Invariance of Frequency Difference in Nonresonant Entrainment of Detuned Oscillators Induced by Common White Noise

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We studied the entrainment of two uncoupled detuned limit cycle oscillators subjected to a common external white Gaussian noise. We found a novel type of entrainment behavior for a general class of oscillators: as noise intensity increases, the phases of the two oscillators come to be almost always locked with each other although there is no frequency locking in the sense that the mean frequency difference remains the same as the natural frequency difference. We also show that a common noise induces a macroscopic oscillation with no frequency locking in a population of detuned oscillators.

§1. Introduction

Entrainment is a key mechanism for the emergence of order and coherence in a variety of physical systems consisting of oscillatory elements. It is one of the fundamental themes of nonlinear physics to explore the possible types of entrainments and clarify their fundamental properties. It is well known that a common external periodic input may lead to entrainment between two independent slightly detuned oscillators when their natural frequencies are in resonance with the input frequency (e.g., see Refs. 1) and 2)). A fundamental property of this resonant entrainment is that both frequencies and phases of the two oscillators become locked with each other.

Recent works have shown that not only a periodic signal but also a noiselike signal can give rise to entrainment between two independent oscillators.[3)–9)] Experimental evidence of this phenomenon has been found for systems as diverse as neuronal networks,[3)] ecological systems[4)] and lasers.[5)] The entrainment by a noiselike signal is nonresonant in the sense that there is no resonance relationship between the oscillator and the noise. Therefore, we call it *nonresonant entrainment.* In order to fully understand the emergence of order and coherence in the real world, it is essential to clarify the fundamental properties of nonresonant entrainment.

The nonresonant entrainment between two independent and *nondetuned* oscillators has been analyzed in the phase approximation of the dynamics.[6)–8)] It was shown that when a weak Gaussian noise is applied, stable phase locking is achieved and the two oscillators maintain the same mean frequency. In this sense, the behavior is similar to that for a periodic input.

In real systems, any two oscillators will have different natural frequencies, i.e., some finite value of detuning will necessarily exist. Thus, it is crucial to consider the case of detuned oscillators. This case has been studied in Ref. 9), which also used phase approximation: it has been shown that the nonresonant entrainment induced by a common noise also occurs between detuned oscillators but this entrainment is imperfect in the sense that phase slips occur intermittently. However, details of this nonresonant entrainment in detuned oscillators have not yet been fully understood. In particular, no discussion has been made about the frequency locking properties of the two oscillators in Ref. 9). It is still unclear whether frequency locking occurs as in the case of periodic driving signals when a sufficiently large noise strength is applied. On this point, we show that as noise intensity increases, there is no frequency locking in the sense that the mean frequency difference stays the same as the natural frequency difference even though the phases of the two oscillators come to be almost always locked with each other. This type of phase locking with an invariance of frequency difference is a universal characteristic of nonresonant entrainment. We emphasize that the invariance of frequency difference is a fundamental property of nonresonant entrainment, which is different from that of a resonant one.

It is important to be aware of this property when investigating a population of oscillatory systems driven by common noise. We demonstrate this with an example of a qualitatively novel type of emergence of macroscopic order in a population of oscillators with a distribution of frequencies. We show that a common noise induces a macroscopic oscillation even though there is no frequency locking. Such phenomena should be observable in experimental systems.

This paper is organized as follows. In §2, we describe a noise-driven oscillator model and present theoretical results based on phase approximation. In §3, we show some numerical results to demonstrate our theoretical results on nonresonant entrainment. In §4, we show numerical results concerning the emergence of macroscopic order in a population of oscillators. Finally, conclusions are drawn in §5.

§2. Model and theoretical analysis

Let $\boldsymbol{X}_i \in \mathbb{R}^N$ be a state variable vector and consider the equation

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + \delta \boldsymbol{F}_i(\boldsymbol{X}_i) + \boldsymbol{G}(\boldsymbol{X}_i)\xi(t), \quad i = 1, 2, \tag{2·1}$$

where $\boldsymbol{F}$ is an unperturbed vector field, $\delta\boldsymbol{F}_1$ and $\delta\boldsymbol{F}_2$ are small deviations from it, $\boldsymbol{G}$ is a vector function, and $\xi(t)$ is the white Gaussian noise such that $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = 2D\,\delta(t-s)$, where $\langle \cdots \rangle$ denotes averaging over the realizations of ξ and δ is Dirac's delta function. We call the constant $D > 0$ the noise intensity. The noise-free unperturbed system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ is assumed to have a limit cycle $\boldsymbol{X}_0(t)$ with a frequency ω. We employ the Stratonovich interpretation for Eq. (2·1).

We introduce the phase variables ϕ_i, $i = 1, 2$ using the unperturbed system. Let us assume the case of weak noise, i.e., $0 < D \ll 1$, to apply the phase reduction method. Recently, it has been shown in Ref. 10) that the noise-driven oscillator (2·1) can be reduced into the Ito-type stochastic differential equation

$$\dot{\phi}_i = \omega + \delta\omega_i(\phi_i) + D\left[Z(\phi_i)Z'(\phi_i) + Y(\phi_i)\right] + Z(\phi_i)\xi(t), \tag{2·2}$$

where $\delta\omega_i$ is the frequency variation due to $\delta\boldsymbol{F}_i$, the prime denotes differentiation with respect to ϕ_i, and Z is defined by $Z(\phi) = \left(\mathrm{grad}_{\boldsymbol{X}}\,\phi|_{\boldsymbol{X}=\boldsymbol{X}_0(\phi)}\right) \cdot \boldsymbol{G}(\boldsymbol{X}_0(\phi))$. By definition, $Z(\phi)$ is a periodic function, i.e., $Z(\phi) = Z(\phi + 2\pi)$. We assume that Z is three times continuously differentiable and not a constant. The function $Y(\phi)$ is also a periodic function satisfying $Y(\phi) = Y(\phi + 2\pi)$. An explicit functional form of $Y(\phi)$ is not important for the present discussion and it is given in Appendix A. We refer the readers to Ref. 10) for details of the derivation of Eq. (2·2).

