Introduction to Supergravities in Diverse Dimensions*

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(1998年2月25日受理)

Abstract

Supergravities in four and higher dimensions are reviewed. We discuss the action and its local symmetries of $N = 1$ supergravity in four dimensions, possible types of spinors in various dimensions, field contents of supergravity multiplets, non-compact bosonic symmetries, non-linear sigma models, duality symmetries of antisymmetric tensor fields and super p-branes.

1. Introduction

Recently interest in supergravities in various space-time dimensions has been much increased due to their relevance to string dualities [1]. String dualities sometimes relate string theories at strong coupling and those at weak coupling, and are extremely useful to understand non-perturbative properties of string theories. However, at present understanding of string theories, it is difficult to show dualities directly in full string theories. Massless sectors of superstring theories are described by supergravities, for which complete field theoretic formulations are known at the classical level. One may try to obtain information about string dualities by studying supergravities.

One of the purposes of this paper is to explain the basic ideas of supergravities to those who are not familiar with supergravities. We try to make the discussions pedagogical and often present explicit calculations. Another purpose is to collect relevant formulae together in one place, which may be useful when one discusses string dualities. They include possible types of spinors in various dimensions, field contents of supergravity multiplets, bosonic symmetries of supergravities, etc. We do not discuss quantum properties of supergravities such as ultraviolet divergences or anomalies.

Supergravities are field theories which have the local supersymmetry. A transformation parameter of the rigid supersymmetry is a constant spinor ϵ^α. (For a review of the rigid

*An expanded version of a review talk at YITP workshop on Supersymmetry, 27 – 30 March, 1996
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supersymmetry see ref. [2], for instance.) To construct theories which have the local supersymmetry we introduce a gauge field $\psi_\mu^\alpha(x)$, which has a vector index μ in addition to the spinor index α. The transformation law of the local supersymmetry is $\delta_Q \psi_\mu^\alpha(x) = \partial_\mu \epsilon^\alpha + \cdots$, where the transformation parameter $\epsilon^\alpha(x)$ is an arbitrary function of the space-time coordinates x^μ. Such a field $\psi_\mu^\alpha(x)$ is the Rarita-Schwinger field representing a spin $\frac{3}{2}$ particle. However, that is not all we need. The anticommutation relation of supercharges Q^α produces the translation generators P_a:

$$\{Q^\alpha, \bar{Q}_\beta\} = (\gamma^a)^\alpha{}_\beta P_a. \tag{1.1}$$

Therefore, we expect that gauging of supersymmetry leads to gauging of translation. Since the local translation is the general coordinate transformation, we also need the gravitational field $g_{\mu\nu}(x)$ as a gauge field. To summarize, supergravities are theories, which are invariant under the local supersymmetry transformation as well as the general coordinate transformation. They contain the gravitational field $g_{\mu\nu}(x)$ and the Rarita-Schwinger field $\psi_\mu^\alpha(x)$.

In the next section we discuss supergravities in four dimensions in some detail. To generalize these results to higher dimensions we first discuss what types of spinor representations are possible in general dimensions in sect. 3. Then, possible superalgebras and supergravity multiplets in higher dimensions are given in sect. 4. Properties of higher dimensional supergravities are discussed in sect. 5. In sects. 6 and 7 we discuss subjects related to bosonic non-compact symmetries appearing in supergravities. In sect. 6 we explain how scalar fields are described by G/H non-linear sigma models with a non-compact Lie group G and its maximal compact subgroup H. In sect. 7 we discuss duality symmetries, which transform field strength of antisymmetric tensor fields into their duals. The non-compact group G acts on the antisymmetric tensor fields as duality transformations. Finally, in sect. 8 we briefly discuss super p-branes (p-dimensionally extended objects), which are closely related to supergravities. The vielbein formulation of gravity is summarized in Appendix A. A proof of the local supersymmetry invariance of $N = 1$ supergravity in four dimensions is given in Appendix B. We have not tried to give complete references to the original papers. For more complete references see ref. [3]. Other useful review papers on supergravities are refs. [4], [5].

2. Supergravities in four dimensions

In this section we shall consider supergravities in $d = 4$ space-time dimensions. Massless irreducible representations of the $N = 1$ superalgebra consist of two states with helicities differing by $\frac{1}{2}$. In particular, we have representations of helicities $(2, \frac{3}{2})$ and $(-\frac{3}{2}, -2)$, which correspond to a pair of fields $(g_{\mu\nu}(x), \psi_\mu^\alpha(x))$. Therefore, there is a possibility of constructing a supergravity which contains only these two fields. Such a theory was explicitly constructed in ref. [6].

The field content of the $d=4$, $N=1$ supergravity is the vierbein (tetrad) $e_\mu{}^a(x)$ and a Majorana Rarita-Schwinger field $\psi_\mu(x)$. The vierbein is related to the metric as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(+1,-1,-1,-1)$ is the flat Minkowski metric. The vielbein formalism of gravity is reviewed in Appendix A. The Rarita-Schwinger field satisfies the Majorana condition $\psi_\mu^c (\equiv C\bar{\psi}_\mu^T) = \psi_\mu$, where C is the charge conjugation matrix satisfying

$$C^{-1}\gamma^a C = -\gamma^{aT}, \qquad C^T = -C. \tag{2.1}$$

The Lagrangian consists of the Einstein term and the Rarita-Schwinger term

$$\mathcal{L} = -\frac{1}{4} e\hat{R} - \frac{1}{2} ie\bar{\psi}_\mu \gamma^{\mu\nu\rho}\hat{D}_\nu \psi_\rho, \tag{2.2}$$

where $e = \det e_\mu{}^a$ and γ's with multiple indices are antisymmetrized products of gamma matrices with unit strength

$$\gamma^{\mu\nu\rho} = \frac{1}{3!}\left(\gamma^\mu\gamma^\nu\gamma^\rho \pm \text{permutations of } \mu\nu\rho\right). \tag{2.3}$$

The curvature and the covariant derivative are defined by

$$\begin{aligned}
\hat{R} &= e_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab}, \\
\hat{R}_{\mu\nu}{}^{ab} &= \partial_\mu \hat{\omega}_\nu{}^{ab} - \partial_\nu \hat{\omega}_\mu{}^{ab} + \hat{\omega}_\mu{}^a{}_c \hat{\omega}_\nu{}^{cb} - \hat{\omega}_\nu{}^a{}_c \hat{\omega}_\mu{}^{cb}, \\
\hat{D}_\nu \psi_\rho &= \left(\partial_\nu + \frac{1}{4}\hat{\omega}_\nu{}^{ab}\gamma_{ab}\right)\psi_\rho.
\end{aligned} \tag{2.4}$$

The spin connection $\hat{\omega}_\mu{}^{ab}$ used here is given by

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{1}{2} i\bar{\psi}_a \gamma_\mu \psi_b - \frac{1}{2} i\bar{\psi}_\mu \gamma_a \psi_b + \frac{1}{2} i\bar{\psi}_\mu \gamma_b \psi_a, \tag{2.5}$$

where $\omega_{\mu ab}$ is the spin connection without torsion given in eq. (A.15). The spin connection (2.5) has a torsion depending on the Rarita-Schwinger field

$$\hat{D}_\mu e_\nu{}^a - \hat{D}_\nu e_\mu{}^a = -i\bar{\psi}_\mu \gamma^a \psi_\nu. \tag{2.6}$$

If one wishes, it is also possible to express the Lagrangian using the torsionless spin connection $\omega_{\mu ab}$ but with explicit 4-fermi terms

$$\mathcal{L} = -\frac{1}{4} eR - \frac{1}{2} ie\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + (\text{4-fermi terms}), \tag{2.7}$$

where R and D_ν are defined by using the torsionless spin connection.

The Lagrangian (2.2) is invariant under three kinds of local symmetries up to total divergences:

(i) general coordinate transformations

$$\begin{aligned} \delta_G(\xi) e_\mu{}^a &= -\xi^\nu \partial_\nu e_\mu{}^a - \partial_\mu \xi^\nu e_\nu{}^a, \\ \delta_G(\xi) \psi_\mu &= -\xi^\nu \partial_\nu \psi_\mu - \partial_\mu \xi^\nu \psi_\nu, \end{aligned} \tag{2.8}$$

(ii) local Lorentz transformations

$$\begin{aligned} \delta_L(\lambda) e_\mu{}^a &= -\lambda^a{}_b e_\mu{}^b, \\ \delta_L(\lambda) \psi_\mu &= -\frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_\mu, \end{aligned} \tag{2.9}$$

(iii) local supertransformations

$$\begin{aligned} \delta_Q(\epsilon) e_\mu{}^a &= -i\bar{\epsilon}\gamma^a \psi_\mu, \\ \delta_Q(\epsilon) \psi_\mu &= \hat{D}_\mu \epsilon \equiv \left(\partial_\mu + \frac{1}{4} \hat{\omega}_\mu{}^{ab} \gamma_{ab} \right) \epsilon, \end{aligned} \tag{2.10}$$

where the transformation parameters $\xi^\mu(x)$, $\lambda^a{}_b(x)$ ($\lambda^{ab} = -\lambda^{ba}$) and $\epsilon_\alpha(x)$ ($\epsilon^c = \epsilon$) are arbitrary functions of the space-time coordinates x^μ. The invariance under the bosonic transformations (i), (ii) is manifest. The invariance under the local supertransformations (iii) is shown in Appendix B.

These local transformations satisfy the following commutator algebra:

$$\begin{aligned} [\delta_G(\xi_1), \delta_G(\xi_2)] &= \delta_G(\xi_1 \cdot \partial \xi_2 - \xi_2 \cdot \partial \xi_1), \\ [\delta_L(\lambda_1), \delta_L(\lambda_2)] &= \delta_L([\lambda_1, \lambda_2]), \\ [\delta_G(\xi), \delta_L(\lambda)] &= \delta_L(\xi \cdot \partial \lambda), \\ [\delta_G(\xi), \delta_Q(\epsilon)] &= \delta_Q(\xi \cdot \partial \epsilon), \\ [\delta_L(\lambda), \delta_Q(\epsilon)] &= \delta_Q(\tfrac{1}{4} \lambda^{ab} \gamma_{ab} \epsilon), \\ [\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_G(\xi) + \delta_L(\xi \cdot \hat{\omega}) + \delta_Q(\xi \cdot \psi), \qquad \xi^\mu = i\bar{\epsilon}_2 \gamma^\mu \epsilon_1. \end{aligned} \tag{2.11}$$

These commutation relations except the last one can be easily shown. The last commutation relation is shown in Appendix B. To obtain the last commutation relation one has to use field equations derived from the Lagrangian (2.2). In this sense the algebra closes only on-shell. In the present theory it is possible to close the commutator algebra off-shell by introducing an appropriate set of auxiliary fields, which have no dynamical degrees of freedom. Theories with off-shell algebra are more convenient, although not indispensable, when one fixes a gauge of local symmetries and when one couples matter supermultiplets. For general supergravities (those with extended supersymmetry and/or in higher dimensions) such an off-shell formulation is not known.

One can couple matter supermultiplets to the supergravity multiplet ($e_\mu{}^a$, ψ_μ). There are two kinds of matter supermultiplets in the $d=4$, $N=1$ supersymmetry. A chiral multiplet (ϕ, λ) consists of a complex scalar field $\phi(x)$ and a Majorana spinor field $\lambda(x)$. A vector multiplet (A_μ, χ) consists of a vector field $A_\mu(x)$ and a Majorana spinor field $\chi(x)$. The Lagrangian and the supertransformations of matter coupled theories can be obtained by using either of the Noether method [4], the superspace formulation [2], [7] or the tensor calculus [8]. In this paper, however, we do not discuss such matter couplings but concentrate on pure supergravities, which contain only supergravity multiplets.

So far we have considered the $N=1$ supergravity based on the $N=1$ supersymmetry. We can also consider a gauging of the N-extended supersymmetry with N transformation parameters ϵ^i $(i=1,2,\cdots,N)$ [5]. We need N gravitinos $\psi^i_\mu(x)$ $(i=1,2,\cdots,N)$, which transform as $\delta_Q\psi^i_\mu=\partial_\mu\epsilon^i+\cdots$. To make supermultiplets of the extended supersymmetry we also need other fields in addition to the metric and the Rarita-Schwinger fields. Supergravity multiplets of extended supersymmetries are listed in Table 3 of sect. 4. Representations of $N\geq 9$ supersymmetry algebra contain particles with helicities greater than two. Since consistent interacting theories of particles with such high helicities are not known, $N\geq 9$ supergravities have not been constructed.

N-extended supergravities contain $\frac{1}{2}N(N-1)$ vector fields denoted B_μ in Table 3. They are $\mathrm{U}(1)^N$ gauge fields. It is possible to construct theories in which the vector fields are $\mathrm{O}(N)$ non-abelian gauge fields. Such theories are called gauged supergravities [9]. Their Lagrangians contain a cosmological term e proportional to g^2 and a gravitino mass term $\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu$ proportional to g, where g is a gauge coupling constant of the non-abelian gauge fields. In the limit $g\rightarrow 0$ they reduce to the ordinary supergravities. The $N=1$ theory also has a generalization with a cosmological term and a gravitino mass term.

