PHYSICAL REVIEW D **79,** 056004 (2009)

Model building by coset space dimensional reduction in ten dimensions with direct product gauge symmetry

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(Received 5 December 2008; published 9 March 2009)

We investigate ten-dimensional gauge theories whose extra six-dimensional space is a compact coset space, S/R, and whose gauge group is a direct product of two Lie groups. We list candidates of the gauge group and embeddings of R into them. After dimensional reduction of the coset space, we find fermion and scalar representations of $G_{\mathrm{GUT}} \times \mathrm{U}(1)$ with $G_{\mathrm{GUT}} = \mathrm{SU}(5)$, SO(10) and E_6, which accommodate all of the standard model particles. We also discuss possibilities to generate distinct Yukawa couplings among the generations using representations with different dimensions for $G_{\mathrm{GUT}} = \mathrm{SO}(10)$ and E_6 models.

DOI: 10.1103/PhysRevD.79.056004 PACS numbers: 11.10.Kk, 12.10.-g, 12.10.Dm

I. INTRODUCTION

The standard model (SM) has been successful in describing the phenomenology of elementary particle physics up to the energy of order TeV. Not only does it explain experimental results, but it also gives us deeper insight into the fact that gauge symmetry governs the interactions among the particles and its spontaneous breaking increases particle masses. Despite its success, the SM is not a satisfactory model because of the choice of gauge groups and because the particle content is the input of the model, and the parameters in the Higgs and Yukawa sectors, which are responsible for the masses, are not predictable. Grand unification addresses the former points by unifying the gauge symmetries into a single gauge group and the fermions into larger representations. But new scalars are required to break the grand unification symmetry in the same manner as the SM, resulting in the introduction of more free parameters than those in the SM. Therefore, a plausible framework for physics beyond the SM will be a unification of the Higgs and the gauge bosons.

The coset space dimensional reduction (CSDR) scheme is one of the attractive approaches in this regard [1–6]. This scheme introduces a compact extra-dimensional space which has the structure of a coset of Lie groups, S/R. The Higgs field and the gauge field of the SM are merged into a gauge field of a gauge group G in the higher-dimensional spacetime. The SM fermions are unified into a representation of this gauge group. The particle content surviving in four-dimensional theory is determined by the identification of the gauge transformation as a rotation within the extra-dimensional space. The four-dimensional gauge symmetries are determined by embedding R into G. Since the Higgs originates from extra-dimensional components of the gauge field, the Higgs and Yukawa sectors in the four-dimensional Lagrangian are uniquely determined. Furthermore, as shown in Refs. [7–10], it is possible to obtain chiral fermions when the total dimension D of the spacetime is even. The chiral fermions can be obtained even from (pseudo)real representations in $D = 8n + 2$ ($D = 8n + 6$) [7,10].

The case $D = 10$ is the most interesting because the superstring theory, which is a candidate of a unified theory including gravity, suggests that this world exists in ten-dimensional spacetime. Thus, CSDR models of $D = 10$ can bridge the superstring theory and the SM. In this spirit, many works have been done in ten dimension, but no realistic model has emerged yet [3,11–18]. A major obstacle to building realistic models is the difficulty to obtain all the SM fermions. One of the critical reasons for this difficulty is the smallness of the SO(6) spinor representation. Another reason is the small degree of freedom in embedding R into G. These facts strongly restrict the fermion representations surviving in four dimensions.

In this paper, we introduce a new freedom to the embedding of R into G by allowing G to be a direct product of two Lie groups in order to overcome the latter difficulty. We have more candidates for G and the embeddings of R into them, providing more possibilities to obtain the SM fermions. Furthermore, one of the gauge groups can be responsible for the four-dimensional gauge symmetry, while the other can be identified with a family symmetry [19–22], which generates a flavor structure in the Yukawa couplings. Thus, it is worthwhile to study the CSDR scheme with direct product gauge groups in ten dimensions. We exhaustively search for fermion content in the SM and in the grand unified theories (GUTs) with SU(5), SO(10), and E_6, limiting the dimension of a fermion representation to less than 1025.

This paper is organized as follows. In Sec. II, we briefly recapitulate the scheme of CSDR for the case with a gauge group of ten-dimensional gauge theory which has direct product structure, and the construction of the four-

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dimensional theory by the scheme. In Sec. III, we obtain the combinations of the coset space S/R and the gauge group G of the ten-dimensional theory. We first obtain the phenomenologically plausible coset space S/R, and then we restrict the possible gauge group G for each S/R. In Sec. IV, we exhaustively list the viable models in four dimensions. Section V is devoted to a summary and discussions.

II. CSDR SCHEME WITH A DIRECT PRODUCT GAUGE GROUP

In this section, we briefly recapitulate the scheme of coset space dimensional reduction in ten dimensions with a direct product gauge group [3].

We begin with a gauge theory defined in ten-dimensional spacetime M^{10} with a gauge group $G = G_1 \times G_2$, where G_1 and G_2 are simple Lie groups. Here M^{10} is a direct product of a four-dimensional spacetime M^4 and a compact coset space S/R, where S is a compact Lie group and R is a Lie subgroup of S. The dimension of the coset space S/R is thus $6 \equiv 10 - 4$, implying $\dim S - \dim R = 6$. This structure of extra-dimensional space requires the group R to be embedded into the group SO(6), which is a subgroup of the Lorentz group SO(1,9). Let us denote the coordinates of M^{10} by $X^M = (x^\mu, y^\alpha)$, where x^μ and y^α are coordinates of M^4 and S/R, respectively. The spacetime index M runs over $\mu \in \{0, 1, 2, 3\}$ and $\alpha \in \{4, 5, \cdots, 9\}$. We introduce, in this theory, a gauge field $A_M(x, y) = (A_\mu(x, y), A_\alpha(x, y))$, which belongs to the adjoint representation of the gauge group G, and fermions $\psi(x, y)$, which lie in a representation F of G.

