PSU$(2,2|4)$ Transformations of IIB Superstring in AdS$_5 \times$ S^5

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Abstract

The PSU$(2,2|4)$ transformation laws of the IIB superstring theory in the AdS$_5 \times$ S^5 background are explicitly obtained for the light-cone gauge in the Green-Schwarz formalism.

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1. Introduction

The construction and the quantization of the superstring theory in anti de Sitter (AdS) spacetime have been an important subject since the original AdS/CFT correspondence [1, 2, 3] was proposed. Metsaev and Tseytlin [4] constructed the Green-Schwarz type action of the type IIB superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ as a sigma model with a coset target space $\mathrm{PSU}(2,2|4)/[\mathrm{SO}(4,1) \times \mathrm{SO}(5)]$. Then the light-cone gauge-fixing of κ transformations and reparametrizations on the worldsheet were discussed in refs. [5, 6].

In this paper we discuss a global symmetry of the type IIB superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ by using a group theoretical method. The symmetry is represented by the supergroup $\mathrm{PSU}(2,2|4)$. We use the worldsheet action of ref. [5], where the κ symmetry is fixed by the light-cone gauge. We obtain explicit forms of the transformation laws for the symmetry $\mathrm{PSU}(2,2|4)$ in the light-cone gauge. The transformation laws we obtain will be useful in constructing the Noether charges for this symmetry [6]. They are also useful in finding consistent truncations of the theory, which are needed in some recent investigations of the gauge/string correspondence [7, 8, 9, 10].

2. IIB superstring in $\mathbf{AdS}_5 \times \mathbf{S}^5$

The type IIB superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ can be described [4] as a sigma model with a target space $\mathrm{PSU}(2,2|4)/[\mathrm{SO}(4,1) \times \mathrm{SO}(5)]$. The supergroup $\mathrm{PSU}(2,2|4)$ contains a bosonic subgroup $\mathrm{SO}(4,2) \times \mathrm{SO}(6)$, which is the isometry of $\mathrm{AdS}_5 \times \mathrm{S}^5$. Its generators are

$$T^{\hat{I}} = P^a, J^{ab}, D, K^a, J^i{}_j, Q^{\pm i}, S^{\pm i}, \tag{1}$$

where P^a, J^{ab}, D, K^a are SO(4,2) generators, $J^i{}_j$ are $\mathrm{SU}(4) \sim \mathrm{SO}(6)$ generators, and $Q^{\pm i}, S^{\pm i}$ are fermionic generators. Here, $a, b, \cdots = 0, 1, 2, 3$ and $i, j, \cdots = 1, 2, 3, 4$ denote SO(3,1) and SU(4) indices. The (anti-)commutation relations of these generators are given in ref. [5], whose conventions we use throughout this paper. The generators of the subalgebra $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ are

$$J^{ab}, \quad \hat{J}^{4a} = K^a + \frac{1}{2}P^a, \quad J^{A'B'} = -\frac{1}{2}(\gamma^{A'B'})^j{}_i J^i{}_j, \tag{2}$$

where $A', B' = 1,2,3,4,5$ are SO(5) indices and $\gamma^{A'}$ are SO(5) gamma matrices. We use the light-cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}}(x^3 \pm x^0)$, $x = \frac{1}{\sqrt{2}}(x^1 + ix^2)$, $\bar{x} = \frac{1}{\sqrt{2}}(x^1 - ix^2)$ and define $P = P^x$, $\bar{P} = P^{\bar{x}}$, $K = K^x$, $\bar{K} = K^{\bar{x}}$.

We choose a representative of the coset space PSU$(2,2|4)$/[SO(4,1) $\times$ SO(5)] as

$$\begin{aligned} G = &\exp(x^a P^a) \exp(\theta^{-i} Q_i^+ + \theta_i^- Q^{+i} + \theta^{+i} Q_i^- + \theta_i^+ Q^{-i}) \\ &\times \exp(\eta^{-i} S_i^+ + \eta_i^- S^{+i} + \eta^{+i} S_i^- + \eta_i^+ S^{-i}) \exp(\phi D) \\ &\times \exp\left(\frac{1}{2} i y^{A'} (\gamma^{A'})^i{}_j J^j{}_i\right), \end{aligned} \tag{3}$$

where $\theta_i^{\pm} = (\theta^{\pm i})^\dagger$, $\eta_i^{\pm} = (\eta^{\pm i})^\dagger$, $Q_i^{\pm} = (Q^{\pm i})^\dagger$, $S_i^{\pm} = (S^{\pm i})^\dagger$. The variables x^a, ϕ, $y^{A'}$, $\theta^{\pm i}$, $\eta^{\pm i}$ are coordinates of the coset space. We then fix the κ symmetry by the light-cone gauge condition [5]

$$\theta^{+i} = \eta^{+i} = 0 \tag{4}$$

and put $\theta^{-i} = \theta^i$, $\eta^{-i} = \eta^i$ for simplicity. The left-invariant Cartan one-forms $L^{\hat{I}}$ are defined by

$$\begin{aligned} G^{-1} dG &= L^{\hat{I}} T^{\hat{I}} \\ &= L_P{}^a P^a + \frac{1}{2} L^{ab} J^{ab} + L_D D + L_K{}^a K^a + L^j{}_i J^i{}_j + L_Q^{-i} Q_i^+ + L_{Qi}^- Q^{+i} \\ &\quad + L_Q^{+i} Q_i^- + L_{Qi}^+ Q^{-i} + L_S^{-i} S_i^+ + L_{Si}^- S^{+i} + L_S^{+i} S_i^- + L_{Si}^+ S^{-i}. \end{aligned} \tag{5}$$

Using the explicit forms of the Cartan one-forms the world-sheet action in the light-cone gauge was obtained in ref. [5].