3. Spinors in higher dimensions

To construct supergravities in higher dimensions we need to know what kinds of spinors we can define in each dimension [10]. We consider spinor representations of the group $\mathrm{SO}(t,s)$ with an invariant metric

$$\eta_{ab}=\mathrm{diag}(\underbrace{+1,\cdots,+1}_{t},\underbrace{-1,\cdots,-1}_{s}),\qquad d=t+s. \tag{3.1}$$

The d-dimensional Minkowski space-time corresponds to the case $t=1$, $s=d-1$. Gamma matrices γ^a $(a=1,2,\cdots,d)$ of $\mathrm{SO}(t,\,s)$ satisfy the anticommutation relation

$$\{\gamma^a,\gamma^b\}=2\eta^{ab}. \tag{3.2}$$

Matrices γ^a are hermitian for $a = 1, \cdots, t$ and antihermitian for $a = t+1, \cdots, d$. The smallest matrices satisfying this anticommutation relation are $2^{[\frac{d}{2}]} \times 2^{[\frac{d}{2}]}$, where $[x]$ is the largest integer not larger than x. An explicit representation of the gamma matrices can be constructed as tensor products of 2×2 matrices. In even dimensions (e.g., $t = 2n$, $s = 0$) we can use the following tensor products of n 2×2 matrices:

$$
\begin{aligned}
\gamma^1 &= \sigma^1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
\gamma^2 &= \sigma^2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
\gamma^3 &= \sigma^3 \otimes \sigma^1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
\gamma^4 &= \sigma^3 \otimes \sigma^2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
&\vdots \\
\gamma^{2k+1} &= \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{k} \otimes \sigma^1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
\gamma^{2k+2} &= \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{k} \otimes \sigma^2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\
&\vdots \\
\gamma^{2n-1} &= \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^1, \\
\gamma^{2n} &= \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^2,
\end{aligned}
\tag{3.3}
$$

where $\mathbf{1}$ is the 2×2 unit matrix and σ^i $(i = 1, 2, 3)$ are the Pauli matrices. Gamma matrices in odd dimensions $(d = t + s = 2n + 1)$ can be constructed by using those in even $(d = t + (s-1) = 2n)$ dimensions. We can use gamma matrices γ^a $(a = 1, \cdots, 2n)$ of SO(t, $s-1$) for the first $2n$ gamma matrices of SO(t, s). The last matrix γ^{2n+1} can be taken as

$$
\gamma^{2n+1} = (-1)^{\frac{1}{4}(s-t)} \, i \, \gamma^1 \gamma^2 \cdots \gamma^{2n}. \tag{3.4}
$$

Spinors of SO(t,s) have $2^{[\frac{d}{2}]}$ complex components in general and transform under the Lorentz transformation as

$$
\delta_L \psi = -\frac{1}{4} \lambda^{ab} \gamma_{ab} \psi. \tag{3.5}
$$

As we will discuss below, we can reduce the number of independent components of spinors by imposing Weyl and/or Majorana conditions. These conditions must be consistent with the Lorentz transformation law (3.5). Spinors satisfying these conditions are called Weyl spinors and Majorana spinors respectively. General spinors without any condition are called Dirac spinors. To discuss supersymmetry it is convenient to use spinors with the smallest number of independent components in each dimension.

Weyl spinors are those having a definite chirality in even dimensions. We define the chirality of spinors as an eigenvalue of the matrix

$$\bar{\gamma} = (-1)^{\frac{1}{4}(s-t)}\gamma^1\gamma^2\cdots\gamma^d \tag{3.6}$$

satisfying

$$\bar{\gamma}^2 = 1, \qquad \{\bar{\gamma}, \gamma^a\} = 0. \tag{3.7}$$

This matrix $\bar{\gamma}$ is a generalization of γ_5 in four dimensions. Weyl spinors with positive (or negative) chirality are defined by

$$\bar{\gamma}\psi = \psi \qquad (\text{or } \ \bar{\gamma}\psi = -\psi)\,. \tag{3.8}$$

It is easy to see that these conditions are consistent with eq. (3.5). The matrix $\bar{\gamma}$ and therefore Weyl spinors can be defined in any even dimensions. In odd dimensions $\bar{\gamma}$ is proportional to the unit matrix and one cannot define Weyl spinors.

Majorana spinors are those satisfying a certain kind of reality condition. The Majorana condition is

$$\psi^c = \psi, \tag{3.9}$$

where the superscript ^{c} represents a charge conjugation. We shall discuss the charge conjugation for $d = t + s =$ even and for $d = t + s =$ odd separately.

Let us first consider the case $d = t + s =$ even. The matrices $\pm(\gamma^a)^*$ satisfy the same anticommutation relation as γ^a. Then, it can be shown that there exist matrices B_+ and B_-, which relate $\pm(\gamma^a)^*$ to γ^a by similarity transformations

$$\begin{aligned} (\gamma^a)^* &= B_+\gamma^a B_+^{-1}, \\ -(\gamma^a)^* &= B_-\gamma^a B_-^{-1}. \end{aligned} \tag{3.10}$$

The charge conjugation is defined by using one of these matrices as

$$\psi^c = B_+^{-1}\psi^* \qquad \text{or} \qquad \psi^c = B_-^{-1}\psi^*. \tag{3.11}$$

It can be shown that $B_\pm$ satisfy

$$B_\pm^* B_\pm = \epsilon_\pm(t,s)\,\mathbf{1}, \qquad \epsilon_\pm(t,s) = \sqrt{2}\cos\left[\frac{\pi}{4}(s - t \pm 1)\right]. \tag{3.12}$$

We have summarized the values of $\epsilon_\pm$ in Table 1.

$s-t$	1	2	3	4	5	6	7	8
ϵ_+	No	-1	-1	-1	No	$+1$	$+1$	$+1$
ϵ_-	$+1$	$+1$	No	-1	-1	-1	No	$+1$

Table 1: The values of $\epsilon_\pm$.

The reason why this operation is called the charge conjugation can be seen as follows. When a spinor field ψ satisfies the Dirac equation in the presence of an electromagnetic gauge field A_μ

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0, \tag{3.13}$$

the charge conjugated field ψ^c in eq. (3.11) satisfies

$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu \pm m)\psi^c = 0. \tag{3.14}$$

Thus, the sign in front of the charge e has changed. In the mass term the upper sign + is for B_+ and the lower − is for B_-. The definition of the charge conjugation in eq. (3.11) is equivalent to the usual one using the charge conjugation matrix C. To show this we introduce the Dirac conjugate of ψ as

$$\bar{\psi} = \psi^\dagger A, \qquad A = \gamma^1\gamma^2\cdots\gamma^t. \tag{3.15}$$

Then, eq. (3.11) can be rewritten as

$$\psi^c = C_+\bar{\psi}^T \qquad \text{or} \qquad \psi^c = C_-\bar{\psi}^T \qquad \left(C_\pm = B_\pm^{-1}A^{-1T}\right). \tag{3.16}$$

The charge conjugation matrices $C_\pm$ satisfy

$$\begin{aligned} \gamma^{aT} &= \pm(-1)^{t+1}C_\pm^{-1}\gamma^a C_\pm, \\ C_\pm^\dagger C_\pm = \mathbf{1}, &\qquad C_\pm^T = (\pm 1)^t(-1)^{\frac{1}{2}t(t-1)}\epsilon_\pm C_\pm. \end{aligned} \tag{3.17}$$

The usual four-dimensional charge conjugation matrix C used in sect. 2 is C_-.

For the Majorana condition (3.9) to be consistent we must have $(\psi^c)^c = \psi$, which is equivalent to $B_+^* B_+ = 1$ or $B_-^* B_- = 1$. Therefore, for $t+s =$ even, Majorana spinors can be defined only when $\epsilon_+(t,s) = 1$ or $\epsilon_-(t,s) = 1$. Sometimes, spinors satisfying eq. (3.9) with the charge conjugation defined by using the matrix B_+ are called pseudo Majorana spinors, while those using B_- are called Majorana spinors. From Table 1 we can see in which dimensions (pseudo) Majorana spinors can be defined.

d	W	M	pM	MW	pMW
2	o	o	o	o	o
3		o			
4	o	o			
5					
6	o				
7					
8	o		o		
9			o		
10	o	o	o	o	o
11		o			

Table 2: Possible types of spinors in d-dimensional Minkowski space-time ($t = 1$, $s = d-1$). W, M, pM, MW, pMW denote Weyl, Majorana, pseudo Majorana, Majorana-Weyl and pseudo Majorana-Weyl spinors respectively.

The charge conjugation for $d = t+s =$ odd is defined by using the matrices $B_\pm$ used in even dimensions. Recall that gamma matrices in d (odd) dimensions can be constructed from those in $d-1$ (even) dimensions. The first $d-1$ matrices γ^a $(a = 1, 2, \cdots, d-1)$ are taken to be those of $d-1$ dimensions. The last matrix γ^d is taken to be $i\bar{\gamma}$ $(\bar{\gamma})$ if $a = d$ is a space-like (time-like) direction. Then the matrices $B_\pm$ used in $d-1$ dimensions satisfy

$$\begin{aligned} B_\pm \gamma^a B_\pm^{-1} &= \pm(\gamma^a)^* \qquad (a = 1, \cdots, d-1), \\ B_\pm \gamma^d B_\pm^{-1} &= (-1)^{\frac{1}{2}(s-t+1)}(\gamma^d)^*. \end{aligned} \tag{3.18}$$

When $(-1)^{\frac{1}{2}(s-t+1)} = 1$ $((-1)^{\frac{1}{2}(s-t+1)} = -1)$, the signs on the right hand side are the same for all γ^a $(a = 1, \cdots, d)$, and we can use B_+ (B_-) to define the charge conjugation. As in the case $d =$ even, the charge conjugation must satisfy $(\psi^c)^c = \psi$ to define (pseudo) Majorana spinors. Possible $B_\pm$ and corresponding $\epsilon_\pm$ are listed in Table 1.

We can also define (pseudo) Majorana-Weyl spinors, which satisfy both of the (pseudo) Majorana condition $\psi^c = \psi$ and the Weyl condition $\bar{\gamma}\psi = \psi$ (or $\bar{\gamma}\psi = -\psi$). (Pseudo) Majorana-Weyl spinors are possible only if these two conditions are consistent, i.e., ψ and ψ^c have the same chirality. In general, when a spinor ψ has a chirality $+$ $(-)$, the charge conjugated spinor ψ^c has a chirality $(-1)^{\frac{1}{2}(s-t)}$ $(-(-1)^{\frac{1}{2}(s-t)})$. Therefore, (pseudo) Majorana-Weyl spinors can be defined only when $s - t = 0$ mod 4. In particular, they can be defined in $d = 2$ mod 4 for Minkowski signature $t = 1$, $s = d-1$.

Possible types of spinors in various dimensions with Minkowski signature $t = 1$, $s = d-1$ are summarized in Table 2. This table is periodic in dimensions d with a period 8.

When $(\psi^c)^c = -\psi$, we cannot impose the (pseudo) Majorana condition $\psi^c = \psi$ and we have to use Dirac spinors (or Weyl spinors in even dimensions). Alternatively, we can introduce even numbers of spinors ψ^i $(i=1,2,\cdots,2n)$ and impose the condition

$$\psi^i = \Omega^{ij}(\psi^j)^c, \tag{3.19}$$

where $\Omega^{ij}=-\Omega^{ji}$ is a constant antisymmetric matrix. Spinors satisfying such a condition are called symplectic (pseudo) Majorana spinors. $2n$ symplectic (pseudo) Majorana spinors are equivalent to n Dirac spinors. Sometimes it is more convenient to use symplectic (pseudo) Majorana spinors than Dirac spinors, especially when the theory has a symplectic symmetry.

4. Superalgebras and supergravity multiplets

Field contents of supergravities are determined by irreducible representations of the super Poincaré algebras [11]. The super Poincaré algebras consist of generators of translations P_a, generators of Lorentz transformations M_{ab}, supercharges $Q^{\alpha i}$, generators of automorphism group T^A and "central" charges Z^{ij}. Nonvanishing (anti) commutation relations besides $\{Q,Q\}$ are, in addition to the usual commutators of the Poincaré algebra,

$$\begin{aligned}
&[M_{ab},Q^i]=\frac{1}{2}\gamma_{ab}Q^i, \qquad [T^A,Q^i]=(t^A)^i{}_jQ^j, \\
&[T^A,Z^{ij}]=(t^A)^i{}_kZ^{kj}+(t^A)^j{}_kZ^{ik}, \qquad [T^A,T^B]=f^{AB}{}_CT^C,
\end{aligned} \tag{4.1}$$

where t^A and $f^{AB}{}_C$ are representation matrices and the structure constant of the Lie algebra of the automorphism group.

The automorphism group K and the form of anticommutators $\{Q,Q\}$ depend on the spinor type of Q^i.