The extra-dimensional space S/R admits S as an isometric transformation group. We impose on $A_M(X)$ and $\psi(X)$ the following symmetry under this transformation in order to carry out the dimensional reduction [2,23–27]. Consider a coordinate transformation which acts trivially on x and gives rise to an S transformation on y as $(x, y) \to (x, sy)$, where $s \in S$. We require that the transformation of $A_M(X)$ and $\psi(X)$ under this coordinate transformation should be compensated by a gauge transformation. This symmetry makes the ten-dimensional Lagrangian invariant under the S transformation and therefore independent of the coordinate y of S/R. The dimensional reduction is then carried out by integrating over the coordinate y to obtain the four-dimensional Lagrangian. The four-dimensional theory consists of the gauge fields A_μ, fermions ψ, and, in addition, the scalar fields originating from A_α. The gauge group reduces to a subgroup H of the original gauge group G.

The gauge symmetry and particle content of the four-dimensional theory are substantially constrained by the CSDR scheme. We provide below the prescriptions to identify the four-dimensional gauge group H and its representations for the particle content.

First, the gauge group of the four-dimensional theory H is easily identified as

$$H = C_G(R), \tag{1}$$

where $C_G(R)$ denotes the centralizer of R in $G = G_1 \times G_2$ [2]. Thus the four-dimensional gauge group H is determined by embedding R into G. We then assume that R also has direct product structure $R = R_1 \times R_2$ which can be embedded into G_1 and G_2. Here, R_1 and R_2 are not necessarily simple. We also assume that the four-dimensional gauge group H is obtained only from G_1 up to U(1) factors. This assumption ensures the coupling unification if H is the gauge group of the SM. These conditions imply

$$G = G_1 \times G_2, \tag{2}$$

$$R = R_1 \times R_2, \tag{3}$$

$$G_1 \supset H \times R_1, \tag{4}$$

$$G_2 \supset R_2, \tag{5}$$

up to U(1) factors.

Second, the representations of H for the scalar fields are specified by the following prescription. Let us decompose the adjoint representation of S according to the embedding $S \supset R_1 \times R_2$ as

$$\mathrm{adj}\, S = (\mathrm{adj} R_1, \mathbf{1}) + (\mathbf{1}, \mathrm{adj} R_2) + \sum_s (r_{1s}, r_{2s}), \tag{6}$$

where r_{1s} and r_{2s} are representations of R_1 and R_2, respectively. We then decompose the adjoint representations of G_1 and G_2 according to the embeddings $G_1 \supset H \times R_1$ and $G_2 \supset R_2$, respectively;

$$\mathrm{adj}\, G_1 = (\mathrm{adj} H, \mathbf{1}) + (\mathbf{1}, \mathrm{adj} R_1) + \sum_g (h_g, r_{1g}), \tag{7}$$

$$\mathrm{adj}\, G_2 = \mathrm{adj} R_2 + \sum_g r_{2g}, \tag{8}$$

where r_{1g} and r_{2g} denote representations of R_1 and R_2, and h_g denotes representations of H. The decomposition of $\mathrm{adj}\, G$ thus becomes

$$\begin{aligned} \mathrm{adj}\, G &= (\mathrm{adj} G_1, \mathbf{1}) + (\mathbf{1}, \mathrm{adj} G_2) \\ &= (\mathrm{adj} H, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathrm{adj} R_1, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathrm{adj} R_2) \\ &\quad + \sum_g (h_g, r_{1g}, \mathbf{1}) + \sum_g (\mathbf{1}, \mathbf{1}, r_{2g}). \end{aligned} \tag{9}$$

The representation of the scalar fields is h_g, whose corresponding $(r_{1g}, \mathbf{1})$ in the decomposition, Eq. (9), are contained also in the set $\{(r_{1s}, r_{2s})\}$ in Eq. (6). Note that the trivial representations $\mathbf{1}$ also remain in four dimensions if the corresponding $(\mathbf{1}, r_{2g})$ of Eq. (9) are also contained in the set $\{(r_{1s}, r_{2s})\}$ in Eq. (6).

Third, the representation of H for the fermion fields is determined as follows [28]. Let the group R be embedded into the Lorentz group SO(6) in such a way that the vector representation $\mathbf{6}$ of SO(6) is decomposed as $\mathbf{6} = \sum_s (r_{1s}, r_{2s})$, where r_{1s} and r_{2s} are the representations obtained in the decomposition, Eq. (6). This embedding specifies a decomposition of the Weyl spinor representations $\mathbf{4}(\bar{\mathbf{4}})$ of SO(6) under $\mathrm{SO}(6) \supset R_1 \times R_2$ as

$$\mathbf{4} = \sum_i (\sigma_{1i}, \sigma_{2i}) \left(\bar{\mathbf{4}} = \sum_i (\bar{\sigma}_{1i}, \bar{\sigma}_{2i}) \right), \tag{10}$$

where $\sigma_{1i}(\bar{\sigma}_{1i})$ and $\sigma_{2i}(\bar{\sigma}_{2i})$ are irreducible representations of R_1 and R_2. We then decompose the SO(1,9) Weyl spinor $\mathbf{16}$ according to $(\mathrm{SU}(2) \times \mathrm{SU}(2))(\approx \mathrm{SO}(1,3)) \times \mathrm{SO}(6)$ as

$$\mathbf{16} = (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}), \tag{11}$$

where the $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ correspond to left- and right-handed spinors, respectively. We now decompose a representation F of the gauge group G. We take F_1 and F_2 to be representations of G_1 and G_2 for the fermions in ten-dimensional spacetime. Decompositions of F_1 and F_2 are

$$F_1 = \sum_f (h_f, r_{1f}), \tag{12}$$

$$F_2 = \sum_f r_{2f}, \tag{13}$$

under $G_1 \supset H \times R_1$ and $G_2 \supset R_2$. Therefore, the decomposition of F becomes

$$F = \sum_f (h_f, r_{1f}, r_{2f}). \tag{14}$$

The representations for the left-handed (right-handed) fermions are the h_f whose corresponding (r_{1f}, r_{2f}) are found in $\{(\sigma_{1i}, \sigma_{2i})\}(\{(\bar{\sigma}_{1i}, \bar{\sigma}_{2i})\})$ which are obtained in Eq. (10). Note that a phenomenologically acceptable model needs chiral fermions in four dimensions as the SM does. The chiral fermions are obtained most straightforwardly when we introduce a complex representation of G as F [7–10]. More interesting is the possibility of obtaining them if F is a real representation, provided $\mathrm{rank}\, S = \mathrm{rank}\, R$ [29]. A pair of Weyl fermions appears in the same representation in this case, and one of the pair is eliminated by imposing the Majorana condition on the Weyl fermions [7,10]. We thus apply the CSDR scheme to complex or real representations of gauge group G for fermions.

