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Non-Perturbative Approach to the Effective Potential of the $\lambda\phi^4$ Theory at Finite Temperature

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We construct a non-perturbative method to calculate the effective potential of the $\lambda\phi^4$ theory at finite temperature. We express the derivative of the effective potential with respect to the mass squared in terms of the full propagator. We reduce this equation to a partial differential equation for the effective potential using an approximation. We numerically solve this equation and obtain the effective potential non-perturbatively. We find that the phase transition is second order, as it should be. We determine several critical exponents.

§1. Introduction

It is often expected that broken symmetries are restored at high temperature.[1] The temperature-induced phase transition should be observed in relativistic heavy ion collisions, the interior of neutron stars, and the early stage of the universe. We may study new physics through phase transitions at high temperature.

It is, however, very difficult to examine such phase transitions. For example, perturbation theory often breaks down at high temperature. As is well-known in finite temperature field theories, higher order contributions of the loop expansion are enhanced for Bose fields by many interactions in the thermal bath.[2),3)] In the $\lambda\phi^4$ theory, physical quantities are expanded in terms of $\lambda T^2/m^2$ and $\lambda T/m$ at finite temperature. The ordinary loop expansion is improved by resumming the daisy diagram, which includes all the higher order contributions of $\mathcal{O}\left(\left(\lambda T^2/m^2\right)^n\right)$.[4)-13)] The loop expansion parameter is $\lambda T/m$ after the resummation. This implies that the perturbation theory breaks down at $T \gtrsim m/\lambda$.[9)] Around the critical temperature, the ratio m/T is always of $\mathcal{O}(\lambda)$, so a non-perturbative analysis is necessary to study the phase transition in $\lambda\phi^4$ theory.[5)]

A variety of methods are used to investigate the phase transition, for example, lattice simulations,[14)-19)] the C.J.T. method,[20),21)] ε-expansion,[22)] effective three dimensional theory,[23)-27)] the gap equation method,[28)] non-perturbative renormalization group methods,[29)-33)] and so on. Dispite these numerous methods, we need additional methods to study the phase transition, since those which exist at present are applicable to only limited situations.

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In Ref. 34) a new non-perturbative approach was suggested to avoid the infrared divergence which appears in the pressure.[35] The authors of that work differentiated the generating functional with respect to the mass squared and found an infrared finite expression for the pressure in thermal equilibrium.

In the present paper we employ the idea of Ref. 34) and develop a new method to calculate the effective potential. Differentiating the effective potential with respect to the mass squared, we express the derivative in terms of the full propagator. We construct a partial differential equation for the effective potential by approximating the full propagator. We calculate the effective potential beyond perturbation theory by solving this equation.

In §2 we consider the $\lambda\phi^4$ theory at finite temperature and find an exact expression for the derivative of the effective potential $\partial V/\partial m^2$. We approximate this expression and obtain a partial differential equation for the effective potential. We determine reasonable initial conditions to solve this equation. In §3 we solve this equation and obtain an effective potential numerically. We obtain the susceptibility, field expectation value, and specific heat from this potential. We determine several critical exponents by observing the T dependence of several quantities. Section 4 is devoted to the concluding remarks.

§2. Evolution equation for the effective potential

As mentioned in §1, the loop expansion loses its validity at high temperature. We thus need a non-perturbative method to calculate the effective potential. The effective potential, in general, satisfies the relation

$$V(m^2) = \int_{M^2}^{m^2} \left(\frac{\partial V}{\partial m^2}\right) dm^2 + V(M^2). \tag{1}$$

Once we know $\partial V/\partial m^2$ and $V(M^2)$, we can calculate the effective potential for arbitrary m^2. Following this idea, we construct an evolution equation for the effective potential of the $\lambda\phi^4$ theory at finite temperature. In the following we determine $\partial V/\partial m^2$ and an appropriate initial condition $V(M^2)$.

Following the standard procedure of dealing with the Matsubara Green function,[36] we introduce a temperature into the theory. The generating functional at finite temperature is given by

$$Z_T = \int D[\phi] \exp\left(\int_0^{1/T} d\tau \int d^3\boldsymbol{x} \mathcal{L}_{\mathrm{E}}\right). \tag{2}$$

Here $\mathcal{L}_{\mathrm{E}}$ is Euclidean Lagrangian density. We consider the $\lambda\phi^4$ theory which is defined by the Lagrangian density

$$\mathcal{L}_{\mathrm{E}} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \mathcal{L}_{\mathrm{ct}} + J\phi, \tag{3}$$

where $\mathcal{L}_{\mathrm{ct}}$ represents the counter term, and J is an external source function. If m^2 is negative, the scalar field ϕ develops a non-vanishing field expectation value at

$T=0$. It is expected that the field expectation value decreases as T increases and that a phase transition takes place at the critical temperature T_c. We can explore the properties of this phase transition by studying the effective potential at finite temperature.