(a) $d=4,8$ mod 8

The supercharges are Weyl spinors with positive chirality Q^i_+ $(i=1,2,\cdots,N)$. Their charge conjugations have negative chirality $(Q^i_+)^c=Q_{-i}$, where the charge conjugation matrix $C=C_-$ $(C=C_+)$ is used for $d=4$ $(d=8)$ mod 8. The automorphism group is $\mathrm{K}=\mathrm{U}(N)$. Anticommutators of the supercharges are

$$\begin{aligned}
\{Q^i_+,Q^T_{-j}\}&=\frac{1}{2}(1+\bar{\gamma})\gamma^aCP_a\delta^i_j, \\
\{Q^i_+,Q^{jT}_+\}&=\frac{1}{2}(1+\bar{\gamma})CZ^{ij},
\end{aligned} \tag{4.2}$$

where $Z^{ij}=-Z^{ji}$ for $d=4$ mod 8 and $Z^{ij}=Z^{ji}$ for $d=8$ mod 8.

(b) $d=10$ mod 8

The supercharges are Majorana-Weyl spinors with positive chirality Q^i_+ $(i=1,2,\cdots,N_+)$ and Majorana-Weyl spinors with negative chirality Q^i_- $(i=1,2,\cdots,N_-)$. The automorphism group is $\mathrm{K} = \mathrm{SO}(N_+) \times \mathrm{SO}(N_-)$. Anticommutators of the supercharges are

$$\begin{aligned}\{Q_+^i, Q_+^{jT}\} &= \frac{1}{2}(1+\bar{\gamma})\gamma^a C_- P_a \delta^{ij},\\ \{Q_-^i, Q_-^{jT}\} &= \frac{1}{2}(1-\bar{\gamma})\gamma^a C_- P_a \delta^{ij},\\ \{Q_+^i, Q_-^{jT}\} &= \frac{1}{2}(1+\bar{\gamma}) C_- Z^{ij}.\end{aligned} \tag{4.3}$$

(c) $d = 6 \bmod 8$

The supercharges are symplectic Majorana-Weyl spinors with positive chirality Q_+^i ($i = 1, 2, \cdots, N_+$) and symplectic Majorana-Weyl spinors with negative chirality Q_-^i ($i = 1, 2, \cdots, N_-$). They satisfy $\Omega_+^{ij}(Q_+^j)^c = Q_+^i$, $\Omega_-^{ij}(Q_-^j)^c = Q_-^i$, where $\Omega_\pm^{ij}$ are antisymmetric matrices. The numbers N_+ and N_- must be even. The automorphism group is K $= \mathrm{USp}(N_+) \times \mathrm{USp}(N_-)$. Anticommutators of the supercharges are

$$\begin{aligned}\{Q_+^i, Q_+^{jT}\} &= \frac{1}{2}(1+\bar{\gamma})\gamma^a C_- P_a \Omega_+^{ij},\\ \{Q_-^i, Q_-^{jT}\} &= \frac{1}{2}(1-\bar{\gamma})\gamma^a C_- P_a \Omega_-^{ij},\\ \{Q_+^i, Q_-^{jT}\} &= \frac{1}{2}(1+\bar{\gamma}) C_- Z^{ij}.\end{aligned} \tag{4.4}$$

(d) $d = 9, 11 \bmod 8$

The supercharges are (pseudo) Majorana spinors Q^i ($i = 1, 2, \cdots, N$). The automorphism group is $\mathrm{K} = \mathrm{SO}(N)$ and anticommutators of the supercharges are

$$\{Q^i, Q^{jT}\} = \gamma^a C P_a \delta^{ij} + C Z^{ij}, \tag{4.5}$$

where $C = C_+$, $Z^{ij} = Z^{ji}$ for $d = 9 \bmod 8$ and $C = C_-$, $Z^{ij} = -Z^{ji}$ for $d = 11 \bmod 8$.

(e) $d = 5, 7 \bmod 8$

The supercharges are symplectic (pseudo) Majorana spinors Q^i ($i = 1, 2, \cdots, N$). They satisfy $\Omega^{ij}(Q^j)^c = Q^i$, where Ω^{ij} is an antisymmetric matrix. The number N must be even. The automorphism group is $\mathrm{K} = \mathrm{USp}(N)$ and anticommutators of the supercharges are

$$\{Q^i, Q^{jT}\} = \gamma^a C P_a \Omega^{ij} + C Z^{ij}, \tag{4.6}$$

where $C = C_+$, $Z^{ij} = -Z^{ji}$ for $d = 5 \bmod 8$ and $C = C_-$, $Z^{ij} = Z^{ji}$ for $d = 7 \bmod 8$.

Particles appearing in supergravities belong to irreducible representations of these super Poincaré algebras. All states in an irreducible representation are obtained by applying components of the supercharges with helicity $\frac{1}{2}$ $Q_{\frac{1}{2}}$ on the lowest helicity state $|h_{\min}\rangle$. A state $Q_{\frac{1}{2}}Q_{\frac{1}{2}}\cdots Q_{\frac{1}{2}}\,|h_{\rm min}\rangle$ with n $Q_{\frac{1}{2}}$'s has helicity $h = h_{\rm min} + \frac{1}{2}n$. (For details, see ref. [11].) When the supercharges have too many components, all representations contain particles with helicity $|h| > 2$. However, consistent interacting theories with helicity > 2 are not known. Hence, we should consider algebras which have representations with helicity ≤ 2. Then, there are only a finite number of possible (d, N). In particular, the spacetime dimension must be $d \leq 11$. Supermultiplets for these (d, N) were given in ref. [11]. Representations which contain graviton and gravitinos are called supergravity multiplets. They are massless representations of the algebra, which satisfy

$$P_a P^a = 0, \qquad Z^{ij} = 0. \tag{4.7}$$

Field contents corresponding to supergravity multiplets in various dimensions are listed in Table 3. In addition to these supergravity multiplets there can exist massless and massive matter supermultiplets containing particles with spins ≤ 1. We do not discuss matter supermultiplets in this paper.

As a check of the field contents in Table 3 one can count the numbers of bosonic and fermionic degrees of freedom in a supergravity multiplet, which should be the same. The physical degrees of freedom of each fields are most easily obtained in the light-cone gauge, where only transverse components of the fields are physical. We find that the numbers of physical degrees of freedom are

$$\begin{aligned}
e_\mu{}^a &:\ \tfrac{1}{2}(d-2)(d-1)-1, \\
B_{\mu\nu\rho\sigma} &:\ {}_{d-2}\mathrm{C}_4 = \tfrac{1}{24}(d-2)(d-3)(d-4)(d-5), \\
B_{\mu\nu\rho} &:\ {}_{d-2}\mathrm{C}_3 = \tfrac{1}{6}(d-2)(d-3)(d-4), \\
B_{\mu\nu} &:\ {}_{d-2}\mathrm{C}_2 = \tfrac{1}{2}(d-2)(d-3), \\
B_\mu &:\ d-2, \\
\phi &:\ 1, \\
\psi_\mu &:\ \tfrac{1}{2}(d-2-1)\,2^{[\frac{d}{2}]}, \\
\lambda &:\ \tfrac{1}{2}\,2^{[\frac{d}{2}]}.
\end{aligned} \tag{4.8}$$

The number -1 for $e_\mu{}^a$ and ψ_μ comes from the (γ-)traceless conditions on graviton and gravitino. The factor $\frac{1}{2}$ for spinor fields is due to the fact that their field equations are first order differential equations. The numbers of the bosonic and fermionic degrees of freedom in each supergravity multiplet are indeed the same and are given in the last column of the table.

d	N	spinors	fields	n
11	1	M	$e_\mu{}^a$, ψ_μ, $B_{\mu\nu\rho}$	128
10	(1,1)	MW	$e_\mu{}^a$, $\psi_{+\mu}$, $\psi_{-\mu}$, $B_{\mu\nu\rho}$, $B_{\mu\nu}$, B_μ, λ_+, λ_-, ϕ	128
	(2,0)	MW	$e_\mu{}^a$, $2\,\psi_{+\mu}$, $B^{(+)}_{\mu\nu\rho\sigma}$, $2\,B_{\mu\nu}$, $2\,\lambda_-$, $2\,\phi$	128
	(1,0)	MW	$e_\mu{}^a$, $\psi_{+\mu}$, $B_{\mu\nu}$, λ_-, ϕ	64
9	2	pM	$e_\mu{}^a$, $2\,\psi_\mu$, $B_{\mu\nu\rho}$, $2\,B_{\mu\nu}$, $3\,B_\mu$, $4\,\lambda$, $3\,\phi$	128
	1	pM	$e_\mu{}^a$, ψ_μ, $B_{\mu\nu}$, B_μ, λ, ϕ	56
8	2	pM	$e_\mu{}^a$, $2\,\psi_\mu$, $B_{\mu\nu\rho}$, $3\,B_{\mu\nu}$, $6\,B_\mu$, $6\,\lambda$, $7\,\phi$	128
	1	pM	$e_\mu{}^a$, ψ_μ, $B_{\mu\nu}$, $2\,B_\mu$, λ, ϕ	48
7	4	sM	$e_\mu{}^a$, $4\,\psi_\mu$, $5\,B_{\mu\nu}$, $10\,B_\mu$, $16\,\lambda$, $14\,\phi$	128
	2	sM	$e_\mu{}^a$, $2\,\psi_\mu$, $B_{\mu\nu}$, $3\,B_\mu$, $2\,\lambda$, ϕ	40
6	(4,4)	sMW	$e_\mu{}^a$, $4\,\psi_{+\mu}$, $4\,\psi_{-\mu}$, $5\,B_{\mu\nu}$, $16\,B_\mu$, $20\,\lambda_+$, $20\,\lambda_-$, $25\,\phi$	128
	(4,2)	sMW	$e_\mu{}^a$, $4\,\psi_{+\mu}$, $2\,\psi_{-\mu}$, $5\,B^{(+)}_{\mu\nu}$, $B^{(-)}_{\mu\nu}$, $8\,B_\mu$, $10\,\lambda_+$, $4\,\lambda_-$, $5\,\phi$	64
	(2,2)	sMW	$e_\mu{}^a$, $2\,\psi_{+\mu}$, $2\,\psi_{-\mu}$, $B_{\mu\nu}$, $4\,B_\mu$, $2\,\lambda_+$, $2\,\lambda_-$, ϕ	32
	(4,0)	sMW	$e_\mu{}^a$, $4\,\psi_{+\mu}$, $5\,B^{(+)}_{\mu\nu}$	24
	(2,0)	sMW	$e_\mu{}^a$, $2\,\psi_{+\mu}$, $B^{(+)}_{\mu\nu}$	12
5	8	spM	$e_\mu{}^a$, $8\,\psi_\mu$, $27\,B_\mu$, $48\,\lambda$, $42\,\phi$	128
	6	spM	$e_\mu{}^a$, $6\,\psi_\mu$, $15\,B_\mu$, $20\,\lambda$, $14\,\phi$	64
	4	spM	$e_\mu{}^a$, $4\,\psi_\mu$, $6\,B_\mu$, $4\,\lambda$, ϕ	24
	2	spM	$e_\mu{}^a$, $2\,\psi_\mu$, B_μ	8
4	8	M	$e_\mu{}^a$, $8\,\psi_\mu$, $28\,B_\mu$, $56\,\lambda$, $70\,\phi$	128
	6	M	$e_\mu{}^a$, $6\,\psi_\mu$, $16\,B_\mu$, $26\,\lambda$, $30\,\phi$	64
	5	M	$e_\mu{}^a$, $5\,\psi_\mu$, $10\,B_\mu$, $11\,\lambda$, $10\,\phi$	32
	4	M	$e_\mu{}^a$, $4\,\psi_\mu$, $6\,B_\mu$, $4\,\lambda$, $2\,\phi$	16
	3	M	$e_\mu{}^a$, $3\,\psi_\mu$, $3\,B_\mu$, λ	8
	2	M	$e_\mu{}^a$, $2\,\psi_\mu$, B_μ	4
	1	M	$e_\mu{}^a$, ψ_μ	2

Table 3: Supergravity multiplets. $e_\mu{}^a$, ψ_μ, $B_{\mu\cdots}$, λ and ϕ represent vielbein, Rarita-Schwinger fields, antisymmetric tensor fields, spin $\frac{1}{2}$ spinor fields and scalar fields respectively. The subscripts $\pm$ on spinor fields denote chiralities. The superscripts $(\pm)$ on antisymmetric tensor fields mean that they are (anti-)self-dual. The numbers of fields are counted by real fields for bosonic fields and (symplectic/pseudo) Majorana(-Weyl) spinors for fermionic fields. The last column n denotes bosonic (= fermionic) physical degrees of freedom.