Coset space S/R of our interest should satisfy $\mathrm{rank}\, S = \mathrm{rank}\, R$ to generate chiral fermions in four dimensions [29]. This condition limits the possible S/R to the coset spaces collected in Table I [3]. The R of coset (i) in Table I with the subscript "max" indicates that this is the maximal regular subgroup of the S. There, the correspondence between the subgroup of R and the subgroup of S is clarified by the brackets in R. For example, the coset space (iv) suggests a direct product of $\mathrm{Sp}(4)/\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2)/\mathrm{U}(1)$.

TABLE I. A complete list of six-dimensional coset spaces S/R with $\mathrm{rank}\, S = \mathrm{rank}\, R$ [3]. The brackets in R clarify the correspondence between the subgroup of R and the subgroup of S. The factor of R with the subscript "max" indicates that this factor is a maximal regular subgroup of S.

Number	S/R
(i)	$\mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathrm{max}}$
(ii)	$\mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\text{non-max}}$
(iii)	$\mathrm{SU}(4)/\mathrm{SU}(3) \times \mathrm{U}(1)$
(iv)	$\mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$
(v)	$\mathrm{G}_2/\mathrm{SU}(3)$
(vi)	$\mathrm{SO}(7)/\mathrm{SO}(6)$
(vii)	$\mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$
(viii)	$\mathrm{SU}(3) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{U}(1)] \times \mathrm{U}(1)$
(ix)	$(\mathrm{SU}(2)/\mathrm{U}(1))^3$

Here we mention the effect of gravity. When we include the effect of gravity and consider dynamics of an extra space, we would find it difficult to obtain stable extra space. This is the common difficulty of extra-dimensional models, and some works have been done on this point. For example, it is discussed in terms of radion fields, which are the scalar fields originating from higher-dimensional components of the metric after compactification [30,31]. The effect of gravity on the CSDR scheme is also discussed in [4,32]. Although we agree that the effect of gravity is important, we do not discuss the effect of gravity here since it is beyond the scope of this paper.

III. CANDIDATES OF THE COSET SPACE S/R AND THE GAUGE GROUP G

In this section we obtain the combinations of the coset space S/R and the gauge group G of the ten-dimensional theory. We first obtain the coset space S/R, and then we restrict the possible gauge group G for each S/R.

We select the coset space S/R from the ones listed in Table I by the following two criteria. First, R should be a direct product of subgroups R_1 and R_2 for us to have new freedom to embed R into G. This criterion excludes the candidates of S/R, (v) and (vi) in Table I.

Second, the four-dimensional gauge group obtained by Eq. (1) should be that of the SM or a GUT with at most one extra U(1) gauge group, i.e. the SM-like gauge group $G_{\mathrm{SM}}(\times \mathrm{U}(1))$, where $G_{\mathrm{SM}} \equiv \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, or a GUT-like gauge group $G_{\mathrm{GUT}}(\times \mathrm{U}(1))$, where G_{GUT} is either SU(5), SO(10), or E_6. This criterion excludes the candidates (vii)–(ix) in Table I for the following reasons.

(1) We note that the U(1)'s in R are also part of its centralizer, i.e. part of H. We thus exclude the candidate (ix) since we consider the H's which have at most two U(1) factors.

TABLE II. The decompositions of the vector representation **6** and the spinor representation **4** of SO(6) under R, which are listed as (i)–(iv) in Table I. The representations of r_s in Eq. (6) and σ_i in Eq. (10) are listed in the columns "Branches of **6**" and "Branches of **4**," respectively.

S/R		Branches of **6**	Branches of **4**
(i)	SU(2)(U(1))	$\mathbf{3}(2)$, $\mathbf{3}(-2)$	$\mathbf{1}(3)$, $\mathbf{3}(-1)$
(ii)	SU(2)(U(1))	$\mathbf{1}(2)$, $\mathbf{1}(-2)$, $\mathbf{2}(1)$, $\mathbf{2}(-1)$	$\mathbf{2}(1)$, $\mathbf{1}(0)$, $\mathbf{1}(-2)$
(iii)	SU(3)(U(1))	$\mathbf{3}(-4)$, $\bar{\mathbf{3}}(4)$	$\mathbf{1}(-6)$, $\mathbf{3}(2)$
(iv)	(SU(2), SU(2))(U(1))	$(\mathbf{2},\mathbf{2})(0)$, $(\mathbf{1},\mathbf{1})(2)$, $(\mathbf{1},\mathbf{1})(-2)$	$(\mathbf{2},\mathbf{1})(1)$, $(\mathbf{1},\mathbf{2})(-1)$

TABLE III. The embedding of R into $G = G_1 \times G_2$ for the coset spaces (i) and (ii).

(i) $\mathrm{Sp}(4)/[\mathrm{SU}(2)\times \mathrm{U}(1)]_{\mathrm{max}}$ and (ii) $\mathrm{Sp}(4)/[\mathrm{SU}(2)\times \mathrm{U}(1)]_{\mathrm{non\text{-}max}}$
(a) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{SU}(2)$, $G_2 \supset \mathrm{U}(1)$
(b) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$

(2) Similarly, as long as we consider the GUT-like and G_{SM} gauge groups, we do not need to consider the candidates (vii) and (viii).

