No. 14

Kohei Marumo*

A Non-parametric Method for Calculating
Conditional Stressed Value at Risk

24 March 2017

*marumo@mail.saitama-u.ac.jp. Faculty of Economics, Saitama University, JAPAN. The author thanks Mr. Yasunari Inamura (Bank of Japan) for inspiring discussions and suggestions. The views expressed here does not represent those of Bank of Japan and all possible mistakes are due to the author.

Abstract

We consider the Value at Risk (VaR) of a portfolio under stressed conditions. In practice, the stressed VaR (sVaR) is commonly calculated using the data set that includes the stressed period. It tells us how much the risk amount increases if we use the stressed data set.

In this paper, we consider the VaR under stress scenarios. Technically, this can be done by deriving the distribution of profit or loss conditioned on the value of risk factors. We use two methods; the one that uses the linear model and the one that uses the Hermite expansion discussed by Marumo and Wolff (2013, 2016). Numerical examples shows that the method using the Hermite expansion is capable of capturing the non-linear effects such as correlation collapse and volatility clustering, which are often observed in the markets.
Keywords: Conditional distribution; Hermite expansion; Linear model; Non-linear effect.

1 Introduction

Value at Risk (VaR) and stress test are common tools for measuring risk of a portfolio and are used as the benchmark for the capital requirement in financial institutions. In addition to these two, a risk measure called the stressed Value at Risk (sVaR) is often discussed (Hong, 2017; Basel Committee on Banking Supervision, 2013; European Banking Authority, 2012).

The sVaR considers the VaR under the stressed market conditions. In practice, this is particularly done by using the market data from the period that includes September to November 2008 financial crisis (Gibart, 2012). This implies that the only difference between the VaR and sVaR is that we use the data set with the larger volatility for calculation. Further, we usually use around two years' historical data for VaR calculations, while many financial crises lasts only a few months. This means that the data set as a whole may not represent the stressed market conditions.

In this paper, we consider the VaR under stress scenarios on risk factors. This can be compared to the stress tests, which considers the loss under stress scenarios. Technically, the VaR under a scenario can be calculated from the distribution of profit or loss conditioned on the risk factor's value. A naïve way of deriving this conditional distribution is to use the linear model. This method essentially uses only first and second moments, and is not capable of capturing non-linear effects such as correlation collapse and volatility clustering, which are often observed in the markets. We consider the application of the Hermite expansion discussed by Marumo and Wolff (2013, 2016) to the calculation of conditional VaR. The Hermite expansion approximates the target density function by the Normal density multiplied by the linear combination of the Hermite polynomials. It is capable of expressing the higher order moments, and hence we suppose that it captures non-linear effects.

Compared to the VaR under stress scenarios, the stressed VaR can be considered as the unconditional VaR, calculated using the data that includes the stressed period. We expect that the VaR under stress scenarios which takes the non-linear effects into accounts can be more informative and contribute to a better understanding on the risk of our portfolios under stressed market conditions.

In the remainder of this paper, we review the theoretical background of the methods used in the paper in Section 2, and show the numerical examples in Section 3. Section 4 concludes.

2 Methodology

In this section we review theoretical background of the method used in this paper.

2.1 VaR and sVaR

As we review later, the VaR is calculated directly from the profit or loss distribution of the portfolio. On the other hand, the sVaR is supposed to be the hypothetical VaR calculated for the stressed market conditions. According to Gibart (2012), this is usually done by estimating the profit or loss distributions using the data that include financial crisis periods, typically September to November 2008. In this sense, the VaR calculated using a data set that includes these two months is the sVaR.

In this paper, we use the data with crisis period, and we further try to capture the risk under the stress by using the scenarios on risk factors.

2.2 VaR under scenario

Let R be the random variable which denotes the return on the portfolio, and $\boldsymbol{X} = (X_1, \ldots, X_p)'$ be a random vector of the risk factors, such as the stock index return or the interest rate change.

Let $f(r, \boldsymbol{x})$, $r \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^p$ be the joint density of R and $\boldsymbol{X}$, and $f_{\boldsymbol{X}}(\boldsymbol{x})$ be the marginal density of $\boldsymbol{X}$. Then, the density of R under the scenario $\boldsymbol{X} = \boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{R}^p$, is given as the conditional density;

$$f_R(r|\boldsymbol{X} = \boldsymbol{x}) = \frac{f(r, \boldsymbol{x})}{f_{\boldsymbol{X}}(\boldsymbol{x})}.$$

Hence, the lower α-quantile under the scenario is given as r_α which satisfies

$$F_R(r_\alpha|\boldsymbol{X} = \boldsymbol{x}) = \int_{-\infty}^{r_\alpha} f_R(r|\boldsymbol{X} = \boldsymbol{x})\mathrm{d}r = 1 - \alpha.$$

Then the $100\alpha\%$-VaR of the portfolio is calculated as $-S_0(\mathrm{e}^{r_\alpha} - 1) \simeq -S_0 r_\alpha$, where S_0 is the present value of the portfolio.

2.3 Methods for deriving conditional density

In the procedure outlined above, the key step is the derivation of the conditional density. Here, we consider the following two methods; the use of linear model and the use of Hermite expansion.

