Level statistics of the q-deformed anharmonic oscillator system

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Abstract

We investigate the level statistics for the q-deformed anharmonic oscillator system, and found that the q-deformation of the Lie algebra weaken the Wigner distribution which is reflection of the underlying classical chaotic structure.

1 Introduction

The classical chaos has a rigid foundation. To investigate the signature of chaos, we can use the Poincare section, Lyapunov exponent, Kolmogorov entropy and so on. The essence of the classical chaos is the infinite division of the phase space by the horse-shoe map. On the other hand, the definition of the chaos in the quantum system is still ambiguous, and it lacks enough theoretical foundation [1]-[11]. At a glance, the role of $\hbar$ in the quantum system seems to be incompatible with the infinite division of the phase space in the classical chaos. There are, however, many numerical researches on the low-dimensional simple physical system for last decade. Some of them analyze the quantum system by the level statistics, such as the nearest level spacing P(s) and number variance $\Sigma_3(L)$. They reveal that there is the universal correspondence between classical chaos and quantum one except the pathological system [9]. Most of the classically chaotic system give the Wigner distribution for P(s) when they are quantized, and the classically regular system gives the Poisson distribution. The order-to-chaos transition of the quantum system can be described by the Brody distribution, Berry-Robnik one [5], Izrailev one [8] and one of band random matrices. In our recent research, we analysed this transition more quantitatively.

Recently quantum algebra becomes to be popular. We can use it to enlarge the quantum mechanics algebraically. Especially using the q-deformed harmonic oscillator system is easy. This may give us a new view-point. What we did in this paper is to investigate the correspondence between the classical system and its q-deformed quantum system numerically.

2 The level statistics and quantum chaos

There exist a lot of papers, which deal with the level statistics as the tool for investigating the quantum chaos. They reveal the universal correspondence. The main point is the statistical behavior of the energy levels of the quantum system which exhibits the chaos as the classical system. The next interesting point is the statistical one between the order-to-chaos transition. Such transition was clarified by Seligman et.al.[1]. They used the anharmonic oscillator system for the numerical investigation. The order-to-chaos transition is controlled by the parameter of the coupling strength λ of the anharmonic term.

$$H = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + V_{12}(X_1, X_2), \tag{1}$$

$$V_{12}(X_1, X_2) = V_1(X_1) + V_2(X_2) + \lambda V_{12}(|X_1 - X_2|), \tag{2}$$

and,

$$V_i(X_i) = \alpha_i X_i^2 + \beta_i X_i^4 + \gamma_i X_i^6. \tag{3}$$

The potential is determined by the several parameters. These parameters are written in ref [1]. What they calculated are the chaotic volume and the kolmogorov entropy as the measure of the classical system, and the level statistics as the one of quantum system. They showed, by the several numerical examples, that such measure are strongly correlated, and that they gradually vary from order to chaos under the controlling parameter λ (not necessarily monotonous). Moreover they revealed that such transition can be simulated by the band random matrices they advocated. We consider their results as the first step of the further study, which will inquire more direct relation.

3 The level statistics for the q-deformed anharmonic oscillator

When we consider the correspondence between the quantum system and classical one, we may get more insight of its correspondence, if we can use more general system which includes the conventional quantum system. As a candidate, there is a quantum algebra, which changes the uncertainty relation and the deviation from the quantum mechanics is measured by one parameter q (real C-number). (We use it only as the tool. It might be that our approach has nothing to do with the essence of the quantum algebra.) For numerical calculation, we use the q-deformed harmonic oscillator system. We summarize the q-deformed harmonic oscillators system in the appendix. The uncertainty relation of the q-deformed harmonic system is eq.(11).

For the numerical experiments, we used the Seligman's anharmonic oscillator system in previous section. We choose the same parameter set. From their analysis, we fixed the coupling constant which enables the system most chaotic classically and quantum mechanically. Then we examined the level statistics by varying q. The results we got are following two points.

- When q deviates from 1, the level statistics changes from Wigner distribution to Poissonian one rapidly.
- When we analyze the level statistics at each energy region, we get more Poissonian distribution according to the energy.

We think, these phenomena are closely related to the new uncertainty relation, which induces the exponential-increase of the uncertainty with each energy level and q. From the connection of the chaos, the increase of the $\hbar$ introduces the suppression of the manifestation of the chaos.

4 Summary

We study the level statistics for the q-deformed anharmonic oscillator system, and get the results that the Wigner distribution is weaken by new uncertainty relation. We think, further studies of quantum algebra and quantum chaos are needed. Especially, it is important to clarify the classical counterpart of the q-deformed harmonic oscillators [14].

Appendix The review of the q-deformed harmonic oscillator

We summarize the q-deformed harmonic oscillator [12] [13]. We consider the q-creation operator $a_q^\dagger$, the q-annihilation operator a_q and the q-vacuum

$|0\rangle_q$ which is defined by the $a_q|0\rangle_q = 0$. The quantum algebra is constructed as follows,

$$a_q^\dagger a_q - \frac{1}{q} a_q a_q^\dagger = q^{N_q} \tag{4}$$

$$[N_q, a_q^\dagger] = a_q^\dagger \tag{5}$$

$$[N_q, a_q] = -a_q. \tag{6}$$

The N_q is the q-number operator, which is not equivalent to $a_q^\dagger a_q$. We can construct the Hilbert space by the

$$|n\rangle_q = \frac{1}{\sqrt{[n]_q!}} (a_q^\dagger)^n |0\rangle_q \tag{7}$$

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \tag{8}$$

In this way, we can define the q-deformed harmonic oscillator, which reduce to the normal harmonic oscillator if q is equate to unity. The analogous operators of coordinate and momentum are defined by

$$X_q \equiv \sqrt{\frac{\hbar}{2m\omega}} (a_q^\dagger + a_q) \tag{9}$$

$$P_q \equiv \sqrt{\frac{m\hbar\omega}{2}} (a_q^\dagger - a_q). \tag{10}$$

The uncertainty relation changes as,

$$[X_q, P_q] = i\hbar \frac{cosh\frac{\gamma}{2}(2n+1)}{cosh\frac{\gamma}{2}} \tag{11}$$

where $\gamma = lnq$.

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