We define the average $\overline{\delta\omega_i} = \int_0^{2\pi} \delta\omega_i(\phi)d\phi/2\pi$ and can assume the case of $\overline{\delta\omega_1} > \overline{\delta\omega_2}$ without loss of generality. In the following theoretical analysis, we assume that the detuning between the two oscillators is much smaller than unity: i.e., $\overline{\delta\omega_1} - \overline{\delta\omega_2} \ll 1$. It is possible to choose D such that $\overline{\delta\omega_1} - \overline{\delta\omega_2} \ll D \ll 1$ under this assumption. Note that for such a choice of D, both of the two conditions $0 < D \ll 1$ and $(\overline{\delta\omega_1} - \overline{\delta\omega_2})/D \ll 1$ hold. The first condition ensures the validity of the phase approximation. Combined with the first condition, the second condition indicates that it is valid to take the limit $(\overline{\delta\omega_1} - \overline{\delta\omega_2})/D \to 0$ within the phase approximation. We will consider this limit later.

The mean frequency $\mathit{\Omega}_i$ of the ith oscillator is defined by $\mathit{\Omega}_i = \lim_{T\to\infty} \int_0^T \dot{\phi}_i(t)dt/T$. This can be calculated by replacing the time average with the ensemble average: i.e., $\mathit{\Omega}_i = \langle \dot{\phi}_i \rangle$. From Eq. (2·2), we have

$$\mathit{\Omega}_i = \omega + \langle \delta\omega_i(\phi_i) \rangle + D\langle\, Z(\phi_i)Z'(\phi_i) + Y(\phi_i)\,\rangle, \tag{2·3}$$

where we used the fact $\langle Z(\phi_i)\xi(t)\rangle = \langle Z(\phi_i)\rangle\langle\xi(t)\rangle = 0$ since the correlation between ϕ_i and ξ vanishes in the Ito equation. For an arbitrary function $A(\phi)$, the ensemble average can be calculated using the steady probability distribution $P_i(\phi_i)$ for ϕ_i, which is determined by the Fokker-Planck equation for Eq. (2·2): i.e., $\langle A \rangle = \int_0^{2\pi} A(\phi)P_i(\phi)d\phi$. The steady distribution P_i can be obtained as $P_i(\phi_i) = 1/2\pi + O(\sigma_i, D/\omega)$, where $\sigma_i = \max_{0\le\phi<2\pi} |\delta\omega_i(\phi)/\omega|$. Since $\delta\boldsymbol{F}_i$ is small, σ_i is a small parameter. Thus, we can use the approximation $P_i \simeq 1/2\pi$ for a small D in Eq. (2·3) and then obtain

$$\mathit{\Omega}_i \simeq \omega + \overline{\delta\omega_i} + D\overline{Y}, \tag{2·4}$$

where $\overline{Y} = \int_0^{2\pi} Y(\phi)d\phi/2\pi$. Since the white Gaussian noise has no characteristic frequency, intuitively, one might expect that the noise causes no change in frequency. However, equation (2·4) shows that it changes $\mathit{\Omega}$: it depends on the sign of $\overline{Y}$ whether $\mathit{\Omega}$ increases or decreases with increasing D.

The mean frequency difference is given by

$$\mathit{\Omega}_1 - \mathit{\Omega}_2 = \overline{\delta\omega_1} - \overline{\delta\omega_2}, \tag{2·5}$$

from Eq. (2·4). Equation (2·5) indicates that the mean frequency difference is independent of the noise intensity D and its constant value is given by the natural frequency difference, although the mean frequency itself does change in each oscillator. The two oscillators are detuned, i.e., $\overline{\delta\omega_1} \neq \overline{\delta\omega_2}$. Thus, on average, the phase difference $|\phi_1 - \phi_2|$ increases in proportion to time t. It can be concluded that a common white Gaussian noise does not cause frequency locking between the two oscillators.

We calculate the steady probability distribution of the phase difference between the two oscillators. Let $P(t, \phi_1, \phi_2)$ be the joint probability distribution of ϕ_1 and ϕ_2. The Fokker-Planck equation corresponding to Eq. (2·2) is given by

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial \phi_i} \left[\{\omega + \delta\omega_i(\phi_i) + DY(\phi_i) + DZ(\phi_i)Z'(\phi_i)\} P \right] + D \sum_{i,j=1}^{2} \frac{\partial^2}{\partial \phi_i \partial \phi_j} \left[Z(\phi_i) Z(\phi_j) P \right]. \tag{2·6}$$