(3) The candidates (vii) and (viii) do not allow $H = G_{\mathrm{SM}} \times \mathrm{U}(1)$ either, because the hypercharge of the SM should be reproduced by a certain linear combination of two U(1)'s in R, which should be matched to the spinor representation of SO(6). The dimension of the SO(6) spinor representation is four, and thus no more than four different values of U(1) charges are available. On the other hand, the fermion content of the SM has five different values of U(1) charges. Hence, this case never reproduces the hypercharges of the SM fermions.

(4) Because of the above three reasons the candidates (i)–(iv) allow neither G_{SM} nor G_{GUT} as H.

TABLE IV. The embedding of R into $G = G_1 \times G_2$ for the coset space (iii).

(iii) $\mathrm{SU}(4)/\mathrm{SU}(3)\times \mathrm{U}(1)$
(a) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{SU}(3)$, $G_2 \supset \mathrm{U}(1)$
(b) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(3)$

TABLE V. The embedding of R into $G = G_1 \times G_2$ for the coset space (iv).

(iv) $\mathrm{Sp}(4)\times \mathrm{SU}(2)/[\mathrm{SU}(2)\times \mathrm{SU}(2)]\times \mathrm{U}(1)$
(a) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{SU}(2)$, $G_2 \supset \mathrm{SU}(2)\times \mathrm{U}(1)$
(b) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{SU}(2)\times \mathrm{SU}(2)$, $G_2 \supset \mathrm{U}(1)$
(c) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)\times \mathrm{SU}(2)$
(d) $G_1 \supset (G_{\mathrm{SM}}$ or $G_{\mathrm{GUT}}) \times \mathrm{SU}(2)\times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$

To summarize, the possible model requires the coset space S/R listed in (i)–(iv) of Table I, with either $H = G_{\mathrm{SM}} \times \mathrm{U}(1)$ or $H = G_{\mathrm{GUT}} \times \mathrm{U}(1)$. In Table II we show the embedding of R in SO(6) for these coset spaces. The representations of r_s in Eq. (6) and σ_i in Eq. (10) are listed in the columns "Branches of **6**" and "Branches of **4**," respectively. The embedding of R into the higher-dimensional gauge group $G = G_1 \times G_2$ is listed in Tables III, IV, and V. These embeddings are straightforwardly obtained by decomposing gauge group G into its regular subgroup which contains an R subgroup of G. A detailed discussion about the embeddings is summarized in [6]. For each embedding of R, the candidates of G are summarized in Tables VI, VII,

TABLE VI. The candidates of the gauge groups G_1 and G_2 for each of the coset spaces (i) and (ii) in Table I. The top row indicates the assigned number of S/R in Table I and the embedding of R assigned in Table III. The leftmost column indicates H.

	(i)-(a) and (ii)-(a)	(i)-(b) and (ii)-(b)
$\mathrm{SU}(3)\times \mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{U}(1)$	$G_1 = \mathrm{SO}(10)$, SO(11), Sp(10)	$G_1 = \mathrm{SU}(6)$, SO(10), SO(11), Sp(10)
	$G_2 = \mathrm{SU}(2)$, U(1)	$G_2 = \mathrm{SU}(2)$
$\mathrm{SU}(5)\times \mathrm{U}(1)$	$G_1 =$ No candidate	$G_1 = \mathrm{SU}(6)$, SO(10), SO(11), Sp(10)
	$G_2 = \mathrm{SU}(2)$, U(1)	$G_2 = \mathrm{SU}(2)$
$\mathrm{SO}(10)\times \mathrm{U}(1)$	$G_1 = \mathrm{SO}(13)$	$G_1 = \mathrm{SO}(12)$, SO(13), E_6
	$G_2 = \mathrm{SU}(2)$, U(1)	$G_2 = \mathrm{SU}(2)$
$E_6 \times \mathrm{U}(1)$	$G_1 =$ No candidate	$G_1 = E_7$
	$G_2 = \mathrm{SU}(2)$, U(1)	$G_2 = \mathrm{SU}(2)$

TABLE VII. The allowed candidates of the gauge groups G_1 and G_2 for the coset space (iii) in Table I. The top row indicates the assigned number of S/R in Table I and the embedding of R assigned in Table IV. The leftmost column indicates H.

	(iii)-(a)	(iii)-(b)
SU(3) × SU(2) × U(1) × U(1)	$G_1 = E_6$	G_1 = SU(6), SO(10), SO(11), Sp(10)
	G_2 = SU(2), U(1)	G_2 = G_2, SU(3)
SU(5) × U(1)	G_1 = No candidate	G_1 = SU(6), SO(10), SO(11), Sp(10)
	G_2 = SU(2), U(1)	G_2 = G_2, SU(3)
SO(10) × U(1)	G_1 = No candidate	G_1 = SO(12), SO(13), E_6
	G_2 = SU(2), U(1)	G_2 = G_2, SU(3)
E_6 × U(1)	$G_1 = E_8$	$G_1 = E_7$
	G_2 = SU(2), U(1)	G_2 = G_2, SU(3)

TABLE VIII. The allowed candidates of the gauge groups G_1 and G_2 for the coset space (iv) in Table I. The top row indicates the assigned number of S/R in Table I and the embedding of R assigned in Table V. The leftmost column indicates H.

	(iv)-(a)	(iv)-(b)
SU(3) × SU(2) × U(1) × U(1)	G_1 = SO(10), SO(11), Sp(10)	G_1 = SO(13), Sp(12)
	G_2 = SU(3), Sp(4), G_2	G_2 = SU(2), U(1)
SU(5) × U(1)	G_1 = No candidate	G_1 = No candidate
	G_2 = SU(3), Sp(4), G_2	G_2 = SU(2), U(1)
SO(10) × U(1)	G_1 = SO(13)	G_1 = SO(14), SO(15)
	G_2 = SU(3), Sp(4), G_2	G_2 = SU(2), U(1)
E_6 × U(1)	G_1 = No candidate	G_1 = No candidate
	G_2 = SU(3), Sp(4), G_2	G_2 = SU(2), U(1)

VIII, and IX. Note that all the candidates of G in Tables VI, VII, VIII, and IX are subgroups of SO(32) or $E_8 \times E_8$ which are required by superstring theory.