3. PSU$(2,2|4)$ transformations

According to the general theory of the nonlinear realization [11, 12] the PSU$(2,2|4)$ transformation of the representative (3) is

$$G \to G' = gGh^{-1}(g), \tag{6}$$

where g is an arbitrary element of PSU$(2,2|4)$, and $h(g)$ is a compensating SO(4,1) $\times$ SO(5) transformation which is chosen such that G' has a form in eq. (3). After the light-cone gauge fixing of the κ symmetry (4) we also need a compensating κ transformation. An infinitesimal PSU$(2,2|4)$ transformation is thus written as

$$G^{-1}\delta G = G^{-1}\epsilon G - \sigma(\epsilon) + G^{-1}\delta_\kappa G, \tag{7}$$

where ϵ is an arbitrary element of the PSU$(2,2|4)$ algebra

$$\begin{aligned}\epsilon = \xi^a P^a + \frac{1}{2}\lambda^{ab} J^{ab} + \Lambda D + \zeta^a K^a + v^j{}_i J^i{}_j + \epsilon^{-i} Q_i^+ + \epsilon_i^- Q^{+i} \\ +\epsilon^{+i} Q_i^- + \epsilon_i^+ Q^{-i} + \beta^{-i} S_i^+ + \beta_i^- S^{+i} + \beta^{+i} S_i^- + \beta_i^+ S^{-i}\end{aligned} \tag{8}$$

and $\sigma(\epsilon)$ is a compensating SO(4,1) $\times$ SO(5) transformation

$$\sigma(\epsilon) = \frac{1}{2}\tilde{\lambda}^{ab} J^{ab} + \tilde{\xi}^a \hat{J}^{4a} + \frac{1}{2}\tilde{v}^{A'B'} J^{A'B'}. \tag{9}$$

The last term in eq. (7) is a compensating κ transformation. The parameters $\tilde{\lambda}^{ab}$, $\tilde{\xi}^a$, $\tilde{v}^{A'B'}$ and those of the κ transformation depend on ϵ.

The general κ transformation has a form [4]

$$\begin{aligned}G^{-1}\delta_\kappa G = \tilde{\kappa}_Q^{-i} Q_i^+ + \tilde{\kappa}_{Qi}^- Q^{+i} + \tilde{\kappa}_Q^{+i} Q_i^- + \tilde{\kappa}_{Qi}^+ Q^{-i} \\ +\tilde{\kappa}_S^{-i} S_i^+ + \tilde{\kappa}_{Si}^- S^{+i} + \tilde{\kappa}_S^{+i} S_i^- + +\tilde{\kappa}_{Si}^+ S^{-i} + (J^{ab},\ J^{A'B'} \text{ terms}).\end{aligned} \tag{10}$$

The coefficients in the present convention are given by

$$\begin{aligned}\tilde{\kappa}_Q^{+i} &= 2i\left[\hat{L}_\mu{}^+ \kappa_S^{\mu-i} + \hat{L}_\mu{}^{\bar{x}} \kappa_S^{\mu+i} + i\hat{L}_\mu{}^4 \kappa_Q^{\mu+i} - L_\mu{}^{A'}(\gamma^{A'})^i{}_j \kappa_Q^{\mu+j}\right], \\ \tilde{\kappa}_Q^{-i} &= 2i\left[\hat{L}_\mu{}^- \kappa_S^{\mu+i} - \hat{L}_\mu{}^x \kappa_S^{\mu-i} + i\hat{L}_\mu{}^4 \kappa_Q^{\mu-i} - L_\mu{}^{A'}(\gamma^{A'})^i{}_j \kappa_Q^{\mu-j}\right], \\ \tilde{\kappa}_S^{+i} &= -2i\left[2\hat{L}_\mu{}^+ \kappa_Q^{\mu-i} + 2\hat{L}_\mu{}^x \kappa_Q^{\mu+i} + i\hat{L}_\mu{}^4 \kappa_S^{\mu+i} + L_\mu{}^{A'}(\gamma^{A'})^i{}_j \kappa_S^{\mu+j}\right], \\ \tilde{\kappa}_S^{-i} &= -2i\left[2\hat{L}_\mu{}^- \kappa_Q^{\mu+i} - 2\hat{L}_\mu{}^{\bar{x}} \kappa_Q^{\mu-i} + i\hat{L}_\mu{}^4 \kappa_S^{\mu-i} + L_\mu{}^{A'}(\gamma^{A'})^i{}_j \kappa_S^{\mu-j}\right],\end{aligned} \tag{11}$$

where $\mu = 0, 1$ is a world index on the worldsheet and $\kappa_Q^{\mu\pm i}$, $\kappa_S^{\mu\pm i}$ on the right-hand sides are independent transformation parameters. The $\hat{L}_\mu{}^a$, $\hat{L}_\mu{}^4$, $L_\mu{}^{A'}$ are the pullbacks of the following one-forms to the worldsheet

$$\hat{L}^a = L_P^a - \frac{1}{2}L_K^a, \qquad \hat{L}^4 = -L_D, \qquad L^{A'} = -\frac{1}{2}i(\gamma^{A'})^i{}_j L^j{}_i. \tag{12}$$