If we introduce the new variables $\varphi_i,\ i = 1, 2$ defined by $\varphi_i = \phi_i - \omega t$, Eq. (2·6) is rewritten as

$$\frac{\partial \tilde{P}}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial \varphi_i} \left[\{\delta\omega_i(\varphi_i + \omega t) + DY(\varphi_i + \omega t) + DZ(\varphi_i + \omega t)Z'(\varphi_i + \omega t)\} \tilde{P} \right] + D \sum_{i,j=1}^{2} \frac{\partial^2}{\partial \varphi_i \varphi_j} \left[Z(\varphi_i + \omega t) Z(\varphi_j + \omega t) \tilde{P} \right], \tag{2·7}$$

where $\tilde{P}(t, \varphi_1, \varphi_2) = P(t, \varphi_1 + \omega t, \varphi_2 + \omega t)$. For small D and $\delta\omega_i$, the right-hand side of Eq. (2·7) is small and it is expected that $\tilde{P}$ varies slowly with time. Therefore, we can perform the time-averaging with respect to t over the period $[0, 2\pi/\omega]$. The time-averaged Fokker-Planck equation is obtained as

$$\frac{\partial \tilde{P}}{\partial t} = -\sum_{i=1}^{2} \left[(\,\overline{\delta\omega}_i + D\overline{Y}\,) \frac{\partial \tilde{P}}{\partial \varphi_i} + D\Gamma(0) \frac{\partial^2 \tilde{P}}{\partial \varphi_i^2} \right] + 2D \frac{\partial^2}{\partial \varphi_1 \varphi_2} \left[\Gamma(\varphi_1 - \varphi_2) \tilde{P} \right], \tag{2·8}$$

where $\Gamma(\theta) = \int_0^{2\pi} Z(\phi) Z(\phi + \theta) d\phi / 2\pi$. Let θ and ψ be defined by $\theta = \varphi_1 - \varphi_2$ and $\psi = \varphi_1 + \varphi_2$, respectively. These two variables are related to the original phases ϕ_1 and ϕ_2 as $\theta = \phi_1 - \phi_2$ and $\psi = \phi_1 + \phi_2 - 2\omega t$. The variable θ measures the phase difference between the two oscillators. If we change the independent variables from $(t, \varphi_1, \varphi_2)$ to (t, θ, ψ) in Eq. (2·8), we have

$$\frac{\partial Q}{\partial t} = -(\,\overline{\delta\omega}_1 - \overline{\delta\omega}_2\,) \frac{\partial Q}{\partial \theta} - (\,\overline{\delta\omega}_1 + \overline{\delta\omega}_2\,) \frac{\partial Q}{\partial \psi} + D \frac{\partial^2}{\partial \theta^2} [u(\theta) Q] + D \frac{\partial^2}{\partial \psi^2} [v(\theta) Q], \tag{2·9}$$

where $Q(t, \theta, \psi) = \tilde{P}(t, (\psi + \theta)/2, (\psi - \theta)/2)$, $u(\theta) = 2\{\Gamma(0) - \Gamma(\theta)\}$, and $v(\theta) = 2\{\Gamma(0) + \Gamma(\theta)\}$.

It is generally possible that Z has a period smaller than 2π. Since Z is not constant, we suppose that $Z(\phi) = Z(\phi + 2\pi/n)$, where n is a positive integer. It can be shown that $u(\theta) \geq 0$ for any $\theta \in [0, 2\pi)$. The zero points s_m of u are given by $s_m = 2\pi m/n,\ m = 0, 1, \ldots, n-1$, where $s_0 = 0$. Equation (2·9) has the steady solution $Q_s(\theta)$ such that it is a continuous function of θ only and satisfies the two conditions (i) $Q_s(\theta) = Q_s(\theta + 2\pi)$ and (ii) $\int_0^{2\pi} Q_s(\theta) d\theta = 1$. In fact, if we substitute the form $Q_s(\theta)$ into Eq. (2·9), we have

$$D \frac{d}{d\theta} [u(\theta) Q_s] - (\,\overline{\delta\omega}_1 - \overline{\delta\omega}_2\,)\, Q_s = C, \tag{2·10}$$

where C is an integration constant, which is to be determined from conditions (i) and (ii). From Eq. (2·10), the solution Q_s is obtained in each interval (s_m, s_{m+1}) as

$$Q_s(\theta) = \frac{\varepsilon}{2\pi u(\theta)} \int_\theta^{s_{m+1}} \exp\left[-\varepsilon \int_\theta^x \frac{1}{u(y)} dy\right] dx, \tag{2·11}$$

where $\varepsilon = (\overline{\delta\omega}_1 - \overline{\delta\omega}_2)/D > 0$. The right-hand side of Eq. (2·11) has singularities at the zero points of u. The Q_s for each s_m is given by $Q_s(s_m) = \lim_{\theta\to s_m} Q_s(\theta) = 1/2\pi$ for any $\varepsilon > 0$ (Appendix B).

We consider the limit $\varepsilon \to 0$. Note that taking this limit within the phase approximation is valid because of the assumption $\overline{\delta\omega}_1 - \overline{\delta\omega}_2 \ll D \ll 1$. Assume that $\theta \in (s_m, s_{m+1})$, i.e., θ is an arbitrary regular point. It can be shown that $\lim_{\varepsilon\to 0} Q_s(\theta) = 0$ holds due to the factor ε in the numerator. This implies that the probability has to concentrate at the singular points $s_m,\ m = 0, 1, \dots, n-1$ because Q_s satisfies condition (ii). Thus, Q_s in the limit $\varepsilon \to 0$ is given by

$$Q_s(\theta) = \frac{1}{n} \sum_{m=0}^{n-1} \delta(\theta - s_m), \tag{2·12}$$

where δ is Dirac's delta function. Equation (2·12) indicates that multiple peaks exist if Z has a period smaller than 2π, i.e., $n > 1$. The existence of multiple peaks has been pointed out in the case of nondetuned oscillators.[8)]

We explain how Q_s converges to Eq. (2·12) as ε goes to zero. For small positive ε, Q_s has narrow and sharp peaks near $\theta = s_m,\ m = 0, 1, \dots, n-1$ while Q_s is close to zero in regions other than the neighborhoods of these singular points. Let θ_m^* be θ such that Q_s takes a maximum over the interval $[s_m, s_{m+1})$, where $m = 0, 1, \dots, n-1$, i.e., θ_m^* represents the positions of the peaks of Q_s. The peak position θ_m^* depends on ε. It is clear that $\theta_m^* \neq s_m$ for any $\varepsilon > 0$ since $Q_s(s_m) = 1/2\pi$. In fact, if $\theta_m^* = s_m$, then Q_s has to be the uniform distribution $Q_s(\theta) = 1/2\pi$. As ε approaches zero, the peaks of Q_s become narrower and higher, and their positions θ_m^* converge to the singular points $s_m,\ m = 0, 1, \dots, n-1$ (see Figs. 1(c) and 2).