The use of linear model is one of the simplest way. It essentially takes only first and second moments into accounts, while the Hermite expansion uses the higher order moments and is capable of capturing non-linear structures such as the correlation collapse and volatility clustering which are often observed in the markets under stress.

Use of linear model

We assume a linear relation between r and $\boldsymbol{X}$ of the form:

$$R = \beta_0 + \boldsymbol{X}'\boldsymbol{\beta} + \varepsilon,$$

where β_0 and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ are the parameters which can be estimated, for instance, by the OLS, and ε is a random variable which is uncorrelated with $\boldsymbol{X}$. It is often assumed that ε has the Normal distribution with mean 0 and constant variance σ^2.

Under this setting, the conditional distribution of R is simply the normal distribution with mean $\beta_0 + \boldsymbol{x}'\boldsymbol{\beta}$ and variance σ^2. Thus, there is no technical challenge in calculating VaR under scenarios.

Use of Hermite expansion

We consider the application of the method introduced by Marumo and Wolff (2013, 2016).

Let us consider smoothing the empirical distribution function given the historical observations $(R(i), \boldsymbol{X}'(i))'$, $i = 1, \ldots, N$, where i is the time step. We are aware of the possible existence of the serial dependence structures; however, here we work on the unconditional distribution. This can be justified by the popularity of the historical simulation (HS) method, which uses the unconditional empirical distribution, among the large majority of commercial banks (Pérignon and Smith, 2010). We deal with the serial dependence structure later in the numerical examples.

According to Marumo and Wolff (2013, 2016), the joint density function can be estimated by smoothing the empirical distribution function using the Hermite expansion as

$$\begin{aligned}\hat{f}(r, \boldsymbol{x}) = {} & \phi(r)\phi(x_1)\cdots\phi(x_p) \\ & \times \sum_{k_r + k_1 + \cdots + k_p \leq n} c_{k_r, k_1, \ldots, k_p} \mathrm{He}_{k_r}(r)\mathrm{He}_{k_1}(x_1)\cdots\mathrm{He}_{k_p}(x_p),\end{aligned} \quad (1)$$

where $\phi(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$ is the density function of the standard Normal distribution,

$$\mathrm{He}_k(x) = \frac{1}{\sqrt{k!}} \frac{1}{\phi(x)} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \phi(x)$$

are the modified Hermite polynomials, and $c_{k_r,k_1,\ldots,k_p}$ are the real coefficients given by

$$\begin{aligned}
c_{k_r,k_1,\ldots,k_p} &= \frac{1}{1+s\{k_r(k_r+1)+\sum_{u=1}^{p} k_u(k_u+1)\}} \frac{\tilde{c}^{2+}_{k_r,k_1,\ldots,k_p}}{\hat{c}_{k_r,k_1,\ldots,k_p}}, \qquad (2)\\
\hat{c}_{k_r,k_1,\ldots,k_p} &= \frac{1}{N}\sum_{i=1}^{N} \mathrm{He}_{k_r}(R(i))\mathrm{He}_{k_1}(X_1(i))\cdots\mathrm{He}_{k_p}(X_p(i)),\\
\hat{b}^2_{k_r,k_1,\ldots,k_p} &= \frac{1}{N}\sum_{i=1}^{N} \mathrm{He}^2_{k_r}(R(i))\mathrm{He}^2_{k_1}(X_1(i))\cdots\mathrm{He}^2_{k_p}(X_p(i)),\\
\tilde{c}^{2+}_{k_r,k_1,\ldots,k_p} &= \max((N\hat{c}^2_{k_r,k_1,\ldots,k_p} - \hat{b}^2_{k_r,k_1,\ldots,k_p})/(N-1), 0),
\end{aligned}$$

where $0 < s \leq \infty$ is the parameter for smoothness and $n \geq 0$ is the degree of expansion. If $\hat{c}_{k_r,k_1,\ldots,k_p} = 0$ then $c_{k_r,k_1,\ldots,k_p}$ can be defined as 0. See Appendix for conversion properties.

In practice, we can standardise the variables so that the sample means equal to 0, sample variances to 1, and sample correlation coefficients to 0, before applying the Hermite expansion, in order to obtain better approximation quality. See Marumo and Wolff (2013, 2016). It has also been shown by Marumo and Wolff (2016) that the density in Equation (1) is convergent for $s \neq 0$, and that the convergence is slower with smaller s.

The marginal density $\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})$ can be estimated similarly, and hence the density under the scenario $\boldsymbol{X} = \boldsymbol{x}$ is given by

$$\hat{f}_R(r|\boldsymbol{X} = \boldsymbol{x}) = \frac{\hat{f}(r, \boldsymbol{x})}{\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})}.$$

2.4 Case with scenario on one risk factor

For illustration, we discuss the case with scenario on one risk factor. This is the simplest case where we deal with the joint distribution of (R, X_1) and consider the conditional distribution of R under the scenario $X_1 = x_1$.

For simplicity, we hereafter denote the risk factor by X instead of X_1.

Use of linear model

The portfolio return under the scenario $X = x$ can be expressed as

$$R = \beta_0 + \beta_1 x + \varepsilon,$$

a simple regression model. We can estimate β_0, β_1 and $\sigma^2 = V(\varepsilon)$, for instance, by the OLS.