For a general variation of the variables $X^M = (x^a, \phi, y^{A'}, \theta^i, \eta^i)$ the variation of G in eq. (3) is given by

$$\begin{aligned}G^{-1}\delta G &= \delta X^M L_M{}^{\hat{I}} T^{\hat{I}} \\ &= e^\phi \delta x^+ P^- + e^\phi\left[\delta x^- - \frac{1}{2}i(\theta^i\delta\theta_i + \theta_i\delta\theta^i)\right]P^+ + e^\phi \delta x \bar{P} + e^\phi \delta\bar{x} P \\ &\quad +e^{-\phi}\left[\frac{1}{4}(\eta^2)^2\delta x^+ + \frac{1}{2}i(\eta^i\delta\eta_i + \eta_i\delta\eta^i)\right]K^+ + \delta\phi D\end{aligned}$$

$$
\begin{aligned}
&+\left[(\delta U U^{-1})^i{}_j + i\left(\tilde{\eta}^i\tilde{\eta}_j - \frac{1}{4}\eta^2\delta^i_j\right)\delta x^+\right] J^j{}_i + e^{\frac{1}{2}\phi}\left(\widetilde{\delta\theta}^i + i\tilde{\eta}^i\delta x\right) Q^+_i \\
&+e^{\frac{1}{2}\phi}\left(\widetilde{\delta\theta}_i - i\tilde{\eta}_i\delta\bar{x}\right) Q^{+i} - ie^{\frac{1}{2}\phi}\tilde{\eta}^i\delta x^+ Q^-_i + ie^{\frac{1}{2}\phi}\tilde{\eta}_i\delta x^+ Q^{-i} \\
&+e^{-\frac{1}{2}\phi}\left(\widetilde{\delta\eta}^i + \frac{1}{2}i\eta^2\tilde{\eta}^i\delta x^+\right) S^+_i + e^{-\frac{1}{2}\phi}\left(\widetilde{\delta\eta}_i - \frac{1}{2}i\eta^2\tilde{\eta}_i\delta x^+\right) S^{+i}, \qquad (13)
\end{aligned}
$$

where we have used the explicit forms of the Cartan one-form in the light-cone gauge given in ref. [5]. $U^i{}_j$ is the SU(4) matrix determined by the coordinates $y^{A'}$

$$
U = \cos\frac{|y|}{2} + i\gamma^{A'}n^{A'}\sin\frac{|y|}{2}, \qquad (14)
$$

where $|y|^2 = y^{A'}y^{A'}$, $n^{A'} = y^{A'}/|y|$, and $\tilde{\theta}^i = U^i{}_j\theta^j$, $\tilde{\theta}_i = \theta_j(U^{-1})^j{}_i$, etc. The compensating transformations in eq. (7) are chosen such that the total transformation (7) has this form.

We are now ready to obtain explicit forms of the PSU(2,2|4) transformations. We first compute the first term in eq. (7). Useful formulae to do this are listed in Appendix. Then, we choose compensating transformations in the second and the third terms such that the total transformations take the form in eq. (13). Comparing the results of these computations and eq. (13) we obtain the PSU(2,2|4) transformations of the variables X^M.

The transformations for P^a, D, J^{+-}, J^{+x}, $J^{x\bar{x}}$, $J^{A'B'}$ and Q^+ do not need compensating κ transformations and are easy to obtain. They were already given in ref. [6]. We give them here for completeness.

- P^a transformations:

$$
\delta x^a = \xi^a, \qquad \delta(\text{others}) = 0. \qquad (15)
$$

- D transformations:

$$
\delta x^a = -\Lambda x^a, \quad \delta\phi = \Lambda, \quad (U^{-1}\delta U)^i{}_j = 0, \quad \delta\theta^i = -\frac{1}{2}\Lambda\theta^i, \quad \delta\eta^i = \frac{1}{2}\Lambda\eta^i. \quad (16)
$$

- J^{+-} transformations:

$$
\begin{aligned}
&\delta x^+ = -\lambda^{-+}x^+, \quad \delta x^- = \lambda^{-+}x^-, \quad \delta x = \delta\phi = (U^{-1}\delta U)^i{}_j = 0, \\
&\delta\theta^i = \frac{1}{2}\lambda^{-+}\theta^i, \quad \delta\eta^i = \frac{1}{2}\lambda^{-+}\eta^i. \qquad (17)
\end{aligned}
$$

- J^{+x} and $J^{+\bar{x}}$ transformations:

$$
\delta x^- = \lambda^{-\bar{x}}x + \lambda^{-x}\bar{x}, \qquad \delta x = -\lambda^{-x}x^+, \qquad \delta(\text{others}) = 0. \qquad (18)
$$

- $J^{x\bar{x}}$ transformations:

$$\delta x = -\lambda^{\bar{x}x} x, \quad \delta\theta^i = -\frac{1}{2}\lambda^{\bar{x}x}\theta^i, \quad \delta\eta^i = \frac{1}{2}\lambda^{\bar{x}x}\eta^i, \quad \delta(\text{others}) = 0. \tag{19}$$

- $J^i{}_j$ transformations:

$$\begin{aligned} \delta x^a = \delta\phi = 0, \qquad \delta\theta^i = -v^i{}_j\theta^j, \qquad \delta\eta^i = -v^i{}_j\eta^j, \\ (U^{-1}\delta U)^i{}_j = v^i{}_j + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j. \end{aligned} \tag{20}$$

- Q^{+i} and Q^+_i transformations:

$$\delta x^- = \frac{1}{2}i\epsilon^-_i\theta^i + \frac{1}{2}i\epsilon^{-i}\theta_i, \qquad \delta\theta^i = \epsilon^{-i}, \qquad \delta(\text{others}) = 0. \tag{21}$$

The transformations for K^+ do not need a compensating κ transformation either.

- K^+ transformations:

$$\begin{aligned} \delta x^a &= \zeta^-\left(x^+x^a - \frac{1}{2}x\cdot x\,\eta^{a+} - \frac{1}{2}e^{-2\phi}\eta^{a+}\right), \qquad \delta\phi = -\zeta^- x^+, \\ \delta\eta^i &= -\zeta^- x^+\eta^i, \qquad \delta(\text{others}) = 0. \end{aligned} \tag{22}$$

The compensating SO(5) transformation with the parameter $\tilde{v}^{A'B'}$ in eq. (20) is not yet fixed. We will determine it and obtain the PSU$(2,2|4)$ transformation of the independent variables $y^{A'}$ in sect. 4.