In the $\lambda\phi^4$ theory, the derivative of the effective potential $\partial V/\partial m^2$ is expressed by the full propagator of the scalar field (see Appendix A),

$$\frac{\partial V}{\partial m^2}=\frac{\partial V_{\rm tree}}{\partial m^2}+\frac{\partial V_1}{\partial m^2}+\frac{\partial V_2}{\partial m^2}+\frac{\partial V_{\rm ct}}{\partial m^2}, \tag{4}$$

where $V_{\rm tree}$ is the tree part:

$$\frac{\partial V_{\rm tree}}{\partial m^2}\equiv\frac{1}{2}\bar{\phi}^2. \tag{5}$$

Here $\bar{\phi}$ is the vacuum expectation value of ϕ with the external source J. The non-perturbative effects are contained in V_1 and V_2:

$$\frac{\partial V_1}{\partial m^2}\equiv\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dp_0\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\frac{1}{e^{p_0/T}-1}, \tag{6}$$

$$\frac{\partial V_2}{\partial m^2}\equiv\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty}dp_0\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}. \tag{7}$$

The quantity $V_{\rm ct}$ in Eq. (4) is the counter term part,

$$\begin{aligned}\frac{\partial V_{\rm ct}}{\partial m^2}\equiv(Z_mZ_\phi-1)\Bigg[&\frac{1}{2}\bar{\phi}^2\\
&+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dp_0\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\frac{1}{e^{p_0/T}-1}\\
&+\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty}dp_0\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\Bigg].\end{aligned} \tag{8}$$

Here, $\Pi=\Pi(\boldsymbol{p}^2,-p_0^2,\bar{\phi},m^2,T)$ describes the full self-energy, and Z_m and Z_ϕ are renormalization constants defined in the Appendix (Eq. (24)). The third term, $\partial V/\partial m^2$, on the right-hand side of Eq. (4) is divergent. This divergence is removed by the counter term (8) after the usual renormalization procedure is adopted at $T=0$.*) The counter term which is determined at $T=0$ removes the ultra-violet divergence even at finite temperature.[7),12),37)]

We give the initial condition at $M^2\sim\mathcal{O}(T^2)$, where the loop expansion is valid, ($\lambda T/M\sim\lambda\ll1$). We calculate $V(M^2)$ within perturbation theory up to one-loop order. After renormalization with the $\overline{MS}$ scheme at the renormalization scale $\bar{\mu}$, the one-loop effective potential becomes

$$V(M^2)=V_{\rm tree}(M^2)+V_1(M^2)+V_2(M^2)+V_{\rm ct}(M^2), \tag{9}$$

*) In the perturbation theory, we expand Π, Z_m and Z_ϕ with respect to the loop number $\hbar$: $\Pi=\pi^{(1)}\hbar+\pi^{(2)}\hbar^2+\cdots,\quad Z_m=1+z_m^{(1)}\hbar+z_m^{(2)}\hbar^2+\cdots,\quad Z_\phi=1+z_\phi^{(1)}\hbar+z_\phi^{(2)}\hbar^2+\cdots$. We determine Z_m and Z_ϕ to subtract the divergences at each order. At leading order, the divergence on the right-hand side of Eq. (7) with $\Pi=0$ is canceled by $\frac{1}{2}\,z_m^{(1)}z_\phi^{(1)}\,\bar{\phi}^2$.

where $V_{\rm tree}$, V_1 and $V_2 + V_{\rm ct}$ are given by

$$V_{\rm tree}(M^2) = \frac{1}{2}M^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4, \tag{10}$$

$$V_1(M^2) = \frac{T}{2\pi^2}\int_0^\infty dr r^2 \log\left[1 - \exp\left(-\frac{1}{T}\sqrt{r^2 + M^2 + \frac{\lambda}{2}\bar{\phi}^2}\right)\right], \tag{11}$$

$$V_2(M^2) + V_{\rm ct}(M^2) = \frac{\left(M^2 + \frac{\lambda}{2}\bar{\phi}^2\right)^2}{64\pi^2}\left[\log\left(\frac{M^2 + \frac{\lambda}{2}\bar{\phi}^2}{\bar{\mu}^2}\right) - \frac{3}{2}\right]. \tag{12}$$

Note that we need not re-sum the daisy diagram, which has only a negligible contribution of $\mathcal{O}(\lambda)$ for $M^2 \sim \mathcal{O}(T^2)$. In Eq. (12), $V_2(M^2)$ is actually divergent, and $V_{\rm ct}(M^2)$ subtract the divergence.

In order to investigate the temperature-induced phase transition, we consider the theory with non-vanishing field expectation value at $T = 0$ (i.e., m^2 takes a negative value, $m^2 = -\mu^2$). We calculate $V(-\mu^2)$ with the effective potential (9) by

$$V(-\mu^2) = \int_{M^2}^{-\mu^2}\left(\frac{\partial V_{\rm tree}}{\partial m^2} + \frac{\partial V_1}{\partial m^2} + \frac{\partial V_2}{\partial m^2} + \frac{\partial V_{\rm ct}}{\partial m^2}\right) dm^2 + V_{\rm tree}(M^2) + V_1(M^2) + V_2(M^2) + V_{\rm ct}(M^2). \tag{13}$$

For $m^2 \ll T^2$ the contribution from $\partial V_1/\partial m^2$ is enhanced by the Bose factor. The contribution from V_1 can be the same order as that from the tree part around the critical temperature.

The quantity $V_2 + V_{\rm ct}$ will have a negligible contribution:

$$\int_{M^2}^{-\mu^2}\left(\frac{\partial V_2}{\partial m^2} + \frac{\partial V_{\rm ct}}{\partial m^2}\right) dm^2 + V_2(M^2) + V_{\rm ct}(M^2) = V_2(-\mu^2) + V_{\rm ct}(-\mu^2). \tag{14}$$

We can show that $V_2(-\mu^2) + V_{\rm ct}(-\mu^2)$ is actually small at the leading order of the loop expansion. At the one-loop level with daisy diagram resummation we find

$$V_2(-\mu^2) + V_{\rm ct}(-\mu^2) = \frac{\left(-\mu^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi\right)^2}{64\pi^2}\left[\log\left(\frac{-\mu^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi}{\bar{\mu}^2}\right) - \frac{3}{2}\right]. \tag{15}$$

The self-energy satisfies $\Pi \sim \mu^2$ around the critical temperature for a second-order or weakly first-order phase transition. [5] Because we are interested in the effective potential in the region of small $\bar{\phi}$ only to investigate the phase structure, we neglect (14) in the following calculations.