The distribution of R is given by $\mathrm{N}(\hat{\beta}_0 + \hat{\beta}_1 x, \hat{\sigma}^2)$, where the symbols with $\hat{}$ are the estimators. Here, the information added to the unconditional VaR is the linear correlation coefficient between the portfolio return and risk factor.

Use of Hermite expansion

Suppose that the historical observations $\{R(i)\}$ and $\{X(i)\}$, are standardised so that the sample means and variances are 0 and 1, respectively. Let $\hat{\rho}$ be the sample correlation coefficient between $\{R(i)\}$ and $\{X(i)\}$. Then

$$Z(i) = \frac{R(i) - \hat{\rho}X(i)}{\sqrt{1-\hat{\rho}^2}}, \ i = 1, \dots, N, \tag{3}$$

are uncorrelated with $\{X(i)\}$.

By applying the Hermite expansion, we can estimate the joint density of (Z, X) by

$$\hat{f}_Z(z, x) = \phi(z)\phi(x) \sum_{k+l \le n} c_{k,l} \mathrm{He}_k(z)\mathrm{He}_l(x),$$

where $c_{k,l}$ are given by

$$\begin{aligned}
c_{k,l} &= \frac{1}{1 + s\{k(k+1) + l(l+1)\}} \frac{\tilde{c}^{2+}_{k,l}}{\hat{c}_{k,l}} \\
\hat{c}_{k,l} &= \frac{1}{N} \sum_{i=1}^{N} \mathrm{He}_k(Z(i))\mathrm{He}_l(X(i)) \\
\hat{b}^2_{k,l} &= \frac{1}{N} \sum_{i=1}^{N} \mathrm{He}^2_k(Z(i))\mathrm{He}^2_l(X(i)) \\
\tilde{c}^{2+}_{k,l} &= \max((N\hat{c}^2_{k,l} - \hat{b}^2_{k,l})/(N-1), 0),
\end{aligned}$$

for $\hat{c}_{k,l} \neq 0$, and $c_{k,l} = 0$, otherwise.

The joint density of (R, X) is given by

$$\hat{f}(r, x) = \frac{1}{\sqrt{1-\hat{\rho}^2}} \hat{f}_Z\left(\frac{r - \hat{\rho}x}{\sqrt{1-\hat{\rho}^2}}, x\right)$$

and the marginal density function of X, by

$$\hat{f}_X(x) = \phi(x) \sum_{l=0}^{n} c_{0,l} \mathrm{He}_l(x).$$

Hence, the conditional density function of R is given by

$$\begin{aligned}
\hat{f}_R(r|X = x) &= \frac{\hat{f}(r, x)}{\hat{f}_X(x)} \\
&= \frac{1}{\sqrt{1-\hat{\rho}^2}} \phi\left(\frac{r - \hat{\rho}x}{\sqrt{1-\hat{\rho}^2}}\right) \sum_{k=0}^{n} c_k(x) \mathrm{He}_k\left(\frac{r - \hat{\rho}x}{\sqrt{1-\hat{\rho}^2}}\right),
\end{aligned}$$

where

$$c_k(x) = \frac{\sum_{l=0}^{n-k} c_{k,l}\mathrm{He}_l(x)}{\sum_{l=0}^{n} c_{0,l}\mathrm{He}_l(x)}.$$

Using the identity

$$\int_{-\infty}^{t} \phi(u)\mathrm{He}_k(u)\mathrm{d}u = \frac{1}{\sqrt{k}}\phi(t)\mathrm{He}_{k-1}(t),$$

for $k = 1, 2, \ldots$, and

$$\int_{-\infty}^{t} \phi(u)\mathrm{He}_0(x)\mathrm{d}u = \int_{-\infty}^{t} \phi(u)\mathrm{d}u = \Phi(t),$$

the conditional distribution function is calculated as

$$\hat{F}_R(r|X = x) = \Phi\left(\frac{r - \hat{\rho}x}{\sqrt{1 - \hat{\rho}^2}}\right) + \phi\left(\frac{r - \hat{\rho}x}{\sqrt{1 - \hat{\rho}^2}}\right)\sum_{k=1}^{n}\frac{c_k(x)}{\sqrt{k}}\mathrm{He}_{k-1}\left(\frac{r - \hat{\rho}x}{\sqrt{1 - \hat{\rho}^2}}\right).$$

The lower α-quantile under the condition $X = x$ can be found by solving $\hat{F}_R(r|X = x) = 1 - \alpha$ for r.

3 Numerical Examples

3.1 Data and parameters

As an example, we consider measuring the risk of the US Sovereign Bond Portfolio, one of the Japanese investment trusts managed by Shinkin Asset Management Co., Ltd., and use its daily reference price series. This fund invests in the US sovereign bonds, and is yen-denominated. We thus expect that it is affected by the US financial markets as well as foreign exchange markets [1].

The observation period is from 1 August 2008 to 30 July 2010, which includes the financial crisis in September to November 2008. In this sense, the VaR calculated using the data from this observation period can be considered as sVaR (see Section 2.1). The total number of observations is $N = 484$.