5. Supergravities in higher dimensions

Lagrangians and local supertransformation laws of fields of supergravities in various dimensions were explicitly obtained by the Noether's method or by dimensional reductions from higher dimensional theories. Supergravity in the highest space-time dimensions is the $d=11$, $N=1$ theory [12]. The field content is the vielbein $e_\mu{}^a$, a Majorana Rarita-Schwinger field ψ_μ and a real third rank antisymmetric tensor field $B_{\mu\nu\rho}$. The Lagrangian has a relatively simple form

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}eR-\frac{1}{2}ie\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho-\frac{1}{48}eF_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\\ &+\frac{1}{96}e\left(\bar{\psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu+12\,\bar{\psi}^\alpha\gamma^{\beta\gamma}\psi^\delta\right)F_{\alpha\beta\gamma\delta}\\ &+\frac{2}{144^2}\,\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4\mu\nu\rho}F_{\alpha_1\cdots\alpha_4}F_{\beta_1\cdots\beta_4}B_{\mu\nu\rho}\ +\ (\text{4-fermi terms}),\end{aligned}\tag{5.1}$$

where $F_{\mu\nu\rho\sigma}=4\partial_{[\mu}B_{\nu\rho\sigma]}$ is the field strength of the antisymmetric tensor. This Lagrangian is invariant under the local supertransformation

$$\begin{aligned}\delta_Q e_\mu{}^a &= -i\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta_Q B_{\mu\nu\rho}=\frac{3}{2}\,\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]},\\ \delta_Q\psi_\mu &= D_\mu\epsilon+\frac{i}{144}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu-8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)\epsilon F_{\alpha\beta\gamma\delta}+(\text{3-fermi terms})\end{aligned}\tag{5.2}$$

in addition to the general coordinate and the local Lorentz transformations. It is also invariant under the local gauge transformation of the antisymmetric tensor field

$$\delta_g e_\mu{}^a=0, \qquad \delta_g\psi_\mu=0, \qquad \delta_g B_{\mu\nu\rho}=3\,\partial_{[\mu}\Lambda_{\nu\rho]} \qquad (\Lambda_{\mu\nu}=-\Lambda_{\nu\mu}).\tag{5.3}$$

In ten dimensions there are three types of supergravities: $(N_+,N_-)=(1,1)$, $(2,0)$, $(1,0)$. The field contents are given in Table 3. The $(1,1)$ supergravity [13] can be obtained from the $d=11$ theory by a dimensional reduction and is vector-like (left-right symmetric). This theory is a massless sector of the type IIA superstring theory. The $(2,0)$ supergravity [14] is a chiral (left-right asymmetric) theory. It contains a fourth rank antisymmetric tensor field $B^{(+)}_{\mu\nu\rho\sigma}$, whose field strength satisfies a self-duality condition $F_{\mu\nu\rho\sigma\tau}=\frac{1}{5!}e\epsilon_{\mu\nu\rho\sigma\tau\alpha\beta\gamma\delta\eta}F^{\alpha\beta\gamma\delta\eta}$. Because of this self-dual field a Lorentz covariant action of this theory is not known although field equations were explicitly obtained. It has a non-compact symmetry SU(1,1) and the scalar fields are described by an SU(1,1)/U(1) non-linear sigma model. This theory is a massless sector of the type IIB superstring theory. The $(1,0)$ supergravity [15] is a chiral theory. There exists a matter supermultiplet (A_μ,χ_+), where A_μ is a gauge field and χ_+ is a spin $\frac{1}{2}$ Majorana-Weyl spinor field, both of which are in the adjoint representation of a certain gauge group. This theory is a massless sector of the type I superstring theory and the heterotic string theory.

Supergravities in $d < 10$ dimensions, whose field contents are given in Table 3, can be obtained from $d = 11$ or $d = 10$ supergravity by dimensional reductions and truncations of fields. Their general structure is as follows. Field contents are the vielbein, Rarita-Schwinger fields, antisymmetric tensor gauge fields, spin $\frac{1}{2}$ spinor fields and scalar fields. When scalar fields are present, the theory has a rigid non-compact symmetry G. The scalar fields are described by a G/H non-linear sigma model, where H is a maximal compact subgroup of G. For instance, two scalar fields in the $d = 4$, $N = 4$ theory are described by the SU(1,1)/U(1) sigma model. In even dimensions ($d = 2n$) G acts on $(n-1)$-th antisymmetric tensor fields $B_{\mu_1 \cdots \mu_{n-1}}$ as duality transformations. In this case G is a symmetry of equations of motion but not of the action. In the following two sections we will discuss these structures in detail.

6. Non-linear sigma models

Scalar fields appearing in supergravities are described by a G/H non-linear sigma model, where G is a non-compact Lie group and H is a maximal compact subgroup of G. The G/H non-linear sigma model is a theory of G/H-valued scalar fields, which is invariant under rigid G transformations. In this section we shall review how to construct G/H non-linear sigma models [16], [17].

We represent the scalar fields by a G-valued scalar field $V(x)$ and require local H invariance. Since we do not introduce independent H gauge fields, the H part of $V(x)$ can be gauged away and physical degrees of freedom are on a coset space G/H. The rigid G transformations act on $V(x)$ from the left

$$V(x) \to gV(x) \qquad (g \in G), \tag{6.1}$$

while the local H transformations act from the right

$$V(x) \to V(x)h^{-1}(x) \qquad (h(x) \in H). \tag{6.2}$$

To construct the action we decompose the Lie algebra $\mathbf{G}$ of G as

$$\mathbf{G} = \mathbf{H} + \mathbf{N}, \tag{6.3}$$

where $\mathbf{H}$ is the Lie algebra of H and $\mathbf{N}$ is its orthogonal complement in $\mathbf{G}$. The orthogonality is defined with respective to trace in a certain representation: $\mathrm{tr}(\mathbf{H}\,\mathbf{N}) = 0$. It can be easily shown that

$$[\mathbf{H},\mathbf{H}] \subset \mathbf{H}, \qquad [\mathbf{H},\mathbf{N}] \subset \mathbf{N}. \tag{6.4}$$

The **G**-valued field $V^{-1}\partial_\mu V$ is decomposed as

$$V^{-1}\partial_\mu V = Q_\mu + P_\mu, \qquad Q_\mu \in \mathbf{H}, \quad P_\mu \in \mathbf{N}. \tag{6.5}$$

By using eq. (6.4) the transformation laws of Q_μ and P_μ under the local H transformations (6.2) are found to be

$$\begin{aligned} Q_\mu &\to h Q_\mu h^{-1} + h\partial_\mu h^{-1}, \\ P_\mu &\to h P_\mu h^{-1}, \end{aligned} \tag{6.6}$$

while they are invariant under the rigid G transformations (6.1). We see that Q_μ transforms as an H gauge field, while P_μ is covariant under the H transformations. From eq. (6.5) P_μ can be expressed as

$$P_\mu = V^{-1}\left(\partial_\mu V - V Q_\mu\right) \equiv V^{-1} D_\mu V, \tag{6.7}$$

where D_μ is the H-covariant derivative on V.

By using these quantities we can construct an action which is invariant under the rigid G and the local H transformations. The kinetic term of the scalar fields is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\,\mathrm{tr}(P_\mu P^\mu) \\ &= \frac{1}{2}\,\mathrm{tr}(V^{-1} D_\mu V V^{-1} D^\mu V). \end{aligned} \tag{6.8}$$

This action is quadratic in derivatives of V and is manifestly invariant under the rigid G and the local H transformations. The H-connection Q_μ can be used to define the covariant derivatives on other fields transforming under the local H. For instance, when a spinor field $\psi(x)$ transforms under the local H transformations as $\psi(x) \to h(x)\psi(x)$, the covariant derivative is

$$D_\mu \psi = \left(\partial_\mu + Q_\mu\right)\psi. \tag{6.9}$$

We can also use P_μ to construct H-invariant terms in the action such as

$$\bar{\psi}\gamma^\mu P_\mu \psi. \tag{6.10}$$

We can describe the theories in terms of physical fields by fixing a gauge for the local H symmetry. For instance, we can choose a gauge

$$V(x) = \mathrm{e}^{\Phi(x)}, \tag{6.11}$$

d	N	G	H
10	(1,1)	GL(1, **R**)	1
	(2,0)	SL(2, **R**)	SO(2)
	(1,0)	G(1, **R**)	1
9	2	GL(2, **R**)	SO(2)
	1	GL(1, **R**)	1
8	2	SL(3, **R**) × SL(2, **R**)	SO(3) × SO(2)
	1	GL(1, **R**)	1
7	4	SL(5, **R**)	SO(5)
	2	GL(1, **R**)	1
6	(4,4)	SO(5,5)	SO(5) × SO(5)
	(4,2)	SO(5,1)	SO(5)
	(2,2)	GL(1, **R**)	1
5	8	$E_{6(+6)}$	USp(8)
	6	$SU^*(6)$	USp(6)
	4	USp(4) × GL(1, **R**)	USp(4)
4	8	$E_{7(+7)}$	SU(8)
	6	$SO^*(12)$	U(6)
	5	SU(5,1)	U(5)
	4	SU(4) × SL(2, **R**)	U(4)

Table 4: G and H in supergravities.

where $\Phi(x)$ is an N-valued field, which represents physical degrees of freedom. The G transformations (6.1) break this gauge condition. To preserve the gauge the transformation g must be accompanied by a compensating H transformation $h(x;g)$. Therefore, the G transformations of $\Phi(x)$ are given by

$$e^{\Phi(x)} \to e^{\Phi'(x)} = g\, e^{\Phi(x)} h^{-1}(x;g). \tag{6.12}$$

The compensating transformation $h(x;g)$ is chosen such that $\Phi'(x)$ belongs to N. The transformation $\Phi(x) \to \Phi'(x)$ is a non-linear realization of (6.1). When $g \in$ H, we can take $h(x;g) = g$ and g is linearly realized.

The groups G and H appearing in supergravities are listed in Table 4. One can check that the dimension of the coset space G/H is equal to the number of scalar fields in each theory. As an example of G/H sigma models in supergravities let us consider the case G = SL(2, **R**) $\sim$ SU(1,1) and H = SO(2) $\sim$ U(1). This sigma model appears in the $d = 10$, $(2,0)$ and $d = 4$, $N = 4$ supergravities. The SU(1,1)-valued scalar field $V(x)$ is parametrized by two complex scalar fields $\phi_0(x)$, $\phi_1(x)$ as

$$V(x) = \begin{pmatrix} \phi_0(x) & \phi_1^*(x) \\ \phi_1(x) & \phi_0^*(x) \end{pmatrix}, \qquad |\phi_0|^2 - |\phi_1|^2 = 1. \tag{6.13}$$

The rigid SU(1,1) and the local U(1) transformations are given by eqs. (6.1) and (6.2) respectively with

$$g = \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix}, \qquad h(x) = \begin{pmatrix} \mathrm{e}^{i\theta(x)} & 0 \\ 0 & \mathrm{e}^{-i\theta(x)} \end{pmatrix}, \tag{6.14}$$

where $|a|^2 - |b|^2 = 1$. The quantities in eq. (6.5) are obtained as

$$\begin{aligned} Q_\mu &= (\phi_0^* \partial_\mu \phi_0 - \phi_1^* \partial_\mu \phi_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ P_\mu &= \begin{pmatrix} 0 & (\phi_0 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_0)^* \\ \phi_0 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_0 & 0 \end{pmatrix}. \end{aligned} \tag{6.15}$$

Therefore, the Lagrangian (6.8) becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \mathrm{tr}\,(P_\mu P^\mu) \\ &= |\phi_0 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_0|^2 \\ &= \frac{\partial_\mu z \partial^\mu z^*}{(1 - |z|^2)^2} \qquad \left(z \equiv \phi_1^* (\phi_0^*)^{-1} \right). \end{aligned} \tag{6.16}$$

The variable z is U(1)-invariant and represents physical degrees of freedom. It transforms under SU(1,1) as

$$z \longrightarrow \frac{az + b^*}{bz + a^*}. \tag{6.17}$$

It can be easily seen that the Lagrangian (6.16) is invariant under this transformation.

7. Duality symmetries

7.1 Duality symmetry in the free Maxwell theory

In this section we discuss duality symmetries appearing in supergravities in even dimensions. Duality symmetries are generalizations of the electric-magnetic duality in the Maxwell theory. Let us first consider the free Maxwell theory as a simple example to explain what duality symmetries are. The free Maxwell equations consist of the equation of motion and the Bianchi identity

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{7.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{7.2}$$

The set of equations (7.1) is invariant under rigid general linear transformations

$$\delta\begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} \qquad (A,\ B,\ C,\ D \in \mathbf{R}). \tag{7.3}$$

Such transformations, which mix $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ are called duality transformations. We have to take into account the fact that $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ are not independent but are related by the duality operation in eq. (7.2). By taking a dual of the upper equation of eq. (7.3) and using an identity $\tilde{\tilde{F}} = -F$, we obtain $\delta\tilde{F} = A\tilde{F} - BF$, which should coincide with the lower equation. Therefore, the transformation parameters must satisfy $D = A$ and $C = -B$, and the symmetry transformation is

$$\delta\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} F \\ \tilde{F} \end{pmatrix}. \tag{7.4}$$

We can diagonalize the transformation matrix by using a complex basis

$$\delta\begin{pmatrix} F + i\tilde{F} \\ F - i\tilde{F} \end{pmatrix} = \begin{pmatrix} A - iB & 0 \\ 0 & A + iB \end{pmatrix}\begin{pmatrix} F + i\tilde{F} \\ F - i\tilde{F} \end{pmatrix}. \tag{7.5}$$

We see that the group of the duality transformations is $\mathrm{GL}(1, \mathbf{C})$.

