THREE FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS WITH CONSTANTS IN COMPLETE METRIC SPACES

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Abstract. We prove three fixed point theorems for generalized contractions with constants in complete metric spaces, which are generalizations of very recent fixed point theorems due to Suzuki. We also raise one problem concerning the constants.

1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all natural numbers. For an arbitrary set A, we also denote by $\sharp A$ the number of elements of A.

The Banach contraction principle [1] plays a very important role in nonlinear analysis and has many generalizations; see [2–4, 6–8, 10–15, 17–20] and others. We also know that the principle cannot characterize the metric completeness of underlying spaces while Kannan's fixed point theorem does; see [5, 9, 16]. Very recently, Suzuki [19] proved the following fixed point theorem, which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

Theorem 1 ([19]). *Define a nonincreasing function θ from $[0,1)$ onto $(\frac{1}{2},1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}(\sqrt{5}-1), \\ \dfrac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1) \le r \le \frac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{1}$$

Then for a metric space (X,d), the following are equivalent:

(i) *X is complete.*

(ii) *Every mapping T on X satisfying the following has a fixed point:*
- *There exists $r \in [0,1)$ such that $\theta(r)\, d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le r\, d(x,y)$ for all $x,y \in X$.*

(iii) *There exists $r \in (0,1)$ such that every mapping T on X satisfying the following has a fixed point:*
- *$\frac{1}{10000}\, d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le r\, d(x,y)$ for all $x,y \in X$.*

Remark. For every $r \in [0,1)$, $\theta(r)$ is the best constant.

2000 *Mathematics Subject Classification.* Primary 54H25, Secondary 54C60, 54E50.

Key words and phrases. contraction, fixed point, the Banach contraction principle, Nadler's fixed point theorem.

The second author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

It is a very natural and significant question of whether or not we can generalize Theorem 1 as we have generalized the Banach contraction principle. In this paper, we shall extend Theorem 1 in two directions — set-valued mappings and commuting mappings. We also discuss the Meir-Keeler fixed point theorem. We remark that the proofs of our results are not obvious.

The authors are very attracted by $\theta(r)$ because $\theta(r)$ does not seem to be natural. We know $\theta(r)$ is the best constant because of the existence of counterexamples. To find an intuitive reason is another motivation. However, we have not found such a reason yet. On the contrary, we have to raise one problem concerning $\theta(r)$.

2. Two Generalizations

In this section, we prove two generalizations of Theorem 1.

Let (X, d) be a metric space. We denote by $\mathrm{CB}(X)$ the family of all nonempty closed bounded subsets of X. Let $H(\cdot,\cdot)$ be the *Hausdorff metric*, i.e.,

$$H(A, B) = \max\left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A) \right\} \quad \text{for } A, B \in \mathrm{CB}(X),$$

where $d(x, B) = \inf_{y\in B} d(x, y)$.

We first prove the following, which is a generalization of both Theorem 1 and the Nadler fixed point theorem (Corollary 1).

Theorem 2. *Define a strictly decreasing function η from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by*

$$\eta(r) = \frac{1}{1 + r}.$$

Let (X, d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$. Assume that there exists $r \in [0, 1)$ such that

$$\eta(r)\, d(x, Tx) \le d(x, y) \quad \textit{implies} \quad H(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof. Take a real number r_1 with $0 \le r < r_1 < 1$. Then, for each $u = u_0 \in X$ and $u_1 \in Tu$, we have $\eta(r)\, d(u, Tu) \le \eta(r)\, d(u, u_1) \le d(u, u_1)$. From the assumption,

$$d(u_1, Tu_1) \le H(Tu, Tu_1) \le r\, d(u, u_1)$$

holds. So, there exists $u_2 \in Tu_1$ such that $d(u_1, u_2) \le r_1\, d(u, u_1)$. Thus, we have a sequence $\{u_n\}$ in X such that $u_n \in Tu_{n-1}$ and $d(u_{n-1}, u_n) \le r_1\, d(u_{n-2}, u_{n-1})$. We have

$$\sum_{n=1}^{\infty} d(u_{n-1}, u_n) \le \sum_{n=1}^{\infty} {r_1}^{n-1}\, d(u, u_1) < \infty$$

and hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges strongly to some point $z \in X$.

