \tag{35}$$

Solution 3:

$$M_\Delta = \frac{-24\sqrt{2} i Y_{\Delta^2 A} \alpha b}{a},$$

$$M_A = \frac{-24 i Y_{\Delta^2 A} b}{\sqrt{2} a\alpha}(-a^2+3b^2+\sqrt{a^4-10a^2b^2+9b^4}),$$

$$Y_\Delta = \frac{-432 i Y_{\Delta^2 A} \alpha(-3a^2+3b^2+\sqrt{a^4-10a^2b^2+9b^4})}{-a^3+3ab^2+a\sqrt{a^4-10a^2b^2+9b^4}},$$

$$Y_{\Delta A^2} = \frac{-36 i Y_{\Delta^2 A}}{a\alpha}(-3a^2+3b^2+\sqrt{a^4-10a^2b^2+9b^4}). \tag{36}$$

In other words, once $M_\Delta, M_A, Y_\Delta$, and $Y_{\Delta A^2}$ are set to be one of these solutions, the VEV's of $a, b$, and $\alpha$ can be chosen at our will and one (1,1,8,0) mode becomes massless.

Because we require also that one (1,3,1,0) mode be massless, determinant of the mass matrix for it $[\equiv M(1,3,1,0)]$ must be zero

$$\begin{aligned}\det M(1,3,1,0) = &- \frac{Y_\Delta Y_{\Delta A^2} a^2}{36} - 16\, i Y_{\Delta A^2} Y_{\Delta^2 A} a\alpha - \frac{Y_{\Delta A^2}^2 \alpha^2}{6} \\ &- \frac{Y_\Delta Y_{\Delta A^2} a b}{18\sqrt{3}} + 16\sqrt{3} i Y_{\Delta A^2} Y_{\Delta^2 A} \alpha b \\ &+ 1152\, Y_{\Delta^2 A}^2 b^2 + \frac{Y_\Delta M_A a}{6\sqrt{6}} \\ &+ 16\sqrt{6} i Y_{\Delta^2 A} M_A \alpha + \frac{Y_\Delta M_A b}{9\sqrt{2}} - \frac{Y_{\Delta A^2} M_\Delta a}{\sqrt{6}} \\ &+ M_A M_\Delta \\ = &\ 0.\end{aligned} \tag{37}$$

Using (37) and (34)–(36), we obtain following equations which determine relations between $a$ and $b$ corresponding to a set of above solutions, respectively.

Solution 1:

$$a^2(-3\, a^2 + 7\sqrt{3} a b - 6\, b^2) = 0.$$

Solution 2:

$$\begin{aligned}&-15\, a^6 + 62\sqrt{3} a^5 b + 237\, a^4 b^2 - 280\sqrt{3} a^3 b^3 - 249\, a^2 b^4 \\ &\quad + 234\sqrt{3} a b^5 + 27\, b^6 \\ &= (33\, a^4 - 50\sqrt{3} a^3 b - 78\, a^2 b^2 + 78\sqrt{3} a b^3 + 9\, b^4) \\ &\quad \times \sqrt{a^4 - 10\, a^2 b^2 + 9\, b^4}.\end{aligned}$$

Solution 3:

$$\begin{aligned}&15\, a^6 - 62\sqrt{3} a^5 b - 237\, a^4 b^2 + 280\sqrt{3} a^3 b^3 + 249\, a^2 b^4 \\ &\quad - 234\sqrt{3} a b^5 - 27\, b^6 \\ &= (33\, a^4 - 50\sqrt{3} a^3 b - 78\, a^2 b^2 + 78\sqrt{3} a b^3 + 9\, b^4) \\ &\quad \times \sqrt{a^4 - 10\, a^2 b^2 + 9\, b^4}.\end{aligned}$$

Numerically, $a$ and $b$ must satisfy the following relations, respectively.

Solution 1:

$$a = \begin{cases} b/\sqrt{3}, \\ 2\sqrt{3} b. \end{cases} \tag{38}$$

Solution 2:

$$a = \begin{cases} -0.987293 b, \\ (-0.120361 - 0.724007 i) b, \\ (-0.120361 + 0.724007 i) b, \\ 5.11238 b. \end{cases} \tag{39}$$

Solution 3:

$$a=\begin{cases}-3.13416b,\\ -0.0643986b,\\ (1.10047-0.0616122i)b,\\ (1.10047+0.0616122i)b.\end{cases}\tag{40}$$

The solution 1 is the exact solution and the others are exact up to $O(\epsilon)$.

In other words, if $a$ and $b$ satisfy these relations, one (1,3,1,0) mode becomes massless.

Other requirements that two (2,2,1,0) modes, one (1,3,1,−6) + H.c. mode, one (2,1,3,1) + H.c. mode, and one (2,1,1,−3) + H.c. mode be massless are easily satisfied by tuning parameters such as $M_\Phi, M_H, Y_{H\Phi\Delta}$, $Y_{H\bar{\Phi}\Delta}$, and so on.

To make (1,3,1,−6) + H.c. mode massless, from the mass term for it (see Appendix C),

$$M_\Phi=-\left(\frac{\sqrt{6}Y_{\Phi A}\alpha}{10}+\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}+\frac{Y_{\Phi\Delta}b}{10\sqrt{2}}\right).\tag{41}$$

To make two (2,2,1,0) modes massless, we tune parameters $M_H, M_\Phi, Y_{H\Phi\Delta}$, and $Y_{H\bar{\Phi}\Delta}$ so that the eigenvalue equation for the mass matrix of (2,2,1,0),

$$\lambda^3-M_H\lambda^2+\left[-\frac{Y_{H\bar{\Phi}\Delta}^2b^2}{10}-\frac{Y_{H\Phi\Delta}^2b^2}{10}-\left(\frac{Y_{\Phi\Delta}b}{15\sqrt{2}}+M_\Phi\right)^2\right]\lambda$$
$$-\left(\frac{Y_{\Phi\Delta}b}{15\sqrt{2}}+M_\Phi\right)\left[M_H\left(\frac{Y_{\Phi\Delta}b}{15\sqrt{2}}+M_\Phi\right)\right.$$
$$\left.+\frac{Y_{H\bar{\Phi}\Delta A^2}Y_{H\Phi\Delta}b^2}{5}\right]=0,\tag{42}$$

has two zero solutions [exactly these two solutions may have at most $O(\epsilon)$ solution].[3] The way of getting two zero eigenvalues is to tune the zeroth and first terms of λ zero. More exactly, the zeroth term must be at most $O(\epsilon^2)$ and the first term must be at most $O(\epsilon)$.

To satisfy these constraints

$$M_\Phi+\frac{Y_{\Phi\Delta}b}{15\sqrt{2}}=O(\epsilon),$$
$$Y_{H\Phi\Delta}\sim Y_{H\bar{\Phi}\Delta}=O(\sqrt{\epsilon}).\tag{43}$$

Equation (41) and the first equation of (43) lead

$$Y_{\Phi A}=-\frac{\sqrt{3}a+b}{6\sqrt{3}\alpha}Y_{\Phi\Delta}\tag{44}$$

up to $O(\epsilon)$.

[3]Implicitly, it is assumed that the mass matrix for (2,2,1,0) is Hermite, that is, all parameters appearing in the mass matrix are real.

Finally, to make one (2,1,3,−1) + H.c. mode and one (2,1,1,3) + H.c. mode massless, for example, we can switch only couplings with subscript 4 on and tune

$$Y_{\psi\Delta}=\frac{7}{16\sqrt{3}}iY_{\psi A}\alpha/b,\tag{45}$$

$$M_\Psi=-\frac{3}{4\sqrt{6}}iY_{\psi A}\alpha-\frac{7}{4\sqrt{2}}iY_{\psi A}\frac{a}{b}\alpha.\tag{46}$$

#### 4. Check mass matrices

Now, we know the necessary condition for the parameters realizing the spectrum (1). Then we check all the mass matrices to examine whether these parameters really produce the spectrum (1).

Solution 1: The solution 1 does not produce the spectrum (1), because by substituting the solution 1 (34) into the mass matrix of (2,2,6,2) multiplet, this multiplet is calculated to be massless.