The above-mentioned profile of Q_s clearly shows that the phase locking states, where $\theta \bmod 2\pi \simeq s_m$, are achieved for a large fraction of time during the time evolution when the noise intensity D is relatively large with respect to the natural frequency difference $\overline{\delta\omega}_1 - \overline{\delta\omega}_2$. Let δ be a small positive constant and U_δ be the δ-neighborhood defined by $U_\delta = \cup_{m=0}^{n-1}(s_m - \delta, s_m + \delta)$, where mod 2π is taken for $s_0 - \delta$. We identify the phase locking state by the condition $\theta \in U_\delta$. As shown in Eq. (2·5), the present entrainment is not characterized by the coincidence of the mean frequencies of the two oscillators. Therefore, as a measure of entrainment, we introduce the phase locking time ratio μ defined by

$$\mu = \lim_{T\to\infty} \frac{T_L}{T}, \tag{2·13}$$

where T_L represents the total time length for which $\theta \in U_\delta$ happens during the period T. This ratio can also be expressed in terms of Q_s by $\mu = \int_{U_\delta} Q_s(\theta) d\theta$, where the integral is taken over the set U_δ. Equation (2·12) shows that $\mu \to 1$ in the limit $\varepsilon = (\overline{\delta\omega_1} - \overline{\delta\omega_2})/D \to 0$.

A phase locking state cannot continue for the infinite time but phase slips have to happen because $\Omega_1 \neq \Omega_2$. Equation (2·5) has shown that $\Omega_1 - \Omega_2$ is constant. This implies that the average number of phase slips, which happen in a unit time interval, does not decrease with increasing D but is constant. In other words, the average inter-phase-slip interval is constant. On the other hand, the probability for $\theta \notin U_\delta$ decreases and converges to zero as D increases: the phases come to be *almost always* locked with each other. These two facts imply that the time needed for one phase slip decreases and converges to zero as D increases, i.e., the velocity $\dot{\theta}$ during a phase slip motion increases. We emphasize that the almost-always phase locking with the invariance of mean frequency difference is an essential feature of the nonresonant entrainment in detuned oscillators. This property is very different from that of resonant entrainment by a periodic signal, where the average inter-phase-slip interval diverges and the mean frequencies become identical to each other as the signal intensity approaches the threshold for entrainment.

§3. Numerical examples

In order to demonstrate the above analytical results, we show numerical results for an example described by the Ito stochastic differential equations

$$\dot{\phi}_i = \omega_i + D\sin(\phi_i)\cos(\phi_i) + \sin(\phi_i)\,\xi(t), \quad i = 1, 2, \tag{3·1}$$

where ω_i, $i = 1, 2$ are slightly different constants. This corresponds to the case $Z = \sin\phi_i$, $Y = 0$ in Eq. (2·2).

Figure 1(a) shows the mean frequency difference $\Delta\Omega = \Omega_1 - \Omega_2$ plotted as a function of D, where $\omega_1 = 1$ and five different values of ω_2 are employed. Except in the case $\omega_1 = \omega_2 = 1$, $\Delta\Omega$ is not zero. It is clearly shown that $\Delta\Omega$ is independent of D and takes a constant value, which equals the natural frequency difference $\omega_1 - \omega_2$. This coincides with the analytical result of Eq. (2·5) .

The time evolution of the phase difference $\theta = \phi_1 - \phi_2$ is shown for three different values of D in Fig. 1(b), where $\omega_1 = 1$ and $\omega_2 = 0.98$. These results clearly show that the phases are locked near $\theta \simeq 2\pi n$, $n \in \mathbb{Z}$ and phase slips occur intermittently. It should be noted that the time needed for a single phase slip becomes shorter as D increases, in other words, $\dot{\theta}$ during the phase slip motion becomes larger. This observation is in agreement with the analytical result.

The numerically obtained distribution $Q_s(\theta)$ is shown in Fig. 1(c) for three different values of D, where $\omega_1 = 1$ and $\omega_2 = 0.98$. The analytical results of Eq. (2·11) are also shown for the corresponding values of $\varepsilon = (\omega_1 - \omega_2)/D$. A good agreement between them is confirmed. It is seen that Q_s is close to the uniform distribution for small D or large ε. In contrast, the distribution has a sharp peak near $\theta = 0$ for a large D or a small ε. The peak in Q_s becomes narrower and higher and its position approaches $\theta = 0$ as D increases. It is also seen that the peak is not centered at $\theta = 0$ but is shifted to the positive direction, i.e., the phase ϕ_1 of the larger-natural-frequency oscillator is kept advanced with respect to ϕ_2 even in the