We next show

$$d(z, Tx) \le r\, d(z, x) \quad \text{for all } x \in X \setminus \{z\}.$$

Since $u_n \to z$, there exists $\nu \in \mathbb{N}$ such that $d(z, u_n) \le \frac{1}{3}\, d(z, x)$ for all $n \in \mathbb{N}$ with $n \ge \nu$. Then we have

$$\begin{aligned} \eta(r)\, d(u_n, Tu_n) &\le d(u_n, Tu_n) \le d(u_n, u_{n+1}) \\ &\le d(u_n, z) + d(u_{n+1}, z) \end{aligned}$$

$$\leq \frac{2}{3}\, d(x,z) = d(x,z) - \frac{1}{3}\, d(x,z)$$
$$\leq d(x,z) - d(u_n,z) \leq d(u_n,x)$$

and hence $H(Tu_n, Tx) \leq r\, d(u_n, x)$. So, it follows that $d(u_{n+1}, Tx) \leq r\, d(u_n, x)$ for $n \in \mathbb{N}$ with $n \geq \nu$. Letting n tend to ∞, we obtain $d(z, Tx) \leq r\, d(z, x)$ for all $x \in X \setminus \{z\}$.

We next prove that $H(Tx, Tz) \leq r\, d(x, z)$ for all $x \in X$. If $x = z$, then it obviously holds. So we assume $x \neq z$. Then for every $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that $d(z, y_n) \leq d(z, Tx) + \frac{1}{n}\, d(x, z)$. We have

$$\begin{aligned}
d(x,Tx) &\leq d(x, y_n) \\
&\leq d(x,z) + d(z,y_n) \\
&\leq d(x,z) + d(z,Tx) + \frac{1}{n}\, d(x,z) \\
&\leq d(x,z) + r\, d(x,z) + \frac{1}{n}\, d(x,z) \\
&= \Big(1 + r + \frac{1}{n}\Big)\, d(x,z)
\end{aligned}$$

for $n \in \mathbb{N}$ and hence $\frac{1}{1+r}\, d(x, Tx) \leq d(x, z)$. From the assumption, we have $H(Tx, Tz) \leq r\, d(x, z)$.

Finally, since

$$d(z, Tz) = \lim_{n\to\infty} d(u_{n+1}, Tz) \leq \lim_{n\to\infty} H(Tu_n, Tz) \leq \lim_{n\to\infty} r\, d(u_n, z) = 0$$

and Tz is closed, we obtain $z \in Tz$. □

Remark. For each $r \in \{0\} \cup \big[\frac{1}{\sqrt{2}}, 1\big)$, $\eta(r)$ is the best constant because $\theta(r) = \eta(r)$. However, we do not know that $\eta(r)$ is best for $r \in \big(0, \frac{1}{\sqrt{2}}\big)$. So we have to raise the following problem:

" Is $\eta(r)$ the best constant for $r \in \big(0, \frac{1}{\sqrt{2}}\big)$? "

When the second author was proving Theorem 1, he had guessed $\eta(r)$ is best from his intuition. However, his guess was false. If we could obtain an affirmative answer to our new problem, we could say that set-valued mappings are more natural than single-valued mappings in some sense.

It is obvious that the Nadler fixed point theorem follows directly from Theorem 2.

Corollary 1 (Nadler [13])**.** *Let (X, d) be a complete metric space and let T be a mapping from X into* $\mathrm{CB}(X)$*. If there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \leq r\, d(x, y) \quad \textit{for all}\; x, y \in X,$$

then there exists $z \in X$ such that $z \in Tz$.

Next, we generalize Theorem 1 as Jungck in [8] generalized the Banach contraction principle.