As for the risk factors, we consider the interest rate (log-difference of the US ten years treasury constant maturity rate), foreign exchange rate (log-difference of the USD/JPY exchange rate), and stock index (log-return on S&P 500). See Table 1.

We consider the scenarios under which the risk factor takes the value in the range $\pm$ three times the volatility (standard deviation of the log-difference), and observe how the conditional quantiles are changed.

We use the smoothness parameter $s = .4$, which is large enough for the approximation to be stable within the set of scenarios. As for the degree of expansion, we set $n = 100$.

[1] See `http://www.skam.co.jp/fund/detail/id=327` for the detailed description and source data. Since the investment trust is dynamically managed, investing in this trust is not equivalent to investing in the US sovereign bond markets directly.

	Mean ($\times 10^{-4}$)	Std. Dev. ($\times 10^{-2}$)	Skewness ($\times 10^{-1}$)	Kurtosis	Cor. Coef.
Portfolio Return	-2.953	0.777	4.011	7.366	(1.000)
10Y TB	-9.895	2.860	-5.900	6.791	0.084
USD/JPY	-5.121	0.895	-7.640	7.360	0.543
SP500	-7.150	2.112	-4.596	7.113	0.391

Table 1: Summary statistics of the portfolio return and log-differences of the risk factors. The Cor. Coef. column shows the sample correlation coefficient between the portfolio return and the log-difference of the risk factor.

3.2 Conditional VaR

As reviewed in Section 2.2, VaR can be approximated by $-S_0 r_\alpha$, where r_α is the α-quantile of the portfolio return. In this Section, we exhibit the results in terms of quantiles scaled by the volatility; that is, we have

$$\text{VaR per currency unit of portfolio} = -\text{scaled quantile} \times 0.777 \times 10^{-2},$$

where 0.777×10^{-2} is the volatility of the of the portfolio (see Table 1).

VaR under scenario on risk factor

Tables 2 to 4 and Figures 1 to 3 show the conditional quantiles of the portfolio return for the scenarios.

.

From these Tables and Figures, we find that the conditional quantiles by the Hermite expansion and the those by linear model agree within $\pm$ one volatility change in the risk factor, while the quantiles by the Hermite expansion are more conservative in the tail around two to three times the volatility. This is consistent with the rule of thumb which claims that the correlation can collapse in the tail events.

We also observe that the conditional quantiles are more conservative than unconditional ones at around minus three times the volatility in all three cases. This suggests that the unconditional VaR may not be conservative enough in the stressed market conditions.

X	-3	-2	-1	0	1	2	3
Lower 99%tile							
Hermite	-2.893	-2.628	-2.452	-2.304	-2.238	-2.549	-2.724
Linear	-2.573	-2.489	-2.405	-2.321	-2.237	-2.152	-2.068
Uncond. (HS)				-2.725			
Uncond. (Gaussian)				-2.326			
Lower 97.5%tile							
Hermite	-2.446	-2.207	-2.048	-1.912	-1.858	-2.037	-2.363
Linear	-2.207	-2.123	-2.039	-1.955	-1.871	-1.787	-1.703
Uncond. (HS)				-1.952			
Uncond. (Gaussian)				-1.960			

Table 2: Scenarios on the change in the US ten years treasury constant maturity rate and conditional quantiles of the portfolio. X corresponds to the value of the risk factor, scaled by the standard deviation. For instance, the column with $X = -3$ corresponds to the quantile of the portfolio return under the condition that the risk factor is dropped by three times its volatility (the standard deviation shown in Table 1). The unconditional quantiles calculated by the HS method and by Gaussian approximation are also shown. See Figure 1.

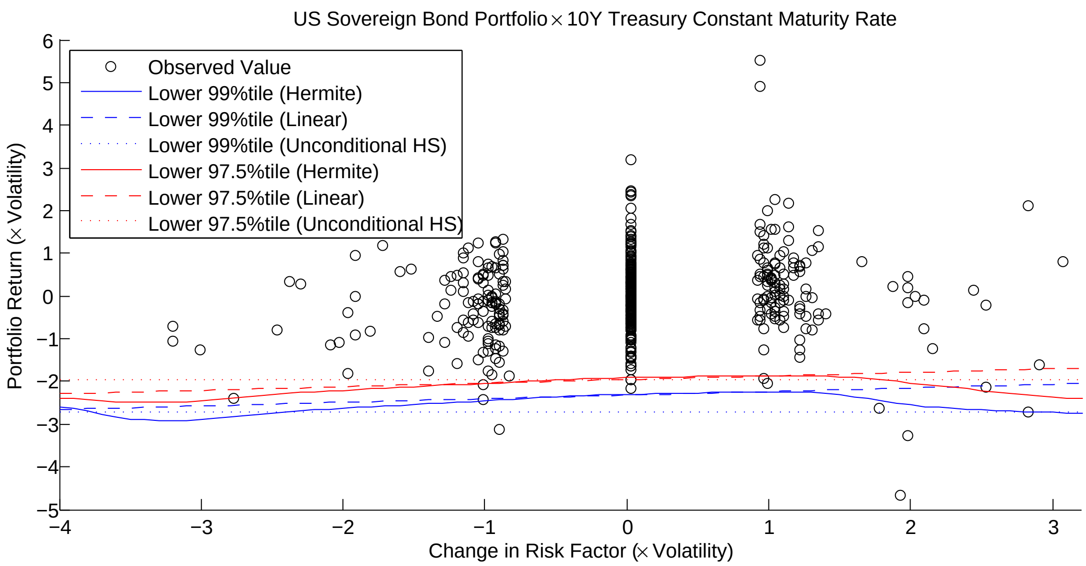

Figure 1: Change in the US ten years treasury constant maturity rate and the portfolio return. The axes are scaled by the corresponding volatilities. The horizontal axis corresponds to the scenario on the risk factor.