Solution 2: First to see whether the solution 2 , (35) with a relation between $a$ and $b$ (39), is usable, we substitute (39) into (35).

$$\left.\begin{matrix}M_\Delta\\ M_A\end{matrix}\right\}=\begin{cases}\begin{cases}24.3089i\sqrt{2}Y_{\Delta^2A}\alpha\\ 19.1441i\sqrt{2}Y_{\Delta^2A}b^2/\alpha\end{cases}\\ \begin{cases}(32.2574+5.36258i)\sqrt{2}Y_{\Delta^2A}\alpha\\ (-4.78842+0.510831i)\sqrt{2}Y_{\Delta^2A}b^2/\alpha\end{cases}\\ \begin{cases}(-32.2574+5.36258i)\sqrt{2}Y_{\Delta^2A}\alpha\\ (4.78842+0.510831i)\sqrt{2}Y_{\Delta^2A}b^2/\alpha\end{cases}\\ \begin{cases}-4.69449i\sqrt{2}Y_{\Delta^2A}\alpha\\ 103.023i\sqrt{2}Y_{\Delta^2A}b^2/\alpha,\end{cases}\end{cases}\tag{47}$$

$$Y_{\Delta A^2}=\begin{cases}-13.6527iY_{\Delta^2A}b/\alpha\\ (37.7632-7.13352i)Y_{\Delta^2A}b/\alpha\\ (-37.7632-7.13352i)Y_{\Delta^2A}b/\alpha\\ 677.159iY_{\Delta^2A}b/\alpha,\end{cases}\tag{48}$$

$$Y_\Delta=\begin{cases}-104.016iY_{\Delta^2A}\alpha/b\\ (-1560.23-131.862i)Y_{\Delta^2A}\alpha/b\\ (1560.23-131.862i)Y_{\Delta^2A}\alpha/b\\ -185.139iY_{\Delta^2A}\alpha/b.\end{cases}\tag{49}$$

In each of these equations, four expressions correspond to the four relations between $a$ and $b$ in (39), respectively.

As we required that Yukawa couplings are not too big [see the statement below (7)], only the first expression of the solution 2 is meaningful. This means that only the first relation between $a$ and $b$ in (39) is meaningful.

By substituting (35) with the first equation of (39), it is easy to check that all multiplets other than those in (1) have their mass of $O(M_U)$ which spread around $M_U$ up to one order of magnitude and multiplets in (1) are massless. Therefore, this solution can be a solution of our scenario.

Solution 3: First, we substitute (40) (relation between $a$ and $b$) into (36) to see an explicit form of solution 3.

$$\left.\begin{matrix} M_\Delta \\ M_A \end{matrix}\right\} = \begin{cases} \begin{cases} 7.65756i\sqrt{2}Y_{\Delta^2 A}\alpha \\ -15.8066i\sqrt{2}Y_{\Delta^2 A}b^2/\alpha \end{cases} \\ \begin{cases} 372.679i\sqrt{2}Y_{\Delta^2 A}\alpha \\ 1115.98i\sqrt{2}Y_{\Delta^2 A}b^2/\alpha \end{cases} \\ \begin{cases} (1.21719-21.7407i)\sqrt{2}Y_{\Delta^2 A}\alpha \\ (17.3100-22.7812i)\sqrt{2}Y_{\Delta^2 A}b^2/\alpha \end{cases} \\ \begin{cases} (-1.21719-21.7407i)\sqrt{2}Y_{\Delta^2 A}\alpha \\ (-17.3100-22.7812i)\sqrt{2}Y_{\Delta^2 A}b^2/\alpha, \end{cases} \end{cases} \tag{50}$$

$$Y_{\Delta A^2} = \begin{cases} -273.079iY_{\Delta^2 A}b/\alpha \\ 3343.29iY_{\Delta^2 A}b/\alpha \\ (56.3660+10.8904i)Y_{\Delta^2 A}b/\alpha \\ (-56.3660+10.8904i)Y_{\Delta^2 A}b/\alpha, \end{cases} \tag{51}$$

$$Y_\Delta = \begin{cases} 793.766iY_{\Delta^2 A}\alpha/b \\ 6698.93iY_{\Delta^2 A}\alpha/b \\ (241.144-102.803i)Y_{\Delta^2 A}\alpha/b \\ (-241.144-102.803i)Y_{\Delta^2 A}\alpha/b. \end{cases} \tag{52}$$

In each of these equations, four expressions correspond to the four relations between $a$ and $b$ in (40), respectively.

By the same way, as we picked up only the first expression from four cases in solution 2, the last two relations between $a$ and $b$ in (40) are meaningful.

By substituting (36) with the third or fourth equation of (40), it is easy to check that all multiplets other than those in (1) have their mass of $O(M_U)$, which spread around $M_U$ up to one order of magnitude, and multiplets in (1) are massless. Therefore, this solution can be a solution of our scenario too.

### B. Second step

In this section we find a parameter region which produces our scenario exactly.

#### *1. Deviation from the previous solutions*

Because the accuracy of the previous calculation is $O(\epsilon)$, all parameters besides $b,\alpha$, and $Y_{\Delta^2 A}$ can deviate from the value which is obtained at the previous section and therefore we can expand the deviation in the power of $\epsilon$ as

$$a = a_0 + \sum_{i=1} a_i \epsilon^i, \tag{53}$$

$$M_\Delta = M_{\Delta 0} + \sum_{i=1} M_{\Delta i}\epsilon^i, \tag{54}$$

$$M_A = M_{A0} + \sum_{i=1} M_{Ai}\epsilon^i, \tag{55}$$

$$Y_\Delta = Y_{\Delta 0} + \sum_{i=1} Y_{\Delta i}\epsilon^i, \tag{56}$$

$$Y_{\Delta A^2} = Y_{\Delta A^2 0} + \sum_{i=1} Y_{\Delta A^2 i}\epsilon^i, \tag{57}$$

$$\beta = \sum_{i=1} \beta_i \epsilon^i, \tag{58}$$

$$c = \sum_{i=1} c_i \epsilon^i. \tag{59}$$

In these expressions, variables with subscript 0 stand for those which are obtained in the previous section.

Substituting (53)–(59) into the $F$-flat condition (17)–(22), we get the following relations.

From (17), (18), and (20) we get

$$M_{\Delta 1} = -\frac{M_{\Delta 0}}{a_0}a_1,$$

$$M_{A1} = \frac{b^3}{9\sqrt{2}\alpha^2}Y_{\Delta 1} + \frac{24\sqrt{2}iY_{\Delta^2 A}b}{\alpha}\left(1+2\frac{b^2}{a_0^2}\right)a_1,$$

$$Y_{\Delta A^2 1} = (b^2/6\alpha^2)Y_{\Delta 1} + \frac{144iY_{\Delta^2 A}}{\alpha}\left(1+\frac{b^2}{a_0^2}\right)a_1. \tag{60}$$

We obtain the relation between $\beta_1$ and $c_1$ by substituting (53)–(59) with (60) into (19) and (21) as follows:

First we note (19) and (21) can be rewritten

$$M(1,3,1,0)\begin{pmatrix}\beta \\ c\end{pmatrix} = -\frac{1}{10}\begin{pmatrix}2Y_{\Phi A} \\ Y_{\Phi\Delta}\end{pmatrix}\phi\bar{\phi} \tag{61}$$

and therefore

$$\begin{pmatrix}\beta \\ c\end{pmatrix} = -\frac{1}{10}M(1,3,1,0)^{-1}\begin{pmatrix}2Y_{\Phi A} \\ Y_{\Phi\Delta}\end{pmatrix}\phi\bar{\phi}, \tag{62}$$

where $M(1,3,\underline{1},0)$ is a mass matrix for (1,3,1,0), and by assumption $\phi,\bar{\phi}=O(\epsilon)$.

Let us decompose the inverse of $M(1,3,1,0)$:

$$M(1,3,1,0)^{-1} = \det[M(1,3,1,0)]^{-1}[A+O(\epsilon)]. \tag{63}$$

Since, by assumption there is one massless mode in (1,3,1,0) up to $O(\epsilon)$, $\det(M(1,3,1,0))=O(\epsilon)$ and the first row in $A$ is parallel to the second row in $A$, that is

$$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}, \tag{64}$$

where $A\equiv(a_{ij})$.

Then up to the leading order of $\epsilon$,

$$\beta = \frac{a_{21}}{a_{11}}c; \tag{65}$$

namely, as an exact relation

$$\beta_1 = \frac{a_{21}}{a_{11}}c_1 \tag{66}$$

is obtained.