Because the effective potential is a generating function of n-point functions with zero external momentum, neglect of momentum dependence in Π allows us to replace as follows in Eq. (4):

$$m^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi(0, 0, \bar{\phi}, m^2, T) \to \frac{\partial^2 V}{\partial\bar{\phi}^2}. \tag{16}$$

We can take into account the super-daisy diagrams correctly in this approximation because they are produced by momentum-independent parts of Π.*) We obtain a partial differential equation for the effective potential by integrating over p_0 and the angle variables in Eq. (4). We obtain

$$\frac{\partial V}{\partial m^2}=\frac{1}{2}\bar{\phi}^2+\frac{1}{4\pi^2}\int_0^\infty drr^2\frac{1}{\sqrt{r^2+\frac{\partial^2 V}{\partial\bar{\phi}^2}}}\frac{1}{\exp\left(\frac{1}{T}\sqrt{r^2+\frac{\partial^2 V}{\partial\bar{\phi}^2}}\right)-1}. \tag{17}$$

§3. Numerical results

We calculated the effective potential by solving the partial differential equation (17) with the initial condition for $V_{\text{tree}}+V_1$ in Eq. (9). We solved the equation numerically and thus determined the phase structure of the $\lambda\phi^4$ theory.

3.1. *Analytic continuation*

The integral in (17) is well defined in the region where $\partial^2 V/\partial\bar{\phi}^2$ is real and positive. The effective potential $V(\bar{\phi})$ is, however, complex for small $\bar{\phi}$ below the critical temperature, $T<T_c$. We must determine the analytic continuation in order to calculate the effective potential in that region.

To make the analytic continuation, we change the variable of integration r to z through the identification

$$z=\sqrt{\frac{r^2}{T^2}+Z^2}-Z, \tag{18}$$

and rewrite the differential equation (17) as

$$\frac{\partial V}{\partial m^2}=\frac{1}{2}\bar{\phi}^2+\frac{T^2}{4\pi^2}\int_0^\infty dz\frac{\sqrt{z(z+2Z)}}{e^{z+Z}-1}. \tag{19}$$

Here Z is a double-valued function which is given by $Z=\sqrt{(1/T^2)(\partial^2 V/\partial\bar{\phi}^2)}$.

The imaginary part of the effective potential is interpreted as the decay rate of the unstable state.[39] It is natural that we assume such an imaginary part to be negative. The imaginary part of $\partial V/\partial m^2$ should be positive, in order to make the imaginary part of the effective potential negative. We must select the branch of $Z=\sqrt{(1/T^2)(\partial^2 V/\partial\bar{\phi}^2)}$ for which the imaginary part of $\partial V/\partial m^2$ is positive. We calculate the effective potential in this branch, and its imaginary part is always negative, as we will see in the next subsection.

3.2. *Numerical results*

Fixing the initial condition for $V_{\text{tree}}+V_1$ in Eq. (9) by setting $M^2=T^2$, we numerically solved Eq. (19) and obtained the effective potential at $m^2=-\mu^2$. We use the explicit differencing method.[40] In this subsection we give the effective potential and calculate critical exponents.

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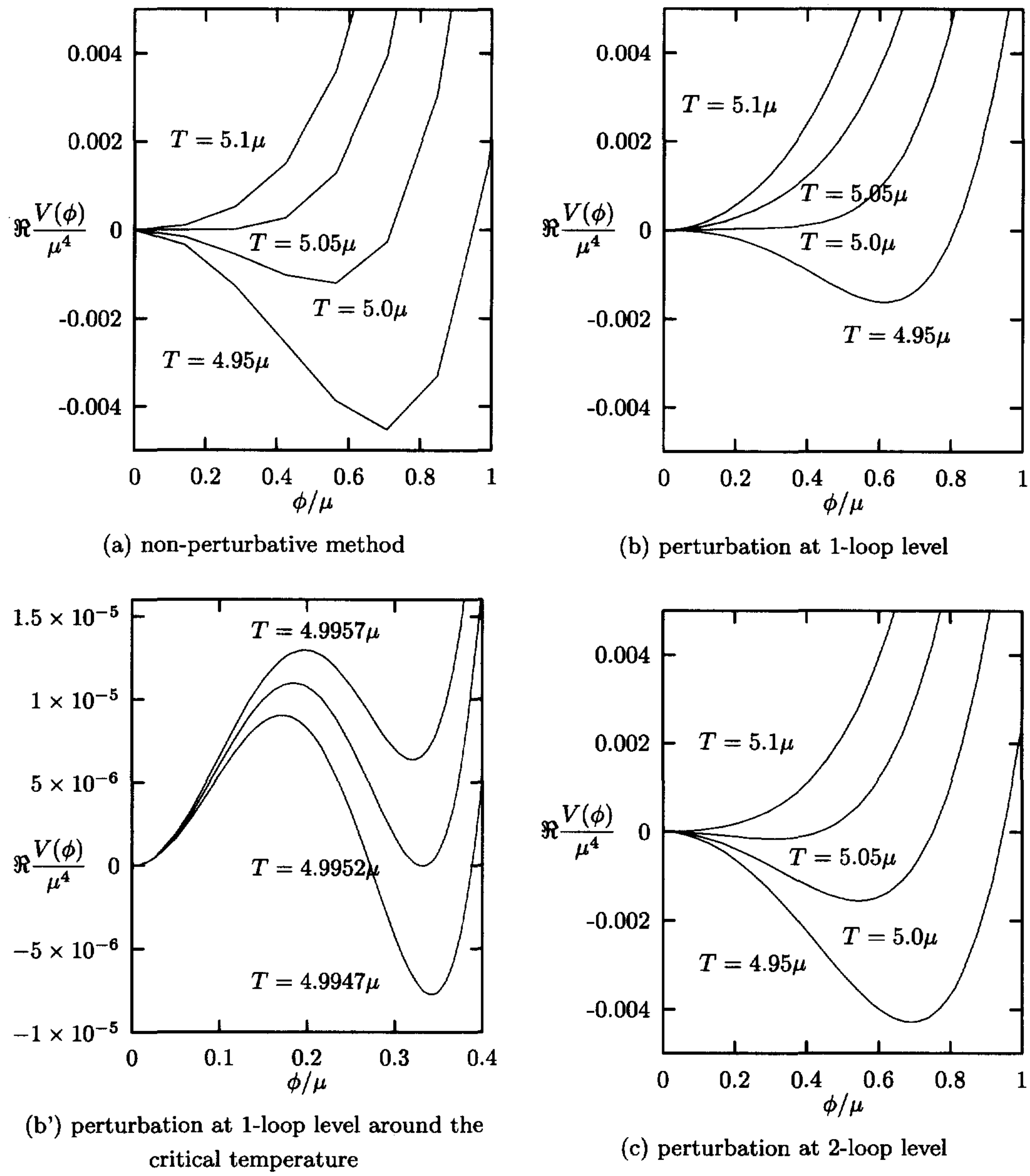
*) This is the first approximation of a systematic calculation.[38]