The representation F_1 of G_1 for the fermions should be either complex or real but not pseudoreal, since the fermions of pseudoreal representations do not allow the Majorana condition when $D = 10$ and they induce doubled fermion content after the dimensional reduction [7]. Table X lists the candidate groups G_1 and their complex and real representations whose dimensions are no more than 1024. The representations in this table are the candidates of F_1. The groups SU(7) and SO(13) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any S/R or embedding of R in Tables III, IV, and V.

The representation F_2 of G_2 has to be real as does F_1 to impose the Majorana condition. Without this condition, F_2 can be any representation. We limited ourselves to the case $\dim F = \dim F_1 \times \dim F_2 < 1025$ since larger representations yield numerous higher-dimensional representations of the fermions under $G_{\rm SM} \times$ U(1) and $G_{\rm GUT} \times$ U(1).

IV. RESULTS

Now we are ready to investigate the representations for fermions and scalars in four dimensions. We first note that

TABLE IX. The allowed candidates of the gauge groups G_1 and G_2 for the coset space (iv) in Table I. The top row indicates the assigned number of S/R in Table I and the embedding of R assigned in Table V. The leftmost column indicates H.

	(iv)-(c)	(iv)-(d)
SU(3) × SU(2) × U(1) × U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10)	G_1 = SU(7), SO(12), SO(13), Sp(12), E_6
	G_2 = G_2, Sp(4)	G_2 = SU(2)
SU(5) × U(1)	G_1 = SU(6), SO(10), SO(11), Sp(10)	G_1 = SU(7), SO(13), Sp(12), E_6
	G_2 = G_2, Sp(4)	G_2 = SU(2)
SO(10) × U(1)	G_1 = SO(12), SO(13), E_6	G_1 = SO(14), SO(15), E_7
	G_2 = G_2, Sp(4)	G_2 = SU(2)
E_6 × U(1)	$G_1 = E_7$	$G_1 = E_8$
	G_2 = G_2, Sp(4)	G_2 = SU(2)

TABLE X. The complex and real representations of the possible gauge groups [33]. The groups SU(7) and SO(13) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any of S/R and for embedding of R in Table III, IV, and V

Group	Complex representations	Real representations
SU(6)	**6**, **15**, **21**, **56**, **70**, **84**, **105**, **105**′, **120**, **126**, **210**, **210**′, **252**, **280**, **315**, **336**, **384**, **420**, **462**, **490**, **504**, **560**, **700**, **720**, **792**, **840**, **840**′, **840**″, **896**, ⋯	**35**, **175**, **189**, **405**, ⋯
SO(11)		**11**, **55**, **65**, **165**, **275**, **320**, **330**, **429**, **462**, **935**, ⋯
SO(12)		**12**, **66**, **77**, **220**, **352**, **462**, **495**, **560**, **792**, ⋯
SO(14)	**64**, **832**, ⋯	**14**, **91**, **104**, **364**, **546**, **896**, ⋯
SO(15)		**15**, **105**, **119**, **128**, **455**, **665**, ⋯
F_4		**26**, **52**, **273**, **324**, ⋯
E_6	**27**, **351**, **351**′ ⋯	**78**, **650**, ⋯
E_7		**133** ⋯
E_8		**248**, ⋯

we need an R_2 singlet in the SO(6) vector to obtain the Higgs candidate h_g [cf. Eq. (9) and the discussion below]. We can thus exclude the candidates (i) and (iii) of S/R in Table I (cf. Table II). In Tables XI, XII, and XIII, we list the possible candidates of G_1, G_2, (F_1, F_2), and the corresponding representations of four-dimensional scalars and fermions for each H, which are either $G_{SM} \times U(1)$, $SU(5) \times U(1)$, $SO(10) \times U(1)$, or $E_6 \times U(1)$. The representations of four-dimensional fermions are classified into A, B, and C. The representations of class A are the *standard representations*: $\bar{\mathbf{5}}$ and **10** for SU(5), **16** for SO(10), and **27** for E_6, which lead to the SM fermions after GUT breaking. The representations of class B lead to both the SM fermions and non-SM fermions after GUT breaking. The representations of class C lead only to non-SM fermions after GUT breaking.

A. $H = G_{SM} \times U(1)$

We investigate all combinations of S/R, G_1, and G_2 in Tables VI, VII, VIII, and IX, which provide $H = G_{SM} \times U(1)$ in four dimensions. We obtain a representation which is identified as the SM Higgs doublet in four dimensions from the following cases.

(1) R embedded as (ii)-(b), $G_1 = SU(6)$ and $G_2 = SU(2)$.
(2) R embedded as (ii)-(b), $G_1 = SO(11)$ and $G_2 = SU(2)$.
(3) R embedded as (iv)-(c), $G_1 = SU(6)$ and $G_2 = G_2$.

TABLE XI. The models for $H = SU(5) \times U(1)$ which include the SM Higgs doublet and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified as A, B, and C. Class A contains only the SM fermions; class B contains both the SM fermions and extra fermions; class C contains only extra fermions.