To see this explicitly, we follow the above calculation in the case of the first relation of solution 2:

$$\det(M(1,3,1,0))=\left(-26423.4Y_{\Delta^2 A}^2 b a_1 + \frac{16.1727 i Y_{\Delta^2 A} Y_{\Delta 1} b^3}{\alpha}\right)\epsilon + O(\epsilon^2)$$

as we expected the determinant is $O(\epsilon)$.

$A$ is calculated to be

$$A=\begin{pmatrix} 72.3850 i Y_{\Delta^2 A}\alpha, & -39.5148 i Y_{\Delta^2 A} b \\ -39.5148 i Y_{\Delta^2 A} b, & 21.5710 i Y_{\Delta^2 A} b^2/\alpha \end{pmatrix}.$$

Apparently, $A$ satisfies (64).

Then

$$\beta_1 = -1.83185\frac{\alpha}{b}c_1 \tag{67}$$

is obtained.

### 2. Determination of input parameters of the theory

Though we can determine the parameters in the power of $\epsilon$ order by order, instead of doing so, we will give the parameters of the theory in terms of the VEV's because the purpose of the paper is to find a parameter region for the theory, $M$'s and $Y$'s, which leads to the spectrum (1). As we will see, by the VEV's $a,b,c,\alpha,$ and $\beta,$ we can express the input parameters of the theory.

To do this, first we see the $F$-flat conditions (17)–(21). These equations can be rewritten

$$C\begin{pmatrix} M_\Delta \\ M_A \\ Y_\Delta \\ Y_{\Delta A^2} \\ Y_{\Delta^2 A}\end{pmatrix} = -\begin{pmatrix} [1/(10\sqrt{6})]Y_{\Phi\Delta} \\ [1/(10\sqrt{2})]Y_{\Phi\Delta} \\ (1/10)Y_{\Phi\Delta} \\ (\sqrt{6}/10)Y_{\Phi A} \\ (1/5)Y_{\Phi A}\end{pmatrix}\phi\bar{\phi}, \tag{68}$$

where

$$C=\begin{pmatrix} a, & 0, & \frac{1}{12\sqrt{6}}c^2, & -\frac{1}{2\sqrt{6}}\beta^2, & 24\sqrt{2}i\alpha b \\ b, & 0, & \frac{1}{18\sqrt{2}}b^2+\frac{1}{18\sqrt{2}}c^2, & -\frac{1}{3\sqrt{2}}\alpha^2, & 24\sqrt{2}ia\alpha+24\sqrt{2}i\beta c \\ c, & 0, & \frac{1}{6\sqrt{6}}ac+\frac{1}{9\sqrt{2}}bc, & -\frac{1}{\sqrt{6}}\alpha\beta, & 16\sqrt{6}i\alpha c+24\sqrt{2}ib\beta \\ 0, & \alpha, & 0, & -\frac{\sqrt{2}}{3}\alpha b-\frac{1}{\sqrt{6}}\beta c, & 24\sqrt{2}iab+8\sqrt{6}ic^2 \\ 0, & \beta, & 0, & -\frac{1}{\sqrt{6}}\alpha c-\frac{1}{\sqrt{6}}a\beta, & 24\sqrt{2}ibc \end{pmatrix}. \tag{69}$$

As we know from the previous argument that $b,c,$ and $\alpha$ can be chosen freely and $a$ and $\beta$ are given by

$$a=a_0+a_1\epsilon,$$
$$\beta=\beta_1\epsilon+\beta_2\epsilon^2, \tag{70}$$

where $a_0$ is given by the first equation of (39) or one of the last two equations of (40) and $\beta_1$ is given by (66). Note that higher orders in (53) and (58) can be absorbed into $a_1$ and $\beta_2$, respectively.

Then, the input parameters are reduced to

$$\begin{pmatrix} M_\Delta \\ M_A \\ Y_\Delta \\ Y_{\Delta A^2} \\ Y_{\Delta^2 A}\end{pmatrix} = -C^{-1}\begin{pmatrix} [1/(10\sqrt{6})]Y_{\Phi\Delta} \\ [1/(10\sqrt{2})]Y_{\Phi\Delta} \\ (1/10)Y_{\Phi\Delta} \\ (\sqrt{6}/10)Y_{\Phi A} \\ (1/5)Y_{\Phi A}\end{pmatrix}\phi\bar{\phi}. \tag{71}$$

For example, in the case of solution 2,

$$C^{-1}=(\det C)^{-1}C'\epsilon$$

$$\det C = (-3.76350 i\alpha^2 b^4\beta_2 c_1 - 2.25347 i\alpha^3 b^2 a_1 c_1^2)\epsilon^3 + \mathrm{O}(\epsilon)^4$$

$$C' = \begin{pmatrix} 0, & 0, & -2.68018 i\alpha^3 b^3 c_1, & 0, & -1.08826 i\alpha^4 b^2 c_1 \\ 0, & 0, & -2.11074 i\alpha b^5 c_1, & 0, & -0.857040 i\alpha^2 b^4 c_1 \\ 0, & 0, & 8.10927 i\alpha^3 b^2 c_1, & 0, & 3.29268 i\alpha^4 b c_1 \\ 0, & 0, & 1.06439 i\alpha b^4 c_1, & 0, & 0.432184 i\alpha^2 b^3 c_1 \\ 0, & 0, & -0.0779620\alpha^2 b^3 c_1, & 0, & -0.0316556\alpha^3 b^2 c_1 \end{pmatrix} + O(\epsilon).$$

From this equation, it is easy to see that all parameters are of order $\epsilon^0$ and they satisfy the first solution of the solution 2.

Finally, from (22), $M_\Phi$ is determined:

$$M_\phi = -Y_{\Phi A}\left(\frac{\sqrt{6}\alpha}{10} + \frac{\beta}{5}\right) - Y_{\Phi\Delta}\left(\frac{a}{10\sqrt{6}} + \frac{b}{10\sqrt{2}} + \frac{c}{10}\right). \quad (72)$$

### 3. Check mass matrices

The multiplets in (1), besides one (2,2,1,0), must decouple at $M_{\nu_R}$,that is, they must acquire mass of $O(M_{\nu_R})$.

From now on, we check whether they have mass of $O(M_{\nu_R})$.

First, we note one $(2,1,3,-1)$ + H.c. and (2,1,1,3) + H.c. can have masses of $O(M_{\nu_R})$ by the following two reasons: (1) Parameters $Y_{\psi\Delta}$ and $M_\Psi$ may deviate from the value given by (45) and (46), respectively.[4] (2) There exist couplings with $c$ and $\beta$.

Then, we see the mass matrix for (2,2,1,0). Under SM, it has a quantum number $(2,1,\pm 1/2)$. (2,2,1,6) + H.c. also includes the same component. Then the mass matrix is

$$M(2,1,\pm 1/2) = \begin{pmatrix} \tilde{M}_\Delta, & x, & y, & 0 \\ x', & M_H, & u, & v \\ 0, & u, & 0, & w-z \\ y', & v, & w+z, & 0 \end{pmatrix}, \quad (73)$$

where

$$\tilde{M}_\Delta = M(2,2,1,6) + \frac{1}{12} Y_\Delta c + 24 i Y_{A\Delta^2}\beta,$$

$$x = -\frac{1}{\sqrt{5}} Y_{H\bar{\Phi}\Delta}\bar{\phi} = O(\epsilon^{3/2}),$$

$$x' = -\frac{1}{\sqrt{5}} Y_{H\Phi\Delta}\phi = O(\epsilon^{3/2}),$$

$$y = -\frac{1}{40} Y_{\Phi\Delta}\bar{\phi} = O(\epsilon),$$

$$y' = -\frac{1}{40} Y_{\Phi\Delta}\phi = O(\epsilon),$$

$$u = -\frac{1}{\sqrt{10}} Y_{H\Phi\Delta} b + \frac{1}{2\sqrt{5}} Y_{H\Phi\Delta} c = O(\sqrt{\epsilon}),$$

$$v = \frac{1}{\sqrt{10}} Y_{H\bar{\Phi}\Delta} b + \frac{1}{2\sqrt{5}} Y_{H\bar{\Phi}\Delta} c = O(\sqrt{\epsilon}),$$

$$w = M_\Phi + \frac{Y_{\Phi\Delta} b}{15\sqrt{2}} = O(\epsilon),$$

$$z = \frac{Y_{\Phi\Delta} c}{30} + \frac{Y_{\Phi A}\beta}{10} = O(\epsilon). \quad (74)$$

$M(2,2,1,6)$ is given in the Appendix C. Orders of $x, y, \dots$ are followed from (43).