Other transformations need compensating κ transformations and are more involved.

- J^{-x} and $J^{-\bar{x}}$ transformations:

$$\begin{aligned} \delta x^+ &= \lambda^{+\bar{x}}x + \lambda^{+x}\bar{x}, \\ \delta x^- &= \frac{1}{4}ie^{-\frac{3}{2}\phi}(\eta^i\hat{\kappa}^-_{Si} + \eta_i\hat{\kappa}^{-i}_S) + \frac{1}{2}ie^{-\frac{1}{2}\phi}(\theta^i\hat{\kappa}^-_{Qi} + \theta_i\hat{\kappa}^{-i}_Q) \\ &\quad -\frac{1}{4}ie^{-2\phi}(\lambda^{+x}\theta_i\eta^i + \lambda^{+\bar{x}}\theta^i\eta_i)\eta^2, \\ \delta x &= -\lambda^{+x}\left(x^- - \frac{1}{2}i\theta^2 + \frac{1}{4}ie^{-2\phi}\eta^2\right), \\ \delta\phi &= -\frac{1}{2}(\lambda^{+\bar{x}}\theta^i\eta_i - \lambda^{+x}\theta_i\eta^i), \\ (U^{-1}\delta U)^i{}_j &= \lambda^{+\bar{x}}\theta^i\eta_j + \lambda^{+x}\theta_j\eta^i - (\text{trace part}) + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j, \\ \delta\theta^i &= e^{-\frac{1}{2}\phi}\hat{\kappa}^{-i}_Q - \frac{1}{4}\lambda^{+x}e^{-2\phi}\eta^2\eta^i, \\ \delta\eta^i &= e^{\frac{1}{2}\phi}\hat{\kappa}^{-i}_S + \frac{1}{2}\lambda^{+\bar{x}}\eta^2\theta^i + \lambda^{+x}\theta_j\eta^j\eta^i, \end{aligned} \tag{23}$$

where we have defined

$$\hat{\kappa}_Q^{\pm i} = (U^{-1})^i{}_j \tilde{\kappa}_Q^{\pm j}, \qquad \hat{\kappa}_S^{\pm i} = (U^{-1})^i{}_j \tilde{\kappa}_S^{\pm j}. \tag{24}$$

To obtain the form (13) we need to choose the parameters of the κ transformation as

$$\hat{\kappa}_Q^{+i} = \lambda^{+\bar{x}} e^{\frac{1}{2}\phi}\theta^i, \qquad \hat{\kappa}_S^{+i} = \lambda^{+x} e^{-\frac{1}{2}\phi}\eta^i. \tag{25}$$

As we will see in sect. 4 $\hat{\kappa}_Q^{-i}$, $\hat{\kappa}_S^{-i}$ in the transformation (23) are determined from these conditions. Similarly, we obtain other transformations and conditions on the parameters of the κ transformations as follows.

- K and $\bar{K}$ transformations:

$$\begin{aligned}
\delta x^+ &= (\bar{\zeta}x + \zeta\bar{x})x^+, \\
\delta x^- &= \frac{1}{4}ie^{-\frac{3}{2}\phi}(\eta^i\hat{\kappa}_{Si}^- + \eta_i\hat{\kappa}_S^{-i}) + \frac{1}{2}ie^{-\frac{1}{2}\phi}(\theta^i\hat{\kappa}_{Qi}^- + \theta_i\hat{\kappa}_Q^{-i}) \\
&\quad + \bar{\zeta}x\left(x^- + \frac{1}{2}i\theta^2\right) + \zeta\bar{x}\left(x^- - \frac{1}{2}i\theta^2\right) + \frac{1}{4}ie^{-2\phi}x^+(\bar{\zeta}\theta^i\eta_i - \zeta\theta_i\eta^i)\eta^2, \\
\delta x &= \bar{\zeta}x^2 - \zeta x^+\left(x^- - \frac{1}{2}i\theta^2\right) - \frac{1}{2}\zeta e^{-2\phi}\left(1 + \frac{1}{2}ix^+\eta^2\right), \\
\delta\phi &= -(\bar{\zeta}x + \zeta\bar{x}) - \frac{1}{2}x^+(\bar{\zeta}\theta^i\eta_i - \zeta\theta_i\eta^i), \\
(U^{-1}\delta U)^i{}_j &= x^+(\bar{\zeta}\,\theta^i\eta_j + \zeta\,\theta_j\eta^i) - (\text{trace part}) + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j, \\
\delta\theta^i &= e^{-\frac{1}{2}\phi}\hat{\kappa}_Q^{-i} + (\bar{\zeta}x + \zeta\bar{x})\theta^i + \frac{1}{2}i\zeta e^{-2\phi}\left(1 + \frac{1}{2}ix^+\eta^2\right)\eta^i, \\
\delta\eta^i &= e^{\frac{1}{2}\phi}\hat{\kappa}_S^{-i} + i\bar{\zeta}\left(1 - \frac{1}{2}ix^+\eta^2\right)\theta^i - \bar{\zeta}x\eta^i + \zeta x^+\theta_j\eta^j\eta^i,
\end{aligned} \tag{26}$$

$$\hat{\kappa}_Q^{+i} = e^{\frac{1}{2}\phi}\bar{\zeta}x^+\theta^i, \qquad \hat{\kappa}_S^{+i} = e^{-\frac{1}{2}\phi}\zeta x^+\eta^i. \tag{27}$$