X	-3	-2	-1	0	1	2	3
Lower 99%tile							
Hermite	-3.736	-3.536	-2.347	-2.023	-1.389	-1.521	-1.175
Linear	-3.584	-3.042	-2.499	-1.956	-1.413	-0.870	-0.327
Uncond. (HS)				-2.725			
Uncond. (Gaussian)				-2.326			
Lower 97.5%tile							
Hermite	-3.586	-2.972	-2.076	-1.691	-1.045	-0.758	-0.902
Linear	-3.276	-2.733	-2.191	-1.648	-1.105	-0.562	-0.019
Uncond. (HS)				-1.952			
Uncond. (Gaussian)				-1.960			

Table 3: Scenarios on the change in USD/JPY exchange rate and conditional quantiles of the portfolio. X corresponds to the value of the risk factor, scaled by the standard deviation. For instance, the column with $X = -3$ corresponds to the quantile of the portfolio return under the condition that the risk factor is dropped by three times its volatility (the standard deviation shown in Table 1). The unconditional quantiles calculated by the HS method and by Gaussian approximation are also shown. See Figure 2.

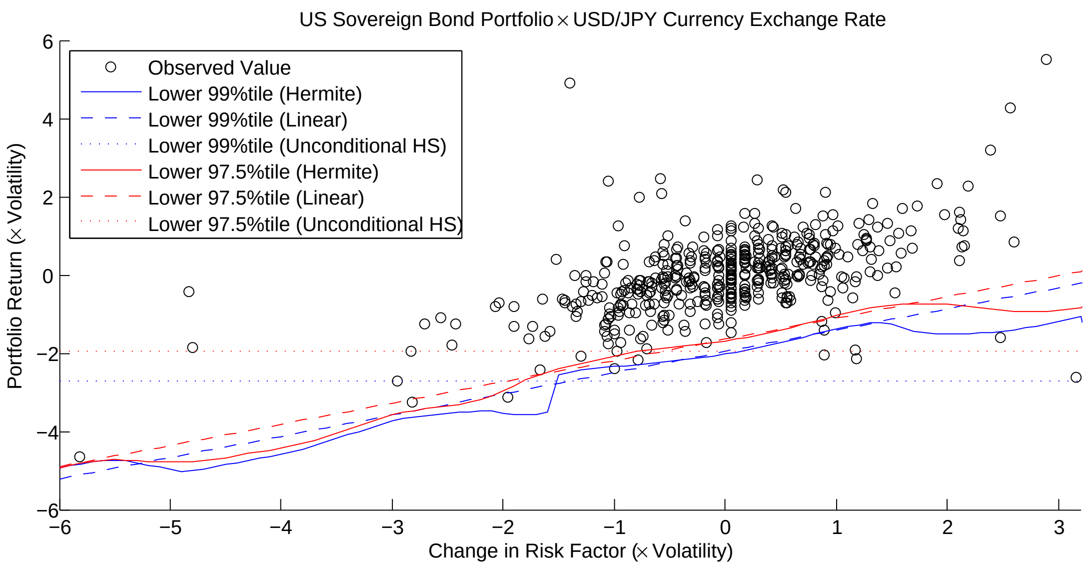

Figure 2: Change in the USD/JPY exchange rate and the portfolio return. The axes are scaled by the corresponding volatilities. The horizontal axis corresponds to the scenario on the risk factor.

X	-3	-2	-1	0	1	2	3
Lower 99%tile							
Hermite	-3.387	-3.075	-2.614	-2.126	-1.737	-1.721	-1.520
Linear	-3.316	-2.925	-2.534	-2.144	-1.753	-1.362	-0.971
Uncond. (HS)				-2.725			
Uncond. (Gaussian)				-2.326			
Lower 97.5%tile							
Hermite	-3.164	-2.693	-2.231	-1.770	-1.374	-1.256	-1.197
Linear	-2.978	-2.588	-2.197	-1.806	-1.415	-1.024	-0.634
Uncond. (HS)				-1.952			
Uncond. (Gaussian)				-1.960			

Table 4: Scenarios on the return on S&P 500 Index and conditional quantiles of the portfolio. X corresponds to the value of the risk factor, scaled by the standard deviation. For instance, the column with $X = -3$ corresponds to the quantile of the portfolio return under the condition that the risk factor is dropped by three times its volatility (the standard deviation shown in Table 1). The unconditional quantiles calculated by the HS method and by Gaussian approximation are also shown. See Figure 3.