Because one $(2,1,\pm 1/2)$ multiplet remains massless after $G_{2231}$ breaks down to the SM group,

$$\begin{aligned} \det(M(2,1,\pm 1/2)) &= \{\tilde{M}_\Delta(z^2 - w^2) + y y'(w-z)\} M_H \\ &\quad + 2\tilde{M}_\Delta u v w \cdots \\ &= 0, \end{aligned} \quad (75)$$

and hence $M_H$ is determined as follows:

$$M_H = \frac{2uvw}{w^2 - z^2} + O(\epsilon). \quad (76)$$

In this case, the higher order terms must be included to have a pair of light Higgs doublets.

Next, let us consider (1,1,8,0). This multiplet becomes (1,8,0) under the SM group and therefore it mixes with $T_{3R}=0$ component of (1,3,8,0) under the SM. Then the mass matrix for (1,8,0) is represented as a $3\times 3$ matrix.

$$M(1,8,0) = \begin{pmatrix} M(1,1,8,0) & \text{mixing} \\ \text{mixing} & M(1,3,8,0) \end{pmatrix}. \quad (77)$$

After $G_{2231}$ breaks down to the SM group, there is a correction of $O(M_U\epsilon \sim M_{\nu_R})$ to the mass matrices $M(1,1,8,0)$ and $M(1,3,8,0)$ because parameters appearing in them are different by $O(\epsilon)$ from those calculated in the previous section. It is directly calculated using (71) [or equivalently (53)–(57) and (60)] that one of the eigenvalues of $M(1,1,8,0)$ is of

[4]Though $(2,1,3,-1)$ + H.c. has a same quantum number under the SM group as an NG mode associated with the breakdown of SO(10) the SM group [see Table (23)], it does not mix with others because the VEV of $\psi=0$ and therefore this NG mode does not consist of it. (2,1,1,3) + H.c. has the same quantum number as that of (2,2,1,0) under the SM group but by the same reason they do not mix with (2,2,1,0). See the superpotential (5)–(8).

$O(M_U)$ which has already been suggested at the previous section and the other is $O(M_{\nu_R})$. As $M(1,3,8,0)$ is $O(M_U)$, even though there is a correction of $O(M_{\nu_R})$, $M(1,3,8,0)$ is still $O(M_U)$. Contributions of $c$ and $\beta$ to the mass matrix (77) appear at mixing terms between (1,1,8,0) and (1,3,8,0)[5] and they are of $O(M_{\nu_R})$. Then $M(1,8,0)$ takes the form

$$\begin{pmatrix} O(M_U) & 0 & O(M_U\epsilon) \\ 0 & O(M_U\epsilon) & O(M_U\epsilon) \\ O(M_U\epsilon) & O(M_U\epsilon) & O(M_U) \end{pmatrix}. \tag{78}$$

Apparently, two eigenvalues are of $O(M_U)$ and the other is of $O(M_{\nu_R})$. This fact suggests that the lightest element of (1,1,8,0) under $G_{2231}$ decouples at the scale $M_{\nu_R}$.

Finally, we check the mass of (1,3,1,0) and (1,3,1,−6) + H.c. Under the SM, (1,3,1,0) is decomposed into one neutral singlet and a pair of charged singlet with hypercharge $Y=\pm 1$. (1,3,1,−6) + H.c. becomes two neutral singlets, a pair of $Y=\pm 1$ and a pair of $Y=\pm 2$ singlets. Then, $Y=\pm 1$ component of them will mix with each other.

Mass for $Y=\pm 2$ component takes the form

$$\begin{aligned} &Y_{\Phi A}\left(\frac{\sqrt{6}\alpha}{10}-\frac{\beta}{5}\right)+Y_{\Phi\Delta}\left(\frac{a}{10\sqrt{6}}+\frac{b}{10\sqrt{2}}-\frac{c}{10}\right)+M_\phi \\ &=-\frac{2}{5}Y_{\Phi A}\beta-\frac{1}{5}Y_{\Phi\Delta}c, \end{aligned} \tag{79}$$

where (72) is used.

From this equation, obviously the $Y=\pm 2$ component has a mass of $O(M_{\nu_R})$.

Mass matrix of $Y=\pm 1$ component is

$$\begin{pmatrix} -\frac{Y_{\Delta A^2}a}{\sqrt{6}}+M_A, & -\frac{Y_{\Delta A^2}\alpha}{\sqrt{6}}+24\,i\sqrt{2}Y_{\Delta^2 A}b, & -\frac{Y_{\Phi A}\phi}{5} \\ -\frac{Y_{\Delta A^2}\alpha}{\sqrt{6}}+24\,i\sqrt{2}Y_{\Delta^2 A}b, & \frac{Y_\Delta a}{6\sqrt{6}}+16\,i\sqrt{6}Y_{\Delta^2 A}\alpha+\frac{Y_\Delta b}{9\sqrt{2}}+M_\Delta, & -\frac{Y_{\Phi\Delta}\phi}{10} \\ -\frac{Y_{\Phi A}\bar{\phi}}{5}, & -\frac{Y_{\Phi\Delta}\bar{\phi}}{10}, & -\frac{Y_{\Phi A}\beta}{5}-\frac{Y_{\Phi\Delta}c}{10} \end{pmatrix}. \tag{80}$$

Since it is an NG mode associated with the breakdown of $G_{2231}$ to $G_{231}$, there is one massless mode. It is easy to see that this matrix has 0 eigenvalue because 1st row $\times\beta/\phi+$2nd row$\times c/\phi+$3rd row$=0$ using the $F$-flat conditions (19) and (21). It is also explicitly calculated that one eigenvalue is of $O(M_U)$ and the other is of $O(M_{\nu_R})$.

## V. SUMMARY

As we saw, by constructing the input parameters for the theory using (71), (72), (74), and (76) from the desired values of VEV's $a,b,c,\alpha,\beta,\phi$, and $\bar{\phi}$ which satisfy (10) and (70), we can have particles (1) in the intermediate region. They decouple from the spectrum at $M_{\nu_R}$ except a pair of what we call Higgs doublets.

It means that it is possible to construct a SUSY SO(10) GUT with an intermediate scale consistent with the gauge unification. It suggests also that the right-handed neutrinos acquire mass through a renormalizable coupling, and it can be understood as a reflection of the breakdown of $G_{2231}$ to $G_{231}$.

There are many variations for a SUSY SO(10) GUT with an intermediate scale because there are many candidates for the particle content which exist in the intermediate region and we have many variations for content of SO(10) multiplets which contain one of the candidates.

For example, we can replace (2,2,1,0) by (2,1,1,3) + H.c. in the spectrum (1) and vice versa, because their contribution to the running of the gauge coupling relevant to $G_{231}$ is the same.

When we remove one (2,2,1,0) from the spectrum (1) and add one (2,1,1,3) + H.c. to it, by adding a pair of SO(10) multiplets $\mathbf{16}+\overline{\mathbf{16}}$ which contains (2,1,1,3) + H.c. under $G_{2231}$, we can have such a spectrum at the intermediate region. At that time, while we have to tune couplings relevant to SO(10) multiplets $\mathbf{16}+\overline{\mathbf{16}}$, we can release the constraint (43) [or equivalently (74)].

Of course, there are quite different types of content for the candidates. Using them, we can construct quite a different SO(10) GUT with an intermediate scale.

Though the gauge unification by the MSSM is a very attractive idea, to take into account a right-handed neutrino mass, we should consider a possibility of a GUT with an intermediate symmetry.


## ACKNOWLEDGMENTS

The author wishes to thank T. Kugo, M. Bando, and T. Takahashi for valuable comments and discussion.


[5]There is no contribution of $c$ or $\beta$ to $M(1,1,8,0)$ and $M(1,3,8,0)$. The reason is as follows. Under $G_{2231}$, $c$ and $\beta$ are contained in (1,3,1,0). Because $(1,3,1,0)(1,1,8,0)^2$ contains no singlet, neither $c$ nor $\beta$ couple to $(1,1,8,0)^2$. Though $(1,3,1,0)(1,3,8,0)^2$ can appear, as there is no three-point coupling of $T_{3R}=0$ component of SU(2) triplet, neither $c$ nor $\beta$ couple to $T_{3R}=0$ component of (1,3,8,0).

## APPENDIX A: THE REASON WHY WE NEED A MULTIPLET (1,3,1,0)

Here, we show the reason why we need a multiplet (1,3,1,0) in the intermediate region.