Theorem 3. *Define a function θ by* (1)*. Let (X, d) be a complete metric space. Let S and T be mappings on X satisfying the following*:

(a) *S is continuous*;
(b) *$T(X) \subset S(X)$*;

(c) *S and T commute.*

Suppose that there exists $r \in [0,1)$ such that

$$\theta(r)\, d(Sx,Tx) \le d(Sx,Sy) \quad \textit{implies} \quad d(Tx,Ty) \le r\, d(Sx,Sy)$$

for all $x, y \in X$. Then there exists a unique common fixed point of S and T.

Remark. $\theta(r)$ is the best constant for every r.

Proof. By (b), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$. Since $\theta(r) \le 1$, $\theta(r)\, d(Sx,Tx) = \theta(r)\, d(Sx,SIx) \le d(Sx,SIx)$ holds. From the assumption, we have

$$d(SIx,SIIx) = d(Tx,TIx) \le r\, d(Sx,SIx) \tag{2}$$

for $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = I^n u$ for all $n \in \mathbb{N}$. Then we note $u_{n+1} = Iu_n$ and $Su_{n+1} = Tu_n$ for $n \in \mathbb{N}$. By (2), we have

$$\begin{aligned} d(Su_n,Su_{n+1}) &= d(SIu_{n-1},SIIu_{n-1}) \le r\, d(Su_{n-1},SIu_{n-1}) \\ &= r\, d(Su_{n-1},Su_n) \le \cdots \le r^n\, d(Su_0,Su_1) \end{aligned}$$

and hence $\sum_{n=1}^{\infty} d(Su_n,Su_{n+1}) < \infty$. So, $\{Su_n\}$ is a Cauchy sequence in X and by the completeness of X, there exists a point $z \in X$ such that $Su_n \to z$.

We next show

$$d(Tx,z) \le r\, d(Sx,z) \quad \text{for all } x \in X \text{ with } Sx \ne z. \tag{3}$$

Since $Su_n \to z$, there exists $\nu_1 \in \mathbb{N}$ such that $d(Su_n,z) \le \frac{1}{3} d(Sx,z)$ for all $n \in \mathbb{N}$ with $n \ge \nu_1$. Then we have

$$\begin{aligned} \theta(r)\, d(Su_n,Tu_n) &\le d(Su_n,Tu_n) = d(Su_n,Su_{n+1}) \\ &\le d(Su_n,z) + d(Su_{n+1},z) \\ &\le \frac{2}{3}\, d(Sx,z) = d(Sx,z) - \frac{1}{3}\, d(Sx,z) \\ &\le d(Sx,z) - d(Su_n,z) \le d(Su_n,Sx) \end{aligned}$$

and hence $d(Tu_n,Tx) \le r\, d(Su_n,Sx)$ for $n \in \mathbb{N}$ with $n \ge \nu_1$. Therefore we have

$$\begin{aligned} d(Tx,z) &= \lim_{n\to\infty} d(Tx,Su_n) = \lim_{n\to\infty} d(Tx,Tu_{n-1}) \\ &\le \lim_{n\to\infty} r\, d(Sx,Su_{n-1}) = r\, d(Sx,z) \end{aligned}$$

for $x \in X$ with $Sx \ne z$.

Let us prove that z is a fixed point of S. In the case where $\sharp\{n : d(Su_n,Tu_n) > d(Su_n,SSu_n)\} = \infty$, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $d(Su_{n_j},Tu_{n_j}) > d(Su_{n_j},SSu_{n_j})$. Then we have

$$\begin{aligned} d(Sz,z) &= \lim_{j\to\infty} d(SSu_{n_j},z) \le \lim_{j\to\infty} \left\{ d(SSu_{n_j},Su_{n_j}) + d(Su_{n_j},z) \right\} \\ &\le \lim_{j\to\infty} \left\{ d(Su_{n_j},Tu_{n_j}) + d(Su_{n_j},z) \right\} \\ &= \lim_{j\to\infty} \left\{ d(Su_{n_j},Su_{n_j+1}) + d(Su_{n_j},z) \right\} = 0. \end{aligned}$$