To discuss the duality symmetry we have not studied the invariance of the action but that of the field equations. The reason is that the duality transformations are consistent only on-shell. Since the independent variables of the theory is A_μ, $\delta F_{\mu\nu}$ in eq. (7.4) should be derived from δA_μ:

$$\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = AF_{\mu\nu} + B\tilde{F}_{\mu\nu}. \tag{7.6}$$

The integrability of this equation requires $\partial_\mu(A\tilde{F} + B\tilde{\tilde{F}})^{\mu\nu} = 0$, i.e., $\partial_\mu F^{\mu\nu} = 0$. Thus, the equation of motion must be satisfied. Even if we ignore this point and formally consider the transformation (7.4) off-shell, the action $-\frac{1}{4}\int d^4x F_{\mu\nu}^2$ is not invariant. Therefore, to construct theories invariant under duality transformations it is easier to study the covariance of equations of motion.

7.2 Duality symmetries in higher dimensions

We shall study duality symmetries of interacting theories in general even dimensions $d = 2n$ [18], [17], [19]. We consider theories of $(n-1)$-th rank antisymmetric tensor fields

$B^I_{\mu_1\cdots\mu_{n-1}}(x)$ $(I=1,\cdots,M)$ interacting with other fields $\phi_i(x)$. The field strengths of the tensor fields and their duals are defined as

$$F^I_{\mu_1\cdots\mu_n} = n\,\partial_{[\mu_1} B^I_{\mu_2\cdots\mu_n]},$$
$$\tilde{F}^{I\mu_1\cdots\mu_n} = \frac{1}{n!} e^{-1} \epsilon^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n} F^I_{\nu_1\cdots\nu_n}. \tag{7.7}$$

In d dimensions the duality operation satisfies

$$\tilde{\tilde{F}} = \epsilon F, \qquad \epsilon = \begin{cases} +1 & \text{for } d=4k+2, \\ -1 & \text{for } d=4k. \end{cases} \tag{7.8}$$

We assume that the Lagrangian has a form

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\phi, \partial\phi, F) \\ &= \frac{1}{2n!}\epsilon e K_{1IJ}(\phi) F^I_{\mu_1\cdots\mu_n} F^{J\mu_1\cdots\mu_n} + \frac{1}{2n!}\epsilon e K_{2IJ}(\phi) F^I_{\mu_1\cdots\mu_n} \tilde{F}^{J\mu_1\cdots\mu_n} \\ &\quad + e F^I_{\mu_1\cdots\mu_n} O_I^{\mu_1\cdots\mu_n}(\phi,\partial\phi) + \mathcal{L}'(\phi,\partial\phi),\end{aligned} \tag{7.9}$$

where $K_{1IJ} = K_{1JI}$, $K_{2IJ} = -\epsilon K_{2JI}$. The fields $B^I_{\mu_1\cdots\mu_{n-1}}$ appear only through their field strengths $F^I_{\mu_1\cdots\mu_n}$. The Lagrangians of supergravities are of this type. We require duality symmetries in this theory, and obtain conditions on the functions K_1, K_2, O and possible duality symmetry groups.

The equations of motion for $B^I_{\mu_1\cdots\mu_{n-1}}$ and the Bianchi identities are

$$\partial_{\mu_1}\left(e\tilde{G}_I^{\mu_1\cdots\mu_n}\right) = 0, \qquad \partial_{\mu_1}\left(e\tilde{F}^{I\mu_1\cdots\mu_n}\right) = 0, \tag{7.10}$$

where the dual of antisymmetric tensors $G_{I\mu_1\cdots\mu_n}$ are defined by

$$\tilde{G}_I^{\mu_1\cdots\mu_n} = \frac{n!}{e}\frac{\partial\mathcal{L}}{\partial F^I_{\mu_1\cdots\mu_n}}. \tag{7.11}$$

(For the free Maxwell theory, $G^{\mu\nu} = \tilde{F}^{\mu\nu}$.) These equations are invariant under transformations

$$\delta\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}, \qquad \delta\phi^i = \xi^i(\phi), \tag{7.12}$$

where A, B, C, D are constant $n\times n$ real matrices and $\xi^i(\phi)$ are functions of ϕ^i. As in the Maxwell theory these constants are not independent. We shall obtain the conditions that these constants should satisfy by studying (i) the covariance of the definition of G (7.11) and (ii) the covariance of the equations of motion for ϕ^i.

Let us first study the covariance of the definition of G. By eq. (7.11) G is expressed in terms of F and ϕ. Therefore, the transformation of G can be derived from those of F and ϕ. From eq. (7.11) we obtain

$$\delta\tilde{G}_I = \frac{n!}{e}\frac{\partial\delta\mathcal{L}}{\partial F^I} - \tilde{G}_J A^J{}_I - \tilde{G}_J B^{JK}\frac{\partial G_K}{\partial F^I}. \tag{7.13}$$

This should coincide with the transformation given in the lower equations in eq. (7.12). By equating these two transformation laws we obtain

$$\begin{aligned}\frac{\partial}{\partial F^I}\left(n!\delta\mathcal{L} - \frac{1}{2}eF^J C_{JK}\tilde{F}^K - \frac{1}{2}e\tilde{G}_J B^{JK} G_K\right) - \left(A^J{}_I + D_I{}^J\right) n!\frac{\partial\mathcal{L}}{\partial F^J} \\ = \frac{1}{2}e\left(C_{IJ} + \epsilon C_{JI}\right)\tilde{F}^J + \frac{1}{2}e\tilde{G}_J\left(B^{JK} + \epsilon B^{KJ}\right)\frac{\partial G_K}{\partial F^I}.\end{aligned} \tag{7.14}$$

When there exist nontrivial interactions, this equation gives conditions on the transformation parameters

$$A^I{}_J + D_J{}^I = \alpha\delta^I_J, \qquad B^{IJ} = -\epsilon B^{JI}, \qquad C_{IJ} = -\epsilon C_{JI}, \tag{7.15}$$

where α is an arbitrary constant, as well as a condition on the variation of the Lagrangian

$$\frac{\partial}{\partial F^I}\left(\delta\mathcal{L} - \frac{1}{2n!}eF^I C_{IJ}\tilde{F}^J - \frac{1}{2n!}e\tilde{G}_I B^{IJ} G_J - \alpha\mathcal{L}\right) = 0. \tag{7.16}$$

The equation of motion for ϕ^i is

$$E_i \equiv \left(\frac{\partial}{\partial\phi^i} - \partial_\mu\frac{\partial}{\partial(\partial_\mu\phi^i)}\right)\mathcal{L} = 0. \tag{7.17}$$

The covariance of this equation under the duality transformations (7.12)

$$\delta E_i = -\frac{\partial\xi^j}{\partial\phi^i}E_j \tag{7.18}$$

requires another condition on the variation of the Lagrangian

$$\left(\frac{\partial}{\partial\phi^i} - \partial_\mu\frac{\partial}{\partial(\partial_\mu\phi^i)}\right)\left(\delta\mathcal{L} - \frac{1}{2n!}e\tilde{G}_I B^{IJ} G_J\right) = 0. \tag{7.19}$$

Now we can find out possible duality groups by studying eqs. (7.15), (7.16) and (7.19). Comparing eqs. (7.16) and (7.19) we find $\alpha = 0$. Then, the conditions on the parameters (7.15) can be written as

$$X^T\Omega + \Omega X = 0, \tag{7.20}$$

where

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}. \tag{7.21}$$

For $d = 4k$ ($\epsilon = -1$) Ω is an antisymmetric matrix and the above condition implies that the group of duality transformations is $\mathrm{Sp}(2M, \mathbf{R})$ or its subgroup. On the other hand, for $d = 4k + 2$ ($\epsilon = +1$) Ω is a symmetric matrix, which can be diagonalized to $\mathrm{diag}(\mathbf{1}, -\mathbf{1})$. Therefore, the group of duality transformations in this case is $\mathrm{SO}(M, M)$ or its subgroup. Eqs. (7.15), (7.16) and (7.19) also restrict the variation of the Lagrangian

$$\begin{aligned}\delta\mathcal{L} &= \frac{1}{2n!} eF^I C_{IJ} \tilde{F}^J + \frac{1}{2n!} e\tilde{G}_I B^{IJ} G_J \\ &= \delta\left(\frac{1}{2n!} eF^I \tilde{G}_I\right).\end{aligned} \tag{7.22}$$

Thus, although the Lagrangian is not invariant under the duality transformations, it transforms in a definite way.

It can be shown that a derivative of the Lagrangian with respect to an invariant parameter λ is invariant under the duality transformations. Indeed, by computing $\frac{\partial}{\partial\lambda}\delta\mathcal{L}$ and using eqs. (7.11), (7.15) we obtain

$$\delta\left(\frac{\partial\mathcal{L}}{\partial\lambda}\right) = \frac{\partial}{\partial\lambda}\left(\delta\mathcal{L} - \frac{1}{2n!} eF^I C_{IJ} \tilde{F}^J - \frac{1}{2n!} e\tilde{G}_I B^{IJ} G_J\right), \tag{7.23}$$

which vanishes by eq. (7.22). Here, we have assumed that ξ^i do not depend on λ. The invariant parameter can be an invariant external field such as the metric. Thus, the energy-momentum tensor obtained as a functional derivative of the Lagrangian with respect to the metric is invariant under the duality transformations.

Let us obtain an explicit form of the Lagrangian which transforms as in eq. (7.22). The Lagrangian satisfying eq. (7.22) can be written as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2n!} eF^I \tilde{G}_I + \mathcal{L}_{\mathrm{inv}}(\phi, \partial\phi, F) \\ &= \frac{1}{2n!} eF^I \tilde{G}_I + \frac{1}{2n!} e(F^I I_I + \epsilon G_I H^I) + \mathcal{L}_{\mathrm{inv}}(\phi, \partial\phi),\end{aligned} \tag{7.24}$$

where n-th antisymmetric tensors $(H^I_{\mu_1\cdots\mu_n}(\phi, \partial\phi), I_{I\,\mu_1\cdots\mu_n}(\phi, \partial\phi))$ transform in the same way as (F^I, G_I), and $\mathcal{L}_{\mathrm{inv}}(\phi, \partial\phi)$ is invariant under the duality transformations. In the second line we have assumed that the duality symmetry group is a maximal one, i.e., $\mathrm{Sp}(2M, \mathbf{R})$ in $d = 4k$ or $\mathrm{SO}(M, M)$ in $d = 4k+2$, for simplicity. When the symmetry group is a subgroup of them, there can be other invariants other than $FI + \epsilon GH$. Substituting eq. (7.24) into eq. (7.11) we obtain a differential equation for $\tilde{G}$

$$(\tilde{G} - I)_I = (F - \epsilon\tilde{H})^J \frac{\partial}{\partial F^I}(\tilde{G} - I)_J. \tag{7.25}$$

To solve this equation we introduce an operation j:

$$j: \quad F \longrightarrow jF \equiv \tilde{F}. \tag{7.26}$$

Then, the solution can be written as

$$jG_I = I_I + \epsilon K_{IJ}(\phi)(F^J - \epsilon j H^J), \tag{7.27}$$

where

$$K_{IJ}(\phi) = K_{1IJ}(\phi) + K_{2IJ}(\phi)j \quad (K_{1IJ} = K_{1JI}, \quad K_{2IJ} = -\epsilon K_{2JI}). \tag{7.28}$$

From the covariance of eq. (7.27) under the duality transformations, K must transform as

$$\delta K = -KA - KBKj + \epsilon Cj + DK. \tag{7.29}$$

Substituting this solution into eq. (7.24) we obtain

$$\mathcal{L} = \frac{1}{2n!}\epsilon e F^I K_{IJ} F^J + \frac{1}{n!} e F^I (I_I - K_{IJ}\tilde{H}^J) - \frac{1}{2n!}\epsilon e \tilde{H}^I (I_I - K_{IJ}\tilde{H}^J) + \mathcal{L}_{\rm inv}(\phi, \partial\phi). \tag{7.30}$$

Thus, if we can find out functions $H^I(\phi, \partial\phi)$, $I^I(\phi, \partial\phi)$, $K_{IJ}(\phi)$ with appropriate transformation properties, we have an explicit form of the Lagrangian.

7.3 Compact duality symmetry

Let us consider a special case of $K = 1$. In this case we will see that the duality symmetry group must be a compact group. From eq. (7.29) the transformation parameters must satisfy

$$A = D, \qquad B = \epsilon C. \tag{7.31}$$

For $d = 4k$ these conditions imply

$$X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \qquad A^T = -A, \quad B^T = B. \tag{7.32}$$

The transformation law becomes

$$\delta \begin{pmatrix} F + iG \\ F - iG \end{pmatrix} = \begin{pmatrix} A - iB & 0 \\ 0 & (A - iB)^* \end{pmatrix} \begin{pmatrix} F + iG \\ F - iG \end{pmatrix}. \tag{7.33}$$

Since $A-iB$ is anti-hermitian, the duality symmetry group is $\mathrm{U}(M)$, which is a maximal compact subgroup of $\mathrm{Sp}(M,M)$, or its subgroup.

