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FABRIC TENSOR FOR DISCONTINUOUS GEOLOGICAL MATERIALS

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ABSTRACT

Geometrical property (fabric) of discontinuity in geological materials is discussed in terms of (1) position and density, (2) shape and dimension and (3) orientation of related discontinuities such as joint, fault and discrete particle. By taking into account these geometrical elements, a unique measure called fabric tensor F_{ij} is definitely introduced to embody the fabric concept without loss of generality.

The first invariant of F_{ij} is important as an index measure to evaluate the crack intensity which is related to the number and dimension of cracks. Porosity of granite is shown to be an index measure equivalent to the first invariant of F_{ij}. According to uniaxial compressive tests on gypsum plaster samples with two-dimensionally oriented cracks and granite samples, the logarithm of the first invariant of F_{ij} is linearly related to their uniaxial compressive strength.

A measure Γ which is related to the second invariant of the deviatoric part of F_{ij} shows a distance from an isotropic fabric. So, it is expected to be an index to measure the degree of anisotropy due to preferred alignment of discontinuity.

The principal axes of F_{ij} are identical to the principal axes of fabric anisotropy. There is no doubt that Γ and the principal axes are important in the analysis of anisotropic-discontinuous geological materials.

Key words : anisotropy, compressive strength, fabric, fault, granular material, joint, rock mass, soil structure (**IGC** : F 3/D 6)

INTRODUCTION

Discontinuity (e. g., fault, joint and fissure) is of widespread occurrence in rock masses. Granular materials (e. g., sand, gravel and rockfill) are also composed of discrete particles. It is quite reasonable to say that the discontinuity is a common character in these geological materials which makes their theoretical analysis very difficult. There are so many published theories to deal with the strength and constitutive equation of geological materials by taking into account their discontinuity. Unfortunately, however, these were not always successful in the representation of discontinuity. Since discontinuity is usually very complicated in usual geological situations, it seems almost impossible to grasp its exact character without losing generality.

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Geometrical property of discontinuity in the geological materials is simply called "fabric" in this paper. The purpose of this paper is to give a general definition of fabric which determines the mechanical properties of geological materials.

FABRIC FOR CRACKED ROCK MASSES

Elements to Define Fabric

Faults, joints and fissures in rock masses should be distinguished in geological sense (e. g., Price, 1966). For our purpose, *crack* is enough to designate all these discontinuities to avoid genetic implication. In the light of the study by John (1962), Hansagi (1974), Silveira, et al. (1966), Kiraly (1969) and Ogata (1978), it is clear that the geometry of cracks must be described in terms of the following elements at least:

1) Position and density of cracks

Position of cracks is conveniently given by an assembly of points corresponding to their centroids. A mean volume density ρ of cracks is given by

$$\rho=m^{(V)}/V \tag{1}$$

where $m^{(V)}$ is a number of cracks whose centroids are located inside a volume V. Multiplication of ρ by V_1 gives an estimated number of cracks belonging to the volume V_1, if V_1 is large enough.

2) Shape and dimension of cracks

Let us consider a flat crack with an occupied area A. The *crack* consists of *two crack surfaces* each of which has a unit normal vector $\boldsymbol{n}$ (or $-\boldsymbol{n}$). (Note that *crack* and *crack surface* are used in two different meanings.) It seems reasonable to assume that the crack is replaced by an equivalent circle with the same occupied area A (see, Warburbon, 1980). Then, the equivalent circle has a radius r equal to $\sqrt{A/\pi}$.

If we accept the assumption of circularity, the dimension of cracks can be described by a probability density function $f(r)$ of their radii r which must satisfy the following relation:

$$\int_0^\infty f(r)dr=1 \tag{2}$$

Field observations generally suggest that the number of cracks having larger dimension becomes smaller (e. g., Priest and Hudson; 1981). If so, $f(r)$ can be approximated by a negative exponential distribution of

$$f(r)=\lambda e^{-\lambda r} \tag{3}$$

This is one parameter distribution with the mean and standard deviation both equal to $1/\lambda$. The approximation by Eq. (3) is not always necessary in the following discussion, but is useful to simplify equations. For example, the n-th moment of r is calculated as

$$\langle r^n\rangle=\int_0^\infty r^n\lambda e^{-\lambda r}dr=\frac{n!}{\lambda^n} \tag{4}$$

where we adopted the notation

$$\langle \Phi\rangle=\int_0^\infty \Phi(r)f(r)dr \tag{5}$$

for the mean of any function Φ (r), based on the probability density function $f(r)$.

3) Orientation of cracks

A probability density function $E(\boldsymbol{n},\ r)$ is introduced to describe orientation of cracks. $E(\boldsymbol{n},\ r)\ d\Omega\ dr$ gives a fraction of *crack surfaces* whose unit normal vectors $\boldsymbol{n}$s are so oriented to be in a small solid angle $d\Omega$, and whose radii are within a small range from r to $r+dr$. By using the notation shown in Fig. 1, $d\Omega$ is simply written as $\sin\beta\ d\alpha\ d\beta$. $E(\boldsymbol{n},\ r)$ must satisfy

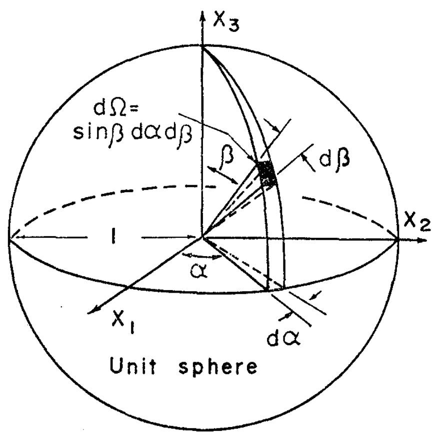

Fig. 1. Unit sphere to define solid angle $d\Omega$

$$\int_0^{\infty}\iint_{\Omega} E(\boldsymbol{n}, r)d\Omega dr=1 \qquad (6)$$

where Ω is a whole solid angle (4π) equivalent to a unit sphere of $0\leqq\alpha\leqq 2\pi$ and $0\leqq\beta\leqq\pi$ (Fig. 1). Since two normal vectors at a crack are opposite in their direction, $E(\boldsymbol{n}, r)$ must be symmetric in the sense of $E(\boldsymbol{n}, r)=E(-\boldsymbol{n}, r)$. If $\boldsymbol{n}$ and r are mutually independent, we get

$$E(\boldsymbol{n}, r)=E(\boldsymbol{n})f(r)$$
$$E(\boldsymbol{n})=E(-\boldsymbol{n}) \qquad (7)$$