This implies $z = Sz$. In the other case, where $\sharp\{n : d(Su_n, Tu_n) > d(Su_n, SSu_n)\} < \infty$, there exists $\nu_2 \in \mathbb{N}$ such that $d(Su_n, Tu_n) \le d(Su_n, SSu_n)$ for all $n \ge \nu_2$. So, $d(Tu_n, TSu_n) \le r\, d(Su_n, SSu_n)$ for all $n \ge \nu_2$. Then we have

$$\begin{aligned} d(Su_n, SSu_n) &= d(Tu_{n-1}, STu_{n-1}) = d(Tu_{n-1}, TSu_{n-1}) \\ &\le r\, d(Su_{n-1}, SSu_{n-1}) \le \cdots \le r^{n-\nu_2}\, d(Su_{\nu_2}, SSu_{\nu_2}) \end{aligned}$$

and hence $\lim_n d(Su_n, SSu_n) = 0$. This implies $z = Sz$. Therefore z is a fixed point of S in both cases.

We next show that

$$d(T^n z, T^{n+1} z) \le r^n\, d(z, Tz) \quad \text{for } n \in \mathbb{N}. \tag{4}$$

We put $T^0 z = z$. Since

$$\theta(r)\, d(ST^{n-1}z, T^n z) \le d(ST^{n-1}z, T^n z) = d(ST^{n-1}z, T^n Sz) = d(ST^{n-1}z, ST^n z),$$

we have

$$d(T^n z, T^{n+1} z) \le r\, d(ST^{n-1}z, ST^n z) = r\, d(T^{n-1}Sz, T^n Sz) = r\, d(T^{n-1}z, T^n z).$$

Using this inequality, we can prove (4).

We shall prove that z is a fixed point of T, dividing the following four cases:

- $0 \le r \le \frac{1}{2}(\sqrt{5} - 1)$
- $\frac{1}{2}(\sqrt{5} - 1) < r < \frac{1}{\sqrt{2}}$
- $\frac{1}{\sqrt{2}} \le r < 1$ and $\sharp\{n : Su_n \ne z\} = \infty$
- $\frac{1}{\sqrt{2}} \le r < 1$ and $\sharp\{n : Su_n \ne z\} < \infty$

In the first case, we note $r^2 + r - 1 \le 0$ and $2\,r^2 < 1$. We can prove that

$$\theta(r)\, d(TTz, TTTz) \le d(TTz, z). \tag{5}$$

If not, then since $d(TTz, z) < d(TTz, TTTz)$, we have by (4)

$$\begin{aligned} d(Tz, z) &\le d(Tz, TTz) + d(TTz, z) < d(Tz, TTz) + d(TTz, TTTz) \\ &\le (r + r^2)\, d(z, Tz) \le d(z, Tz), \end{aligned}$$

which implies a contradiction. Therefore (5) holds. Hence

$$\theta(r)\, d(STTz, TTTz) = \theta(r)\, d(TTz, TTTz) \le d(TTz, z) = d(STTz, Sz)$$

holds. From the assumption, we have

$$d(TTTz, Tz) \le r\, d(STTz, Sz) = r\, d(TTz, z). \tag{6}$$

Arguing by contradiction, we assume that $TTz \ne z$. Then we note that $STTz \ne z$ and $STz = Tz \ne z$. Using (3) two times, we obtain

$$d(TTTz, z) \le r\, d(STTz, z) = r\, d(TTz, z) \le r^2\, d(STz, z) = r^2\, d(Tz, z).$$

Using this inequality and (6), we have

$$\begin{aligned} d(z, Tz) &\le d(z, TTTz) + d(TTTz, Tz) \le r^2\, d(z, Tz) + r\, d(TTz, z) \\ &\le 2\, r^2\, d(z, Tz) < d(z, Tz). \end{aligned}$$