On the other hand, for $d=4k+2$ the conditions in eq. (7.31) imply

$$X=\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \qquad A^T=-A, \quad B^T=-B. \tag{7.34}$$

and

$$\delta\begin{pmatrix} F+G \\ F-G \end{pmatrix}=\begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}\begin{pmatrix} F+G \\ F-G \end{pmatrix}. \tag{7.35}$$

Since $A+B$ and $A-B$ are real and antisymmetric, the duality symmetry group is $\mathrm{SO}(M) \times \mathrm{SO}(M)$, which is a maximal compact subgroup of $\mathrm{SO}(M,M)$, or its subgroup.

As an example of compact duality symmetries let us consider the $d=4$, $N=2$ supergravity [20]. The fields are the vierbein $e_\mu{}^a$, two Majorana Rarita-Schwinger fields ψ_μ^i $(i=1,2)$ and a U(1) gauge field B_μ. The Lagrangian is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}eR-\frac{1}{2}ie\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}D_\nu\psi_\rho^i-\frac{1}{4}eF_{\mu\nu}F^{\mu\nu} \\ &-\frac{1}{2}e\epsilon_{ij}\bar{\psi}_\mu^i(F^{\mu\nu}-i\bar{\gamma}\tilde{F}^{\mu\nu})\psi_\nu^j+(\text{4-fermi terms}).\end{aligned} \tag{7.36}$$

This Lagrangian is invariant under a rigid SU(2) transformation

$$\delta e_\mu{}^a=0, \qquad \delta\psi_\mu^i=(\Sigma^{ij}+i\bar{\gamma}\Lambda^{ij})\psi_\mu^j, \qquad \delta B_\mu=0, \tag{7.37}$$

where Σ^{ij}, Λ^{ij} are are real parameters satisfying $\Sigma^{ij}=-\Sigma^{ji}$, $\Lambda^{ij}=\Lambda^{ji}$ and $\Lambda^{ii}=0$. The equations of motion have an additional symmetry under a rigid U(1) transformation

$$\delta e_\mu{}^a=0, \qquad \delta\psi_\mu^i=-\frac{1}{2}\Lambda i\bar{\gamma}\psi_\mu^i, \qquad \delta\begin{pmatrix} F^{\mu\nu} \\ G^{\mu\nu} \end{pmatrix}=\begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}\begin{pmatrix} F^{\mu\nu} \\ G^{\mu\nu} \end{pmatrix}, \tag{7.38}$$

where Λ is a real parameter. This U(1) transformation acts on the gauge field as a duality transformation. An explicit form of G can be obtained from eq. (7.11)

$$\tilde{G}_{\mu\nu}=-F_{\mu\nu}-\tilde{H}_{\mu\nu}+I_{\mu\nu}, \tag{7.39}$$

where

$$H_{\mu\nu}=-\epsilon_{ij}\bar{\psi}_\mu^i i\bar{\gamma}\psi_\nu^j, \qquad I_{\mu\nu}=-\epsilon_{ij}\bar{\psi}_\mu^i\psi_\nu^j. \tag{7.40}$$

One can easily see that (H, I) transform in the same way as (F, G). Furthermore, the transformation of G in eq. (7.39) derived from δF and $\delta\psi$ correctly reproduces δG in eq. (7.38). Using G the Lagrangian (7.36) can be rewritten in the form (7.24)

$$\mathcal{L} = \frac{1}{4} e F_{\mu\nu}\tilde{G}^{\mu\nu} + \frac{1}{4} e\left(F_{\mu\nu}I^{\mu\nu} - G_{\mu\nu}H^{\mu\nu}\right) + (B_\mu\text{-independent terms}). \qquad (7.41)$$

7.4 Non-compact duality symmetry

We can construct $K_{IJ}(\phi)$ which transforms as in eq. (7.29) by using a G/H non-linear sigma model. Here, G is a duality symmetry group, which we assume to be a maximal one, i.e., Sp$(2M)$ or SO(M, M). H is a maximal compact subgroup of G, i.e., U(M) or SO(M) $\times$ SO(M). We use a G-valued scalar field $V(x)$, which transforms under $\mathrm{G}_{\mathbf{rigid}} \times \mathrm{H}_{\mathbf{local}}$ as in eqs. (6.1), (6.2).

Let us first discuss the case $d = 4k$, G $=$ Sp$(2M)$, H $=$ U(M). It is convenient to use the complex basis, in which the G transformation is

$$\begin{pmatrix} F + iG \\ F - iG \end{pmatrix} \to \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix}\begin{pmatrix} F + iG \\ F - iG \end{pmatrix}, \qquad a^\dagger a - b^\dagger b = 1, \quad a^T b - b^T a = 0. \qquad (7.42)$$

In this basis the scalar field is expressed as

$$V(x) = \begin{pmatrix} \phi_0(x) & \phi_1^*(x) \\ \phi_1(x) & \phi_0^*(x) \end{pmatrix}, \qquad \phi_0^\dagger\phi_0 - \phi_1^\dagger\phi_1 = 1, \quad \phi_0^T\phi_1 - \phi_1^T\phi_0 = 0. \qquad (7.43)$$

Using the components in eq. (7.43) we can construct K transforming as in eq. (7.29)

$$K = (\phi_0^* - \phi_1^*)(\phi_0^* + \phi_1^*)^{-1}, \qquad (7.44)$$

where the imaginary unit i in ϕ_0, ϕ_1 is replaced by the operation j, i.e., $\phi_0 = \mathrm{Re}\,\phi_0 + j\,\mathrm{Im}\,\phi_0$, $\phi_0^* = \mathrm{Re}\,\phi_0 - j\,\mathrm{Im}\,\phi_0$, etc. Note that $j^2 = -1$ in $d = 4k$ as $i^2 = -1$. It is convenient to introduce an H-invariant variable

$$z = \phi_1^*(\phi_0^*)^{-1} = z^T, \qquad (7.45)$$

which transforms under G as

$$z \to (az + b^*)(bz + a^*)^{-1}. \qquad (7.46)$$

Then, K can be expressed as

$$K = \frac{1 - z}{1 + z}. \qquad (7.47)$$

We now turn to $d = 4k+2$ and consider the maximal case G = SO(M,M), H = SO(M) × SO(M). In the Ω-diagonal basis the G transformation is written as

$$\begin{pmatrix} F+G \\ F-G \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F+G \\ F-G \end{pmatrix},$$
$$a^T a - c^T c = 1, \qquad d^T d - b^T b = 1, \qquad a^T b - c^T d = 0. \tag{7.48}$$

The G-valued scalar field is parametrized as

$$V(x) = \begin{pmatrix} \phi_1(x) & \psi_2(x) \\ \psi_1(x) & \phi_2(x) \end{pmatrix},$$
$$\phi_1^T \phi_1 - \psi_1^T \psi_1 = 1, \qquad \phi_2^T \phi_2 - \psi_2^T \psi_2 = 1, \qquad \phi_1^T \psi_2 - \psi_1^T \phi_2 = 0. \tag{7.49}$$

In this case the operation j satisfies $j^2 = 1$, which suggests to introduce the projection operators

$$P_\pm = \frac{1}{2}(1 \pm j). \tag{7.50}$$

A coefficient function K which has the right transformation property in eq. (7.29) is

$$K = (\phi_1 - \psi_1)(\phi_1 + \psi_1)^{-1} P_+ + (\phi_2 - \psi_2)(\phi_2 + \psi_2)^{-1} P_-. \tag{7.51}$$

We define an H-invariant variable

$$z = (\psi_1 (\phi_1)^{-1})^T = \psi_2 (\phi_2)^{-1}, \tag{7.52}$$

which transforms under the G transformation as

$$z \rightarrow (az+b)(cz+d)^{-1}. \tag{7.53}$$

In terms of this variable K can be written as

$$K = \frac{1-z^T}{1+z^T} P_+ + \frac{1-z}{1+z} P_-. \tag{7.54}$$

As an example of non-compact duality symmetries in supergravities let us consider the $d=4$, $N=4$ theory [21], [18]. Among the fields in the theory we are interested in six U(1) gauge fields $B_\mu^{ij} = -B_\mu^{ji}$ $(i,j = 1,\cdots,4)$ and two real scalar fields (see Table 3). The scalar fields are represented as the G/H non-linear sigma model with G = SU(4) × SU(1,1), H = SU(4) × U(1). The SU(4) factors cancel each other in the coset G/H.

G = SU(4) × SU(1,1) acts on the gauge fields as duality transformations. The SU(4) transformation is

$$\begin{pmatrix} (F+iG)^{ij} \\ (F-iG)^{ij} \end{pmatrix} \to \begin{pmatrix} U^{ik}U^{jl} & 0 \\ 0 & U^{*ik}U^{*jl} \end{pmatrix} \begin{pmatrix} (F+iG)^{kl} \\ (F-iG)^{kl} \end{pmatrix}, \tag{7.55}$$

where $U^\dagger U = 1$, $\det U = 1$, while the SU(1,1) transformation is

$$\begin{pmatrix} (F+iG)^{ij} \\ (F-iG)^{ij} \end{pmatrix} \to \begin{pmatrix} a\delta^{i[k}\delta^{l]j} & b^*\frac{1}{2}\epsilon^{ijkl} \\ b\frac{1}{2}\epsilon^{ijkl} & a^*\delta^{i[k}\delta^{l]j} \end{pmatrix} \begin{pmatrix} (F+iG)^{kl} \\ (F-iG)^{kl} \end{pmatrix}, \tag{7.56}$$

where $a, b \in \mathbf{C}$, $|a|^2 - |b|^2 = 1$. The SU(1,1)-valued scalar field is parametrized as

$$V(x) = \begin{pmatrix} \Phi_0(x) & \Phi_1^*(x) \\ \Phi_1(x) & \Phi_0^*(x) \end{pmatrix}, \tag{7.57}$$

where

$$\begin{gathered} \Phi_0^{ijkl}(x) = \phi_0(x)\delta^{i[k}\delta^{l]j}, \qquad \Phi_1^{ijkl}(x) = \phi_1(x)\frac{1}{2}\epsilon^{ijkl}. \\ \phi_0(x), \phi_1(x) \in \mathbf{C}, \quad |\phi_0|^2 - |\phi_1|^2 = 1. \end{gathered} \tag{7.58}$$

The imaginary unit i in ϕ_0 and ϕ_1 is replaced by the duality operation j. In terms of an H-invariant variable

$$\begin{aligned} Z^{ij,kl}(x) &= (\Phi_1^*(x))^{ij,pq}(\Phi_0^*(x)^{-1})^{pq,kl} \\ &= z(x)\frac{1}{2}\epsilon^{ijkl}, \end{aligned} \tag{7.59}$$

where $z(x) = \phi_1^*(x)\phi_0^*(x)^{-1}$ we construct

$$\begin{aligned} K_{ij,kl} &= \left(\frac{1-Z}{1+Z}\right)^{ij,kl} \\ &= \frac{1+z^2}{1-z^2}\delta^{i[k}\delta^{l]j} - \frac{2z}{1-z^2}\frac{1}{2}\epsilon^{ijkl}. \end{aligned} \tag{7.60}$$

Then, the Lagrangian can be written as

$$\mathcal{L} = e\frac{\partial_\mu z \partial^\mu z^*}{(1-|z|^2)^2} - \frac{1}{4}eF_{\mu\nu}^{ij}K_{ij,kl}F^{\mu\nu kl} + \cdots. \tag{7.61}$$

8. Super p-branes

In this section we briefly discuss super p-branes, which are closely related to supergravities. We shall consider a theory of p-dimensionally extended objects moving in d-dimensional space-time. They are generalizations of strings and are called p-branes: 0-branes are point particles, 1-branes are strings, 2-branes are membranes, etc.

Let us first consider bosonic p-branes without supersymmetry. Dynamical variables are $X^\mu(\xi)$ $(\mu = 0, 1, \cdots, d-1)$, which represent a map from $(p+1)$-dimensional world-volume with coordinates ξ^i $(i = 0, 1, \cdots, p)$ to d-dimensional space-time. When space-time is a flat Minkowski space-time, a natural action is the one proportional to the volume of the $(p+1)$-dimensional world volume (Nambu-Goto type action)

$$S[X] = -T \int d^{p+1}\xi \sqrt{|\det h_{ij}|}, \qquad h_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}, \tag{8.1}$$

where T is the p-brane tension of dimension $(\text{length})^{-p-1}$. We will take $T = 1$ in the following for simplicity. h_{ij} is a metric on the world-volume induced by the space-time flat metric $\eta_{\mu\nu}$. This action is invariant under the space-time Poincaré transformations and the world-volume reparametrizations. One can write down another action (Polyakov type action) using an independent metric g_{ij}

$$S'[X, g] = -\int d^{p+1}\xi \left[\frac{1}{2}\sqrt{|g|} g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - \frac{1}{2}(p-1)\sqrt{|g|}\right], \tag{8.2}$$

which are equivalent to the above action (8.1). The field equation of g_{ij} can be solved algebraically: $g_{ij} = h_{ij}$. (For $p = 1$ the general solution is $g_{ij} = \mathrm{e}^\phi h_{ij}$, where ϕ is an arbitrary function of ξ^i.) Substituting this solution into eq. (8.2) we obtain the action (8.1). Therefore, these two actions are equivalent, at least at the classical level. We will use the Nambu-Goto type action (8.1) in the following.