First, we note that we required at least a pair of multiplet (1,3,1,−6) + H.c. ($\equiv \Phi + \bar{\Phi}$) in the intermediate region [6] and hence at this region in the superpotential effectively there must be a term

$$W = M_\Phi \Phi \bar{\Phi}. \tag{A1}$$

Because we consider an SO(10) GUT the mass parameter $M_\Phi$ is, in general, thought to be of $O(M_U)$.

In this case it is, however, impossible that $\Phi$ acquires a VEV. Of course if we tune the parameter $M_\Phi$, to be zero, as there is a flat direction in $D$ term, $\Phi$ can acquire a VEV, but in this case there are two problems: (1) there is no way to determine a magnitude of the VEV of $\Phi$; (2) hypercharge Y $=\pm 2$ component of $\Phi$ cannot have any mass.[6]

Then, we have to add other multiplets. The easiest way to solve the problem (1) is to add a singlet ($\equiv S$).[7] If there is a singlet, the superpotential will have a form

$$W = M_\Phi \Phi \bar{\Phi} + Y_{\Phi S} S \Phi \bar{\Phi} + \frac{1}{2} M_S S^2 + \frac{1}{3!} Y_S S^3 \tag{A2}$$

and $F$-flat conditions are ($\langle \Phi \rangle \equiv \phi$, $\langle S \rangle \equiv s$)

$$\frac{\partial W}{\partial \phi} = (M_\Phi + Y_{\Phi S} s) = 0, \tag{A3}$$

$$\frac{\partial W}{\partial s} = Y_{\Phi S} \phi \bar{\phi} + M_S s + \frac{1}{2} Y_S s^2. \tag{A4}$$

Then VEV's are determined to

$$s = -\frac{M_\Phi}{Y_{\Phi S}}, \tag{A5}$$

$$\phi \bar{\phi} = \frac{M_S M_\Phi}{Y_{\Phi S}} - \frac{1}{2} Y_S \left( \frac{M_\Phi}{Y_{\Phi S}} \right)^2. \tag{A6}$$

Though, as we mention below (A1), $M$'s are thought to be of $O(M_U)$, we can give a VEV of $O(M_{\nu_R})$ to $\Phi$ if coupling constants are fine tuned while $s$ is of $O(M_U)$.

Unfortunately, even after we add a singlet, the problem (2) is not solved because the mass for $Y = \pm 2$ component is

$$M_\Phi + Y_S s = 0 \tag{A7}$$

according to the $F$-flat condition (83). The reason why it is still massless is that no multiplet couples to $\Phi$ which acquires a VEV of $O(M_{\nu_R})$ and distinguishes the component of a $SU(2)_R$ triplet and hence all components of $\Phi$ are still degenerate after $SU(2)_R$ breaking.

This means that to make $Y = \pm 2$ component decouple from the spectrum after $SU(2)_R$ breaking, we have to make a multiplet couple to $\Phi$ which will get a VEV of $O(M_{\nu_R})$ and distinguishes the component of an $SU(2)_R$ triplet, that is, a nonsinglet. It is easy to find what nonsinglet can couple to $\Phi\bar{\Phi}$. From $\Phi\bar{\Phi}$, we have three representations:

$$(1,1,1,0),$$

$$(1,3,1,0),$$

$$(1,5,1,0). \tag{A8}$$

As $SU(2)_R$ nonsinglets are the latter two and (1,5,1,0) is not contained in a relatively smaller representation of SO(10), we have to use (1,3,1,0). Since $T_{3R} = 0$ component of a triplet is an SM singlet, it can get a VEV.

Since (1,3,1,0) is not a singlet under $G_{2231}$, its VEV is at most of $O(M_{\nu_R})$, while because (1,3,1,0) gives a mass of $O(M_{\nu_R})$ to $Y = \pm 2$ component of $\Phi$, even if there are many (1,3,1,0), one of their VEV's must be of $O(M_{\nu_R})$. This implies that at least one of (1,3,1,0) must have a mass of $O(M_{\nu_R})$. In the following, we will see it explicitly.

First, when there are also (1,3,1,0) multiplets ($\equiv B_i$), the superpotential takes the form

$$W = M_\Phi \Phi \bar{\Phi} + Y_{\Phi S} S \Phi \bar{\Phi} + \sum_i Y_i B_i \Phi \bar{\Phi} + \frac{1}{2} M_S S^2 + \frac{1}{3!} Y_S S^3$$
$$+ \frac{1}{2} \sum_{i,j} (M_{ij} + Y_{ij} S) B_i B_j + \frac{1}{3!} \sum_{i,j,k} Y_{ijk} B_i B_j B_k \tag{A9}$$

and $F$-flat conditions are ($\langle B_i \rangle \equiv \beta_i$)

$$\frac{\partial W}{\partial \Phi} = \left( M_\Phi + Y_{\Phi S} s + \sum_i Y_i \beta_i \right) \bar{\Phi} = 0, \tag{A10}$$

$$\frac{\partial W}{\partial S} = Y_{\Phi S} \phi \bar{\phi} + M_S s + \frac{1}{2} Y_S s^2 + \sum_{i,j} Y_{Sij} \beta_i \beta_j = 0, \tag{A11}$$

$$\frac{\partial W}{\partial B_i} = Y_i \phi \bar{\phi} + \sum_{i,j} (M_{ij} + Y_{ij} s) \beta_i = 0. \tag{A12}$$

Note that there is no three-point coupling of $T_3 = 0$ component of SU(2) triplet and hence there is no affect of $Y_{ijk}$.

From (A12), $\beta_i$ is calculated to

$$\beta_i = -(\tilde{M}^{-1})_{ij} a_j \phi \bar{\phi},$$

$$\tilde{M}_{ij} \equiv (M_{ij} + Y_{ij} s). \tag{A13}$$

By assumption, $\phi = O(M_{\nu_R})$ and as we mentioned one of $\beta_i$ also must be of $O(M_{\nu_R})$. These facts imply that in the above equation, $\tilde{M}$ must have at least one eigenvalue of $O(M_{\nu_R})$. Because $\tilde{M}$ is a mass matrix for (1,3,1,0) [see (A9)], it means that at least one of (1,3,1,0) must be massless at the GUT scale.

---

[6] Note that only an NG mode can get a mass through $D$ term. In general, such a component corresponds to a massive gaugino.

[7] Because we consider an SO(10) GUT, there are several singlets though naturally their masses are of $O(M_U)$.

In this case mass for $Y=\pm 2$ is calculated

$$\left( M_{\Phi}+Y_{\Phi S}s-\sum_i a_i\beta_i \right)=-2\sum_i a_i\beta_i=O(M_{\nu_R}), \qquad \text{(A14)}$$

where (A10) is used. Apparently, this component decouples at $M_{\nu_R}$, namely, the problem (2) is solved.

## APPENDIX B: CONSTRUCTION OF REPRESENTATIONS

In this section we briefly review how we construct representations of subgroups contained in SO(10) representations and give the rule for calculating CG coefficients appearing in three-point couplings. However, we do not mention about an SO(10) spinor 16 because it is impossible to understand the meaning of the indices for a spinor in the same way as understanding an SO(10) vector 10 and essentially we do not need to handle them directly in this paper. To see how to handle an SO(10) spinor, see Ref. [12]. When calculating CG coefficient relevant to a spinor the gamma matrices for SO(10) constructed explicitly in the reference are used.

### 1. Meanings of subscripts

For SO(10), the fundamental representation[8] is a ten-dimensional real vector

$$H=(H_i), \quad i=1,\dots,10.$$

It means when we construct a fundamental representation for SO(10), we can use the basis for it

$$H=h_i e_i, \qquad \text{(B1)}$$

where

$$h_i=e_i^{\dagger}H, e_i\equiv \begin{pmatrix} 0\\ \vdots \\ 0\\ 1\\ 0\\ \vdots\\ 0 \end{pmatrix} \}i\text{th component}. \qquad \text{(B2)}$$

Hereafter in this appendix, repeated subscripts are assumed to be contracted.

In this case, index $i$ means nothing but SO(10) vector.