(a) non-perturbative method

(b) perturbation at 1-loop level

(b') perturbation at 1-loop level around the critical temperature

(c) perturbation at 2-loop level

Fig. 1. The behaviour of the effective potential V is shown for fixed $\lambda(=1)$ as the function of the temperature. We find no qualitative change for other values $\lambda(=0.5, 0.1, 0.05)$. We normalize that $V(0) = 0$. Perturbative results (b), (b') and (c) are calculated with $\overline{MS}$ scheme.

We illustrate the behaviour of the effective potential at $\lambda = 1$ in Fig. 1(a). The field expectation value ϕ_c is the minimum point of the effective potential. It seems to disappear smoothly at the critical temperature. We thus find that the phase transition is second order, as it should be.

For comparison, in Figs. 1(b), (b') and (c) we display the effective potential calculated using the perturbation theory at one- and two-loop orders with daisy

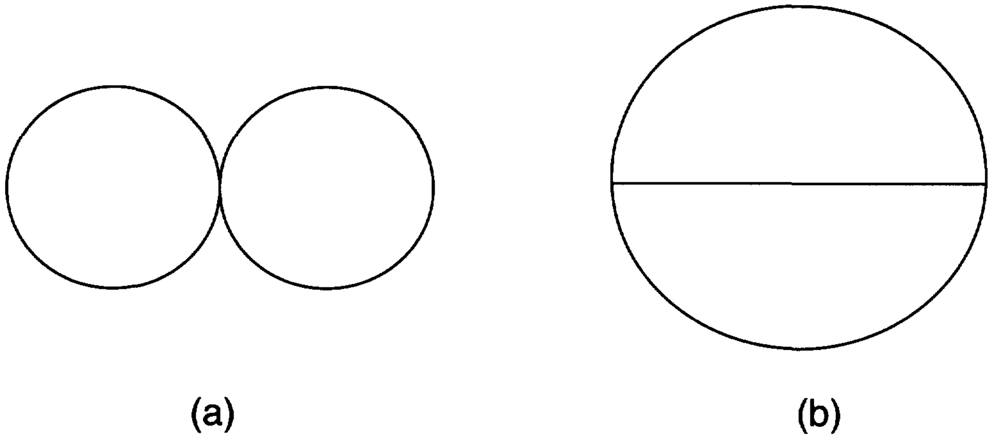


Fig. 2. Two loop diagrams that contribute to the effective potential.

diagram resummation.*) At one loop order, an extremely small gap appears at the critical temperature, as is clearly seen in Fig. 1(b'). The phase transition is thus seen to be first order at one loop order.

This situation is modified at two loop order. Here, we observe no gap, and find that the phase transition is second order, as shown in Fig. 1(c). Though Figs. 1(a) and (c) show similar behaviour, this is purely accidental. The effective potential calculated up to two loop order includes contributions from the graphs shown in Figs. 2(a) and (b), with daisy resummation. On the other hand, we can automatically take into account the contributions from all other graphs within the approximation (16) by solving Eq. (19). Figure 1(a) accidentally coincides with Fig. 1(c).

For $T < T_c$ the effective potential develops a non-vanishing imaginary part for small $\bar{\phi}$. We show this in Fig. 3. It should be noted that the sign of the imaginary part is always negative. This is consistent with the discussion in the previous subsection.

Evaluating the effective potential by varying the temperature T and the coupling constant λ, we obtain the critical temperature as a function of λ, where the field expectation value disappears. We display the phase boundary in the T-λ plane in Fig. 4.

The critical exponents are defined for a second-order phase transition. Around the critical temperature we expect that the susceptibility $\chi \equiv \frac{\partial\bar{\phi}}{\partial J}\Big|_{J=0} = (\frac{\partial^2 V}{\partial\bar{\phi}^2})^{-1}|_{\bar{\phi}=\phi_c}$, the expectation value ϕ_c, and the specific heat $C \equiv \frac{\partial^2 V}{\partial t^2}|_{\bar{\phi}=\phi_c}$ behave as[41]

$$\chi \propto |t|^{-\gamma},\ \phi_c \propto |t|^{+\beta},\ C \propto |t|^{-\alpha}, \tag{20}$$

where $t = (T - T_c)/T$. Analysing the effective potential more precisely we can calculate the critical exponents γ, β and α.