If a function $P(\boldsymbol{n})$ satisfies the condition of $P(\boldsymbol{n})=P(-\boldsymbol{n})$ we get

$$\iint_{\Omega/2} 2P(\boldsymbol{n})E(\boldsymbol{n})d\Omega=\iint_{\Omega} P(\boldsymbol{n})E(\boldsymbol{n})d\Omega$$
$$=\langle P(\boldsymbol{n})\rangle \qquad (8)$$

where $\Omega/2$ is an solid angle (2π) equivalent to an upper hemisphere of $0\leqq\alpha\leqq 2\pi$ and $0\leqq\beta\leqq\pi/2$. In Eq. (8), the symbol of $<P(\boldsymbol{n})>$ is to represent the mean value of $P(\boldsymbol{n})$, based on the probability density function $E(\boldsymbol{n})$. If $P(\boldsymbol{n})>0$ equals to $-P(-\boldsymbol{n})$, then we get

$$\iint_{\Omega/2} 2P(\boldsymbol{n})E(\boldsymbol{n})d\Omega$$
$$=\iint_{\Omega} |P(\boldsymbol{n})|E(\boldsymbol{n})d\Omega=\langle|P(n)|\rangle \qquad (9)$$

If $\boldsymbol{n}$*s* are oriented isotropically, $E(\boldsymbol{n})$ must equal to $1/4\pi$.

Number of Cracks which Cross a Scanning Line

Let's consider a straight scanning line, as being parallel to a unit vector $\boldsymbol{i}$. It is called $\boldsymbol{i}$*-scanning line*. At each crack, two normal vectors are introduced. One of them (symbolized by $\boldsymbol{n}'$) is selected with respect to the scanning line so that it has a direction making an acute angle with the $\boldsymbol{i}$-direction. (The scalar product $\boldsymbol{n}'\cdot\boldsymbol{i}=n_i$ between $\boldsymbol{n}'$ and $\boldsymbol{i}$ must be greater than zero.) The designation of $(\boldsymbol{n}', 2r)$-crack is convenient to identify the crack having $\boldsymbol{n}'$ as a normal vector and $2r$ as a crack diameter.

Let's make a column of length h (Fig. 2). Its center axis accords with the $\boldsymbol{i}$-scanning line, and its cross section corresponds to the

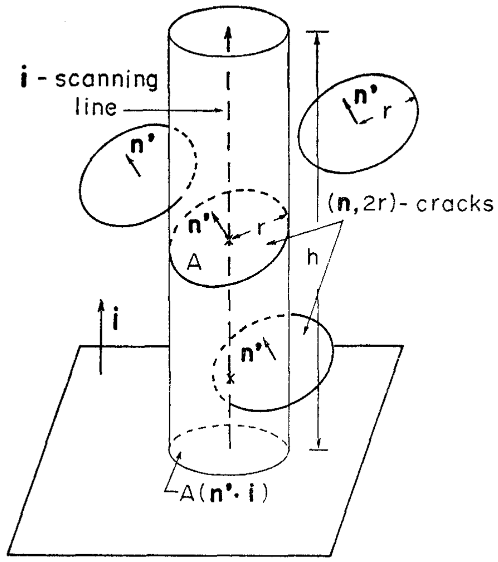

Fig. 2. A column of volume Ah $(\boldsymbol{n}'\cdot\boldsymbol{i})$ whose center line accords with $\boldsymbol{i}$-scanning line (Note that if the centers of $(\boldsymbol{n}, 2r)$-cracks are located inside the column, these must cross the $\boldsymbol{i}$-scanning line)

projection image of the $(\boldsymbol{n}', 2r)$-crack on a plane perpendicular to the $\boldsymbol{i}$-scanning line. Its cross section area is equal to $\pi r^2 n_i$ in which n_i is a component of $\boldsymbol{n}'$ on $\boldsymbol{i}$. If the length h is selected so that the volume $(\pi h r^2 n_i)$ of the column is large enough, the total number of cracks whose centers are located in the column is obtained by multiplying the corresponding volume by ρ.

Since $E(\boldsymbol{n}', r)$ is equals to $2E(\boldsymbol{n}, r)$, $2E(\boldsymbol{n}, r)\,d\Omega dr$ gives the fraction of cracks whose normal vectors fall in $d\Omega$ and whose radii are in a range from r to $r+dr$. Hence, $(\pi\rho h r^2 n_i)\times\{2E(\boldsymbol{n}, r)d\Omega dr\}$ is the number $dN^{(i)}$ of $(\boldsymbol{n}', 2r)$-cracks whose centers are inside the column. It is important to know that if the centers of $(\boldsymbol{n}', 2r)$-cracks are inside the column, those cracks must cross the $\boldsymbol{i}$-scanning line. The number $dN^{(i)}$ calculated by

$$dN^{(i)}=2\pi\rho h r^2 n_i E(\boldsymbol{n}, r)d\Omega dr \qquad (10)$$

is also the number of $(\boldsymbol{n}', 2r)$-cracks which cross the $\boldsymbol{i}$-scanning line. The cracks which cross the $\boldsymbol{i}$-scanning line *are called cracks associated with the* $\boldsymbol{i}$*-scanning line.* The total number of all cracks associated with the $\boldsymbol{i}$-scanning line is estimated by integrating Eq. (10) over $\Omega/2$ and $0 \leqq r < \infty$:

$$N^{(i)} = 2\pi h\rho \int_0^\infty \iint_{\Omega/2} r^2 n_i E(\boldsymbol{n}, r)\, d\Omega\, dr \qquad (11)$$

If the crack orientation $\boldsymbol{n}'$ is independent of its dimension r, we get

$$\frac{N^{(i)}}{h} = 2\pi\rho \int_0^\infty r^2 f(r)\, dr \iint_{\Omega/2} n_i E(\boldsymbol{n})\, d\Omega$$
$$= \pi\rho \langle r^2 \rangle \langle |n_i| \rangle \qquad (12)$$

where $N^{(i)}/h$ means the number of the cracks associated with the unit length of the $\boldsymbol{i}$-scanning line.

Fabric Tensor for Cracked Rock Masses

The unit normal vector $\boldsymbol{n}'$ has been defined at each contact to specify its orientation with reference to a scanning line. A new vector $\boldsymbol{m}$ (called crack vector) is now introduced at each $(\boldsymbol{n}', 2r)$—crack. The direction of $\boldsymbol{m}$ accords completely with $\boldsymbol{n}'$, and its norm is $2r$, not unity. Then we get

$$\boldsymbol{m} = 2r\boldsymbol{n}' \qquad (13)$$

$dN^{(i)}$ of Eq. (10) is the number of $(\boldsymbol{n}', 2r)$—cracks which cross the length h of $\boldsymbol{i}$-scanning line. So, the multiplication of $dN^{(i)}/h$ by $\boldsymbol{m}$ corresponds to the vector sum of all $(\boldsymbol{n}', 2r)$-cracks associated with the unit length of the scanning line.

$$\frac{dN^{(i)}}{h} \cdot \boldsymbol{m} = \{4\pi\rho r^3 n_i E(\boldsymbol{n}, r)\, d\Omega\, dr\}\, \boldsymbol{n}' \qquad (14)$$