- K^- transformations:

$$\begin{aligned}
\delta x^+ &= \zeta^+\left(x^-x^+ - \frac{1}{2}x\cdot x - \frac{1}{2}e^{-2\phi}\right), \\
\delta x^- &= \zeta^+\Big[\frac{1}{2}ie^{-\frac{1}{2}\phi}(\theta^i\hat{\kappa}_{Qi}^- + \theta_i\hat{\kappa}_Q^{-i}) + \frac{1}{4}ie^{-\frac{3}{2}\phi}(\eta^i\hat{\kappa}_{Si}^- + \eta_i\hat{\kappa}_S^{-i}) + (x^-)^2 - \frac{1}{4}(\theta^2)^2 \\
&\quad + \frac{1}{2}e^{-2\phi}\theta^i\eta_i\theta_j\eta^j + \frac{1}{4}ie^{-2\phi}\eta^2(x\theta_i\eta^i + \bar{x}\theta^i\eta_i) + \frac{1}{16}e^{-4\phi}(\eta^2)^2\Big], \\
\delta x &= \zeta^+\left[\left(x^- - \frac{1}{2}i\theta^2\right)x + \frac{1}{2}e^{-2\phi}\left(\theta^i\eta_i + \frac{1}{2}ix\eta^2\right)\right],
\end{aligned}$$

$$
\begin{aligned}
\delta\phi &= \zeta^+\left(-x^- - \frac{1}{2}x\theta_i\eta^i + \frac{1}{2}\bar{x}\theta^i\eta_i\right),\\
(U^{-1}\delta U)^i{}_j &= -i\zeta^+\left(\theta^i\theta_j + ix\eta^i\theta_j - i\bar{x}\theta^i\eta_j - \frac{1}{2}e^{-2\phi}\eta^i\eta_j\right) - (\text{trace part})\\
&\quad +\frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j,\\
\delta\theta^i &= \zeta^+\left[e^{-\frac{1}{2}\phi}\hat{\kappa}_Q^{-i} + \left(x^- - \frac{1}{2}i\theta^2\right)\theta^i - \frac{1}{2}ie^{-2\phi}\left(\theta^j\eta_j + \frac{1}{2}ix\eta^2\right)\eta^i\right],\\
\delta\eta^i &= \zeta^+\left[e^{\frac{1}{2}\phi}\hat{\kappa}_S^{-i} + i\theta_j\eta^j(\theta^i + ix\eta^i) - \frac{1}{2}\bar{x}\eta^2\theta^i + \frac{1}{4}ie^{-2\phi}\eta^2\eta^i\right],
\end{aligned}
\tag{28}
$$

$$
\hat{\kappa}_Q^{+i} = \zeta^+\left(-\bar{x}e^{\frac{1}{2}\phi}\theta^i + \frac{1}{2}ie^{-\frac{3}{2}\phi}\eta^i\right), \qquad \hat{\kappa}_S^{+i} = \zeta^+e^{-\frac{1}{2}\phi}i\left(\theta^i + ix\eta^i\right).
\tag{29}
$$

- Q^{-i} and Q^-_i transformations:

$$
\begin{aligned}
\delta x^+ &= 0, \qquad \delta x = i\epsilon_i^+\theta^i, \qquad \delta\phi = -\frac{1}{2}(\epsilon_i^+\eta^i - \epsilon^{+i}\eta_i),\\
\delta x^- &= \frac{1}{2}ie^{-\frac{1}{2}\phi}(\theta^i\hat{\kappa}^-_{Qi} + \theta_i\hat{\kappa}_Q^{-i}) + \frac{1}{4}ie^{-\frac{3}{2}\phi}(\eta^i\hat{\kappa}^-_{Si} + \eta_i\hat{\kappa}_S^{-i})\\
&\quad +\frac{1}{8}ie^{-2\phi}\eta^2(\epsilon_i^+\eta^i + \epsilon^{+i}\eta_i),\\
(U^{-1}\delta U)^i{}_j &= -(\epsilon_j^+\eta^i + \epsilon^{+i}\eta_j) - (\text{trace part}) + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j,\\
\delta\theta^i &= e^{-\frac{1}{2}\phi}\hat{\kappa}_Q^{-i}, \qquad \delta\eta^i = e^{\frac{1}{2}\phi}\hat{\kappa}_S^{-i} - \epsilon^+{}_j\eta^j\eta^i - \frac{1}{2}\eta^2\epsilon^{+i},
\end{aligned}
\tag{30}
$$

$$
\hat{\kappa}_Q^{+i} = -e^{\frac{1}{2}\phi}\epsilon^{+i}, \qquad \hat{\kappa}_S^{+i} = 0.
\tag{31}
$$

- S^{+i} and S^+_i transformations:

$$
\begin{aligned}
\delta x^+ &= 0, \qquad \delta x = x^+\beta_i^-\theta^i, \qquad \delta\phi = \frac{1}{2}ix^+(\beta_i^-\eta^i + \beta^{-i}\eta_i),\\
\delta x^- &= \frac{1}{2}i\theta^i(e^{-\frac{1}{2}\phi}\hat{\kappa}^-_{Qi} - i\bar{x}\beta_i^-) + \frac{1}{2}i\theta_i(e^{-\frac{1}{2}\phi}\hat{\kappa}_Q^{-i} + ix\beta^{-i})\\
&\quad +\frac{1}{4}ie^{-2\phi}\eta^i\left[e^{\frac{1}{2}\phi}\hat{\kappa}^-_{Si} - \left(1 - \frac{1}{2}ix^+\eta^2\right)\beta_i^-\right]\\
&\quad +\frac{1}{4}ie^{-2\phi}\eta_i\left[e^{\frac{1}{2}\phi}\hat{\kappa}_S^{-i} - \left(1 + \frac{1}{2}ix^+\eta^2\right)\beta^{-i}\right],\\
(U^{-1}\delta U)^i{}_j &= ix^+(\beta_j^-\eta^i - \beta^{-i}\eta_j) - (\text{trace part}) + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j,\\
\delta\theta^i &= e^{-\frac{1}{2}\phi}\hat{\kappa}_Q^{-i} - ix\beta^{-i},\\
\delta\eta^i &= e^{\frac{1}{2}\phi}\hat{\kappa}_S^{-i} + \left(1 - \frac{1}{2}ix^+\eta^2\right)\beta^{-i} + ix^+\beta_j^-\eta^j\eta^i,
\end{aligned}
\tag{32}
$$

$$\hat{\kappa}_Q^{+i} = -ix^+ e^{\frac{1}{2}\phi}\beta^{-i}, \qquad \hat{\kappa}_S^{+i} = 0. \tag{33}$$