*) We have used the equations in Ref. 5) to construct them.

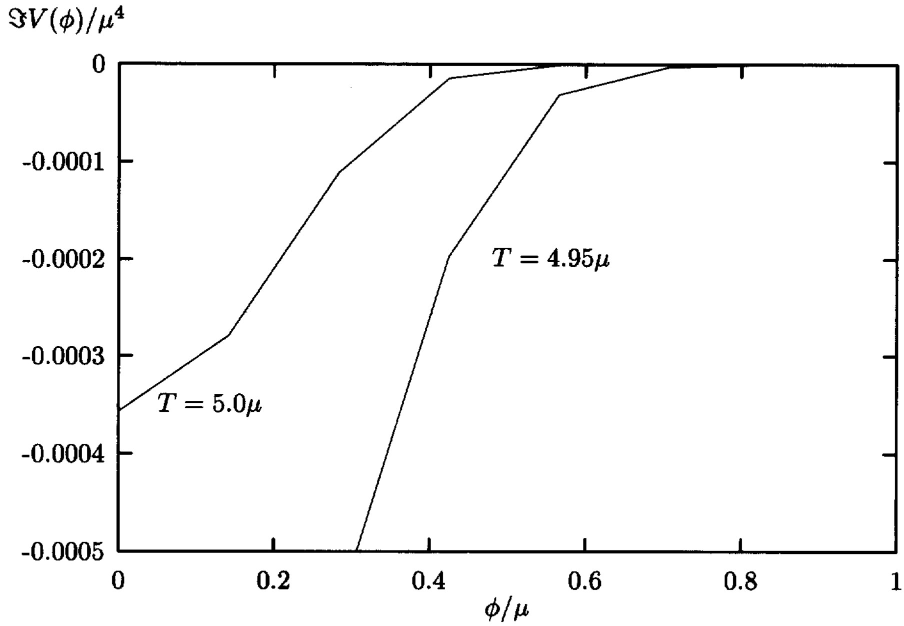


Fig. 3. Imaginary part of the effective potential near the critical temperature.