We now consider a supersymmetric generalization of p-branes. In the case of strings $(p = 1)$ there are two formulations: the Neveu-Schwarz-Ramond formulation and the Green-Schwarz formulation. For general p-branes Neveu-Schwarz-Ramond formulation is not known. A technical reason is that a supersymmetrization of the cosmological term $\sqrt{|g|}$ for $p > 1$ requires the Einstein term $\sqrt{|g|}R$ and the theory becomes more complicated. Moreover, even if one could construct an appropriate action with world-volume local supersymmetry, it is not clear whether it leads to space-time supersymmetry. On the other hand, the Green-Schwarz formulation of p-branes was constructed in refs. [22], [23], which we will discuss in the following.

Dynamical variables of super p-branes in the Green-Schwarz formulation are $Z^M(\xi) = (X^\mu(\xi), \theta^\alpha(\xi))$ $(M = (\mu, \alpha),\ \mu = 0, 1, \cdots, d-1;\ \alpha = 1, \cdots, n)$, which represent a map from $(p+1)$-dimensional world-volume to d-dimensional (extended) superspace. Here, n is a number of independent components of fermionic coordinates of the superspace. The action of super p-branes in Minkowski space-time is

$$S = -\int d^{p+1}\xi \left[\sqrt{|\det h_{ij}|} + \frac{2}{(p+1)!}\epsilon^{i_1\cdots i_{p+1}}\Pi_{i_1}^{A_1}\cdots\Pi_{i_{p+1}}^{A_{p+1}}B_{A_{p+1}\cdots A_1}\right],$$
$$\Pi_i^\mu = \partial_i X^\mu - i\bar\theta\gamma^\mu\partial_i\theta, \qquad \Pi_i^\alpha = \partial_i\theta^\alpha, \qquad h_{ij} = \Pi_i^\mu\Pi_j^\nu\eta_{\mu\nu}, \tag{8.3}$$

where γ^μ are d-dimensional gamma matrices. $B_{A_{p+1}\cdots A_1}(Z)$ $(A = (\mu,\alpha))$ is a $(p+1)$-form on the superspace, whose non-vanishing components of the field strength $H_{A_{p+2}\cdots A_1}$ are

$$H_{\alpha\beta\mu_1\cdots\mu_p} = -i\zeta^{-1}(C^{-1T}\gamma_{\mu_1\cdots\mu_p})_{\alpha\beta}, \qquad \zeta = (-1)^{\frac{1}{4}p(p-1)}. \tag{8.4}$$

The action (8.3) is invariant under the space-time super Poincaré transformations and $(p+1)$-dimensional reparametrizations on the world-volume. The space-time supertransformations are

$$\delta_Q X^\mu = i\bar\epsilon\gamma^\mu\theta, \qquad \delta_Q\theta = \epsilon, \tag{8.5}$$

where ϵ^α is a constant spinor parameter. In the case of superstrings it is also invariant under local fermionic transformations called κ-transformations. This symmetry reduces the fermionic degrees of freedom by half. We require such symmetry also for $p \geq 2$. The κ-transformations are

$$\delta_\kappa X^\mu = i\bar\theta\gamma^\mu\delta_\kappa\theta, \qquad \delta_\kappa\theta = (1+\Gamma)\kappa,$$
$$\Gamma \equiv \frac{\zeta}{(p+1)!\sqrt{|h|}}\epsilon^{i_1\cdots i_{p+1}}\Pi_{i_1}^{\mu_1}\cdots\Pi_{i_{p+1}}^{\mu_{p+1}}\gamma_{\mu_1\cdots\mu_{p+1}}, \tag{8.6}$$

where $\kappa^\alpha(\xi)$ is a parameter of the transformations. The matrix Γ defined above satisfies $\Gamma^2 = 1$ and therefore $\frac{1}{2}(1\pm\Gamma)$ are projection operators. The action (8.3) is invariant under the transformations (8.6) provided that the field strength of $B_{A_{p+1}\cdots A_1}$ is given by eq. (8.4). Thus, the presence of the second term (Wess-Zumino term) in the action (8.3) is required by the κ-invariance.

The field strength $H_{A_{p+2}\cdots A_1}$ given in eq. (8.4) must be a closed $(p+2)$-form. This requires that gamma matrices should satisfy a certain kind of identity. When θ is a Majorana spinor, the identity is

$$(C^{-1}\gamma_{\mu_1})_{(\alpha\beta}(C^{-1}\gamma^{\mu_1\cdots\mu_p})_{\gamma\delta)} = 0. \tag{8.7}$$

A similar identity must be satisfied when θ is a spinor of other type. (For details see ref. [23].) These identities lead to a condition on d, p and n

$$d - p - 1 = \frac{1}{4}n \tag{8.8}$$

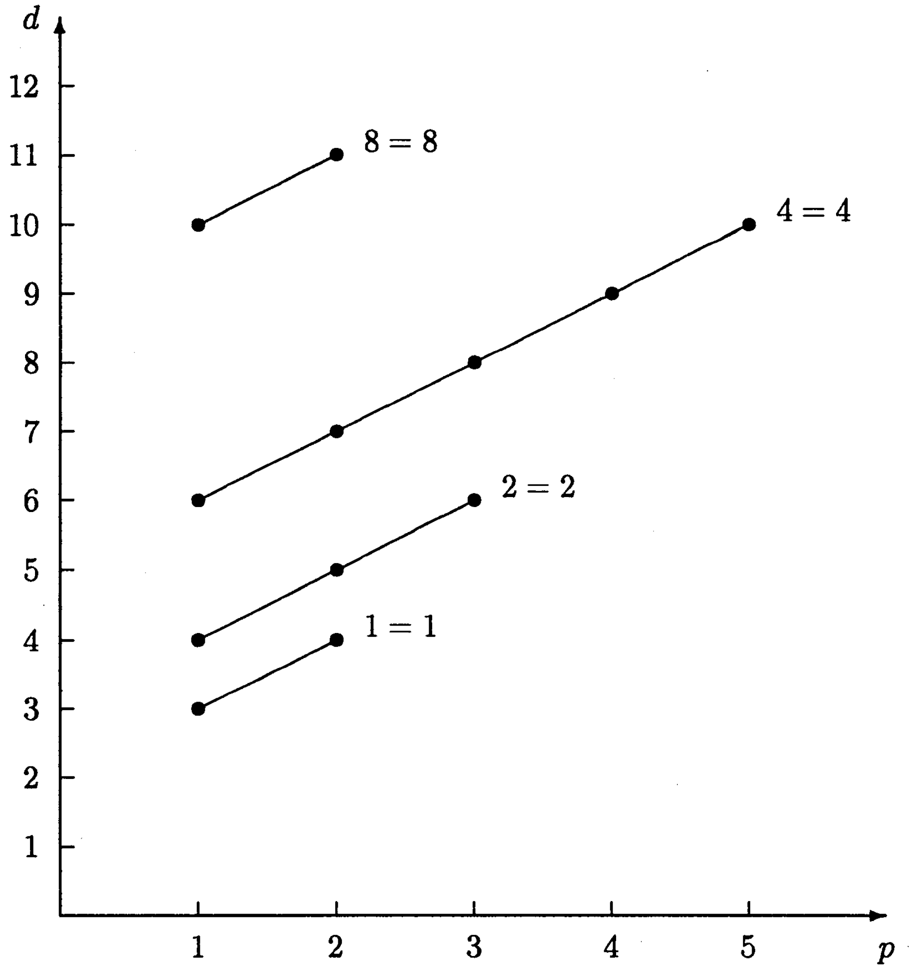

Figure 1: The brane scan.

for $p \geq 2$ and

$$d-2=\frac{1}{4}n \qquad \text{or} \qquad d-2=\frac{1}{2}n \tag{8.9}$$

for $p=1$. The left and the right hand sides of eqs. (8.8) and (8.9) represent bosonic (X^μ) and fermionic (θ) physical degrees of freedom up to gauge degrees of freedom respectively. Eqs. (8.8) and (8.9) are satisfied only for 12 pairs (d,p) shown in Fig. 1. Possible types of supersymmetries are (N_+,N_-) = (1,1), (2,0), (1,0) for $(d,p)=(10,1)$, (N_+,N_-) = (2,2), (4,0), (2,0) for $(d,p)=(6,1)$, $N=2$, 1 for (d,p) = (4,1), (3,1) and the minimal one for other (d,p). The numbers on the right of each sequence in Fig. 1 represent physical degrees of freedom (8.8), (8.9). (A similar analysis for general signature of space-time was given in ref. [24].)

The action (8.3) is the one for a flat Minkowski space-time. One can introduce space-time background fields. The background fields are represented by supervielbein $E_M{}^A$ and super $(p+1)$-form $B_{A_{p+1}\cdots A_1}$, both of which are superfields. The action is given by eq.

(8.3) with Π_i^A replaced by

$$\Pi_i^A = \partial_i Z^M E_M{}^A. \tag{8.10}$$

The κ-invariance of the action imposes constraints on these background fields. For $(d,p) = (11,2)$, $(10,1)$, $(10,5)$ these constraints are shown to be equivalent to field equations of $d = 11$, $N = 1$ and $d = 10$ (1,1) supergravities in superspace. For other values of (d,p) an equivalence to field equations of supergravities is expected but has not yet been analyzed in detail. For string theories ($p = 1$) there is one-to-one correspondence between background fields and massless physical states. It is an interesting open problem whether such relations are present for $p \geq 2$.

The author would like to thank T. Eguchi and N. Sakai for suggesting him to publish this work.

Appendix A. The vielbein formalism

In the usual formulation of gravitational theories the metric tensor $g_{\mu\nu}(x)$ $(\mu,\nu = 0,1,\cdots,d-1)$ is used to describe gravity. Our signature convention of $g_{\mu\nu}$ is $(+,-,\cdots,-)$. The Einstein Lagrangian in this formulation is

$$\mathcal{L} = -\frac{1}{16\pi G}\sqrt{-g}R, \tag{A.1}$$

where G is the gravitational constant and $g = \det g_{\mu\nu}$. In the following we will put $4\pi G = 1$ for simplicity. The scalar curvature R is defined from the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu}{}^\rho{}_\sigma$ as

$$\begin{aligned} &R = g^{\mu\nu}R_{\mu\nu}, \qquad R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu, \\ &R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \end{aligned} \tag{A.2}$$

The Christoffel symbol $\Gamma^\lambda_{\mu\nu}$ is defined by using the metric as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right). \tag{A.3}$$

This form is determined by the metricity condition $D_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu}g_{\rho\nu} - \Gamma^\rho_{\lambda\nu}g_{\mu\rho} = 0$ and the torsionless condition $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

To couple gravity to spinor fields it is more convenient to use the vielbein formulation of gravity. In this formulation we introduce d vectors $e_a{}^\mu(x)$ $(a = 0,1,\cdots,d-1)$ at each point of space-time, which are orthogonal to each other and have a unit length

$$e_a{}^\mu(x)e_b{}^\nu(x)g_{\mu\nu}(x) = \eta_{ab}, \tag{A.4}$$

where $\eta_{ab} = \mathrm{diag}(+1, -1, \cdots, -1)$ is a flat Minkowski metric. We also introduce inverse matrices $e_\mu{}^a(x)$, which satisfy

$$e_\mu{}^a(x)e_a{}^\nu(x) = \delta_\mu^\nu, \qquad e_a{}^\mu(x)e_\mu{}^b(x) = \delta_a^b. \tag{A.5}$$

The fields $e_\mu{}^a(x)$ are called vielbein (vierbein or tetrad in four dimensions, fünfbein in five dimensions, etc.). From eqs. (A.4) and (A.5) we can express the metric in terms of the vielbein

$$g_{\mu\nu}(x) = e_\mu{}^a(x)e_\nu{}^b(x)\eta_{ab}. \tag{A.6}$$

Therefore, we can use the vielbein $e_\mu{}^a(x)$ as dynamical variables representing gravitational degrees of freedom.

For a given metric $g_{\mu\nu}$ the vielbein $e_\mu{}^a$ satisfying eq. (A.6) is not uniquely determined. If $e_\mu{}^a$ satisfies eq. (A.6), then another vielbein

$$e'_\mu{}^a(x) = e_\mu{}^b(x)\Lambda_b{}^a(x) \qquad \left(\Lambda_a{}^c(x)\Lambda_b{}^d(x)\eta_{cd} = \eta_{ab}\right) \tag{A.7}$$

also satisfies eq. (A.6) with the same $g_{\mu\nu}$. The metric tensor has $\frac{1}{2}d(d+1)$ independent components, while the vielbein has d^2 components. The difference $\frac{1}{2}d(d-1)$ is the number of independent components of $\Lambda_a{}^b$. The transformation (A.7) is called a local Lorentz transformation. Since the theories are originally formulated by using only $g_{\mu\nu}$, they should be invariant under the local Lorentz transformations. Thus, gravitational theories in the vielbein formulation have two local symmetries: the general coordinate symmetry and the local Lorentz symmetry.