- S^{-i} and S_i^- transformations:

$$\begin{aligned}
\delta x^+ &= 0, \qquad \delta x = x\beta_i^+\theta^i + \frac{1}{2}ie^{-2\phi}\beta^{+i}\eta_i, \\
\delta x^- &= \frac{1}{4}ie^{-\frac{3}{2}\phi}(\eta^i\hat{\kappa}_{Si}^- + \eta_i\hat{\kappa}_S^{-i}) + \frac{1}{2}ie^{-\frac{1}{2}\phi}(\theta^i\hat{\kappa}_{Qi}^- + \theta_i\hat{\kappa}_Q^{-i}) \\
&\quad + \frac{1}{4}ie^{-2\phi}\left[\left(\theta_j - \frac{1}{2}i\bar{x}\eta_j\right)\eta^j\beta^{+i}\eta_i - \left(\theta^j + \frac{1}{2}ix\eta^j\right)\eta_j\beta^+{}_i\eta^i\right] \\
&\quad + \frac{1}{2}\left(x^- - \frac{1}{2}i\theta^2\right)\beta^+{}_i\theta^i - \frac{1}{2}\left(x^- + \frac{1}{2}i\theta^2\right)\beta^{+i}\theta_i, \\
\delta\phi &= -\frac{1}{2}\beta_i^+(\theta^i - ix\eta^i) + \frac{1}{2}\beta^{+i}(\theta_i + i\bar{x}\eta_i), \\
(U^{-1}\delta U)^i{}_j &= \beta_j^+(\theta^i + ix\eta^i) + \beta^{+i}(\theta_j - i\bar{x}\eta_j) - (\text{trace part}) + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j, \\
\delta\theta^i &= e^{-\frac{1}{2}\phi}\hat{\kappa}_Q^{-i} + \beta_j^+\theta^j\theta^i + i\left(x^- - \frac{1}{2}i\theta^2\right)\beta^{+i} + \frac{1}{2}e^{-2\phi}\beta^{+j}\eta_j\eta^i, \\
\delta\eta^i &= e^{\frac{1}{2}\phi}\hat{\kappa}_S^{-i} + \beta^+{}_j\eta^j(\theta^i + ix\eta^i) - \left(\theta_j - \frac{1}{2}i\bar{x}\eta_j\right)\eta^j\beta^{+i},
\end{aligned} \tag{34}$$

$$\hat{\kappa}_Q^{+i} = -i\bar{x}e^{\frac{1}{2}\phi}\beta^{+i}, \qquad \hat{\kappa}_S^{+i} = -e^{-\frac{1}{2}\phi}\beta^{+i}. \tag{35}$$

4. Compensating SO(5) and κ transformations

Here we fix the compensating SO(5) and κ transformations left undetermined above. Let us first consider the SO(5) transformations. The PSU$(2,2|4)$ transformations of U obtained in eqs. (20), (23), (26), (28), (30), (32), (34) have a form

$$(U^{-1}\delta U)^i{}_j = v^i{}_j + \frac{1}{4}\tilde{v}^{A'B'}(U^{-1}\gamma^{A'B'}U)^i{}_j, \tag{36}$$

where $v^i{}_j$ is a given function of the variables X^M and the transformation parameters, and $\tilde{v}^{A'B'}$ represents a compensating SO(5) transformation. On the other hand, a variation of the independent variables $y^{A'}$ in eq. (14) gives

$$\begin{aligned}
U^{-1}\delta U &= \frac{1}{2}i\gamma^{A'}n^{A'}n^{B'}\delta y^{B'} + \frac{1}{2}i\frac{\sin|y|}{|y|}\gamma^{A'}(\delta^{A'B'} - n^{A'}n^{B'})\delta y^{B'} \\
&\quad + \frac{1}{|y|}\sin^2\frac{|y|}{2}\gamma^{A'B'}n^{A'}\delta y^{B'}.
\end{aligned} \tag{37}$$

We choose the compensating SO(5) transformations such that eq. (36) has the form (37). Decomposing $v^i{}_j$ in eq. (36) as

$$v^i{}_j = \frac{1}{2} i v^{A'} (\gamma^{A'})^i{}_j - \frac{1}{4} v^{A'B'} (\gamma^{A'B'})^i{}_j \tag{38}$$

we find that $\tilde{v}^{A'B'}$ and the PSU$(2,2|4)$ transformations of $y^{A'}$ are given by

$$\begin{aligned}
\tilde{v}^{A'B'} &= v^{A'B'} + 2\tan\frac{|y|}{2} n^{[A'} v^{B']}, \\
\delta y^{A'} &= v^{A'B'} y^{B'} + \left[n^{A'} n^{B'} + \frac{|y|}{\tan|y|} (\delta^{A'B'} - n^{A'} n^{B'}) \right] v^{B'}.
\end{aligned} \tag{39}$$