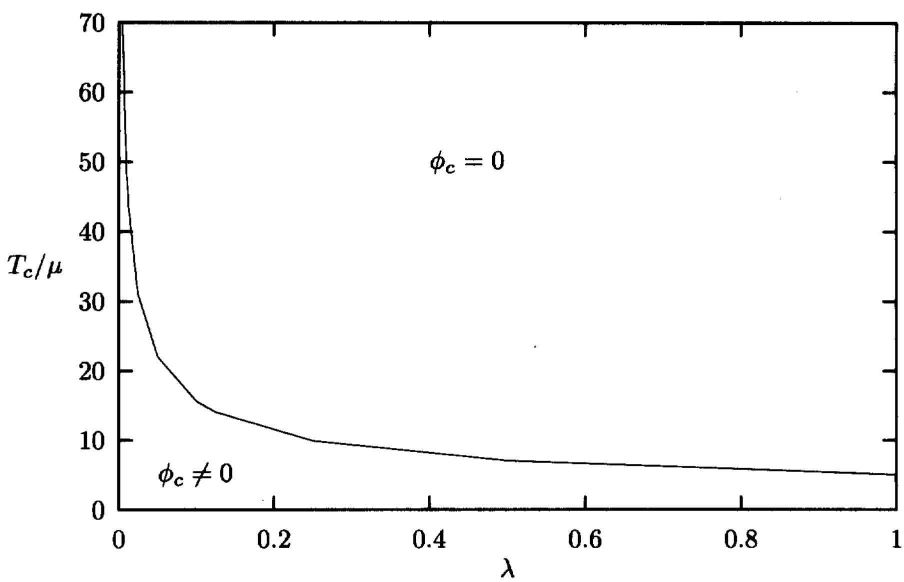


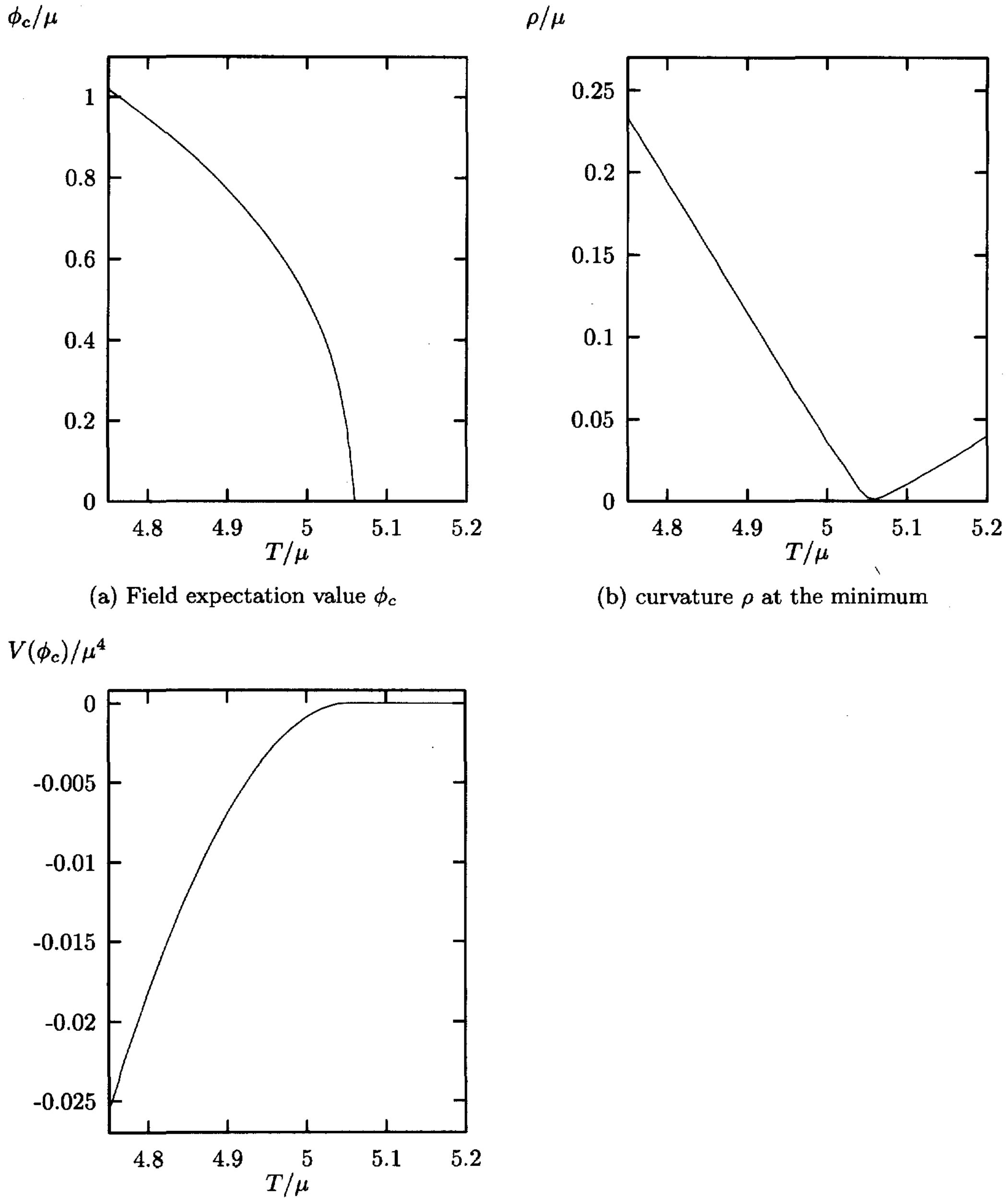
Fig. 4. Phase boundary.

The susceptibility χ satisfies the relation

$$\chi \propto \rho^{-1}, \tag{21}$$

where ρ is the curvature of the effective potential at ϕ_c.

Since the specific heat C is given by the second derivative of the effective poten-



(a) Field expectation value ϕ_c

(b) curvature ρ at the minimum

(c) Minimum of the effective potential $V(\phi_c)$

Fig. 5. Critical behaviour of ϕ_c, ρ and $V(\phi_c)$.

tial around the critical temperature, the effective potential $V(\phi_c)$ behaves as

$$V(\phi_c) \propto |t|^{2-\alpha}. \tag{22}$$

We examined the behaviour of ρ, ϕ_c and $V(\phi_c)$ around the critical temperature and determined the critical exponents γ, β and α numerically. In Fig. 5 the critical behaviour of each ρ, ϕ_c and $V(\phi_c)$ is shown as a function of the temperature.We

Table I. Critical exponents.

	our results	Landau theory	experimental results[42]
β	~ 0.5	0.5	0.33
γ	~ 1	1	1.24
α	~ 0	0	0.11

numerically calculated the critical exponents from these functions. Our numerical results are presented in Table I.*) The critical exponents within our approximation are independent of the coupling constant λ. We note that the results described in the present subsection remain unchanged when the initial mass scale is set to $M^2 = T^2/4$ or $M^2 = 4T^2$.

§4. Conclusion

We constructed a non-perturbative method to investigate the phase structure of the $\lambda\phi^4$ theory. An exact expression for the derivative of the effective potential with respect to mass squared was found in terms of the full propagator at finite temperature. We found a partial differential equation for the effective potential with the replacement (16). We determined suitable initial conditions using the one-loop effective potential in the range where the perturbation theory is reliable. We numerically solved the partial differential equation and thereby obtained an effective potential.

Though we made the approximation (16), we were able to determine that the phase transition of the $\lambda\phi^4$ theory is second order, as it should be. Our method is very interesting because it correctly determined the order of the phase transition. We believe that the approximation (16) may be fairy good.

We determined several critical exponents which roughly agree with those found using the Landau approximation. They are, however, rough values because it is very difficult to solve the nonlinear partial differential equation (17) numerically. We need a more elaborate numerical study to obtain more accurate critical exponents.

The main problem we must address with regard to our non-perturbative method is determining how to improve the approximation of the full propagator. We cannot estimate the error from the approximation (16). We need to improve the approximation of the full propagator in order to determine the corrections to the present result.

Our method is very promising since it can probe the region where the traditional perturbation theory breaks down.

*) Due to the instability of the explicit differencing method, we could not observe the fine structure of the effective potential, and we could only obtain rough values of the critical exponents. We need further numerical study to obtain more precise values.

Acknowledgements

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Appendix A
—— *The Derivative of the Effective Potential in Terms of the Full Propagator* ——

The derivative of the effective potential, $\partial V/\partial m^2$, can be represented by the full propagator. In this appendix we present details of the calculation of $\partial V/\partial m^2$ given in Eq. (4).

We consider the Lagrangian density,

$$\mathcal{L}_{\mathrm{E}} = -\frac{1}{2}\left(\frac{\partial \phi_0}{\partial \tau}\right)^2 - \frac{1}{2}(\nabla\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 + J_0\phi_0, \tag{23}$$

where the suffix 0 denotes the bare quantities.