For our convenience, we can attach an additional meaning to it. SO(10) includes SU(5) $\otimes$ U(1) and SO(6) $\otimes$ SO(4) $\simeq$ SU(4) $\otimes$ SU(2) $\otimes$ SU(2). Under them, the fundamental representation **10** is decomposed into [11]

$$\mathbf{10}=\begin{cases} \mathbf{5}(2)+\overline{\mathbf{5}}(-2) & \text{under SU}(5)\otimes\text{U}(1)\\ (6,1)+(1,4) & \text{under SO}(6)\otimes\text{SO}(4)\\ (6,1,1)+(2,2,1) & \text{under SU}(4)\otimes\text{SU}(2)\otimes\text{SU}(2) \end{cases}$$

Then we can add a meaning of, for example, SO(6) vector to indices 1 to 6 and SO(4) vector to 7 to 10.[9] Hereafter, 0 stands for **10**. In other words, SO(6), an SO(10) subgroup, acts on the indices 1–6 and SO(4) acts on 7–0.

We can add more meaning to indices of an SO(10) vector by giving a meaning $\mathbf{5}(2)$ representation under SU(5) $\otimes$ U(1) to $(1+2i,3+4i,5+6i,7+8i,9+0i)$ and its complex conjugate to $(1-2i,3-4i,5-6i,7-8i,9-0i)$.

What $1+2i$ means is as follows. When we construct a vector representation, we can use a basis $E_{a+bi}$ and its complex conjugate $\overline{E}_{a-bi}\equiv\overline{E_{a+bi}}$, where $b=a+1$ and $a$ is an odd number other than $e_i$ which is introduced at the top of this section:

$$E_{a+bi}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0\\ 1\\ i\\ 0\end{pmatrix}\begin{matrix} \\ \}a\text{th}\\ \}b\text{th}\\ \\ \end{matrix}=\frac{1}{\sqrt{2}}e_a+\frac{i}{\sqrt{2}}e_b, \qquad \text{(B3)}$$

where $1/\sqrt{2}$ is a normalization factor to achieve $E_{a+bi}^{\dagger}E_{a+bi}=1$.

Then,

$$H=h_ie_i=h_{a+bi}E_{a+bi}+h_{a-bi}\overline{E}_{a-bi},$$

where

$$h_{a+bi}=E_{a+bi}^{\dagger}H=\frac{1}{\sqrt{2}}(h_a-h_bi), \qquad \text{(B4)}$$

$h_{a+bi}$ is a component of an SU(5) vector and its U(1) charge is two. As it is easily seen, the component for an SO(10) vector depends on a basis.

Because both SU(5) and SO(6) $\simeq$ SU(4) contain SU(3)$_C$, we can add the meaning of SU(3) **3** and $\overline{\mathbf{3}}$ to the SO(6) vector indices 1 to 6: $(1+2i,3+4i,5+6i)$ is an SU(3) vector 3. By the same way, we can add the meaning of SU(2) **2** and $\overline{\mathbf{2}}$ to the SO(4) $\simeq$ SU(2) $\otimes$ SU(2) vector indices 7–0: $(7+8i,9+0i)$ is an SU(2) vector **2**.

As we will see later, a higher representation is represented as a tensor. By this construction when we consider what representations a higher representation contains under, for example, SO(10) subgroup SU(4), it is sufficient to deal with indices 1 to 6. When considering SU(5) subgroup, we can deal with combinations of SO(10) subscripts $1+2i$ and so on.

### 2. SO(10) representations and representations of subgroups contained in SO(10) representations

The representations **45**, **126** + $\overline{\mathbf{126}}$ and **210** are formulated from the fundamental representation as antisymmetric tensors of 2nd, 5th, and 4th ranks, respectively. By the characteristic of SO(10), 5th rank antisymmetric tensor is decom-

[8]Exactly in a mathematical term what fundamental representation means is identity representation.

[9]In the papers [7,9], the authors give a meaning of SO(6) vector to indices 5–10 and that of SO(4) to 1–4.

posed into two parts, **126** and $\overline{\mathbf{126}}$. Using 10th rank antisymmetric $\epsilon$ tensor ($\equiv \varepsilon_{abcdeijklm}$), it is decomposed into two eigenstates [12]:

$$\frac{i}{5!}\varepsilon_{abcdeijklm}\Phi_{ijklm}=+\Phi_{abcde},$$

$$\frac{i}{5!}\varepsilon_{abcdeijklm}\overline{\Phi}_{ijklm}=-\overline{\Phi}_{abcde}. \quad \text{(B5)}$$

What has a plus eigenvalue is defined to be **126** and the other is to be $\overline{\mathbf{126}}$.

In the same way as an SO(10) vector **10**, we can express these representations using a component and a basis. To express **45** ($\equiv A$), we can take a basis $e_{ij}$ as

$$A=a_{ij}e_{ij}, \quad \text{(B6)}$$

where

$$a_{ij}=\mathrm{tr}Ae_{ij},e_{ij}=[(e_{ij})_{ab}]=\frac{i}{\sqrt{2}}(\delta_{ai}\delta_{bj}-\delta_{aj}\delta_{bi}). \quad \text{(B7)}$$

$a_{ij}$ corresponds to a component of **45** representation. In our notation, subscripts $i,j$ for a component and a basis satisfy that $i>j$.

In a similar manner $\mathbf{126}+\overline{\mathbf{126}}$ ($\equiv \Phi+\overline{\Phi}$) is written as

$$\Phi(\text{or } \overline{\Phi})=\phi_{ijklm}e_{ijklm}, \quad \text{(B8)}$$

where $e_{ijklm}$ is an antisymmetric tensor and only when a combination of indices coincide with subscripts $\{ijklm\}$, it has a value $1/\sqrt{5!}$ or $-1/\sqrt{5!}$ . The sign is defined to make $e_{ijklm}$ antisymmetric. Here $\{ijklm\}$ satisfies $i>j>k>l>m$. Exactly, for $e_{ijklm}$ to be a basis of **126** (or $\overline{\mathbf{126}}$), there is another constraint for it as we explained in (B5), though we do not touch the detail here. Then a component of **126** is given by

$$\phi_{ijklm}=\Phi_{abcde}(e_{ijklm})_{abcde}. \quad \text{(B9)}$$

$1/\sqrt{5!}$ is a necessary normalization factor to express a **126** representation by (B8) and (B9) similar to $1/\sqrt{2}$ in (B3).

In the case of **210** a basis for it becomes 4th rank antisymmetric tensor and its normalization is $1/\sqrt{4!}$. Besides it, **210** ($\equiv \Delta$) is represented in the same way:

$$\Delta=\delta_{ijkl}e_{ijkl},$$

where

$$\delta_{ijkl}=\Delta_{abcd}(e_{ijkl})_{abcd}$$

and $i>j>k>l$.

To construct a representation under subgroups, we use a linear combination of these bases in the same way as when we extract a **5**(2) of the subgroup SU(5) $\otimes$ U(1) from an SO(10) vector we use a basis $E_{a+bi}$.

For example, let us consider $G_{231}$ singlets contained in **126** and $\overline{\mathbf{126}}$. They are SU(5) singlets. Then it is sufficient to deal with SU(5) subscripts $1+2i$ and so on. By the quintality of SU(5), the form of the basis of SU(5) singlets in **126** and $\overline{\mathbf{126}}$ are determined to be $e_{1-2i,3-4i,5-6i,7-8i,9-0i}$, $e_{1+2i,3+4i,5+6i,7+8i,9+0i}$. They are understood in the same way as $E_{1+2i}$ (B3):

$$e_{1-2i,3-4i,5-6i,7-8i,9-0i}=\frac{1}{\sqrt{10}}(e_{13579}-ie_{23579}+\ldots),$$

where $1/\sqrt{10}$ is an extra normalization factor to achieve

$$(e_{1-2i,3-4i,5-6i,7-8i,9-0i})^{*}_{abcde}(e_{1-2i,3-4i,5-6i,7-8i,9-0i})_{abcde}=1$$

similar to $1/\sqrt{2}$ in (B3).

It is easily seen that the former is a basis of **126** and the latter is that of $\overline{\mathbf{126}}$ by making $\varepsilon_{abcdeijklm}$ acting on them or by counting U(1) charge [11]. All other representations of subgroups contained in SO(10) representations are constructed in a similar way.