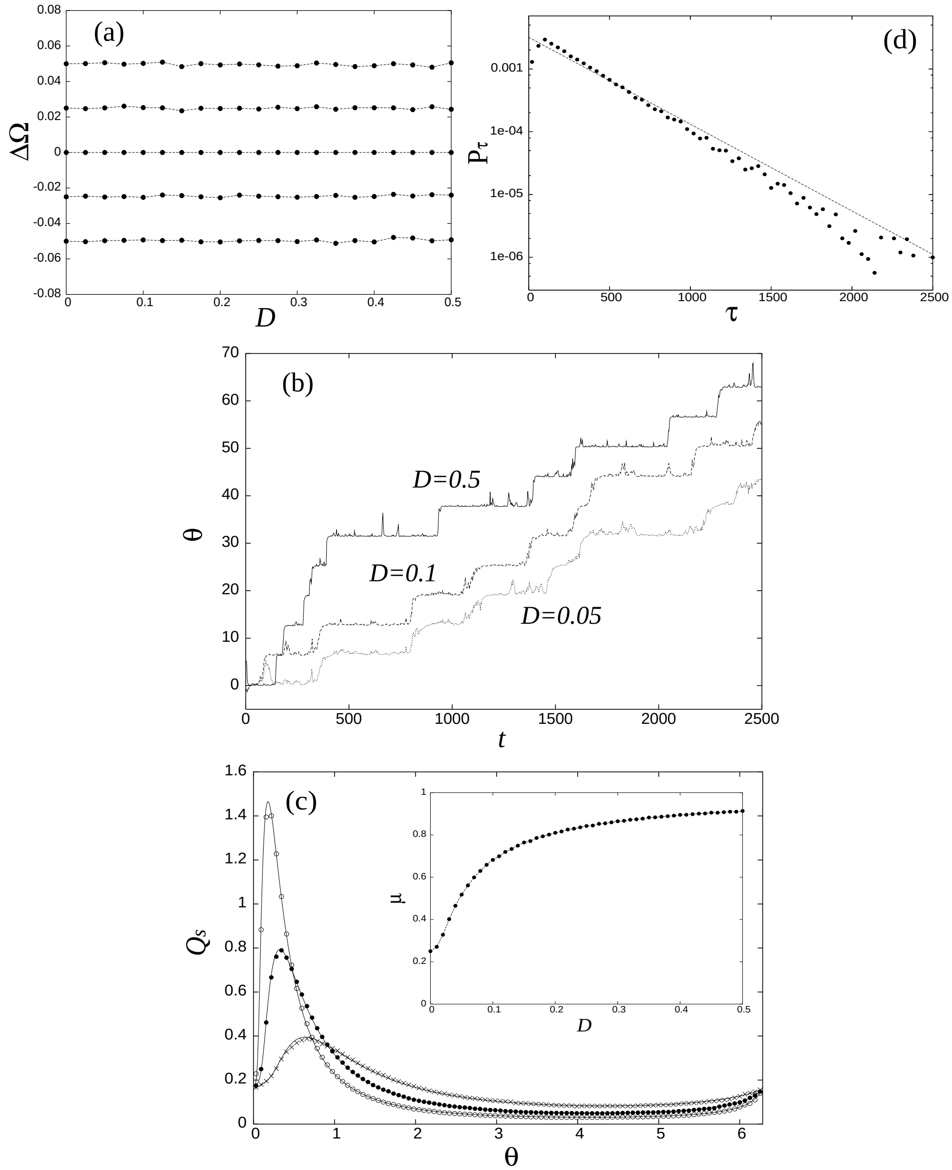

Fig. 1. Entrainment in phase model (3·1): (a) mean frequency difference $\Delta\Omega$ vs D, (b) time evolution of phase difference $\theta = \phi_1 - \phi_2$, (c) probability distribution $Q_s(\theta)$ for $D = 0.02\,(\times)$, $0.05\,(\bullet)$, and $0.1\,(\circ)$, where analytical results are shown by solid line, and (d) distribution of inter-phase-slip interval for $D = 0.2$ with reference line for $P_\tau(\tau) = \lambda \exp[-\lambda\tau]$, $\lambda = (\omega_1 - \omega_2)/2\pi$. The inset in (c) shows μ plotted against D, which is obtained for $\delta = \pi/4$. In (b)-(d), $\omega_1 = 1$ and $\omega_2 = 0.98$.

phase locking state. The inset of Fig. 1(c) shows that the phase locking time ratio μ monotonically increases and approaches unity with increasing D, where the size δ of the neighborhood U_δ is $\delta = \pi/4$. Figure 1(c) clearly demonstrates that the phases are locked for a larger fraction of the time as D increases.

Figure 1(d) shows the probability distribution P_τ of the inter-phase-slip interval

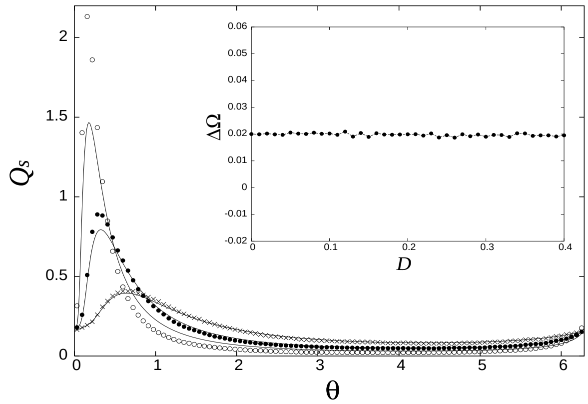

Fig. 2. Entrainment in SL oscillators with $\boldsymbol{G} = (-1, 0)$. Probability distribution $Q_s(\theta)$ is shown for $D = 0.02\,(\times)$, $0.05\,(\bullet)$, and $0.1\,(\circ)$, where analytical results are shown by solid line. The inset shows $\Delta\Omega$ vs D. $(\alpha_1, \beta_1) = (1, 0)$ and $(\alpha_2, \beta_2) = (0.98, 0)$.