$S/R = Sp(4)/[SU(2) \times U(1)]$, $G_1 \supset SU(5) \times U(1)$, $G_2 \supset SU(2)$						
G_1	G_2	(F_1, F_2)	Scalars	A	B	C
SU(6)	SU(2)	(56, 2)	5(6), $\bar{5}(-6)$		15(−3)	35(−3)
		(70, 2)	5(6), $\bar{5}(-6)$	10(−3)	15(−3), 40(−3)	
		(280, 1)	5(6), $\bar{5}(-6)$	$\bar{5}(-6)$	$\overline{70}(-6)$	24(0), 45(−6), 126(0), 24(0), 126(0)
		(405, 1)	5(6), $\bar{5}(-6)$	$\bar{5}(-6)$	$\overline{70}(-6)$	1(0), 24(0), 200(0)
						5(6), 70(6), 1(0), 24(0), 200(0)
		(840, 1)	5(6), $\bar{5}(-6)$		$\overline{45}(6)$	280′(6), 126(0), 224(0), $\overline{105}(6)$, 126(0), 224(0)
SO(11)	SU(2)	(11, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$		1(0)
		(55, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$		1(0), 24(0)
		(65, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$		1(0), 24(0)
		(165, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$	$\overline{45}(-2)$	1(0), 24(0)
		(275, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$	$\overline{70}(-2)$	1(0), 24(0)
		(320, 2)	5(2), $\bar{5}(-2)$	10(−1), 10(−1)	15(−1), 40(−1)	
		(330, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$	$\overline{45}(-2)$	1(0), 24(0), 75(0)
		(429, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$, $\bar{5}(-2)$	$\overline{45}(-2)$, $\overline{70}(-2)$	1(0), 24(0), 24(0)
		(462, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$	$\overline{45}(-2)$, $\overline{50}(-2)$	1(0), 24(0), 75(0)
		(935, 1)	5(2), $\bar{5}(-2)$	$\bar{5}(-2)$	$\overline{70}(-2)$	1(0), 24(0), 200(0)

TABLE XII. The models for $H = \mathrm{SO}(10) \times \mathrm{U}(1)$ which include the SM Higgs and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified as A, B, and C, where fermions in class A are **16** representations of SO(10); class B contains both the SM fermions and extra fermions; class C contains only extra fermions. We can obtain two types of results for fermions from one combination of (G_1, G_2, F) since we have the freedom to change the overall sign of the U(1) charges that appear in the R decomposition of the SO(6) vector and spinor.

$S/R = \mathrm{Sp}(4)/[\mathrm{SU}(2) \times \mathrm{U}(1)]$, $G_1 \supset \mathrm{SO}(10) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$						
G_1	G_2	(F_1, F_2)	Scalars	A	B	C
SO(12)	SU(2)	(12, 1)	10(2), 10(−2)		10(0)	1(2)
					10(0)	1(−2)
		(66, 1)	10(2), 10(−2)		10(2), 45(0)	1(0)
					10(−2), 45(0)	1(0)
		(77, 1)	10(2), 10(−2)		10(2), 54(0)	1(0)
					10(−2), 54(0)	1(0)
		(220, 1)	10(2), 10(−2)		45(2), 10(0), 120(0)	
					45(−2), 10(0), 120(0)	
		(352, 1)	10(2), 10(−2)		54(2), 10(0), 210′(0)	
					54(−2), 1(−2), 10(0), 210′(0)	
		(462, 1)	10(2), 10(−2)		126(2), 210(0)	
					$\overline{126}$(−2), 210(0)	
		(495, 1)	10(2), 10(−2)		120(2), 45(0), 210(0)	
					120(−2), 45(0), 210(0)	
		(560, 1)	10(2), 10(−2)		54(2), 45(2), 10(0), 10(0), 320(0)	1(2)
					54(−2), 45(−2), 10(0), 10(0), 320(0)	1(−2)
		(792, 1)	10(2), 10(−2)		210(2), 120(0), 126(0), $\overline{126}$(0)	
					210(−2), 120(0), 126(0), $\overline{126}$(0)	
E_6	SU(2)	(78, 1)	16(−3), $\overline{16}$(3)	16(−3)	45(0)	1(0)
					45(0)	1(0) $\overline{16}$(3)
		(650, 1)	16(−3), $\overline{16}$(3)	16(3)	$\overline{144}$(3), 45(0), 54(0), 210(0)	1(0)
					144(−3), 45(0), 54(0), 210(0)	1(0) $\overline{16}$(−3)

$S/R = \mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$, $G_1 \supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, $G_2 \supset \mathrm{U}(1)$						
G_1	G_2	(F_1, F_2)	Scalars	A	B	C
SO(14)	SU(2)	(64, 2)	10(0)	16(1), 16(1)		$\overline{16}$(−1), $\overline{16}$(−1)
	U(1)	64(1)	10(0)	16(1), 16(−1)		
		832(1)	10(0)	16(1), 16(−1)	144(1), 144(−1)	
SO(15)	SU(2)	(128, 2)	10(0), 1(0)	16(1), 16(−1)		$\overline{16}$(1), $\overline{16}$(−1)
	U(1)	128(1)	10(0), 1(0)	16(1)		$\overline{16}$(1)

$S/R = \mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$, $G_1 \supset \mathrm{SO}(10) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$						
G_1	G_2	(F_1, F_2)	Scalars	A	B	C
SO(15)	SU(2)	(128, 1)	10(2), 10(−2)	16(1)	$\overline{16}$(1)	
E_7	SU(2)	(133, 1)	10(2), 10(−2)	16(1)		

(4) R embedded as (iv)-(c), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{Sp}(4)$.
(5) R embedded as (iv)-(c), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{G}_2$.
(6) R embedded as (iv)-(c), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{Sp}(4)$.
(7) R embedded as (iv)-(d), $G_1 = \mathrm{Sp}(12)$ and $G_2 = \mathrm{SU}(2)$.
(8) R embedded as (iv)-(d), $G_1 = E_6$ and $G_2 = \mathrm{SU}(2)$.

None of these cases reproduces a whole generation of SM fermions. Therefore, we cannot obtain the SM in four dimensions. The difficulty in obtaining the SM is ultimately due to the smallness of the dimension of the SO(6) spinor representation.

B. $H = \mathbf{SU(5)} \times \mathrm{U}(\mathbf{1})$

We investigate the case of $H = \mathrm{SU}(5) \times \mathrm{U}(1)$ and summarize the results in Table XI. We obtain the representation **5**, which corresponds to the Higgs scalar in the following cases.

TABLE XIII. The models for $H = E_6 \times U(1)$ which include the SM Higgs and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified into A, B, and C, where fermions in class A are **27** representations of E_6; class B contains both the SM fermions and extra fermions; class C contains only extra fermions. We can obtain two types of results for fermions from one combination of (G_1, G_2, F) since we have the freedom to change the overall sign of the U(1) charges which appear in the R decomposition of the SO(6) vector and spinor.