This is a contradiction. Thus we obtain $TTz = z$. By (4),

$$d(Tz, z) = d(Tz, TTz) \le r\, d(z, Tz),$$

which implies $Tz = z$. In the second case, we note $2\,r^2 < 1$. If (5) does not hold, then we obtain by (4)

$$\begin{aligned} d(z,Tz) &\le d(z,TTz) + d(TTz,Tz) < \theta(r)\,d(TTz,TTTz) + d(TTz,Tz) \\ &\le \theta(r)\,r^2\,d(z,Tz) + r\,d(z,Tz) = d(z,Tz), \end{aligned}$$

which implies a contradiction. Therefore (5) holds. As in the first case, we can prove $Tz = z$. In the third case, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $Su_{n_j} \neq z$. By (3), we have

$$\begin{aligned} \theta(r)\,d(Su_{n_j},Tu_{n_j}) &\le \theta(r)\,\big(d(Su_{n_j},z) + d(Tu_{n_j},z)\big) \\ &\le \theta(r)\,\big(d(Su_{n_j},z) + r\,d(Su_{n_j},z)\big) \\ &= d(Su_{n_j},z). \end{aligned}$$

By the assumption, we have $d(Tu_{n_j},Tz) \le r\,d(Su_{n_j},z)$ and hence

$$d(z,Tz) = \lim_{j\to\infty} d(Su_{n_j+1},Tz) = \lim_{j\to\infty} d(Tu_{n_j},Tz) \le \lim_{j\to\infty} r\,d(Su_{n_j},z) = 0.$$

Thus, $Tz = z$ holds. In the fourth case, there exists $\nu_3 \in \mathbb{N}$ such that $Su_n = z$ for $n \ge \nu_3$. In particular, $Su_{\nu_3} = Su_{\nu_3+1} = z$. We have

$$Tz = TSu_{\nu_3} = STu_{\nu_3} = SSu_{\nu_3+1} = Sz = z.$$

We have shown that z is a common fixed point of S and T in all the cases.

We conclude the proof by showing that the common fixed point is unique. Suppose that y is a common fixed point of S and T. Since $\theta(r)\,d(Sz,Tz) = 0 \le d(Sz,Sy)$, we have

$$d(z,y) = d(Tz,Ty) \le r\,d(Sz,Sy) = r\,d(z,y)$$

and hence $z = y$. □

3. The Meir-Keeler's Theorem

Finally, we generalize another result in [19] as Park and Bae in [14] generalized the Meir-Keeler fixed point theorem [12]. See also [20].

Theorem 4. *Let (X,d) be a complete metric space. Let S and T be mappings on X satisfying* (a)–(c) *in Theorem 3. Assume that*

$$\frac{1}{2}\,d(Sx,Tx) < d(Sx,Sy) \quad \textit{implies} \quad d(Tx,Ty) < d(Sx,Sy) \tag{7}$$

for all $x, y \in X$, and that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\frac{1}{2}\,d(Sx,Tx) < d(Sx,Sy) \ \textit{ and } \ d(Sx,Sy) < \varepsilon + \delta(\varepsilon) \quad \textit{imply} \quad d(Tx,Ty) \le \varepsilon \tag{8}$$

for all $x, y \in X$. Then there exists a unique common fixed point of S and T.

Remark. $\frac{1}{2}$ is the best constant because $\theta(r)$ in Theorem 1 is best.