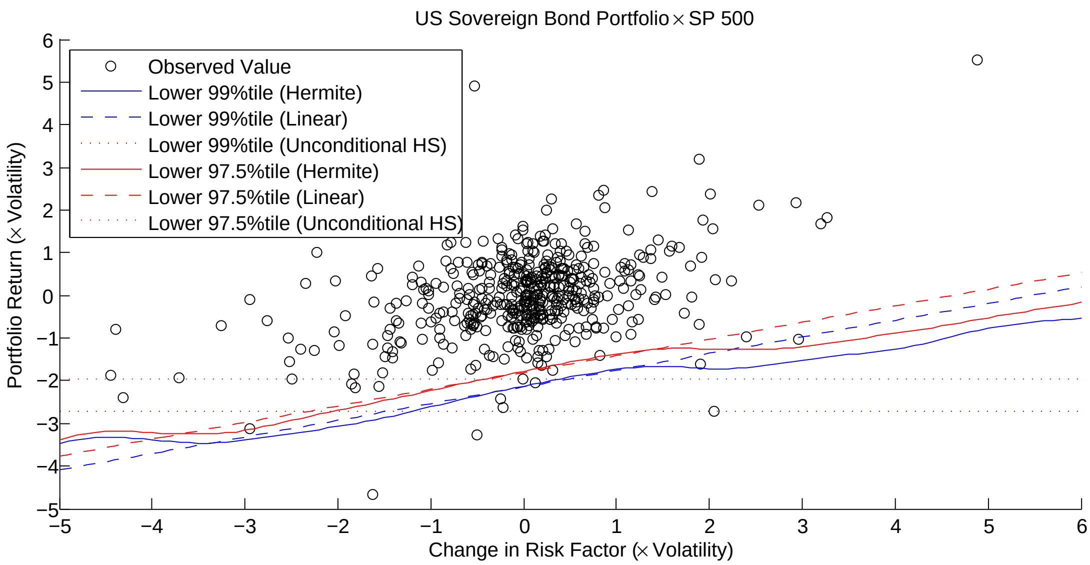

Figure 3: Return on S&P 500 Index and the portfolio return. The axes are scaled by the corresponding volatilities. The horizontal axis corresponds to the scenario on the risk factor.

Scenario on previous day change

Volatility clustering is frequently observed in the financial markets. Loosely speaking, volatility clustering claims that large changes are likely to be followed by large changes, regardless of the directions. Thus, we expect that the conditioning on the previous day return can alter the distribution of next day return. We investigate such non-linear dependence structure using the same data. The auto-covariance for the observed period is -0.1370.

Table 5 and Figure 4 show the quantiles conditioned on the previous day return. We observe that the quantiles calculated by the Hermite expansion are more conservative than those by the linear model. This is consistent with volatility clustering frequently observed in the markets.

X	-3	-2	-1	0	1	2	3
Lower 99%tile							
Hermite	-2.140	-2.416	-2.326	-2.268	-2.348	-2.636	-2.833
Linear	-1.896	-2.033	-2.170	-2.307	-2.444	-2.581	-2.718
Uncond. (HS)				-2.725			
Uncond. (Gaussian)				-2.326			
Lower 97.5%tile							
Hermite	-1.891	-2.002	-1.894	-1.885	-1.977	-2.281	-2.542
Linear	-1.532	-1.669	-1.807	-1.944	-2.081	-2.218	-2.355
Uncond. (HS)				-1.952			
Uncond. (Gaussian)				-1.960			

Table 5: Scenarios on the previous day's return and conditional quantiles of the portfolio. X corresponds to the value of the risk factor, scaled by the standard deviation. For instance, the column with $X = -3$ corresponds to the quantile of the portfolio return under the condition that the risk factor is dropped by three times its volatility (the standard deviation shown in Table 1). The unconditional quantiles calculated by the HS method and by Gaussian approximation are also shown. See Figure 4.

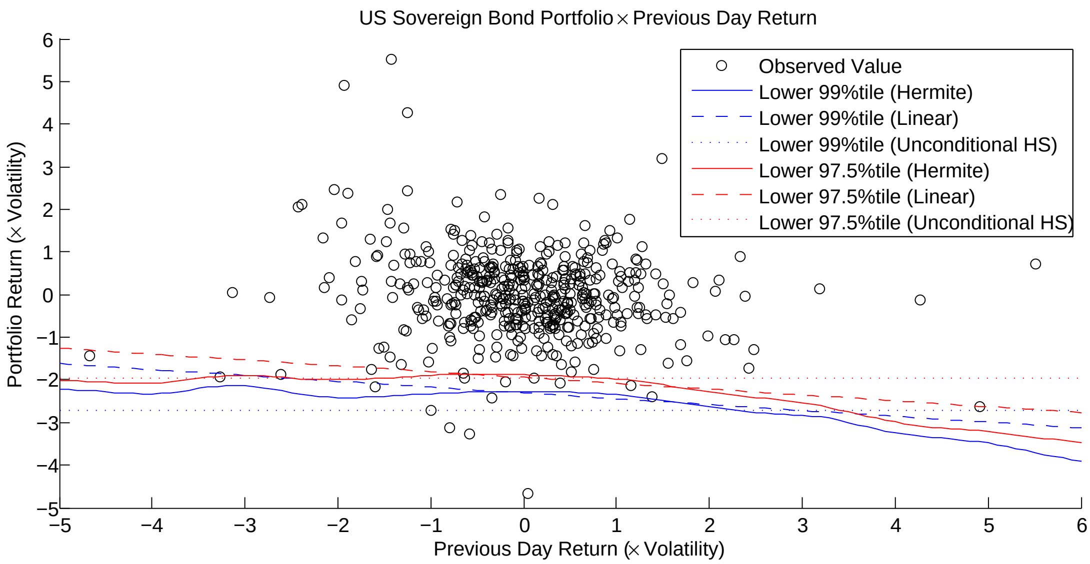

Figure 4: Previous day change and the portfolio return. The axes are scaled by the corresponding volatilities. The horizontal axis corresponds to the scenario on the previous day return.