We adopt a mass-independent renormalization procedure and represent the effective potential as a function of renormalized quantities. The renormalization constants Z and renormalized quantities are introduced through the transformations

$$\begin{aligned} \phi_0 &= Z_\phi^{1/2}\phi, \\ m_0 &= Z_m^{1/2} m, \\ \lambda_0 &= Z_\lambda \lambda, \\ J_0 &= Z_\phi^{-1/2} J. \end{aligned} \tag{24}$$

Using these renormalization constants and renormalized quantities, we separate the Lagrangian density (23) into the tree part $\mathcal{L}_1$ and the counter term part $\mathcal{L}_{\mathrm{ct}}$ as[41),43)]

$$\mathcal{L}_{\mathrm{E}} = \mathcal{L}_1[\phi] + \mathcal{L}_{\mathrm{ct}}[\phi] + (J_1 + J_{\mathrm{ct}})\phi, \tag{25}$$

where $J_1 + J_{\mathrm{ct}} \equiv J$. The Lagrangian densities $\mathcal{L}_1$ and $\mathcal{L}_{\mathrm{ct}}$ are given by

$$\left\{\begin{aligned} \mathcal{L}_1[\phi] &\equiv -\frac{1}{2}\left(\frac{\partial \phi}{\partial \tau}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \\ \mathcal{L}_{\mathrm{ct}}[\phi] &\equiv -\frac{1}{2}(Z_\phi - 1)\left[\left(\frac{\partial \phi}{\partial \tau}\right)^2 + (\nabla\phi)^2\right] - \frac{1}{2}(Z_m Z_\phi - 1)m^2\phi^2 \\ &\quad -\frac{\lambda}{4!}(Z_\lambda Z_\phi^2 - 1)\phi^4. \end{aligned}\right. \tag{26}$$

Here we separate the external source J into J_1 and J_{ct}. These satisfy the following equations:

$$\left\{\begin{aligned} &\left.\frac{\partial \mathcal{L}_1}{\partial \phi}\right|_{\phi=\bar{\phi}} + J_1 = 0, \\ &\qquad \langle\phi\rangle_J = \bar{\phi} = \mathrm{const.} \end{aligned}\right. \tag{27}$$

Here $\langle\phi\rangle_J$ is the field expectation value with the existence of J. We expand the field $\phi(x)$ around the classical background $\bar{\phi}$ as

$$\phi(x) = \bar{\phi} + \eta(x), \tag{28}$$

and then $\eta(x)$ satisfies

$$\langle\eta\rangle_J = 0. \tag{29}$$

In terms of $\bar{\phi}$ and $\eta(x)$, Eq. (26) can be rewritten as

$$\begin{aligned}
\mathcal{L}_1 + J_1\phi = &-\frac{1}{2}m^2\bar{\phi}^2 - \frac{\lambda}{4!}\bar{\phi}^4 + J_1\bar{\phi} - \frac{1}{2}\left[\left(\frac{\partial\eta}{\partial\tau}\right)^2 + (\nabla\eta)^2\right] \\
&-\frac{1}{2}\left(m^2 + \frac{\lambda}{2}\bar{\phi}^2\right)\eta^2 - \frac{\lambda}{3!}\bar{\phi}\eta^3 - \frac{\lambda}{4!}\eta^4 \\
\equiv\ &\mathcal{L}_1(\bar{\phi}) + \mathcal{L}_1'[\eta] + J_1\bar{\phi},
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
\mathcal{L}_{\rm ct} + J_{\rm ct}\phi = &-\frac{1}{2}(Z_m Z_\phi - 1)m^2\bar{\phi}^2 - \frac{\lambda}{4!}(Z_\lambda Z_\phi^2 - 1)\bar{\phi}^4 + J_{ct}\bar{\phi} \\
&\left[-(Z_m Z_\phi - 1)m^2\bar{\phi} - \frac{\lambda}{3!}(Z_\lambda Z_\phi^2 - 1)\bar{\phi}^3 + J_{ct}\right]\eta \\
&-\frac{1}{2}(Z_\phi - 1)\left[\left(\frac{\partial\eta}{\partial\tau}\right)^2 + (\nabla\eta)^2\right] - \frac{1}{2}(Z_\phi Z_m - 1)m^2\eta^2 \\
&-\frac{\lambda}{4}(Z_\lambda Z_\phi^2 - 1)\bar{\phi}^2\eta^2 - \frac{\lambda}{3!}(Z_\lambda Z_\phi^2 - 1)\bar{\phi}\eta^3 - \frac{\lambda}{4!}(Z_\lambda Z_\phi^2 - 1)\eta^4 \\
\equiv\ &\mathcal{L}_{\rm ct}(\bar{\phi}) + K\eta + \mathcal{L}_{\rm ct}'[\eta] + J_{\rm ct}\bar{\phi},
\end{aligned} \tag{31}$$

where K is defined by

$$K \equiv -(Z_m Z_\phi - 1)m^2\bar{\phi} - \frac{\lambda}{3!}(Z_\lambda Z_\phi^2 - 1)\bar{\phi}^3 + J_{\rm ct}. \tag{32}$$

Using Z_T (Eq. (2)), the generating functional $W_T[J]$ for connected Green functions is given by

$$W_T[J] = \log Z_T[J]. \tag{33}$$

The effective action $\Gamma_T(\bar{\phi})$ is defined as the Legendre transformation of $W_T[J]$. In a spacetime with translational invariance, the effective potential $V(\bar{\phi})$ is proportional to the effective action. The effective potential $V(\bar{\phi})$ is

$$-\frac{\Omega}{T}V(\bar{\phi}) = \Gamma_T(\bar{\phi}) = W_T[J] - \frac{\Omega}{T}\bar{\phi}J, \tag{34}$$

where $\Omega = \int d^3\boldsymbol{x}$, and the new variable $\bar{\phi}$ is given by

$$\frac{\delta}{\delta J(y)}W_T[J] = \bar{\phi}(y) = \bar{\phi} = \text{const.} \tag{35}$$