This vector can be projected on a direction given by $\boldsymbol{j}$, with the following projected image $dF_{ij}^{(R)}$:

$$dF_{ij}^{(R)} = 4\pi\rho r^3 n_i n_j E(\boldsymbol{n}, r)\, d\Omega\, dr \qquad (15)$$
$$(i, j = 1, 2, 3)$$

(The reference vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are selected so as to make orthogonal reference axes.) The $\boldsymbol{j}$-component of the total sum of all $\boldsymbol{m}$s associated with the unit length of the $\boldsymbol{i}$-scanning line is then obtained by integrating $dF_{ij}^{(R)}$ over $\Omega/2$ and $0 \leqq r < \infty$ as follows:

$$F_{ij}^{(R)} = 4\pi\rho \int_0^\infty \iint_{\Omega/2} r^3 n_i n_j E(\boldsymbol{n}, r)\, d\Omega\, dr \qquad (16)$$
$$(i, j = 1, 2, 3)$$

If $\boldsymbol{n}'$ and r are mutually independent variables, we get

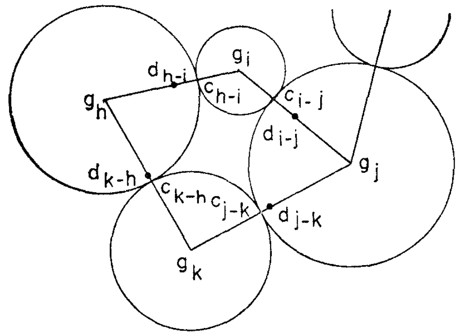

Fig. 3. An assembly of spheres replaced by an assembly of lines (called branches) connecting centers of adjacent particles which are in contact at points ($\boldsymbol{g}_i$=center of sphere; $\boldsymbol{c}_{i-j}$=contact between spheres $\boldsymbol{g}_i$ and $\boldsymbol{g}_j$; $\mathbf{d}_{i-j}$=mid-point of branch $\overline{\boldsymbol{g}_i\boldsymbol{g}_j}$)

$$F_{ij}^{(R)} = 2\pi\rho \langle r^3 \rangle \langle n_i n_j \rangle \qquad (17)$$
$$(i, j = 1, 2, 3)$$

Note that $F_{ij}^{(R)}$ of Eq. (16) is transformed as a second order tensor when the reference axes are rotated. $F_{ij}^{(R)}$ is called *fabric tensor*. Note that the tensorial character is independent of the specific form of the probability density function $E(\boldsymbol{n}, r)$. Note also that $F_{ij}^{(R)}$ is a dimensionless quantity which, as will be shown later, is a favorable character for an index measure of discontinuity in rock masses.

In the definition (Eq. (16)) of the fabric tensor, $E(\boldsymbol{n}, r)$ plays an essential role. However, it is easy to rewrite Eq. (16) in another way without using $E(\boldsymbol{n}, r)$, as follows;

$$F_{ij}^{(R)} = \frac{1}{V} \sum^{m^{(V)}} (2\pi r^3 n_i n_j) \qquad (18)$$
$$(i, j = 1, 2, 3)$$

where the summation must be taken all over cracks $m^{(V)}$ included in a given volume V.

FABRIC FOR GRANULAR MATERIALS

Element to define fabric of spherical granules

For the sake of simplicity, each particle is idealized as a sphere with the same volume. If we accept this simplification, size of particles can be expressed by a probability density function $f(R)$ so that $f(R)dR$ means the fraction in number of spheres with radii ranging from R to $R+dR$.

According to the study by Satake (1978) and Oda, et al. (1980), the assembly of spheres can be replaced by an assembly of lines ($\cdots$, $\overline{g_i g_j}$, $\overline{g_j g_k}$, $\cdots$ in Fig. 3) connecting the centers of adjacent particles which are in contact at points ($\cdots$, c_{i-j}, c_{j-k}, $\cdots$). This replacement is possible on the basic assumption that the corresponding fabric is represented with sufficient accuracy by the distribution and geometrical arrangement of lines. Satake (1978) calls the line connecting the centers of two contacting spheres, *branch*. Branch is also used here. Based on the view point that an assembly of particles is represented by an assembly of branches, (1) density, (2) dimension and (3) orientation of branches must be included at least in the definition of fabric.

1) Density of branches

Since the volume of a sphere with a radius R is $4/3\pi R^3$, and since the number of spheres whose radii range from R to $R+dR$ is given by $n^{(V)}f(R)dR$, the solid volume V_s by all spheres can be estimated by

$$V_s=\int_{R_m}^{R_M}\frac{4}{3}\pi R^3 n^{(V)} f(R)dR=\frac{4}{3}\pi n^{(V)}\langle R^3\rangle \tag{19}$$

where $n^{(V)}$ is the total number of spheres in a total volume $V(=V_s+\text{void volume})$, and R_m and R_M are the minimum and maximum radii of spheres respectively. Furthermore, since the total volume V equals to $(1+e)V_s$, the number $n^{(V)}$ of spheres can be expressed as

$$n^{(V)}=\frac{3V}{4\pi(1+e)\langle R^3\rangle} \tag{20}$$

where e is the void ratio of assembly defined by $(V-V_s)/V_s$.

Associated with each contact there are two contact points, one belonging to each contacting particle. Accordingly, the total number of contacts (not contact points) equals to $1/2\xi n^{(V)}$, in which ξ is a mean number of contact points per a particle (=mean co-ordination number). The volume density δ of contacts is defined as

$$\delta=\frac{\xi n^{(V)}}{2V}=\frac{3\xi}{8\pi(1+e)\langle R^3\rangle} \tag{21}$$

There have been published a number of equations to show the relation between the mean co-ordination number ξ and the corresponding void ratio e; e, g., Smith, Foote and Busang (1929), Field (1963), Gray (1968) and Oda (1977). According to the experiments by Field (1963) and Oda (1977), there is a unique relation between ξ and e, being independent of the grain size distribution $f(R)$. By using the relation, the volume density δ of Eq. (21) can be expressed as a function of e and $\langle R^3\rangle$. Note that position of branches is represented by their mid-points ($\cdots$, d_{i-j}, d_{j-k}, d_{k-h} in Fig. 3). Since the number of contacts is exactly the same as the number of the mid-points, Eq. (21) gives not only the volume density of contacts but also the volume density of the mid-branch points.

2) Dimension of branches

On the basis of statistical consideration, Oda, Nemat-Nasser and Mehrabadi (1980) have given the following equation as a density function $g(l)$ of branch length l:

$$g(l)=\int_{l-R_M}^{l-R_m} f(R)f(l-R)dR \Big/ \int_{2R_m}^{2R_M}\int_{l-R_M}^{l-R_m} f(R)f(l-R)dRdl \tag{22}$$

which yields the distribution of branch length l in terms of the particle size distribution $f(R)$.