Proof. By (b), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$, and $Ix = x$ for all $x \in X$ with $Sx = Tx$. For $x \in X$ with $Sx \neq Tx$, we have $d(Sx,Tx) < 2\,d(Sx,Tx) = 2\,d(Sx,SIx)$. It follows from (7) that $d(Tx,TIx) < d(Sx,SIx)$. Therefore

$$d(SIx,SIIx) < d(Sx,SIx) \quad \text{for all } x \in X \text{ with } Sx \neq SIx. \tag{9}$$

Thus,

$$d(SIx, SIIx) \le d(Sx, SIx) \quad \text{for all } x \in X. \tag{10}$$

Let $u \in X$. Put $u_0 = u$ and $u_n = I^n u$ for all $n \in \mathbb{N}$. By (10), $\{d(Su_n, Su_{n+1})\}$ is a nonincreasing sequence and hence $\{d(Su_n, Su_{n+1})\}$ converges to some $\alpha \ge 0$. Suppose $\alpha > 0$. Then by (9), $\{d(Su_n, Su_{n+1})\}$ is strictly decreasing and hence $d(Su_n, Su_{n+1}) > \alpha$ for $n \in \mathbb{N}$. Take $j \in \mathbb{N}$ with $d(Su_j, Su_{j+1}) < \alpha + \delta(\alpha)$. It follows from (8) that $d(Su_{j+1}, Su_{j+2}) \le \alpha$. This is a contradiction. Therefore we obtain $\alpha = 0$, that is,

$$\lim_{n\to\infty} d(Su_n, Su_{n+1}) = 0. \tag{11}$$

Fix $\varepsilon > 0$ and put $\delta_1 = \min\{\delta(\varepsilon),\ \varepsilon\}$. By (11), we can choose $\nu_1 \in \mathbb{N}$ such that $d(Su_n, Su_{n+1}) < \delta_1$ for all $n \ge \nu_1$. Fix $\ell \in \mathbb{N}$ with $\ell \ge \nu_1$. We shall show that

$$d(Su_\ell, Su_{\ell+m}) < \varepsilon + \delta_1 \tag{12}$$

for $m \in \mathbb{N}$ by induction. If $m = 1$, (12) obviously holds. Suppose that $d(Su_\ell, Su_{\ell+m}) < \varepsilon + \delta_1$ holds for some $m \in \mathbb{N}$. In the case where $d(Su_\ell, Su_{\ell+m}) \le \varepsilon$, we have

$$d(Su_\ell, Su_{\ell+m+1}) \le d(Su_\ell, Su_{\ell+m}) + d(Su_{\ell+m}, S_{\ell+m+1}) < \varepsilon + \delta_1.$$

In the other case, where $\varepsilon < d(Su_\ell, Su_{\ell+m}) < \varepsilon + \delta_1$, since

$$d(Su_\ell, Su_{\ell+1}) < \delta_1 \le \varepsilon < d(Su_\ell, Su_{\ell+m}) < 2\, d(Su_\ell, Su_{\ell+m}),$$

we have $d(Su_{\ell+1}, Su_{\ell+m+1}) \le \varepsilon$ by (8). Hence,

$$d(Su_\ell, Su_{\ell+m+1}) \le d(Su_\ell, Su_{\ell+1}) + d(Su_{\ell+1}, Su_{\ell+m+1}) < \delta_1 + \varepsilon.$$

So, by induction, we obtain (12) for all $m \in \mathbb{N}$. Since ε is arbitrary, we have

$$\lim_{n\to\infty} \sup_{m>n} d(Su_m, Su_n) = 0.$$

This implies that $\{Su_n\}$ is a Cauchy sequence. Since X is complete, $\{Su_n\}$ converges to some point $z \in X$.

Next we show that z is a fixed point of S. In the case where $\sharp\{n : d(Su_n, Tu_n) \ge 2\, d(Su_n, SSu_n)\} = \infty$, there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ such that $d(Su_{n_j}, Tu_{n_j}) \ge 2\, d(Su_{n_j}, SSu_{n_j})$. Then it follows from (a) that

$$\begin{aligned} d(Sz, z) &= \lim_{j\to\infty} d(SSu_{n_j}, z) \le \lim_{j\to\infty} \big(d(SSu_{n_j}, Su_{n_j}) + d(Su_{n_j}, z)\big) \\ &\le \lim_{j\to\infty} \big(\frac{1}{2}\, d(Su_{n_j}, Tu_{n_j}) + d(Su_{n_j}, z)\big) \\ &= \lim_{j\to\infty} \big(\frac{1}{2}\, d(Su_{n_j}, Su_{n_j+1}) + d(Su_{n_j}, z)\big) = 0. \end{aligned}$$