4 Conclusion

We considered the application of the Hermite expansion to the calculation of the conditional VaRs, or equivalently for this case, conditional sVaRs, and compared it with those by the linear model. The numerical examples demonstrated that the sVaRs by two methods agreed with each other at the body of the distribution, while the sVaR by the Hermite expansion was more conservative than that by the linear model in the tails. This suggests that the Hermite expansion is capable of capturing the correlation collapse, which is often observed under the stressed market conditions.

We also applied the methods to the sVaR with conditions on the previous day return of the portfolio, and investigated how these methods capture the serial dependence structure. It was observed that the sVaR by the Hermite expansion was more conservative than that by the linear model. This suggests that the Hermite expansion is capable of capturing the volatility clustering which refers to the phenomenon observed in the market that large changes are likely to be followed by large changes.

The sVaRs by the Hermite expansion under the condition that the risk factor is around three times its volatility were more conservative than unconditional ones. This suggests that the unconditional sVaR may not be conservative enough under the stressed conditions.

By construction, the sVaR by the Hermite expansion depends on the parameters, the smoothness weight s in Equation (2), and the degree of expansion n in Equation (1). The convergence property has been discussed in Marumo and Wolff (2016), and it has been shown that the density function is uniformly convergent for $s \neq 0$ as $n \to \infty$ (see Appendix). Hence, we might choose reasonably large number. With regard to the smoothness weight, however, the criteria for choosing an appropriate value has not been proposed. This is our future work.

References

Basel Committee on Banking Supervision. Fundamental review of the trading book : A revised market risk framework. Consultative Document, Bank for International Settlements, Oct. 2013.

European Banking Authority. EBA guidelines on stressed value at risk (stressed VaR). Technical report, European Banking Authority, 2012.

P. Gibart. Stressed VaR. Technical report, Crédit Agricole, 2012.

K. Hong. Analytical method of computing stressed value-at-risk with conditional value-at-risk. *Journal of Risk*, 19(3):85–106, 2017.

K. Marumo and R. C. Wolff. A non-parametric method for approximating joint densities and copula functions for financial markets. *Saitama Univeristy Working Paper*, (4), 2013.

K. Marumo and R. C. Wolff. On optimal smoothing of density estimators obtained from orthogonal polynomial expansion methods. *Journal of Risk*, 18(3), 2016.

C. Pérignon and D. R. Smith. The level and quality of value-at-risk disclosure by commercial banks. *Journal of Banking and Finance*, 34:362–377, 2010.

A Convergence of Hermite expansion

We outline the proof of the convergence of the Hermite expansion in Equation (1). Although we use the bivariate case here, the general multivariate case can be shown similarly.

A.1 Hermite expansion of unit step function

It has been shown by Marumo and Wolff (2016) that the unit step function has the convergent Hermite expansion of the form

$$\mathbf{1}_{\{X \le x\}} = \Phi(x) + \phi(x) \sum_{k=1}^{\infty} \frac{\mathrm{He}_k(X)}{\sqrt{k}} \mathrm{He}_{k-1}(x), \text{ a. e. } x.$$

Since the bivariate unit step function is the product of two univariate ones,

$$\mathbf{1}_{\{X \le x, Y \le y\}} = \mathbf{1}_{\{X \le x\}} \mathbf{1}_{\{Y \le y\}},$$

it has the convergent Hermite expansion of the form

$$\begin{aligned} \mathbf{1}_{\{X \le x, Y \le y\}} &= \Phi(x)\Phi(y) \\ &+ \Phi(x)\phi(y) \sum_{l=1}^{\infty} \frac{\mathrm{He}_l(Y)}{\sqrt{l}} \mathrm{He}_{l-1}(y) + \phi(x)\Phi(y) \sum_{k=1}^{\infty} \frac{\mathrm{He}_k(X)}{\sqrt{k}} \mathrm{He}_{k-1}(x) \\ &+ \phi(x)\phi(y) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\mathrm{He}_k(X)\mathrm{He}_l(Y)}{\sqrt{kl}} \mathrm{He}_{k-1}(x)\mathrm{He}_{l-1}(y), \text{ a. e. } x, y. \end{aligned}$$