$S/R = \mathrm{Sp}(4)/\mathrm{SU}(2) \times \mathrm{U}(1)$, $G_1 \supset E_6 \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$						
G_1	G_2	(F_1, F_2)	Scalars	A	B	C
E_7	SU(2)	(133, 1)	27(2), $\overline{27}(-2)$	27(2)	78(0)	1(0)
					$\overline{27}(-2)$, 78(0)	1(0)

$S/R = \mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$, $G_1 \supset E_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$, $G_2 \supset \mathrm{SU}(2)$						
G_1	G_2	(F_1, F_2)	Scalars	A	B	C
E_8	SU(2)	F(248, 1)	27(−2), $\overline{27}(2)$	27(1)		
					$\overline{27}(-1)$	

(1) R embedded as (ii)-(b), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{SU}(2)$.
(2) R embedded as (ii)-(b), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{SU}(2)$.
(3) R embedded as (iv)-(c), $G_1 = \mathrm{SU}(6)$ and $G_2 = \mathrm{Sp}(4)$.
(4) R embedded as (iv)-(c), $G_1 = \mathrm{SO}(11)$ and $G_2 = \mathrm{Sp}(4)$.
(5) R embedded as (iv)-(d), $G_1 = E_6$ and $G_2 = \mathrm{SU}(2)$.

As for the fermions, we see that the *standard representations* of SU(5) GUT are not obtained at all for cases 3, 4, and 5, while they are obtained by combining two representations of F in cases 1 and 2. For the example of case 1, we can choose (**70**, **2**) and (**280**, **1**) to obtain all the *standard representations*, $\bar{\mathbf{5}}$ and **10**, in four dimensions, along with the extra fermions of classes B and C.

C. $H = \mathrm{SO}(10) \times \mathrm{U}(1)$

We investigate all the combinations of S/R, G_1, and G_2 for $H = \mathrm{SO}(10) \times \mathrm{U}(1)$. We obtain the representation **10**, which corresponds to the Higgs scalar in the following cases.

(1) R embedded as (ii)-(b), $G_1 = \mathrm{SO}(12)$ and $G_2 = \mathrm{SU}(2)$.
(2) R embedded as (ii)-(b), $G_1 = E_6$ and $G_2 = \mathrm{SU}(2)$.
(3) R embedded as (iv)-(b), $G_1 = \mathrm{SO}(14)$ and $G_2 = \mathrm{SU}(2)$.
(4) R embedded as (iv)-(b), $G_1 = \mathrm{SO}(14)$ and $G_2 = \mathrm{U}(1)$.
(5) R embedded as (iv)-(b), $G_1 = \mathrm{SO}(15)$ and $G_2 = \mathrm{SU}(2)$.
(6) R embedded as (iv)-(b), $G_1 = \mathrm{SO}(15)$ and $G_2 = \mathrm{U}(1)$.
(7) R embedded as (iv)-(c), $G_1 = \mathrm{SO}(12)$ and $G_2 = \mathrm{G}_2$.
(8) R embedded as (iv)-(c), $G_1 = \mathrm{SO}(12)$ and $G_2 = \mathrm{Sp}(4)$.
(9) R embedded as (iv)-(c), $G_1 = E_6$ and $G_2 = \mathrm{SU}(2)$.
(10) R embedded as (iv)-(c), $G_1 = E_6$ and $G_2 = \mathrm{G}_2$.
(11) R embedded as (iv)-(d), $G_1 = \mathrm{SO}(15)$ and $G_2 = \mathrm{SU}(2)$.
(12) R embedded as (iv)-(d), $G_1 = E_7$ and $G_2 = \mathrm{SU}(2)$.

We further obtain the *standard representations* of the fermions which lead to all the SM fermions of one generation in cases 1–6, 8, 11, and 12 (see Table XII).

Case 4 with $F = \mathbf{832}(1)$ is intriguing since we obtain two **16**'s and two **144**'s, each of which leads to a complete set of the SM fermions of one generation. We thus obtain four generations of fermions which can accommodate the three known generations. Furthermore, these representations can form three distinct types of Yukawa couplings: $\mathbf{16} \times \mathbf{16} \times \mathbf{10}$, $\mathbf{144} \times \mathbf{16} \times \mathbf{10}$, and $\mathbf{144} \times \mathbf{144} \times \mathbf{10}$. These couplings may explain the origin of the Yukawa couplings, distinguishing the generations and possibly introducing the mixing among them.

D. $H = E_6 \times \mathrm{U}(1)$

The results for $H = E_6 \times \mathrm{U}(1)$ are listed in Table XIII. We obtain the representation **27**, which corresponds to the Higgs scalar in the following cases.

(1) R embedded as (ii)-(b), $G_1 = E_7$ and $G_2 = \mathrm{SU}(2)$.
(2) R embedded as (iv)-(c), $G_1 = E_7$ and $G_2 = \mathrm{G}_2$.
(3) R embedded as (iv)-(d), $G_1 = E_8$ and $G_2 = \mathrm{SU}(2)$.

The *standard representations* of fermion **27**, which provide all the SM fermions of one generation, are obtained in cases 1 and 3.

Case 1 with $F = (\mathbf{133}, \mathbf{1})$ is interesting since the structure of the SM with three generations may be explained.

The Yukawa coupling of this model needs to be in the form $\overline{\mathbf{27}}(-2) \times \mathbf{27}(2) \times \mathbf{78}(0)$. The fermion representation $\mathbf{27} + \mathbf{78}$ of E_6 contains three generations of $\bar{\mathbf{5}} + \mathbf{10}$ in terms of its SU(5) subgroup, giving the origin of the three known generations. Indeed, this fermion content is analyzed in, for example, nonlinear sigma models, giving a family unification [34] based on a broken E_7 symmetry [35], under which a reproduction of the observed mixing structure among the three generations of fermions has been attempted [36].