This implies that $z = Sz$. In the other case, where $\sharp\{n : d(Su_n, Tu_n) \ge 2\, d(Su_n, SSu_n)\} < \infty$, there exists $\nu_2 \in \mathbb{N}$ such that $d(Su_n, Tu_n) < 2\, d(Su_n, SSu_n)$ for all $n \ge \nu_2$. It follows from (7) that

$$d(Su_{n+1}, SSu_{n+1}) = d(Tu_n, TSu_n) < d(Su_n, SSu_n)$$

for $n \ge \nu_2$, thus, $\{d(Su_n, SSu_n)\}$ is strictly decreasing for large $n \in \mathbb{N}$. This implies that $\{d(Su_n, SSu_n)\}$ converges to some $\beta \ge 0$, and $d(Su_n, SSu_n) > \beta$ for $n \ge \nu_2$. Suppose $\beta > 0$. Then from the definition of β, we can take $j \in \mathbb{N}$ such that $j \ge \nu_2$ and $d(Su_j, SSu_j) < \beta + \delta(\beta)$. From (8) and (c), we obtain $d(Su_{j+1}, SSu_{j+1}) = d(Tu_j, TSu_j) \le \beta$. This is a contradiction. Thus, we obtain $\beta = 0$, that is, $\lim_n d(Su_n, SSu_n) = 0$, which implies that $z = Sz$.

Let us prove $Tz = z$. We consider the following two cases:

- There exists $\nu \in \mathbb{N}$ such that $Su_\nu = Su_{\nu+1}$.
- $Su_n \neq Su_{n+1}$ for all $n \in \mathbb{N}$.

In the first case, we note $u_\nu = u_{\nu+1}$ from the definition of I. Hence $u_n = u_\nu$ for all $n \geq \nu$. Since $Su_n \to z$, we have $Su_n = z$ for $n \geq \nu$. Hence, we obtain

$$Tz = TSu_\nu = STu_\nu = SSu_{\nu+1} = Sz = z.$$

In the second case, we note $Su_n \neq Tu_n$ for $n \in \mathbb{N}$. Suppose $d(Su_n, Su_{n+1}) \geq 2\,d(Su_n, z)$ and $d(Su_{n+1}, Su_{n+2}) \geq 2\,d(Su_{n+1}, z)$ for $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
d(Su_n, Su_{n+1}) &\leq d(Su_n, z) + d(Su_{n+1}, z) \\
&\leq \frac{1}{2}\big(d(Su_n, Su_{n+1}) + d(Su_{n+1}, Su_{n+2})\big) \\
&< d(Su_n, Su_{n+1})
\end{aligned}$$

by (9). This is a contradiction. Therefore we have either

$$d(Su_n, Su_{n+1}) < 2\,d(Su_n, z) \quad \text{or} \quad d(Su_{n+1}, Su_{n+2}) < 2\,d(Su_{n+1}, z)$$

for $n \in \mathbb{N}$. Then, from (7), either

$$d(Tu_n, Tz) < d(Su_n, z) \quad \text{or} \quad d(Tu_{n+1}, Tz) < d(Su_{n+1}, z)$$

holds for $n \in \mathbb{N}$. Since $Tu_n = Su_{n+1}$ and $Su_n \to z$, there exists a subsequence of $\{Su_n\}$ converging to Tz. This implies that $Tz = z$. Therefore, in all the cases, we have shown z is a common fixed point of S and T.

We conclude the proof by showing that the common fixed point is unique. Suppose that y is another common fixed point of S and T. Since $d(Sz, Tz) = 0 < 2\,d(z, y) = 2\,d(Sz, Sy)$, we have

$$d(z, y) = d(Tz, Ty) < d(Sz, Sy) = d(z, y).$$

This is a contradiction. □

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