Next, we shall obtain $\hat{\kappa}_Q^{-i}$, $\hat{\kappa}_S^{-i}$ from the conditions on $\hat{\kappa}_Q^{+i}$, $\hat{\kappa}_S^{+i}$ in eqs. (25), (27), (29), (31), (33), (35), which we write as

$$\hat{\kappa}_Q^{+i} = \tau_Q^i, \qquad \hat{\kappa}_S^{+i} = \tau_S^i. \tag{40}$$

From eq. (11) these conditions are satisfied if we choose the independent κ transformation parameters as

$$\kappa_S^{\mu - i} = -\frac{1}{4} i \frac{\tau_Q^i}{\hat{L}_\mu{}^+}, \qquad \kappa_Q^{\mu - i} = \frac{1}{8} i \frac{\tau_S^i}{\hat{L}_\mu{}^+}, \qquad \kappa_S^{\mu + i} = \kappa_Q^{\mu + i} = 0, \tag{41}$$

where $\mu = +, -$ are indices of the world-sheet light-cone coordinates. Substituting these equations into $\hat{\kappa}_Q^{-i}$, $\hat{\kappa}_S^{-i}$ in eq. (11) we obtain

$$\begin{aligned}
\hat{\kappa}_Q^{-i} &= -\frac{1}{2} \left(\frac{\partial_+ x}{\partial_+ x^+} + \frac{\partial_- x}{\partial_- x^+} \right) \tau_Q^i + \frac{1}{4} i e^{-\phi} \left(\frac{\partial_+ \phi}{\partial_+ x^+} + \frac{\partial_- \phi}{\partial_- x^+} \right) \tau_S^i \\
&\quad + \frac{1}{4} e^{-\phi} \left(\frac{L_+{}^{A'}}{\partial_+ x^+} + \frac{L_-{}^{A'}}{\partial_- x^+} \right) (\gamma^{A'})^i{}_j \tau_S^j, \\
\hat{\kappa}_S^{-i} &= -\frac{1}{2} \left(\frac{\partial_+ \bar{x}}{\partial_+ x^+} + \frac{\partial_- \bar{x}}{\partial_- x^+} \right) \tau_S^i + \frac{1}{2} i e^{-\phi} \left(\frac{\partial_+ \phi}{\partial_+ x^+} + \frac{\partial_- \phi}{\partial_- x^+} \right) \tau_Q^i \\
&\quad - \frac{1}{2} e^{-\phi} \left(n \frac{L_+{}^{A'}}{\partial_+ x^+} + \frac{L_-{}^{A'}}{\partial_- x^+} \right) (\gamma^{A'})^i{}_j \tau_Q^j,
\end{aligned} \tag{42}$$

where we have used the explicit forms of eq. (12) given in ref. [5]

$$\begin{aligned}
\hat{L}_\mu{}^+ &= e^\phi \partial_\mu x^+, \qquad \hat{L}_\mu{}^x = e^\phi \partial_\mu x, \qquad \hat{L}_\mu{}^4 = -\partial_\mu \phi, \\
L_\mu{}^{A'} &= -\frac{1}{2} i (\gamma^{A'})^j{}_i \left[(\partial_\mu U U^{-1})^i{}_j + i \left(\tilde{\eta}^i \tilde{\eta}_j - \frac{1}{4} \eta^2 \delta^i_j \right) \partial_\mu x^+ \right].
\end{aligned} \tag{43}$$

Using these $\hat{\kappa}^-$'s in eqs. (23), (26), (28), (30), (32), (34) we obtain explicit transformation laws.

From eq. (30) we see that the Q^- transformation of x^+ vanishes. This means in particular that the commutator of two Q^- transformations is zero on x^+, which at first sight looks inconsistent with the PSU$(2,2|4)$ algebra

$$\{Q^{-i}, Q_j^-\} = iP^-\delta_j^i. \tag{44}$$

This apparent inconsistency can be resolved as follows. Since we have not fixed a gauge for reparametrizations on the worldsheet, the commutator algebra closes up to a reparametrization. From eqs. (30), (31), (42) the commutator of two Q^- transformations on x, which should vanish according to the PSU$(2,2|4)$ algebra (44), becomes

$$[\delta_{Q^-}(\epsilon_1^+), \delta_{Q^-}(\epsilon_2^+)]x = (\xi^+\partial_+ + \xi^-\partial_-)x, \tag{45}$$

where

$$\xi^\pm = \frac{1}{2\,\partial_\pm x^+}\, i(\epsilon_{2i}^+\epsilon_1^{+i} - \epsilon_{1i}^+\epsilon_2^{+i}). \tag{46}$$

This is a reparametrization with the parameters $\xi^\pm$. As the reparametrization of x^+ with these parameters is

$$(\xi^+\partial_+ + \xi^-\partial_-)x^+ = i(\epsilon_{2i}^+\epsilon_1^{+i} - \epsilon_{1i}^+\epsilon_2^{+i}), \tag{47}$$

the commutator on x^+ can be written as

$$[\delta_{Q^-}(\epsilon_1^+), \delta_{Q^-}(\epsilon_2^+)]x^+ = -i(\epsilon_{2i}^+\epsilon_1^{+i} - \epsilon_{1i}^+\epsilon_2^{+i}) + (\xi^-\partial_- + \xi^+\partial_+)x^+. \tag{48}$$

The first term on the right-hand side is a P^- transformation of x^+ expected from the PSU$(2,2|4)$ algebra (44). Thus, the algebra (44) is satisfied up to a reparametrization.