3) Angular distribution of branches

Two unit normal vectors $\boldsymbol{n}$s for spherical granules which are co-axial with the corresponding branch are considered at each contact. Horne (1965) and Oda (1972) have introduced a probability density function $E(\boldsymbol{n})$ to describe the angular distribution of $\boldsymbol{n}$. $E(\boldsymbol{n})$ is sufficient to describe the angular distribution of branches if the branch length l is an independent variable of the direction $\boldsymbol{n}$. In more general cases, $E(\boldsymbol{n}, l)$ must be used instead of $E(\boldsymbol{n})$.

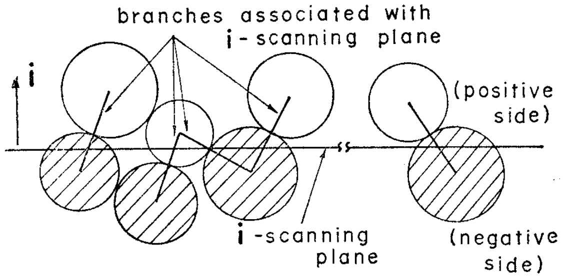

Fig. 4. Number of branches associated with i-scanning line

Number of Branches which Cross a Scanning Plane

Let's consider a scanning plane cutting through an assembly of particles. Since the plane is specified by its unit normal vector $\boldsymbol{i}$, it is called *$\boldsymbol{i}$-scanning plane.*

All particles located in the immediate vicinity of the $\boldsymbol{i}$-scanning plane can be divided into the following two groups (Fig. 4): Group A (hatched) consists of those particles whose centers are located on the negative side of the plane, and Group B (unhatched) consists of those whose centers are located on the positive side of the plane. Branches connecting the centers of the hatched particles with those of the unhatched ones through common contacts is called *branches associated with the $\boldsymbol{i}$-scanning plane.*

According to the definition, the associated branches must cross the $\boldsymbol{i}$-scanning plane at points (see, Fig. 4). Oda et al. (1980) have already given the number $M^{(i)}$ of the associated branches per unit area of the $\boldsymbol{i}$-scanning plane. The number equals to the areal density of the branch intersections. $M^{(i)}$ is given by

$$M^{(i)}=2\delta\int_{2R_m}^{2R_M}\iint_{\Omega/2} ln_i g(l)E(\boldsymbol{n})d\Omega dl$$
$$=\delta\langle l\rangle\langle|n_i|\rangle \tag{23}$$

which corresponds to $N^{(i)}/h$ of Eq. (12) showing the number of the cracks associated with the $\boldsymbol{i}$-scanning line. In the derivation of Eq. (23), l and $\boldsymbol{n}$ are assumed to be mutually independent variables which is quite reasonable for the assembly of spheres.

Fabric Tensor for Granular Materials

Let's consider each associated branch as a unit vector (symbolized by $\boldsymbol{n}'$) from the unhatched particle to the hatched one (Fig. 4). We can make the vector sum of all vectors $\boldsymbol{n}$' s associated with the unit area of the $\boldsymbol{i}$-scanning plane. The vector sum has a projection on a unit vector $\boldsymbol{j}$. The projection $F_{ij}{}^{(G)}$ is calculated by

$$F_{ij}{}^{(G)}=2\delta\int_{2R_m}^{2R_M} lg(l)\,dl\iint_{\Omega/2} n_i n_j E(\boldsymbol{n})d\Omega$$
$$=\delta\langle l\rangle\langle n_i n_j\rangle \tag{24}$$
$$(i,j=1,2,3)$$

This expression was first reported by Oda et al. (1980). Mehrabadi, Nemat-Nasser and Oda (1980) have also said that stress mobilized in a granular mass can be defined in terms of the fabric given by Eq. (24) by taking into account its arrangement of discrete particles.

Similar tensors with the form of

$$J_{ij}=\langle n_i n_j\rangle \tag{25}$$

have been introduced by Satake (1978) as an index to show the fabric anisotropy of granular materials, and by Scheidegger (1965) in the analysis of plane and linear elements in geological body. Gudehus (1968) has also introduced a tensor A_{ij} (Affiniyöt) in order to represent the micro-structure of soils.

FURTHER CONSIDERATION ON THE FABRIC TENSOR

The fabric tensor is symmetric, $F_{ij}=F_{ji}$. Therefore, F_{ij} has three principal values F_1, F_2 and F_3 which are calculated by solving the following determinant (see, e.g., Prager (1961)).

$$|F_{ij}-F\delta_{ij}|=0 \tag{26}$$

where δ_{ij} is Kronecker delta. Their corresponding principal direction μ_i are calculated by

$$\left.\begin{array}{l}(F_{ij}-F\delta_{ij})\,\mu_i=0\\ \mu_i\mu_j=\delta_{ij}\end{array}\right\} \tag{27}$$

Invariants $I_1{}^{(F)}$, $I_2{}^{(F)}$ and $I_3{}^{(F)}$ of the fabric are defined by

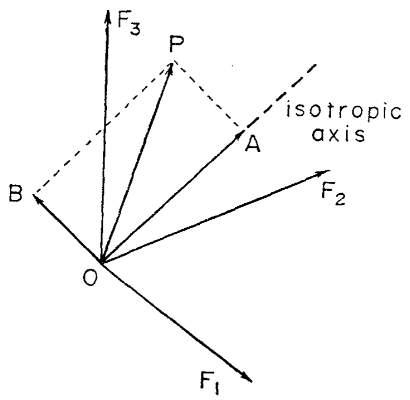

Fig. 5. Principal fabric space

$$\left.\begin{aligned} I_1^{(F)} &= F_1+F_2+F_3 \\ I_2^{(F)} &= -(F_1F_2+F_2F_3+F_3F_1) \\ I_3^{(F)} &= F_1F_2F_3 \end{aligned}\right\} \tag{28}$$

Let's introduce a space of principal values F_1, F_2 and F_3. Since the space is quite similar to the principal stress space, we call it tentatively *principal fabric space*. Then, the fabric character can be represented by a vector $\overrightarrow{\mathrm{OP}}$ in the space (Fig. 5), with components F_1, F_2 and F_3 respectively. Since the straight line of $F_1=F_2=F_3$ passing through the origin means an isotropic fabric, we call it *isotropic axis*.