We have now two kinds of vector indices: $\mu, \nu, \cdots$ and $a, b, \cdots$. To distinguish them the indices $\mu, \nu, \cdots$ are called 'world indices', while $a, b, \cdots$ are called 'local Lorentz indices'. These two kinds of indices are converted into each other by using the vielbein and its inverse, e.g.,

$$A_a(x) = e_a{}^\mu(x)A_\mu(x), \qquad A_\mu(x) = e_\mu{}^a(x)A_a(x). \tag{A.8}$$

Tensor fields with local Lorentz indices transform under the local Lorentz transformations as in eq. (A.7). They also transform under the general coordinate transformations as tensor fields determined by the world indices they have. For instance, the general coordinate (G) and the local Lorentz (L) transformations of a tensor field $T_{\mu a}(x)$ are

$$\begin{aligned} \delta_G T_{\mu a} &= -\xi^\nu \partial_\nu T_{\mu a} - \partial_\mu \xi^\nu T_{\nu a}, \\ \delta_L T_{\mu a} &= -\lambda_a{}^b T_{\mu b}, \end{aligned} \tag{A.9}$$

where $\xi(x)$ and $\lambda^{ab}(x) = -\lambda^{ba}(x)$ are infinitesimal transformation parameters. The transformations of spinor fields are assumed to be

$$\delta_G\psi = -\xi^\mu\partial_\mu\psi,$$
$$\delta_L\psi = -\frac{1}{4}\lambda_{ab}\gamma^{ab}\psi. \tag{A.10}$$

Under the general coordinate transformations they transform as scalars.

To construct an action of spinor fields invariant under the local Lorentz transformations we need a gauge field. It is called a spin connection $\omega_\mu{}^a{}_b(x)$ $(\omega_\mu{}^{ab} = -\omega_\mu{}^{ba})$. The local Lorentz transformation of the spin connection should be

$$\delta_L\omega_\mu{}^{ab} = D_\mu\lambda^{ab} \equiv \partial_\mu\lambda^{ab} + \omega_\mu{}^a{}_c\lambda^{cb} + \omega_\mu{}^b{}_c\lambda^{ac} \tag{A.11}$$

so that the covariant derivative of a spinor field ψ

$$D_\mu\psi = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi \tag{A.12}$$

transforms covariantly. The spinor Lagrangian invariant under the general coordinate and the local Lorentz transformations is

$$\mathcal{L} = ie\bar{\psi}\gamma^\mu D_\mu\psi, \tag{A.13}$$

where $e = \det e_\mu{}^a = \sqrt{-g}$ and $\gamma^\mu = \gamma^a e_a{}^\mu$.

As for the Christoffel symbol, the spin connection is completely determined by the vielbein if we impose the torsionless condition

$$D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \qquad (D_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b). \tag{A.14}$$

(The metricity condition corresponds to the antisymmetry property $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$.) The solution of eq. (A.14) is $\omega_{\mu ab} = \omega_{\mu ab}(e)$, where

$$\omega_{\mu ab}(e) = \frac{1}{2}\left(e_a{}^\nu\Omega_{\mu\nu b} - e_b{}^\nu\Omega_{\mu\nu a} - e_a{}^\rho e_b{}^\sigma e_\mu{}^c\Omega_{\rho\sigma c}\right), \quad \Omega_{\mu\nu a} = \partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}. \tag{A.15}$$

The spin connection (A.15) is related to the Christoffel symbol (A.3) as

$$\partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\mu{}^b - \Gamma^\lambda_{\mu\nu}e_\lambda{}^a = 0. \tag{A.16}$$

The field strength of the spin connection

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b. \tag{A.17}$$

is shown to be related to the Riemann tensor and the scalar curvature as

$$R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu}{}^a{}_b e_a{}^\rho e_\sigma{}^b, \qquad R = e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}. \tag{A.18}$$

Appendix B. Local supersymmetry invariance of $d = 4$, $N = 1$ supergravity

The Lagrangian of $d = 4$, $N = 1$ supergravity consists of two terms

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm E} + \mathcal{L}_{\rm RS}, \\
\mathcal{L}_{\rm E} &= -\frac{1}{4} e\, e_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab} = \frac{1}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e_\rho{}^c e_\sigma{}^d \hat{R}_{\mu\nu}{}^{ab}, \\
\mathcal{L}_{\rm RS} &= -\frac{1}{2} i e e_a{}^\mu e_b{}^\nu e_c{}^\rho \bar{\psi}_\mu \gamma^{abc} \hat{D}_\nu \psi_\rho = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 \hat{D}_\rho \psi_\sigma,
\end{aligned} \tag{B.1}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon^{0123} = +1$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The Riemann tensor $\hat{R}_{\mu\nu}{}^{ab}$ and the covariant derivative $\hat{D}_\mu$ depend on the vierbein $e_\mu{}^a$ only through the spin connection $\hat{\omega}_\mu{}^{ab}$. When the action is viewed as a functional of $e_\mu{}^a$, ψ_μ and $\hat{\omega}_\mu{}^{ab}$, the spin connection (2.5) satisfies an equation

$$\frac{\delta}{\delta \hat{\omega}_{\mu ab}} \int d^4x\, \mathcal{L}(e, \psi, \hat{\omega}) = 0. \tag{B.2}$$

(To show this it is convenient to use the second form of $\mathcal{L}_{\rm E}$ and $\mathcal{L}_{\rm RS}$ in eq. (B.1).) Therefore, when we compute a variation of the Lagrangian under supertransformations, the spin connection need not be varied.

To show the local supersymmetry invariance we need the following formulae involving spinors. For four arbitrary spinors ψ, χ, λ and ϕ the Fierz identity

$$\bar{\psi}\chi\bar{\lambda}\phi = -\frac{1}{4}\left[\bar{\psi}\phi\bar{\lambda}\chi + \bar{\psi}\gamma^a\phi\bar{\lambda}\gamma_a\chi - \frac{1}{2}\bar{\psi}\gamma^{ab}\phi\bar{\lambda}\gamma_{ab}\chi - \bar{\psi}\gamma^a\gamma_5\phi\bar{\lambda}\gamma_a\gamma_5\chi + \bar{\psi}\gamma_5\phi\bar{\lambda}\gamma_5\chi\right] \tag{B.3}$$

is satisfied. Bilinears of two arbitrary Majorana spinors ψ and χ have symmetry properties

$$\begin{aligned}
\bar{\psi}\chi &= \bar{\chi}\psi, \\
\bar{\psi}\gamma^a\chi &= -\bar{\chi}\gamma^a\psi, \\
\bar{\psi}\gamma^{ab}\chi &= -\bar{\chi}\gamma^{ab}\psi, \\
\bar{\psi}\gamma^a\gamma_5\chi &= \bar{\chi}\gamma^a\gamma_5\psi, \\
\bar{\psi}\gamma_5\chi &= \bar{\chi}\gamma_5\psi.
\end{aligned} \tag{B.4}$$

Let us now compute the variation of the Lagrangian (B.1) under the supertransformation (2.10). Using the first form in eq. (B.1) the variation of the Einstein term is

$$\begin{aligned}\delta_Q\mathcal{L}_{\mathrm{E}} &= -\frac{1}{4}\delta_Q(e\,e_a{}^\mu e_b{}^\nu)\hat{R}_{\mu\nu}{}^{ab}\\ &= -\frac{1}{2}ie\bar{\epsilon}\gamma^\mu\psi_a\left(e_b{}^\nu\hat{R}_{\mu\nu}{}^{ab}-\frac{1}{2}e_\mu{}^a\hat{R}\right).\end{aligned}\tag{B.5}$$

On the other hand, using the second form in eq. (B.1) the variation of the Rarita-Schwinger term is

$$\begin{aligned}\delta_Q\mathcal{L}_{\mathrm{RS}} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\delta_Q\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\delta_Q\psi_\sigma\\ &\quad+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\delta_Q e_\nu{}^a\,\bar{\psi}_\mu\gamma_a\gamma_5\hat{D}_\rho\psi_\sigma.\end{aligned}\tag{B.6}$$

By partial integration the first term becomes

$$\begin{aligned}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{D}_\mu\bar{\epsilon}\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\epsilon}\gamma_\nu\gamma_5\hat{D}_\mu\hat{D}_\rho\psi_\sigma-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{D}_\mu e_\nu{}^a\,\bar{\epsilon}\gamma_a\gamma_5\hat{D}_\rho\psi_\sigma\\ &\quad+\text{ total derivative terms.}\end{aligned}\tag{B.7}$$

By using eq. (2.6), the Fierz identity (B.3) and the symmetry properties (B.4) the second term in eq. (B.7) is shown to cancel the third term in eq. (B.6). Then, eq. (B.6) becomes

$$\begin{aligned}\delta_Q\mathcal{L}_{\mathrm{RS}} &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\epsilon}\gamma_\nu\gamma_5[\hat{D}_\mu,\hat{D}_\rho]\psi_\sigma+\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5[\hat{D}_\rho,\hat{D}_\sigma]\epsilon\\ &= \frac{1}{2}ie\bar{\epsilon}\gamma^\mu\psi_a\left(e_b{}^\nu\hat{R}_{\mu\nu}{}^{ab}-\frac{1}{2}e_\mu{}^a\hat{R}\right),\end{aligned}\tag{B.8}$$

up to total derivative terms. In the last equality we have used

$$[\hat{D}_\mu,\hat{D}_\nu]\epsilon=\frac{1}{4}\hat{R}_{\mu\nu}{}^{ab}\gamma_{ab}\epsilon\tag{B.9}$$

and the properties (B.4). Thus, the variations of $\mathcal{L}_{\mathrm{E}}$ and $\mathcal{L}_{\mathrm{RS}}$ cancel each other and the total Lagrangian (2.2) is invariant under the local supertransformation (2.10) up to total derivative terms.

Next let us show the commutator algebra of two local supertransformations in eq. (2.11). We shall first consider the commutator applied on the vierbein

$$\begin{aligned}[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]e_\mu{}^a &= \delta_Q(\epsilon_1)\left(-i\bar{\epsilon}_2\gamma^a\psi_\mu\right)-(1\leftrightarrow 2)\\ &= -i\bar{\epsilon}_2\gamma^a\hat{D}_\mu\epsilon_1+i\bar{\epsilon}_1\gamma^a\hat{D}_\mu\epsilon_2\\ &= -i\hat{D}_\mu\left(\bar{\epsilon}_2\gamma^a\epsilon_1\right),\end{aligned}\tag{B.10}$$

where we have used the second equation in eq. (B.4). Defining $\xi^\nu=i\bar{\epsilon}_2\gamma^\nu\epsilon_1$ we obtain

$$\begin{aligned}[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]e_\mu{}^a &= -\hat{D}_\mu\left(\xi^\nu e_\nu{}^a\right)\\ &= -\partial_\mu\xi^\nu e_\nu{}^a-\xi^\nu\hat{D}_\nu e_\mu-\xi^\nu\left(\hat{D}_\mu e_\nu{}^a-\hat{D}_\nu e_\mu{}^a\right)\\ &= -\partial_\mu\xi^\nu e_\nu{}^a-\xi^\nu\partial_\nu e_\mu-\xi^\nu\hat{\omega}_\nu{}^a{}_b e_\mu{}^b-i\xi^\nu\bar{\psi}_\nu\gamma^a\psi_\mu,\end{aligned}\tag{B.11}$$

where we have used eq. (2.6). This shows the last commutation relation in eq. (2.11) for $e_\mu{}^a$. Similarly, we can compute the commutation relation on the Rarita-Schwinger field. We need the supertransformation of the spin connection $\hat{\omega}_\mu{}^a{}_b$, which can be obtained by applying δ_Q on both sides of eq. (2.6) as

$$\delta_Q(\epsilon)\hat{\omega}_{\mu ab} = \frac{1}{2}i\left(\bar{\epsilon}\gamma_\mu\psi_{ab} - \bar{\epsilon}\gamma_a\psi_{b\mu} + \bar{\epsilon}\gamma_b\psi_{a\mu}\right), \tag{B.12}$$

where $\psi_{\mu\nu} = \hat{D}_\mu\psi_\nu - \hat{D}_\nu\psi_\mu$. By using eqs. (B.12), (B.3) and (B.4) we obtain

$$\begin{aligned}[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]\psi_\mu = {} & [\delta_G(\xi) + \delta_L(\xi\cdot\hat{\omega}) + \delta_Q(\xi\cdot\psi)]\,\psi_\mu \\ & + \frac{1}{16}\xi^\nu\left(\gamma_\nu\mathcal{R}_\mu + 2\gamma_{\mu\nu\lambda}\mathcal{R}^\lambda\right) \\ & + \frac{1}{32}i\,\bar{\epsilon}_2\gamma^{ab}\epsilon_1\left(2\gamma_{ab\mu\nu}\mathcal{R}^\nu - \gamma_{ab}\mathcal{R}_\mu - 2e_{\mu a}\mathcal{R}_b\right),\end{aligned} \tag{B.13}$$

where $\mathcal{R}^\nu = \gamma^{\nu\rho\sigma}\psi_{\rho\sigma}$. The field equation of the Rarita-Schwinger field is $\mathcal{R}^\nu = 0$. Therefore, the commutator algebra closes only on-shell.

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