A.2 Hermite expansion of empirical distribution function

Given the data set $\{(X(1), Y(1)), \dots, (X(N), Y(N))\}$, the (joint) empirical distribution function can be written as $\hat{F}(x, y) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{X(i) \le x, Y(i) \le y\}}$. Using the result from the previous Section, the Hermite expansion of $\hat{F}$ can be expressed as

$$\begin{aligned} \hat{F}(x, y) &= \Phi(x)\Phi(y) \\ &+ \Phi(x)\phi(y) \sum_{l=1}^{\infty} \frac{\hat{c}_{0,l}}{\sqrt{l}} \mathrm{He}_{l-1}(y) + \phi(x)\Phi(y) \sum_{k=1}^{\infty} \frac{\hat{c}_{k,0}}{\sqrt{k}} \mathrm{He}_{k-1}(x) \\ &+ \phi(x)\phi(y) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\hat{c}_{k,l}}{\sqrt{kl}} \mathrm{He}_{k-1}(x)\mathrm{He}_{l-1}(y), \text{ a. e. } x, y, \end{aligned}$$

where

$$\hat{c}_{k,l} = \frac{1}{N} \sum_{i=1}^{N} \mathrm{He}_k(X(i))\mathrm{He}_l(Y(i)).$$

This can be written as

$$\begin{aligned}\hat{F}(x,y) &= \Phi(x)\hat{F}_Y(y) + \hat{F}_X(x)\Phi(y) - \Phi(x)\Phi(y) \\ &\quad + \phi(x)\phi(y)\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\hat{c}_{k,l}}{\sqrt{kl}}\mathrm{He}_{k-1}(x)\mathrm{He}_{l-1}(y), \text{a. e. } x, y,\end{aligned}$$

where $\hat{F}_X$ and $\hat{F}_Y$ are the empirical marginal distribution functions of X and Y, respectively. Thus, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\{\hat{F}(x,y) - \Phi(x)\hat{F}_Y(y) - \hat{F}_X(x)\Phi(y) + \Phi(x)\Phi(y)\}^2}{\phi(x)\phi(y)}\mathrm{d}x\mathrm{d}y \\ &\quad = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\hat{c}_{k,l}^2}{kl}.\end{aligned}$$

We show that the integral in the left hand side is bounded. Let a be a large enough constant. Split the integral by all combinations of $x \le -a, -a < x \le a, a < x$, and $y \le -a, -a < y \le a, a < y$. For $x \le -a$, we have $\hat{F}(x,y) = \hat{F}_X(x) = 0$, so the integral is

$$\begin{aligned}I_1 &= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{-a}\frac{\{-\Phi(x)\hat{F}_Y(y) + \Phi(x)\Phi(y)\}^2}{\phi(x)\phi(y)}\mathrm{d}x\mathrm{d}y \\ &= \int_{x=-\infty}^{-a}\frac{\Phi^2(x)}{\phi(x)}\mathrm{d}x\int_{y=-\infty}^{\infty}\frac{\{\hat{F}_Y(y) - \Phi(y)\}^2}{\phi(y)}\mathrm{d}y.\end{aligned}$$

Since $\Phi(x) - \phi(x)$ is decreasing for $x \le -1$ with $\lim_{x\to-\infty}\{Phi(x) - \phi(x)\} = 0$, we have $0 < \Phi(x) < \phi(x)$ for $x \le -1$. Therefore $\int_{-\infty}^{-a}\frac{\Phi^2}{\phi(x)}\mathrm{d}x < \int_{-\infty}^{-a}\phi(x)\mathrm{d}x = \Phi(-a) < \infty$. Hence we have

$$I_1 < \Phi(-a)\int_{y=-\infty}^{\infty}\frac{\{\hat{F}_Y(y) - \Phi(y)\}^2}{\phi(y)}\mathrm{d}y.$$

Now, split the integral with respect to y. For $y \le -a$, similarly to the case with $x \le -a$, we have $\int_{-\infty}^{-a}\{\hat{F}_Y(y) - \Phi(y)\}^2/\phi(y)\mathrm{d}y$ is bounded. By symmetry the integral for $a < y$ is also bounded. It is trivial to show that the integral for $-a < y \le a$ is bounded. Hence, I_1 is bounded. By symmetry, the integral for $a < x$ is also bounded. The case with $-a < x \le a$ can be shown similarly.

Thus, it has been shown that $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\hat{c}_{k,l}^2/(kl)$ is bounded.

A.3 Convergence of Hermite expansion

As discussed in Marumo and Wolff (2013), the smoothed joint density function is given by the form

$$f^S(x,y) = \phi(x)\phi(y)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}c_{k,l}^S\mathrm{He}_k(x)\mathrm{He}_l(y),$$

where

$$c_{k,l}^S = \frac{\hat{c}_{k,l}}{1 + s\{k(k+1) + l(l+1)\}}.$$

From the fact that $(c_{k,l}^S)^2 < \hat{c}_{k,l}^2/(kl)$ for large k and l, and the result from the previous Section, we have that $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(c_{k,l}^S)^2$ is bounded, which implies that the infinite sum on the right hand side is convergent.

As for the expansion used in this paper, the coefficient is

$$\frac{1}{1 + s\{1 + k(k+1) + l(l+1)\}}\frac{\tilde{c}_{k,l}^{2+}}{\hat{c}_{k,l}},$$

which is smaller than $\frac{N}{N-1}c_{k,l}^S$. This suggests that the Hermite expansion used in this paper is also convergent.