The vector $\overrightarrow{\mathrm{OP}}$ in the space is resolved into two vectors; $\overrightarrow{\mathrm{OP}}=\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}}$. The length of $\overrightarrow{\mathrm{OA}}$ is proportional to the first invariant $I_1^{(F)}$ of the fabric tensor:

$$|\overrightarrow{\mathrm{OA}}|=\frac{1}{\sqrt{3}}I_1^{(F)} \tag{29}$$

On the other hand, another vector $\overrightarrow{\mathrm{OB}}$ which is on the plane of $F_1+F_2+F_3=0$ characterizes the deviatoric part of F_{ij}; that is,

$$D_{ij}=F_{ij}-\frac{I_1^{(F)}}{3}\delta_{ij} \tag{30}$$

It is well known that the length Γ of the vector $\overrightarrow{\mathrm{OB}}$ is related to the second invariant $I_2^{(D)}$ of D_{ij} as

$$\begin{aligned} \Gamma &= \sqrt{2I_2^{(D)}} \\ &= \sqrt{(F_1-F_2)^2+(F_2-F_3)^2+(F_3-F_1)^2} \end{aligned} \tag{31}$$

From the above discussion, it becomes clear that the tensor F_{ij} conveys the following informations each of which is important in the analysis of fabric character of geological materials:

1) $I_1^{(F)}$ (the first invariant of F_{ij}): The isotropic part of F_{ij} is proposed as an index to evaluate the intensity of discontinuity, depending not only on a volume density (ρ or δ) but on a typical dimension ($\langle r^3\rangle$ or $\langle l\rangle$). On an acceptable assumption, $I_1^{(F)}$ is proved to be proportional to porosity p of a body, as follows: A crack with radius r has a corresponding void volume of $\pi r^2 t$ (t=width of crack). The void volume V_v by all cracks in a total volume V is thus estimated by

$$V_v=\int_0^\infty \pi m^{(V)} r^2 t f(r)dr \tag{32}$$

On an assumption of $t=2kr$ where k is a proportional coefficient, porosity p is given by

$$\begin{aligned} p &= \frac{V_v}{V}=2\pi k\rho\int_0^\infty r^3 f(r)dr \\ &= 2\pi k\rho\langle r^3\rangle \\ &= kI_1^{(F)} \end{aligned} \tag{33}$$

2) Γ: By thinking that $\Gamma\,(=\sqrt{2I_2^{(D)}})$ gives the distance of a point P with components F_1, F_2 and F_3 from the isotropic line in the principal fabric space, it is proposed as an index to evaluate the degree of anisotropy of F_{ij}.

3) The last information derived from F_{ij} is concerned with its principal axes. These principal axes can also be considered as the principal axes of fabric anisotropy. In a rather special case in which r and $\boldsymbol{n}$ are mutually independent variables, F_{ij} has the same principal axes as the tensor J_{ij} proposed by Satake (1978) and Scheidegger (1965). It must be emphasized, however, that J_{ij} was formulated by taking into account only the orientation $\boldsymbol{n}$ of discontinuity.

EXPERIMENTAL JUSTIFICATION TO INTRODUCE THE CONCEPT OF FABRIC

Many experimental and theoretical studies

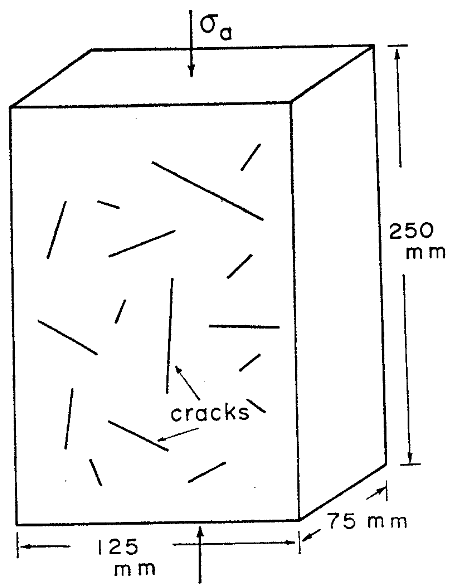

Fig. 6. Gypsum plaster sample with random cracks

have been published to show the cross relation of the fabric of granular materials to their mechanical properties (e. g., Oda, 1972; Arthur and Henzies, 1972; Matsuoka, 1974; Mahmood and Mitchell, 1974; Mulilis, Chan and Seed, 1975; Konishi, 1978; Mehrabadi, et al., 1980; Kanatani, 1981). In particular, Oda (1978) has examined this relation rather extensively with the conclusion that the fabric concept gives us a sound basis to make clear the complex mechanical property of granular materials.

In spite of the importance of fabric concept in the analysis of cracked rock masses (Gerrard, 1977), there has been published few work dealing with the topic from a general point of view. This is because the geometry of cracks is usually too complicated to be identified by a unique measure. In this section, two experimental results are given with special attention to show how powerful the fabric tensor is in the analysis of cracked rock masses.

Uniaxial Compression Tests on Gypsum Plaster Samples with Randomly Oriented Cracks (Onodera, Oda and Ishii, 1972)

Experimental work

Onodera, et al. (1972) reported uniaxial compression tests on gypsum plaster samples with random cracks. Their experiment was done in the following order:

1) In order to make a gypsum plaster sample with a system of two-dimensional cracks, the position and orientation of cracks are arranged in random so that the system of cracks becomes isotropic as well as homogeneous. A random digit is conveniently used for this purpose.

2) Water-gypsum mixture (2 : 3 by weight) is poured into a rectangular prismatic mold with a dimension of $250\times125\times100$ mm^3. Strips ($2r\times100\times0.25$ mm^3) made of picture postcard are inserted into the water-gypsum mixture at the previously selected positions with the previously selected orientations (Fig. 6). Boundaries between the gypsum plaster and the picture postcard are regarded as two-dimensional cracks of length $2r$.

3) After about an hour, the well-hardened water-gypsum mixture is taken out of the mold, and is trimed to make a rectangular prismatic sample of $250\times125\times75$ mm^3. The samples thus made are cured for a week in a constant temperature (50°C) and humidity (38%) bath.

4) Axial compressive stress σ_a increases at a constant loading rate of 9.8 kN/m^2/s to determine the uniaxial compressive strength $(\sigma_a)_f$.

5) Experiment consists of the following three series:

a) a-series: The number $m^{(V)}$ of cracks changes from 10 to 100, with a definite crack length ($2r=10$ mm).

b) b-series: The crack length changes from 10 mm to 50 mm, with a definite number of cracks ($m^{(V)}=20$).

c) c-series: Both of $2r$ and $m^{(V)}$ change so as to give a suitable value of the first invariant $I_1^{(F)}$ of $F_{ij}^{(R)}$.

Result

Since cracks in a sample are arranged to be isotropic as well as two-dimensional, its fabric tensor can be reduced to

$$F_{ij}^{(R)}=\frac{m^{(V)}\langle 4r^2\rangle T}{V}\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix} \qquad (34)$$

where T is the thickness of sample (Fig. 6).