V. SUMMARY AND DISCUSSIONS

We studied the ten-dimensional gauge theories whose extra six-dimensional spacetime is a coset space of Lie groups. We focused on the case where the gauge group is a direct product of two simple Lie groups, and searched for models which lead to phenomenologically promising four-dimensional models after applying coset space dimensional reduction.

We first limited the possible coset space S/R to four types, listed in (i)–(iv) of Table I, by requiring that R should be factored as $R = R_1 \times R_2$. All of these four types have a U(1) factor in R, but this U(1) can never be identified as the hypercharge symmetry of the SM. We thus needed to introduce an extra U(1) in the four-dimensional gauge group H, and we searched for SM-like models or GUT-like ones. The former is the case where $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$, while the latter is where $H = \mathrm{SU}(5) \times \mathrm{U}(1)$, $H = \mathrm{SO}(10) \times \mathrm{U}(1)$, and $H = E_6 \times \mathrm{U}(1)$. We also required that the induced four-dimensional model should include the particle content appropriate for the SM particles. We then found the candidates of the gauge group $G = G_1 \times G_2$ of the ten-dimensional theory and the representations for fermions.

For each of the candidates obtained, we made the complete lists of representations of the scalars and the fermions that constitute the corresponding four-dimensional theory. The results are summarized as follows.

(1) No ten-dimensional model was found to induce the promising model with $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ in the four-dimensional spacetime.

(2) The models which induce an $\mathrm{SU}(5) \times \mathrm{U}(1)$ gauge theory in four-dimensional spacetime were found when $S/R = \mathrm{Sp}(4)/\mathrm{SU}(2) \times \mathrm{U}(1)$. A possible gauge group is either $\mathrm{SU}(6) \times \mathrm{SU}(2)$ or $\mathrm{SO}(11) \times \mathrm{SU}(2)$, and each case has several choices for the representation of the ten-dimensional fermions as listed in Table XI. Many of the fermion representations generate either $\bar{\mathbf{5}}$ or $\mathbf{10}$ of the SU(5) after dimensional reduction. None of them, however, generates both from a single representation, and we thus need at least two fermion representations in the ten-dimensional model as well as in the four-dimensional one.

(3) The models which induce an $\mathrm{SO}(10) \times \mathrm{U}(1)$ gauge theory in four dimensions were found for the three possible choices of S/R, and each choice allows a number of gauge groups as listed in Table XII.

(4) The models which induce an $E_6 \times \mathrm{U}(1)$ gauge theory in four dimensions were found when $S/R = \mathrm{Sp}(4)/\mathrm{SU}(2) \times \mathrm{U}(1)$, $G = E_7 \times \mathrm{SU}(2)$ and $S/R = \mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$, $G = E_8 \times \mathrm{SU}(2)$, as listed in Table XIII.

The fermion representations in four-dimensional theories obtained from the candidate models mentioned above are not limited to the standard ones, i.e. $\bar{\mathbf{5}}$ and $\mathbf{10}$ for $H = \mathrm{SU}(5) \times \mathrm{U}(1)$, $\mathbf{16}$ for $H = \mathrm{SO}(10) \times \mathrm{U}(1)$, and $\mathbf{27}$ for $H = E_6 \times \mathrm{U}(1)$. Some of these extra representations can accommodate the SM particles as well and thus can take part in further model building. The following two models are found to be of particular interest.

(1) $H = \mathrm{SO}(10) \times \mathrm{U}(1)$, $S/R = \mathrm{Sp}(4) \times \mathrm{SU}(2)/[\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1)$, $G = \mathrm{SO}(14) \times \mathrm{U}(1)$, and $F = \mathbf{832}(1)$ (see Table XII). In this case, the fermions in four-dimensional theory include two $\mathbf{16}$'s and two $\mathbf{144}$'s. Since both can include a complete set of the SM fermions of a generation, this case has four generations of fermions and thus can accommodate the three known generations. Besides, this case allows three distinct types of Yukawa couplings: $\mathbf{16} \times \mathbf{16} \times \mathbf{10}$, $\mathbf{144} \times \mathbf{16} \times \mathbf{10}$, and $\mathbf{144} \times \mathbf{144} \times \mathbf{10}$. These three types of couplings can admit different Yukawa couplings, giving rise to the distinction of generations. Hence this model may introduce mixing among the generations.

(2) $H = E_6 \times \mathrm{U}(1)$, $S/R = \mathrm{Sp}(4)/\mathrm{SU}(2)$, $G = E_6 \times \mathrm{SU}(2)$, and $F = (\mathbf{133}, \mathbf{1})$ (see Table XIII). The Yukawa coupling of this model is necessarily of the form $\overline{\mathbf{27}}(-2) \times \mathbf{27}(2) \times \mathbf{78}(0)$. The fermion representation $\mathbf{27} + \mathbf{78}$ of E_6 contains three generations of $\bar{\mathbf{5}} + \mathbf{10}$ in terms of its SU(5) subgroup, giving the origin of the three known generations. Indeed, this fermion content is analyzed in, for example, nonlinear sigma models, giving a family unification [34] based on a broken E_7 symmetry [35], under which a reproduction of the observed mixing structure among the three generations of fermions has been attempted [36].

We leave further analysis, as well as building phenomenological models based on the models mentioned above, for future studies.

ACKNOWLEDGMENTS

The work of T. J. was financially supported by the Sasakawa Scientific Research Grant from the Japan Science Society. The work of T. N. was supported in part by the Grant-in-Aid for the Ministry of Education, Culture, Sports, Science, and Technology, Government of Japan (No. 19010485). The work of J. S. was supported in part by the Grant-in-Aid for the Ministry of Education, Culture, Sports, Science, and Technology, Government of Japan (No. 20025001, No. 20039001, and No. 20540251). The work of T. S. was supported in part by MEC and FEDER (EC), Grant No. FPA2005-01678, and the Generalitat Valencianaunder Grant No. GVPRE/2008/003, No. PROMETEO/2008/004, and No. GV05/267.

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