Appendix

We summarize formulae useful in computing $G^{-1}\epsilon G$. From the formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots \tag{49}$$

we obtain the following identities.

$$e^{-x\cdot P} J^{ab} e^{x\cdot P} = J^{ab} - x^a P^b + x^b P^a,$$

$$
\begin{aligned}
e^{-x\cdot P} D e^{x\cdot P} &= D - x\cdot P,\\
e^{-x\cdot P} K^a e^{x\cdot P} &= K^a - x^a D + x^b J^{ba} + x^a x\cdot P - \frac{1}{2} x\cdot x\, P^a,\\
e^{-x\cdot P} S^{+i} e^{x\cdot P} &= S^{+i} - i x^+ Q^{-i} + i\bar{x} Q^{+i},\\
e^{-x\cdot P} S^{-i} e^{x\cdot P} &= S^{-i} - i x^- Q^{+i} - i x Q^{-i},\\
e^{-\theta\cdot Q^+} J^{+-} e^{\theta\cdot Q^+} &= J^{+-} + \frac{1}{2}\theta\cdot Q^+,\\
e^{-\theta\cdot Q^+} J^{x\bar{x}} e^{\theta\cdot Q^+} &= J^{x\bar{x}} - \frac{1}{2}(\theta^i Q^+_i - \theta_i Q^{+i}) - \frac{1}{2} i\theta^2 P^+,\\
e^{-\theta\cdot Q^+} J^{-x} e^{\theta\cdot Q^+} &= J^{-x} - \theta^i Q^-_i - \frac{1}{2} i\theta^2 P,\\
e^{-\theta\cdot Q^+} D e^{\theta\cdot Q^+} &= D - \frac{1}{2}\theta\cdot Q^+,\\
e^{-\theta\cdot Q^+} K e^{\theta\cdot Q^+} &= K + i\theta^i S^+_i + \frac{1}{2} i\theta^2 J^{+x},\\
e^{-\theta\cdot Q^+} K^- e^{\theta\cdot Q^+} &= K^- - i(\theta^i S^-_i - \theta_i S^{-i}) + \frac{1}{2} i\theta^2 J^{x\bar{x}} - i\theta^j \theta_i J^i{}_j\\
&\quad -\frac{1}{2} i\theta^2(\theta^i Q^+_i - \theta_i Q^{+i}) + \frac{1}{4}(\theta^2)^2 P^+,\\
e^{-\eta\cdot Q^+} J^i{}_j e^{\eta\cdot Q^+} &= J^i{}_j - \theta^i Q^+_j + \theta_j Q^{+i} - i\theta^i\theta_j P^+ - (\text{trace part}),\\
e^{-\theta\cdot Q^+} Q^{+i} e^{\theta\cdot Q^+} &= Q^{+i} + i\theta^i P^+,\\
e^{-\theta\cdot Q^+} Q^{-i} e^{\theta\cdot Q^+} &= Q^{-i} + i\theta^i \bar{P},\\
e^{-\theta\cdot Q^+} S^{+i} e^{\theta\cdot Q^+} &= S^{+i} - \theta^i J^{+\bar{x}},\\
e^{-\theta\cdot Q^+} S^{-i} e^{\theta\cdot Q^+} &= S^{-i} + \frac{1}{2}\theta^i (J^{+-} - J^{x\bar{x}} - D) + \theta^j J^i{}_j\\
&\quad +\frac{1}{2}\theta^2 Q^{+i} + \theta^i\theta^j Q^+_j + \frac{1}{2} i\theta^i\theta^2 P^+,\\
e^{-\eta\cdot S^+} P e^{\eta\cdot S^+} &= P - i\eta_i Q^{+i} + \frac{1}{2} i\eta^2 J^{+x},\\
e^{-\eta\cdot S^+} P^- e^{\eta\cdot S^+} &= P^- - i(\eta^i Q^-_i - \eta_i Q^{-i}) + \frac{1}{2} i\eta^2 J^{x\bar{x}} + i\eta^i\eta_j J^j{}_i\\
&\quad +\frac{1}{2} i\eta^2(\eta^i S^+_i - \eta_i S^{+i}) + \frac{1}{4}(\eta^2)^2 K^+,\\
e^{-\eta\cdot S^+} D e^{\eta\cdot S^+} &= D + \frac{1}{2}\eta\cdot S^+,\\
e^{-\eta\cdot S^+} J^{+-} e^{\eta\cdot S^+} &= J^{+-} + \frac{1}{2}\eta\cdot S^+,
\end{aligned}
$$

$$
\begin{aligned}
e^{-\eta\cdot S^+} J^{x\bar{x}} e^{\eta\cdot S^+} &= J^{x\bar{x}} + \frac{1}{2}(\eta^i S_i^+ - \eta_i S^{+i}) - \frac{1}{2} i\eta^2 K^+, \\
e^{-\eta\cdot S^+} J^{-x} e^{\eta\cdot S^+} &= J^{-x} - \eta_i S^{-i} - \frac{1}{2} i\eta^2 K, \\
e^{-\eta\cdot S^+} J^i{}_j e^{\eta\cdot S^+} &= J^i{}_j - \eta^i S_j^+ + \eta_j S^{+i} + i\eta^i\eta_j K^+ - (\text{trace part}), \\
e^{-\eta\cdot S^+} Q^{+i} e^{\eta\cdot S^+} &= Q^{+i} + \eta^i J^{+x}, \\
e^{-\eta\cdot S^+} Q^{-i} e^{\eta\cdot S^+} &= Q^{-i} - \frac{1}{2}\eta^i (J^{+-} + J^{x\bar{x}} + D) - \eta^j J^i{}_j \\
&\quad -\eta^i\eta^j S_j^+ - \frac{1}{2}\eta^2 S^{+i} + \frac{1}{2} i\eta^i\eta^2 K^+, \\
e^{-\eta\cdot S^+} S^{+i} e^{\eta\cdot S^+} &= S^{+i} - i\eta^i K^+, \\
e^{-\eta\cdot S^+} S^{-i} e^{\eta\cdot S^+} &= S^{-i} - i\eta^i K. \qquad (50)
\end{aligned}
$$

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