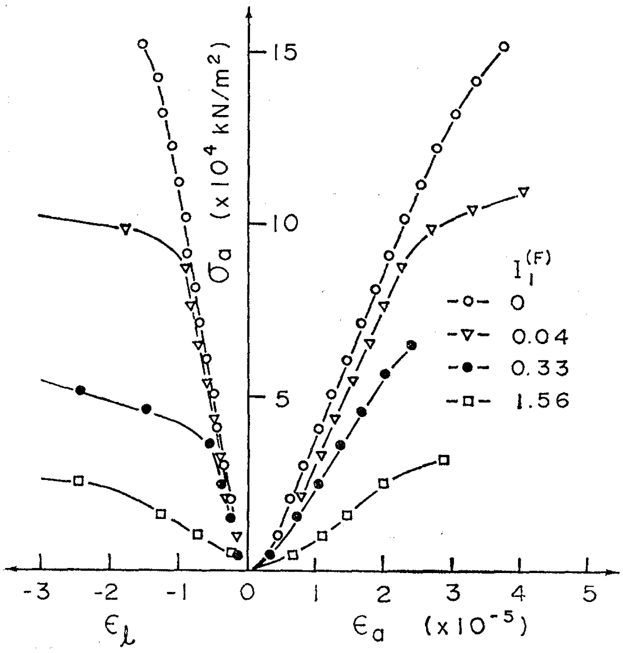

Fig. 7. Relation between axial stress σ_a, axial strain ε_a and lateral strain ε_l for four gypsum plaster samples having different values of first invariant of $F_{ij}^{(R)}$

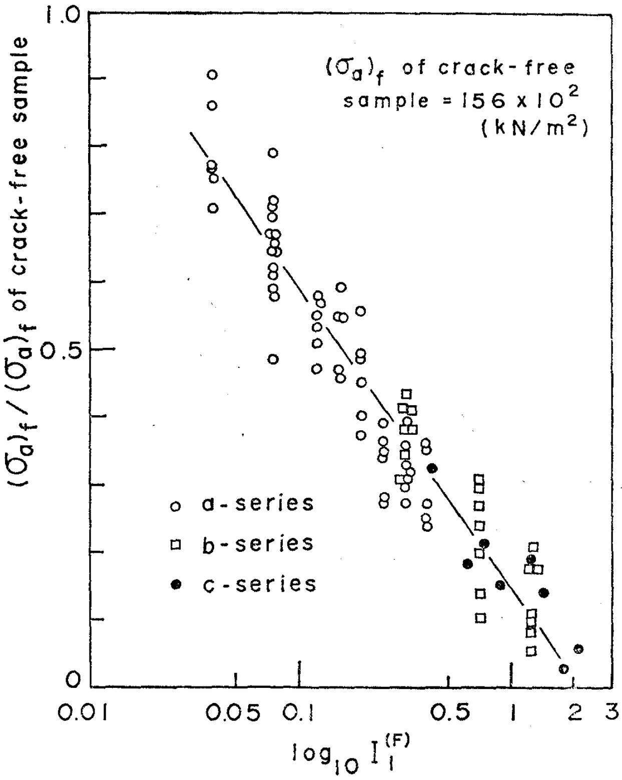

Fig. 8. Linear relation of uniaxial compressive strength $(\sigma_a)_f$ to logarithm of first invariant of $F_{ij}^{(R)}$

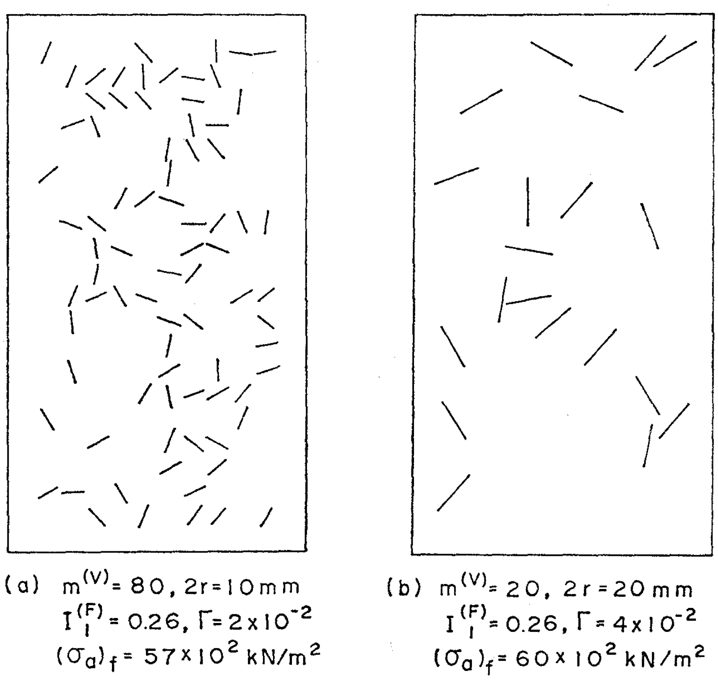

Fig. 9. Two crack systems with different appearance (Note, however, that both have almost the same fabric tensor $F_{ij}^{(R)}$)

In this special case, the first invariant $I_1^{(F)}$ is only a nonvanished measure derived from $F_{ij}^{(R)}$. (Real samples are not ideally isotropic but with some deviatoric component of $D_{ij}^{(F)}$. With a few exceptions, however, the deviatoric component is so small that it can be neglected in the following consideration. The problem of the deviatoric tensor will be discussed in the next paper.)

Fig. 7 shows the relations between axial strain ε_a, lateral strain ε_l and axial stress σ_a for four samples. Each sample has a value of the first invariant different from others. A sample with $I_1^{(R)}=0$ means a crack-free sample. It is no wonder that the crack-free sample has the highest uniaxial compressive strength $(\sigma_a)_f$ and the sharpest stress-strain relation. With the increase of the first invariant $I_1^{(F)}$, the sample loses gradually its strength and stiffness.

Fig. 8 shows the change of the uniaxial compressive strength as a function of $I_1^{(F)}$. (Note that $(\sigma_a)_f$ is normalized by that of the crack-free sample. It must be emphasized that all experimental results about $(\sigma_a)_f$ are plotted in almost the same area, irrespective of the series (a, b, c) of tests. This result strongly supports the idea that the first invariant $I_1^{(F)}$ has a unique physical meaning as an index for the crack geometry. A sample (a) in Fig. 9 seems to have a quite different crack system from a sample (b) because of such different appearance. It must be pointed out, however, that both samples have almost the same uniaxial compressive strength and secant deformation modulus owing to the same value of $I_1^{(F)}=0.256$.

Experiment on weathered granite (Onodera, Yoshinaka and Oda, 1974)

In order to see the effect of weathering of granite on its mechanical property, Onodera, et al. (1974) observed cracks under a microscope on thin sections sliced from the weathered granite. Granite was sampled from Innoshima and Shimotsui at which Honshu-Shikoku bridges were planned to be constructed. Cracks were observed along a scanning line (total scanning being 240 mm).