τ. The result is well described by the exponential distribution $P_\tau(\tau) = \lambda \exp[-\lambda\tau]$, where $\lambda = (\omega_1 - \omega_2)/2\pi$. The exponential distribution with the same λ has been observed for different D values. The agreement becomes better as D increases.

In order to validate the theory based on the phase reduction method, we carried out numerical experiments on the Stuart-Landau (SL) oscillator

$$\dot{x}_i = x_i - \alpha_i y_i - (x_i^2 + y_i^2)(x_i - \beta_i y_i) + G_x \xi(t), \tag{3·2}$$

$$\dot{y}_i = \alpha_i x_i + y_i - (x_i^2 + y_i^2)(\beta_i x_i + y_i) + G_y \xi(t), \tag{3·3}$$

where α_i and β_i are constants, $\boldsymbol{G} = (G_x, G_y)$ is a vector function of (x_i, y_i), and $i = 1, 2$. The natural frequency is given by $\omega_i = \alpha_i - \beta_i$. The Stratonovich interpretation is employed for Eqs. (3·2) and (3·3).

We assume the case $\boldsymbol{G} = (-1, 0)$, in which the SL oscillator is reduced to the Ito type phase model $\dot{\phi}_i = \omega_i + D\sin(2\phi_i) + \sin(\phi_i)\xi(t)$. In Fig. 2, the numerically obtained $Q_s(\theta)$ is shown for three different values of D, where $(\alpha_1, \beta_1) = (1, 0)$ and $(\alpha_2, \beta_2) = (0.98, 0)$, i.e., $\omega_1 = 1$ and $\omega_2 = 0.98$. The analytical results obtained from the phase model are also shown. A sharp peak of Q_s appears near $\theta = 0$. It approaches $\theta = 0$ and becomes narrower and higher as D increases. The agreement between the numerical and analytical results is excellent, particularly in the small D region, where the phase reduction method gives a good approximation. The inset shows the mean frequency difference $\Delta\Omega = \Omega_1 - \Omega_2$ plotted against D. It is clear that $\Delta\Omega$ does not depend on D and its constant value is given by $\omega_1 - \omega_2$. The almost-always phase locking with the invariance of mean frequency difference is clearly confirmed. This also agrees with the theory based on the phase approximation. The agreements in the behaviors of Q_s and $\Delta\Omega$ validate the theory based on the phase model.

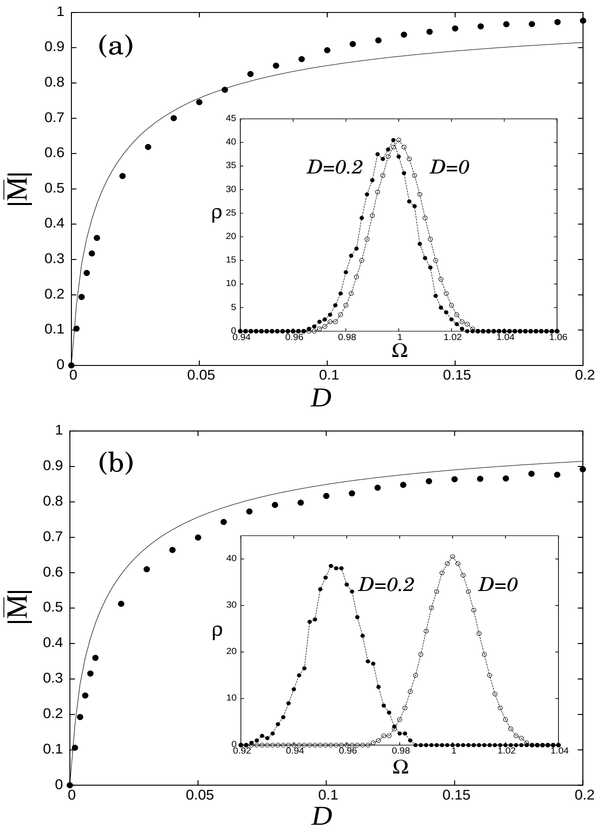

Fig. 3. Order parameter $|\bar{M}|$ vs D for SL oscillators with (a) $\boldsymbol{G} = (-1, 0)$ and (b) $\boldsymbol{G} = (x_i, 0)$. Analytical estimation given by Eq. (4·3) is shown by solid line. The inset shows the frequency distribution ρ for $D = 0$ (○) and 0.2 (●). The parameters are $\omega_0 = 1$, $\sigma = 0.01$, and $N = 1000$.

§4. Emergence of macroscopic rhythm

A common noise can induce macroscopic oscillation in a population of detuned oscillators. As examples, we employ N noise-driven SL oscillators of the form (3·2) and (3·3) with two different $\boldsymbol{G}$s. The first example is $\boldsymbol{G} = (-1, 0)$ with the parameters $\alpha_i = 1$ and different β_i. The second one is $\boldsymbol{G} = (x_i, 0)$ with the parameters $\alpha_i = 2$ and different β_i. In both the examples, the natural frequencies $\omega_i = \alpha_i - \beta_i$ are distributed according to the Gaussian distribution $g(\omega) = \exp[-(\omega - \omega_0)^2/2\sigma^2]/\sqrt{2\pi}\sigma$, where $\omega_0 = 1$ and $\sigma = 0.01$.