### 3. CG coefficient

Using **10**, **45**, **126**, $\overline{\mathbf{126}}$, and **210**, we have following SO(10) singlets [11]:

$$H\Phi\Delta,H\overline{\Phi}\Delta,\Delta^3,\overline{\Phi}\Delta\Phi,\overline{\Phi}A\Phi,A^2\Delta,A\Delta^2.$$

We can get singlets by contracting all indices of tensors:

$$H\Phi\Delta\equiv H_a\Phi_{abcde}\Delta_{bcde},$$

$$H\overline{\Phi}\Delta\equiv H_a\overline{\Phi}_{abcde}\Delta_{bcde},$$

$$\Delta^3\equiv\Delta_{abcd}\Delta_{cdef}\Delta_{efab},$$

$$\overline{\Phi}\Delta\Phi\equiv\overline{\Phi}_{abijk}\Delta_{abcd}\Phi_{cdijk},$$

$$\overline{\Phi}A\Phi\equiv\overline{\Phi}_{aijkl}A_{ab}\Phi_{bijkl},$$

$$A^2\Delta\equiv A_{ab}A_{cd}\Delta_{abcd},$$

$$A\Delta^2\equiv\varepsilon_{abcdefghij}A_{ab}\Delta_{cdef}\Delta_{ghij}.$$

In terms of components of the representations

$$H\Phi\Delta=\frac{1}{\sqrt{5}}h_a\phi_{abcde}\delta_{bcde},$$

$$H\overline{\Phi}\Delta=\frac{1}{\sqrt{5}}h_a\overline{\phi}_{abcde}\delta_{bcde},$$

$$\Delta^3 = \frac{1}{6\sqrt{6}} \delta_{abcd}\delta_{cdef}\delta_{efab},$$

$$\overline{\Phi}\Delta\Phi = \frac{1}{10\sqrt{6}} \bar{\phi}_{abijk}\delta_{abcd}\phi_{cdijk},$$

$$\overline{\Phi}A\Phi = \frac{i}{5\sqrt{2}} \overline{\Phi}_{aijkl}A_{ab}\Phi_{bijkl},$$

$$A^2\Delta = -\frac{1}{\sqrt{6}} a_{ab}a_{cd}\delta_{abcd},$$

$$A\Delta^2 = 24\sqrt{2} i a_{ab}\delta_{cdef}\delta_{ghij},$$

where repeated subscripts are not summed and in the last equation, $abcdefghij$ are different from each other.

Then, we rewrite the superpotential (5) in terms of components, for example,

$$Y_\Delta \Delta^3 = \frac{Y_\Delta}{6\sqrt{6}} \delta_{abcd}\delta_{cdef}\delta_{efab}$$

and so on. Therefore, for components that as an expansion parameter for the perturbation Yukawa coupling = 1, means $Y_\Delta = 6\sqrt{6}$ and so on.

Of course, since a component of an irreducible representation is a linear combination of these components, CG coefficient for an irreducible representation is different from, for example, $1/6\sqrt{6}$ in the case of $\Delta^3$.

For example, let us calculate a CG coefficient for the singlet $\beta$ contained in **45** and $a$ contained in **210** [see the Table (3)]. They are contained in the form $A_{78+90} = \beta e_{78+90}$ and $\Delta_{7890} = a e_{7890}$, respectively. Then

$$\begin{aligned} A_{ab}A_{cd}\Delta_{abcd} &= \beta^2 a (e_{78+90})_{ab}(e_{78+90})_{cd}(e_{7890})_{abcd} \\ &= \beta^2 a \left(\frac{i}{2}\right)^2 \frac{1}{\sqrt{4!}} 2!2!\times 2 \\ &= -\frac{1}{\sqrt{6}} \beta^2 a. \end{aligned}$$

In the second line, $i/2$ comes from an element of $e_{78+90}$ and $1/\sqrt{4!}$ comes from an element of $e_{7890}$. $2!$ comes from a summation between $\{ab\}$ and $\{cd\}$. $\{ab\}$ and $\{cd\}$ are $\{78\}$ or $\{90\}$. The last factor 2 comes from an exchange of $\{78\}$ and $\{90\}$.

## APPENDIX C: MASS MATRICES UNDER $G_{2231}$ AND THEIR EIGENVALUE EQUATIONS

Under $G_{2231}$, the multiplets of our model have mass terms as follows. They are listed following the order of the list (23). Full mass matrices are given with contributions from $c, \beta, \phi$, and after $G_{2231}$ breaks down to $G_{231}$. But these contributions are of order $M_{\nu_R} \sim M_U \epsilon$ and hence if the mass eigenvalue is of $O(M_U)$, they are negligible and we do not need to consider them.

(2,2,1,0) multiplet:

$$M(2,2,1,0) = \begin{pmatrix} M_H, & -\frac{Y_{H\Phi\Delta} b}{\sqrt{10}}, & \frac{Y_{H\bar{\Phi}\Delta} b}{\sqrt{10}} \\ -\frac{Y_{H\Phi\Delta} b}{\sqrt{10}}, & 0, & \frac{Y_{\Phi\Delta} b}{15\sqrt{2}} + M_\Phi \\ \frac{Y_{H\bar{\Phi}\Delta} b}{\sqrt{10}}, & \frac{Y_{\Phi\Delta} b}{15\sqrt{2}} + M_\Phi, & 0 \end{pmatrix}.$$

(1,1,3,−2) + H.c. multiplet:

$$M(1,1,3,2) = \begin{pmatrix} M_H, & \frac{Y_{H\Phi\Delta}(\sqrt{3}a - b)}{\sqrt{30}}, & \frac{Y_{H\bar{\Phi}\Delta}(\sqrt{3}a + b)}{\sqrt{30}} \\ \frac{Y_{H\Phi\Delta}(\sqrt{3}a - b)}{\sqrt{30}}, & 0 & \frac{Y_{\Phi A}\alpha}{5\sqrt{6}} + M_\Phi \\ \frac{Y_{H\bar{\Phi}\Delta}(\sqrt{3}a + b)}{\sqrt{30}}, & -\frac{Y_{\Phi A}\alpha}{5\sqrt{6}} + M_\Phi, & 0 \end{pmatrix}.$$

(3,1,1,0) + H.c. multiplet:

$$M(3,1,1,0)=\begin{pmatrix} M_A+\frac{Y_{\Delta A^2}a}{\sqrt{6}}, & -\frac{Y_{\Delta A^2}\alpha}{\sqrt{6}}-24\,i\sqrt{2}Y_{\Delta^2 A}b \\ -\frac{Y_{\Delta A^2}\alpha}{\sqrt{6}}-24\,i\sqrt{2}Y_{\Delta^2 A}b, & \frac{-Y_\Delta a}{6\sqrt{6}}-16\,i\sqrt{6}Y_{\Delta^2 A}\alpha+\frac{Y_\Delta b}{9\sqrt{2}}+M_\Delta \end{pmatrix}.$$

(1,3,1,0) multiplet:

$$M(1,3,1,0)=\begin{pmatrix} -\frac{Y_{\Delta A^2}a}{\sqrt{6}}+M_A, & -\frac{Y_{\Delta A^2}\alpha}{\sqrt{6}}+24\,i\sqrt{2}Y_{\Delta^2 A}b \\ -\frac{Y_{\Delta A^2}\alpha}{\sqrt{6}}+24\,i\sqrt{2}Y_{\Delta^2 A}b, & \frac{Y_\Delta a}{6\sqrt{6}}+16\,i\sqrt{6}Y_{\Delta^2 A}\alpha+\frac{Y_\Delta b}{9\sqrt{2}}+M_\Delta \end{pmatrix}.$$

(1,1,3,−4) multiplet:

$$M(1,1,3,-4)=\begin{pmatrix} \frac{-Y_{\Delta A^2}b}{3\sqrt{2}}+M_A, & 24\sqrt{2}iY_{\Delta^2 A}a-\frac{Y_{\Delta A^2}\alpha}{3\sqrt{2}} \\ 24\,i\sqrt{2}Y_{\Delta^2 A}a-\frac{Y_{\Delta A^2}\alpha}{3\sqrt{2}}, & \frac{Y_\Delta b}{18\sqrt{2}}+M_\Delta \end{pmatrix}.$$

(1,1,8,0) multiplet:

$$M(1,1,8,0)=\begin{pmatrix} \frac{Y_{\Delta A^2}b}{3\sqrt{2}}+M_A, & 24\,i\sqrt{2}Y_{\Delta^2 A}a-\frac{Y_{\Delta A^2}\alpha}{3\sqrt{2}} \\ 24\,i\sqrt{2}Y_{\Delta^2 A}a-\frac{Y_{\Delta A^2}\alpha}{3\sqrt{2}}, & -\frac{Y_\Delta b}{18\sqrt{2}}+M_\Delta \end{pmatrix}.$$