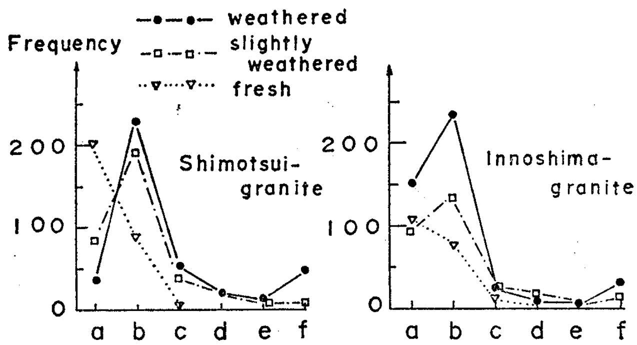

Fig. 10. Progressive change of cracks due to weathering of granite (Onodera, Yoshinaka and Oda, 1974)

They were classified into six classes according to their crack widths as a (0~0.0016 mm), b (0.0016~0.016 mm), c (0.016~0.032 mm), d (0.032~0.048 mm), e (0.048~0.064 mm) and f (more than 0.064 mm). Number of observed cracks belonging to each class are summarized in Fig. 10. It is clear that the width as well as the number of cracks increase with the progress of weathering which was estimated by the chemical analysis. Especially, the rate of cracks belonging to the class f becomes larger when granite is exposed to more severe weathering.

Since granite has isotropic appearance, the density function $E(\boldsymbol{n})$ can be assumed to be isotropic ($E(\boldsymbol{n})=1/4\pi$). Then, the number $N^{(i)}/h$ of cracks associated with the unit length of the scanning line is estimated by

$$N^{(i)}/h=\pi\rho\langle r^2\rangle\int_0^{\pi/2}\int_0^{2\pi}\frac{1}{4\pi}\cos\beta\sin\beta\, d\alpha d\beta$$
$$=\frac{\pi}{4}\rho\langle r^2\rangle \qquad (35)$$

$N^{(i)}/h$ measured under a microscope is shown in Fig. 11 which represent the relation between $N^{(i)}/h$ and p (porosity). Porosity of cracked materials has the equivalent meaning to the first invariant of $F_{ij}^{(R)}$ if the crack width is proportional to the crack length $2r$. (This assumption seems to be reasonable for the weathered granite.) $N^{(i)}/h$ of Shimotsui-Granite is plotted within a rather small range from 1.2 to 1.8, while $N^{(i)}/h$ of Innoshima-granite is linearly related to the porosity. This result suggests that $N^{(i)}/h$ itself is not an adequate index

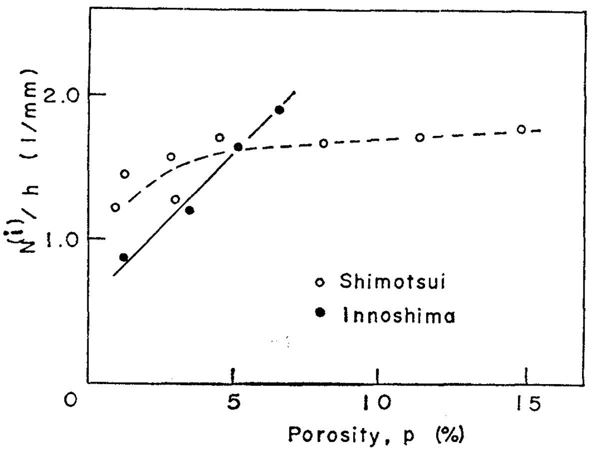

Fig. 11. **Number of cracks crossed by unit length of i-scanning line**

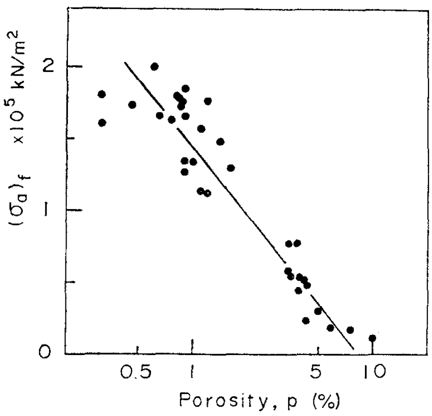

Fig. 12. **Relation between uniaxial compressive strength $(\sigma_a)_f$ and porosity p of granite (modified from Yoshinaka's (1973) data) (Note that this linear relation is quite similar to the relation of Fig. 8)**

measure of cracks.

Yoshinaka (1973) has reported that the uniaxial compressive strength $(\sigma_a)_f$ of granite is given as a linear function of log p (Fig. 12). It is worthy of note that this linearity between $(\sigma_a)_f$ and log p is quite similar to the linearity between $(\sigma_a)_f$ and log $I_1^{(F)}$. This seems to support the idea that porosity of cracked materials is related to the first invariant of fabric tensor.

CONCLUSION

Geometrical property (fabric) of discontinuity in geological materials was discussed in terms of (1) position and density, (2) shape and dimension and (3) orientation of related discontinuities such as joint, fault and discrete particle. Based on the statistical consideration, a unique measure called *fabric tensor* was introduced to describe the fabric.

From the fabric tensor of cracked rock masses, we can obtain the following informations:

1) The first invariant of fabric tensor is important as an index measure to evaluate the crack intensity which is related to the number and dimension of cracks. Porosity of granite is an index having almost the same meaning as the first invariant. According to uniaxial compressive tests on gypsum plaster samples with two-dimensionally oriented cracks and granite samples, the logarithm of first invariant is linearly related to their uniaxial compressive strength.

2) The measure Γ which is related to the second invariant of the deviatoric part shows a distance from an isotropic fabric. So, it is expected to be an index to measure the degree of anisotropy due to the preferred alignment of discontinuity.

3) The principal axes of the fabric tensor are identical to the principal axes of fabric anisotropy. The determination of Γ value as well as the principal axes are indispensable in the analysis of anisotropic, discontinuous materials.

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NOTATION

A = area of crack surface

$E(\boldsymbol{n})$ = density function showing distribution of $\boldsymbol{n}s$ (normals to contacts and cracks)
$E(\boldsymbol{n}, r)$ = density function showing distribution of $\boldsymbol{n}s$ (normals to cracks)
e = void ratio
F_1, F_2, F_3 = principal values of fabric tensor F_{ij}
$F_{ij}^{(R)}$ and $F_{ij}^{(G)}$ = fabric tensors for rock masses and for granular materials respectively
$f(r)$ and $f(R)$ = density functions of r (radious of crack) and of R (radious of spherical particle) respectively
h = length of column
$I_1^{(F)}$, $I_2^{(F)}$ and $I_3^{(F)}$ = first, second and third invariant of fabric tensor F_{ij}
$I_2^{(D)}$ = second invariant of deviatoric tensor D_{ij}
l = length of branch
$N^{(i)}$ = number of cracks associated with $\boldsymbol{i}$-scanning line.
$m^{(V)}$ = number of cracks inside a volume V
$\boldsymbol{m}$ = crack vector defined by $2r\boldsymbol{n}'$
$M^{(i)}$ = number of branches associated with $\boldsymbol{i}$-scanning plane.
$n^{(V)}$ = number of particles inside a volume V
n_i = component of unit vector $\boldsymbol{n}$ on a direction given by $\boldsymbol{i}$
$\boldsymbol{n}$ = unit vector normal to crack surfaces and contact surfaces
$\boldsymbol{n}'$ = unit normal vector defined with respect to $\boldsymbol{i}$-direction $(\boldsymbol{n}'\cdot\boldsymbol{i}\geqq 0)$
p = porosity
R = radius of circular crack
r = radius of spherical particle
V = reference volume
α and β = angles to show unit normal vector $\boldsymbol{n}$
Γ = parameter showing anisotropy of F_{ij} equal to $\sqrt{2I_1^{(D)}}$
δ = volume density of contacts
δ_{ij} = Kronecker delta
ε_a and ε_l = axial and lateral strains
μ_i = direction cosine to show principal axes of F_{ij}
ξ = mean co-ordination number
ρ = volume density of cracks
σ_a = axial stress
Ω = whole solid angle (4π) given by a surface of unit sphere
$\Omega/2$ = half solid angle given by a surface of upper hemisphere