In order to measure macroscopic oscillation, we introduce an order parameter. Consider a noise-driven SL oscillator having the natural natural frequency ω_0 and denote its phase variable by ϕ_0. We employ this oscillator as a reference. Let $\theta_i,\ i = 1, \ldots, N$ be defined by $\theta_i = \phi_i - \phi_0$, which measures the deviation of ϕ_i from the reference oscillator's phase ϕ_0. As mentioned in §2, there is the possibility that an entrainment state consisting of n clusters appears, where n is the number of zero points of $u(\theta)$. Taking into account this possibility, we define the order parameter

M as

$$M = \frac{1}{N} \sum_{i=1}^{N} \exp[\, in\theta_i \,], \tag{4·1}$$

where $n = 1$ for the first example and $n = 2$ for the second one. If the phases evolve collectively, $|M|$ takes a value close to unity.

We numerically computed the time average $\bar{M}$ of the order parameter. Figures 3(a) and (b) show $|\bar{M}|$ plotted against D for the first example $\boldsymbol{G} = (-1, 0)$ and the second example $\boldsymbol{G} = (x_i, 0)$, respectively. It is shown that $|\bar{M}|$ increases and becomes close to unity as D increases, indicating the emergence of macroscopic oscillation. In the inset of each figure, the mean frequency distribution $\rho(\Omega)$ is shown for $D = 0$ and 0.2. In the case of $D = 0$, ρ coincides with the natural frequency distribution g. In both the examples, the profile of ρ is preserved as D increases although its center shifts to the smaller Ω direction in Fig. 3(b). This center shift is due to the term $D\overline{Y}$ in Eq. (2·4). It can be shown that $\overline{Y} = 0$ for the first example and $\overline{Y} = -1/4$ for the second example. Since $\overline{Y} = 0$, no apparent center shift is observed in Fig. 3(a), where a slight center shift may be due to a higher order effect in D. In Fig. 3(b), the center shift is clearly observed and the amount of the center shift is about -0.05, which coincides with the theoretical value $D\overline{Y} = -0.05$. We emphasize that this preservation of ρ profile is a characteristic of the nonresonant entrainment, which is different from the periodic signal case, where the frequency locking occurs.

We calculate the order parameter using the phase approximation theory in §2 and compare it with the numerical results. If we replace the time average of M by the ensemble average, we can calculate $\bar{M}$ as

$$\bar{M} = \int_0^{2\pi} d\theta_1 \dots \int_0^{2\pi} d\theta_N \left(\frac{1}{N} \sum_{i=1}^{N} e^{in\theta_i} \right) P_s(\theta_1, \dots, \theta_N), \tag{4·2}$$

where $P_s(\theta_1, \dots, \theta_N)$ is the steady joint distribution of $\theta_i,\ i = 1, \dots, N$. Although these variables $\theta_i,\ i = 1, \dots, N$ are generally not independent, we employ the approximation $P_s(\theta_1, \dots, \theta_N) \simeq \prod_{i=1}^{N} Q_s(\theta_i; \varepsilon_i)$, where Q_s is the two-body steady distribution given by Eq. (2·11) and we indicated that it depends on the parameter $\varepsilon_i = (\omega_i - \omega_0)/D$. If we use this approximation and consider the limit $N \to +\infty$ in Eq. (4·2), we have

$$\bar{M} = \int_{-\infty}^{+\infty} d\omega \int_0^{2\pi} d\theta \, e^{in\theta} Q_s(\theta; \varepsilon(\omega)) g(\omega), \tag{4·3}$$

where $\varepsilon = (\omega - \omega_0)/D$. In Figs. 3(a) and (b), $|\bar{M}|$ computed by using Eq. (4·3) is shown by a solid line. This theoretical estimation is in good agreement with the numerical results in both examples. The slight discrepancies may be due to the approximation by the two-body steady distribution and phase approximation.

§5. Conclusions

In conclusion, we studied the nonresonant entrainment of two detuned limit cycle oscillators subjected to a common external white Gaussian noise. We showed that their phases come to be almost always locked with each other as noise intensity increases even though no frequency locking occurs in the sense that the mean frequency difference remains identical to the natural frequency difference. Moreover, it has been shown that common noise can induce a macroscopic oscillation in a population of detuned oscillators without changing the frequency distribution profile except for a constant center shift.

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Appendix A
—— *Functional Form of* $Y(\phi)$ ——

Consider the noise-driven oscillator

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{G}(\boldsymbol{X})\xi(t), \tag{A·1}$$

which is Eq. (2·1) with $\delta\boldsymbol{F}_i = \boldsymbol{0}$. The phase coordinate ϕ can be defined using the unperturbed system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ in a neighbourhood U of the limit cycle $\boldsymbol{X}_0$ in phase space. We can define the other $N-1$ coordinates $\boldsymbol{r} = (r_1, \ldots, r_{N-1})$ such that $\det \partial(\phi, \boldsymbol{r})/\partial\boldsymbol{X} \neq 0$ in U. We assume that $\boldsymbol{r} = \boldsymbol{a}$ on the limit cycle, where $\boldsymbol{a} = (a_1, \ldots, a_{N-1})$ is a constant vector. If we perform the transformation $(x_1, \ldots, x_N) \mapsto (\phi, r_1, \ldots, r_{N-1})$, Eq. (A·1) is rewritten into the Stratonovich stochastic differential equations

$$\dot{\phi} = \omega + h(\phi, \boldsymbol{r})\xi(t), \tag{A·2}$$
$$\dot{r}_j = f_j(\phi, \boldsymbol{r}) + g_j(\phi, \boldsymbol{r})\xi(t), \tag{A·3}$$