(2,2,3,2) + H.c. multiplet:

$$M(2,2,3,2)=\begin{pmatrix} M_A, & 8\sqrt{6}iY_{\Delta^2 A}b, & -\frac{Y_{\Delta A^2}\alpha}{3} \\ 8\sqrt{6}iY_{\Delta^2 A}b, & M_\Delta, & 16i\sqrt{3}Y_{\Delta^2 A}\alpha \\ -\frac{Y_{\Delta A^2}\alpha}{3}, & 16\sqrt{3}iY_{\Delta^2 A}\alpha, & \frac{Y_\Delta b}{18\sqrt{2}}+M_\Delta \end{pmatrix}.$$

(3,1,1,6) + H.c. multiplet:

$$M(3,1,1,6)=-\frac{\sqrt{6}Y_{\Phi A}\alpha}{10}-\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}+\frac{Y_{\Phi\Delta}b}{10\sqrt{2}}+M_\Phi.$$

(3,1,3,2) + H.c. multiplet:

$$M(3,1,3,2)=-\frac{Y_{\Phi A}\alpha}{5\sqrt{6}}-\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}+\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_\Phi.$$

(3,1,6,−2) + H.c. multiplet:

$$M(3,1,6,-2)=\frac{Y_{\Phi A}\alpha}{5\sqrt{6}}-\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}-\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_\Phi.$$

(1,3,1,−6) + H.c. multiplet:

$$M(1,3,1,-6)=\frac{\sqrt{6}Y_{\Phi A}\alpha}{10}+\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}+\frac{Y_{\Phi\Delta}b}{10\sqrt{2}}+M_\Phi.$$

$(1,3,3,-2)$ + H.c. multiplet:

$$M(1,3,3,-2)=\frac{Y_{\Phi A}\alpha}{5\sqrt{6}}+\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}+\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_{\Phi}.$$

$(1,3,6,2)$ + H.c. multiplet:

$$M(1,3,6,2)=-\frac{Y_{\Phi A}\alpha}{5\sqrt{6}}+\frac{Y_{\Phi\Delta}a}{10\sqrt{6}}-\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_{\Phi}.$$

$(2,2,3,-4)$ + H.c. multiplet:

$$M(2,2,3,-4)=\begin{pmatrix} \frac{\sqrt{6}Y_{\Phi A}\alpha}{15}+\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_{\Phi}, & 0 \\ 0, & -\frac{\sqrt{6}Y_{\Phi A}\alpha}{15}+\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_{\Phi} \end{pmatrix}.$$

$(2,2,8,0)$ multiplet:

$$M(2,2,8,0)=-\frac{Y_{\Phi\Delta}b}{30\sqrt{2}}+M_{\Phi}.$$

$(3,1,3,-4)$ + H.c. multiplet:

$$M(3,1,3,-4)=-\frac{Y_{\Delta}a}{6\sqrt{6}}-8i\sqrt{6}Y_{\Delta^2 A}\alpha+\frac{Y_{\Delta}b}{18\sqrt{2}}+M_{\Delta}.$$

$(1,3,3,-4)$ + H.c. multiplet:

$$M(1,3,3,-4)=\frac{Y_{\Delta}a}{6\sqrt{6}}+8i\sqrt{6}Y_{\Delta^2 A}\alpha+\frac{Y_{\Delta}b}{18\sqrt{2}}+M_{\Delta}.$$

$(3,1,8,0)$ multiplet:

$$M(3,1,8,0)=-\frac{Y_{\Delta}a}{6\sqrt{6}}+8i\sqrt{6}Y_{\Delta^2 A}\alpha-\frac{Y_{\Delta}b}{18\sqrt{2}}+M_{\Delta}.$$

$(1,3,8,0)$ multiplet:

$$M(1,3,8,0)=\frac{Y_{\Delta}a}{6\sqrt{6}}-8i\sqrt{6}Y_{\Delta^2 A}\alpha-\frac{Y_{\Delta}b}{18\sqrt{2}}+M_{\Delta}.$$

$(2,2,1,6)$ + H.c. multiplet:

$$M(2,2,1,6)=\frac{Y_{\Delta}b}{6\sqrt{2}}+M_{\Delta}.$$

$(2,2,6,-2)$ + H.c. multiplet:

$$M(2,2,6,-2)=-\frac{Y_{\Delta}b}{18\sqrt{2}}+M_{\Delta}.$$

(2,1,3,−1) + H.c. multiplet:

$$M(2,1,3,-1)=\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{6}}iY_{\psi_2 A}\alpha \\ -\frac{1}{\sqrt{6}}iY_{\psi_3 A}\alpha+2Y_{\psi_3\Delta}(\sqrt{6}a+\sqrt{2}b) \\ -\frac{1}{\sqrt{6}}iY_{\psi_4 A}\alpha+2Y_{\psi_4\Delta}(\sqrt{6}a+\sqrt{2}b)+M_\Psi \end{pmatrix}.$$

(1,2,$\overline{3}$,1) + H.c. multiplet:

$$M(1,2,\overline{3},1)=\begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}}iY_{\Psi A2}\alpha \\ \frac{1}{\sqrt{6}}iY_{\Psi A3}\alpha+2Y_{\Psi\Delta 3}(-\sqrt{6}a+\sqrt{2}b) \\ \frac{1}{\sqrt{6}}iY_{\Psi A4}\alpha+2Y_{\Psi\Delta 4}(-\sqrt{6}a+\sqrt{2}b)+M_\Psi \end{pmatrix}.$$

(2,1,1,3) + H.c. multiplet:

$$M(2,1,1,3)=\begin{pmatrix} 0 \\ \sqrt{6}iY_{\Psi A2}\alpha \\ \sqrt{6}iY_{\Psi A3}\alpha+2\sqrt{6}Y_{\Psi\Delta 3}(a-\sqrt{3}b) \\ \sqrt{6}iY_{\Psi A4}\alpha+2\sqrt{6}Y_{\Psi\Delta 4}(a-\sqrt{3}b)+M_\Psi \end{pmatrix}.$$

(1,2,1,−3) + H.c. multiplet:

$$M(1,2,1,-3)=\begin{pmatrix} 0 \\ -\sqrt{6}iY_{\Psi A2}\alpha \\ -\sqrt{6}iY_{\Psi A3}\alpha-2\sqrt{6}Y_{\Psi\Delta 3}(a+\sqrt{3}b) \\ -\sqrt{6}iY_{\Psi A4}\alpha-2\sqrt{6}Y_{\Psi\Delta 4}(a+\sqrt{3}b)+M_\Psi \end{pmatrix}.$$

[1] H. Georgi, in *Particle and Fields*, edited by C. E. Carlson, AIP Conf. Proc. No. 23 (AIP, New York, 1975); H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) **93**, 193 (1975).

[2] U. Amaldi, W. de Boer, and H. Fürstenau, Phys. Lett. B **260**, 447 (1991); P. Langacker and M. Luo, Phys. Rev. D **44**, 817 (1991).

[3] For a review, see M. Fukugita and T. Yanagida, in *Physics and Astrophysics of Neutrinos*, edited by M. Fukugita and A. Suzuki (Springer-Verlag, Tokyo, 1994).

[4] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978); S. P. Mikheev and A. Yu. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985).

[5] T. Yanagida, in *Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe*, Tsukuba, Japan, 1979, edited by A. Sawada and H. Sugawara (KEK Report No. 79-18, Tsukuba, Japan, 1979); M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, Proceedings of the Workshop, Stony Brook, New York, 1979, edited by P. van Nieuwenhuizen and D. Freedman (North-Holland, Amsterdam, 1979).

[6] M. Bando, J. Sato, and T. Takahashi, Phys. Rev. D **52**, 3076 (1995).

[7] D. Lee, Phys. Rev. D **49**, 1417 (1994).

[8] D. Chang, R. N. Mohapatra, and M. K. Parida, Phys. Rev. D **30**, 1052 (1984).

[9] X.-G. He and S. Meljanac, Phys. Rev. D **41**, 1620 (1990).

[10] F. Buccella, J.-P. Derendinger, and C. A. Savoy, in *Unification of the Fundamental Particle Interaction II*, edited by J. Ellis and S. Ferrara (Plenum, New York, 1983).

[11] R. Slansky, Phys. Rep. **79**, 1 (1981).

[12] T. Kugo and J. Sato, Prog. Theor. Phys. **91**, 1217 (1994); H. Georgi, *Lie Algebras in Particle Physics* (Addison-Wesley, Redwood City, 1982).