REFERENCES

1) Arthur, J. R. F. and Menzies, B. K. (1972): "Inherent anisotropy in a sand," Geotechnique, Vol. 22, No. 1, pp. 115-128.
2) Field, W. G. (1963): "Towards the statistical definition of granular mass," Proc. 4th A. and N. Z. Conf. on Soil Mech., pp. 143-148.
3) Gerrard, G. M. (1977): "Background to mathematical modelling in geomechanics: The roles of fabric and stress history," Finite Elements in Geomechanics (Edited by G. Gudehus), John Wiley & Sons, pp. 33-120.
4) Gray, W. A. (1968): The Packing of Solid Particles, Chapman and Hall Ltd.
5) Gudehus, G. (1968): "Gedanken zue Statistischen Boden Mechanik," Der Baningenieur, 43, pp. 320-326.
6) Hansagi, I. (1974): "A method of determining the degree of fissuration of rock," Int. J. Rock Mech. Min. Sci. & Geomech. Abstr., Vol. 11, pp. 379-388.
7) Horne, M. R. (1965): "The behavior of an assembly of rotund, rigid, cohesionless particles (I and II)," Proc. Roy. Soc. A., Vol. 286, pp. 62-97.
8) John, K. W. (1962): "An approach to rock mechanics," Proc. of ASCE, Jour. Soil Mech. Found. Div., Vol. 188, SM 4, pp. 1.
9) Kanatani, K. (1981): "A theory of contact force distribution in granular materials," Powder Technology, Vol. 28, pp. 167-172.
10) Kiraly, L. (1969): "Statistical analysis of fracture (orientation and density)," Geol. Rundsch 59, No. 1, pp. 125-151.
11) Konishi, J. (1978): "A microscopic consideration on deformation and strength behavior of granular material like sand," Doctor Thesis at Kyoto University (in Japanese).
12) Mahmood, A. and Mitchell, J. K. (1974): "Fabric-property relationships in fine granular materials," Clay and Clay Minerals, 22, 516, pp. 397-408.
13) Matsuoka, H. (1974): "A microscopic study on shear mechanism of granular materials," Soils and Foundations, Vol. 14, No. 1, pp. 29-43.
14) Mehrabadi, M. M., Nemat-Nasser, S. and Oda, M. (1980): "On statistical description of stress and fabric in granular materials," Earthquake Research and Engineering Laboratory, Technical Report No. 80-4-29, Dept. of Civil Eng., Northwestern University, Evanston, Ill.
15) Mulilis, J. P., Chan, C. K. and Seed, H. B. (1975): "The effects of method of sample preparation on the cyclic stress-strain behavior of sand," Report No. HERC 75-18, Earth-

quake Engineering Research Center, University of California, Berkely.
16) Oda, M. (1972): "Initial fabrics and their relations to mechanical properties of granular material," Soils and Foundations, Vol. 12, No. 1, pp. 17-36.
17) Oda, M. (1977): "Co-ordination number and its relation to shear strength of granular material," Soils and Foundations, Vol. 17, No. 2, pp. 29-42.
18) Oda, M. (1978): "Significance of fabric in granular mechanics," Proc. U.S.-Japan Seminar on Continuum-Mechanical and Statistical Approaches in the Mechanics of Granular Materials, Gakujutsu Bunken Fukyukai, Tokyo, Japan, pp. 7-26.
19) Oda, M., Nemat-Nasser, S. and Mehrabadi, M. M. (1980): "A statistical study of fabric in random assembly of spherical granules," Earthquake Research and Engineering Laboratory, Technical Report No. 804-29, Dept. of Civil Eng., Northwestern University, Evanston, Ill.
20) Ogata, S. (1978): "Quantitative characterization of distribution parameters of joints in rock masses," Jour. Japan Soc. Engrg. Geol., 19-1, pp. 12-20.
21) Onodera, T., Oda, M. and Ishii, T. (1972): "Strength- and deformation characteristics of jointed material as rock and rock mass," Report of Dept. Found. Engrg., Faculty of Engineering, Saitama University, Vol. 3, pp. 1-17.
22) Onodera, T., Yoshinaka, R. and Oda, M. (1974): "Weathering and its relation to mechanical properties of granite," Proc. 3rd Int. Congr. on Rock Mech., ISRM, Vol. 2, pp. 71-78.
23) Prager, W. (1961): Introduction to Mechanics of Continua, Ginn and Company.
24) Price, N. J. (1966): Fault and Joint Development in Brittle and Semi-brittle Rock, Pergamon Press, London.
25) Priest, S. D. and Hudson, J. A. (1981): "Estimation of discontinuity spacing and trace length using scanline surveys," Int. J. Rock. Mech. Min. Sci. & Geomech. Abstr. Vol. 18, pp. 183-197.
26) Satake, M. (1978): "Constitution of mechanics of granular materials through graph representation," Theoretical and Applied Mechanics, 26, University Tokyo Press, pp. 257-266.
27) Scheidegger, A. E. (1965): "On the statistics of the orientation of bedding planes, grain axes, and similar sedimentrogical data," U.S. Geol. Survey Prof. Paper 525-c, pp. 164-167.
28) Silveira, A. F., Rodrigues F. P. and Grossmann, N. F. (1966): Proc. Ist Cong. Rock Mech., ISRM., Vol. 1, pp. 225-233.
29) Smith, W. O., Foote, P. D. and Busang, P. F. (1929): "Packing of homogeneous spheres," Pys. Rev., 34, pp. 1271-1274.
30) Warburton, P. M. (1980): "A stereological interpretation of joint trace data," Int. J. Rock Mech. Min. Sci. & Geomech. Abstr., Vol. 17, pp. 181-190.
31) Yoshinaka, R. (1973): "Engineering properties of foundation of granite," Kyōryō-to-Kiso, Vol. 7, No. 10, pp. 88-93.