where $j = 1, \ldots, N-1$. The functions h, f_j, and g_j are defined as $h(\phi, \boldsymbol{r}) = (\mathrm{grad}_{\boldsymbol{X}}\phi) \cdot \boldsymbol{G}(\boldsymbol{X}(\phi, \boldsymbol{r}))$, $f_j(\phi, \boldsymbol{r}) = (\mathrm{grad}_{\boldsymbol{X}} r_j) \cdot \boldsymbol{F}(\boldsymbol{X}(\phi, \boldsymbol{r}))$, $g_j(\phi, \boldsymbol{r}) = (\mathrm{grad}_{\boldsymbol{X}} r_j) \cdot \boldsymbol{G}(\boldsymbol{X}(\phi, \boldsymbol{r}))$, where the gradients are evaluated at the point $\boldsymbol{X}(\phi, \boldsymbol{r})$. These functions h, f_j, and g_j are 2π-periodic with respect to ϕ. The functions $Z(\phi)$ and $Y(\phi)$ in Eq. (2·2) are given in the $(\phi, \boldsymbol{r})$ coordinates as

$$Z(\phi) = h(\phi, \boldsymbol{a}), \quad Y(\phi) = \sum_{j=1}^{N-1} \frac{\partial h(\phi, \boldsymbol{a})}{\partial r_j} g_j(\phi, \boldsymbol{a}). \tag{A·4}$$

Since h and g_j are 2π-periodic, $Z(\phi)$ and $Y(\phi)$ are also 2π-periodic: $Z(\phi+2\pi) = Z(\phi)$ and $Y(\phi + 2\pi) = Y(\phi)$. When the term $\delta\boldsymbol{F}_i$ exists, only ω and f_j are modified in Eqs. (A·2) and (A·3), while h and g_j do not change. Therefore, $Z(\phi)$ and $Y(\phi)$ are also the same in the case of $\delta\boldsymbol{F}_i \neq \boldsymbol{0}$.

Appendix B
—— *Proof of* $\lim_{\theta\to s_m} Q_s(\theta) = 1/2\pi$ ——

For simplicity, we assume the case $n = 1$ and show the outline of the proof of $\lim_{\theta\to 0} Q_s(\theta) = 1/2\pi$. In this case, Q_s is given by

$$Q_s(\theta) = \frac{\varepsilon}{2\pi u(\theta)} \int_\theta^{2\pi} \exp\left[-\varepsilon \int_\theta^x \frac{1}{u(y)} dy\right] dx. \tag{B·1}$$

If we introduce the new integration variables $s = x - \theta$ and $t = y - \theta$, then Eq. (B·1) is rewritten as

$$Q_s(\theta) = \frac{\varepsilon}{2\pi u(\theta)} \int_0^{2\pi-\theta} \exp\left[-\varepsilon \int_0^s \frac{1}{u(\theta+t)} dt\right] ds. \tag{B·2}$$

Assume $\theta \simeq 0$ and recall the facts that $u(\theta) \geq 0$ and $u(0) = 0$. When $\theta \simeq 0$ and t is small, $1/u(\theta + t) \gg 1$ holds. Thus, the integrand $\exp[-\varepsilon \int_0^s 1/u(\theta + t)dt]$ is a rapidly decreasing function of s. Because of this fact, it can be proved that, in Eq. (B·2), the dominant contribution to the integral comes from the interval $s \in [0, \theta^{3/2}]$. Therefore, we can obtain

$$Q_s(\theta) \simeq \frac{\varepsilon}{2\pi u(\theta)} \int_0^{\theta^{3/2}} \exp\left[-\varepsilon \int_0^s \frac{1}{u(\theta)} \{1 + R(t)\}\, dt\right] ds, \tag{B·3}$$

where we expanded $1/u(\theta + t)$ with respect to t. The term $R(t)$ is given by $R(t) = -u(\theta)(u'(\eta)/u(\eta)^2) \cdot t$, where $\theta \leq \eta \leq \theta + t$ and η depends on t. We estimate the order of $R(t)$. The function $u(\theta)$ can be expanded as $u(\theta) = (u''(0)/2)\,\theta^2 + o(\theta^2)$, where $u''(0) = \int_0^{2\pi} \{Z'(\phi)\}^2 d\phi/2\pi \neq 0$ whenever $Z(\phi)$ is not a constant. As for η, we can use the evaluation $\eta = \theta$ since $\eta \in [\theta, \theta + \theta^{3/2}]$. In addition, note that t is at most $t = \theta^{3/2}$. If we use the expansion of u and the estimations $\eta = \theta$ and $t = \theta^{3/2}$, then we have $R(t) = O(\theta^{1/2})$. Therefore, the term $R(t)$ is negligible in taking the limit $\theta \to 0$. We remark that this fact can be proved, however, only we have roughly explained it. If we neglect the term $R(t)$, we arrive at

$$\lim_{\theta\to 0} Q_s(\theta) = \lim_{\theta\to 0} \frac{\varepsilon}{2\pi u(\theta)} \int_0^{\theta^{3/2}} \exp\left[-\frac{\varepsilon}{u(\theta)} s\right] ds, \tag{B·4}$$

$$= \lim_{\theta\to 0} \frac{1}{2\pi} \left(1 - \exp\left[-\frac{\varepsilon}{u(\theta)} \theta^{3/2}\right]\right), \tag{B·5}$$

$$= \frac{1}{2\pi}, \tag{B·6}$$

where we used $u(\theta) = (u''(0)/2)\,\theta^2 + o(\theta^2)$ in the last equality.

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