Substituting Eqs. (30) and (31) into Eq. (25), we rewrite the generating functional Z_T as a functional of renormalized quantities:

$$\begin{aligned} Z_T &= \exp\left\{\frac{\Omega}{T}\left[\mathcal{L}_1(\bar{\phi}) + \mathcal{L}_{\text{ct}}(\bar{\phi}) + (J_1 + J_{\text{ct}})\bar{\phi}\right]\right\} \\ &\quad \times \int D[\eta] \exp \int_0^{1/T} d\tau \int d^3\boldsymbol{x}\, \left(\mathcal{L}_1'[\eta] + \mathcal{L}_{\text{ct}}'[\eta] + K\eta\right) \\ &\equiv \exp\left\{\frac{\Omega}{T}\left[\mathcal{L}_1(\bar{\phi}) + \mathcal{L}_{\text{ct}}(\bar{\phi}) + (J_1 + J_{\text{ct}})\bar{\phi}\right]\right\} Z'(K). \end{aligned} \tag{36}$$

Taking into account Eqs. (33) and (34), we obtain the effective potential from Eq. (36):

$$V(\bar{\phi}) = -\left[\mathcal{L}_1(\bar{\phi}) + \mathcal{L}_{\text{ct}}(\bar{\phi})\right] - \frac{T}{\Omega}\log Z'(K). \tag{37}$$

In the mass-independent renormalization procedure, the renormalization constants Z are independent of the mass m. We easily differentiate the effective potential V by the mass square m^2 and obtain

$$\begin{aligned} \frac{\partial V(\bar{\phi})}{\partial m^2} &= \left[\frac{1}{2}\bar{\phi}^2 + \frac{1}{2}(Z_m Z_\phi - 1)\bar{\phi}^2\right] \\ &\quad - \frac{T}{\Omega}\frac{1}{Z'}\int D[\eta]\left[-\frac{1}{2}\eta^2 + \frac{\partial K}{\partial m^2}\eta - \frac{1}{2}(Z_m Z_\phi - 1)\eta^2\right] \\ &\quad \times \exp\int_0^{1/T} d\tau \int d^3\boldsymbol{x}\,\left(\mathcal{L}_1'[\eta] + \mathcal{L}_{\text{ct}}'[\eta] + K\eta\right) \\ &= \left[\frac{1}{2}\bar{\phi}^2 + \frac{1}{2}(Z_m Z_\phi - 1)\bar{\phi}^2 + \frac{1}{2}\langle\eta^2(0)\rangle + \frac{1}{2}(Z_m Z_\phi - 1)\langle\eta^2(0)\rangle\right]. \end{aligned} \tag{38}$$

For $\lambda\phi^4$ theory, the two-point function $\langle\eta^2(0)\rangle$ in Eq. (38) is rewritten as

$$\begin{aligned} \langle\eta^2(0)\rangle &= \int_0^{1/T} d\tau \int d^3\boldsymbol{x}\,\delta^3(\boldsymbol{x})\delta_T(\tau)\langle\eta(\boldsymbol{x},\tau)\eta(\mathbf{o},0)\rangle \\ &= T\sum_{n=-\infty}^{\infty}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\int_0^{1/T} d\tau\int d^3\boldsymbol{x}\, e^{i(\boldsymbol{x}\boldsymbol{p}+\omega_n\tau)}\langle\eta(\boldsymbol{x},\tau)\eta(\mathbf{o},0)\rangle \\ &= T\sum_{n=-\infty}^{\infty}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\omega_n^2 + \boldsymbol{p}^2 + m^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi}, \end{aligned} \tag{39}$$

where $\omega_n = 2\pi n T = i p_0$ due to the periodic boundary condition for Bose fields, and $\Pi = \Pi(\boldsymbol{p}^2, -p_0^2, \bar{\phi}, m^2, T)$ is the full self-energy. Substituting Eq. (39) into Eq. (38), we express the derivative of the effective potential in terms of the full propagator,

$$\begin{aligned} \frac{\partial V}{\partial m^2} &= \frac{1}{2}\bar{\phi}^2 + \frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\omega_n^2 + \boldsymbol{p}^2 + m^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi} \\ &\quad + (Z_m Z_\phi - 1)\left[\frac{1}{2}\bar{\phi}^2 + \frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\omega_n^2 + \boldsymbol{p}^2 + m^2 + \frac{\lambda}{2}\bar{\phi}^2 + \Pi}\right]. \end{aligned} \tag{40}$$

Using the residue theorem, we convert the frequency sum $T\sum_{n=-\infty}^{\infty}$ to contour integrals. As long as Π has no singularity along the imaginary p_0 axis, Eq. (40) naturally separates into a piece which contains a Bose factor and a piece which does not: [12), 37)]

$$\begin{aligned}\frac{\partial V}{\partial m^2} = &\frac{1}{2}\bar{\phi}^2 + \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\frac{1}{e^{p_0/T}-1} \\ &+\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi} \\ &+(Z_m Z_\phi - 1)\left[\frac{1}{2}\bar{\phi}^2\right. \\ &+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\frac{1}{e^{p_0/T}-1} \\ &\left.+\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} dp_0 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\bar{\phi}^2+\Pi}\right].\end{aligned} \tag{41}$$

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