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# Mapping theorems for $C_f$-spaces

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**Abstract**

In this paper we study dimension-raising mapping theorems for $C_f$-spaces.

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## 1 Introduction

In this paper we assume that all spaces are normal unless otherwise stated. We refer the readers to [2] for dimension theory.

If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of a space $X$, then $\mathcal{A} < \mathcal{B}$ means that $\mathcal{A}$ is a refinement of $\mathcal{B}$, i.e. for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$. Notice that $\mathcal{A}$ need not be a cover even if $\mathcal{B}$ is a cover. For a collection $\mathcal{A}$ of subsets of a space $X$ and for $Y \subset X$ we write $\mathcal{A}|Y$ for $\{A \cap Y : A \in \mathcal{A}\}$, $\bigcup \mathcal{A}$ for $\bigcup\{A : A \in \mathcal{A}\}$ and $[\mathcal{A}]^{<\omega}$ for $\{\mathcal{B} : \mathcal{B}$ is a finite subcollection of $\mathcal{A}\}$.

Haver [3] introduced the notion of $C$-spaces for the class of metric spaces. Addis and Gresham [1] extended this notion to normal spaces. A space $X$ is a *$C$-space* if for every countable collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of open covers of $X$ there exists a countable collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$ of collections of pairwise disjoint open subsets of $X$ such that $\mathcal{H}_i < \mathcal{G}_i$ for every $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ covers $X$. Notice that not all finite-dimensional spaces are $C$-spaces (See [1]). In [4] we introduced the notion of the class of $C_f$-spaces which contains all finite-dimensional spaces. A space $X$ is a *$C_f$-space* if for every countable collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of finite open covers of $X$ there exists a countable collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$ of pairwise disjoint collections of open subsets of $X$ such that $\mathcal{H}_i < \mathcal{G}_i$ for every $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ covers $X$. We call $\{\mathcal{H}_i : i \in \mathbb{N}\}$ *$C_f$-refinement* of $\{\mathcal{G}_i : i \in \mathbb{N}\}$. In particular if all $\mathcal{H}_i$ are discrete, then we call $\{\mathcal{H}_i : i \in \mathbb{N}\}$ a *discrete $C_f$-refinement* of $\{\mathcal{G}_i : i \in \mathbb{N}\}$.

It is easily seen that every $C$-space is a $C_f$-space and every $C_f$-space is $A$-weakly infinite-dimensional. Polkowski [5] proved the following theorems.

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**Theorem** [5].

(i) *If* $f: X \to Y$ *is a closed mapping of an A-weakly infinite-dimensional countably paracompact space X onto a space Y and there exists an integer* $k \geq 1$ *such that* $|f^{-1}(y)| \leq k$ *for every* $y \in Y$, *then Y is A-weakly infinite-dimensional.*

(ii) *If* $f: X \to Y$ *is an open mapping of an A-weakly infinite-dimensional space X onto a countably paracompact space Y such that* $|f^{-1}(y)| < \aleph_0$ *for every* $y \in Y$, *then Y is A-weakly infinite-dimensional.*

## 2 Main theorems

A collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of finite open covers of a space $X$ is called *inessential* if there exists a discrete $C_f$-refinement of $\{\mathcal{G}_i : i \in \mathbb{N}\}$ ; if $\{\mathcal{G}_i : i \in \mathbb{N}\}$ is not inessential then it is called *essential*.

The following lemma will play an important role in the proof of our main theorem.

**2.1. Lemma.** *If* $f: X \to Y$ *is a closed mapping of a countably paracompact* $C_f$*-space X onto a countably paracompact (resp. hereditarily normal) non-*$C_f$*-space Y, then there exist a closed non-*$C_f$*-space* $Y_0$ *and disjoint closed subsets* $X_0$, $X_1$ *of X such that* $f(X_0)=f(X_1)=Y_0$.

To prove Lemma 2.1, we need the following Lemmas 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7.

**2.2. Lemma** ([4], Lemma 4.2). *Let X be a countably paracompact* $C_f$*-space. Then for every collection* $\{\mathcal{G}_i : i \in \mathbb{N}\}$ *of finite open covers of X, where* $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, *there exists a discrete* $C_f$*-refinement* $\{\mathcal{H}_i : i \in \mathbb{N}\}$ *of* $\{\mathcal{G}_i : i \in \mathbb{N}\}$, *where* $\mathcal{H}_i = \{H_\lambda : \lambda \in \Lambda_i\}$, *such that* $H_\lambda \subset G_\lambda$ *and* $H_\lambda$ *is an* $F_\sigma$*-set.*

**2.3. Lemma** ([4], Lemma 4.3). *Let E be a closed subset of a space X. For every finite discrete collection* $\mathcal{U}$ *of open subsets of E there exists a discrete collection* $\mathcal{V}$ *of open subsets of X which satisfies* $\mathcal{V}|E = \mathcal{U}$.

**2.4. Lemma** ([4], Lemma 4.4). *Let E be a closed subset of a hereditarily normal space X. For every finite collection* $\mathcal{U}$ *of pairwise disjoint open subsets of E there exists a finite collection* $\mathcal{V}$ *of pairwise disjoint open subsets of X which satisfies* $\mathcal{V}|E = \mathcal{U}$.

**2.5. Lemma** ([4],Theorem 4.5). *If a space X is either countably paracompact or hereditarily normal, and can be represented as the union of a countable collection of closed* $C_f$*-spaces, then X is a* $C_f$*-space.*

**2.6. Lemma.** *Let* $\{\mathcal{G}_i : i \in \mathbb{N}\}$ *be an essential collection of finite open covers of a countably paracompact space X. For every collection* $\{\mathcal{H}_{2i} : i \in \mathbb{N}\}$ *of pairwise disjoint open subsets of X such that* $\mathcal{H}_{2i} < \mathcal{G}_{2i}$, *the space* $L = X - \bigcup_{i \in \mathbb{N}} \bigcup \mathcal{H}_{2i}$ *is not a* $C_f$*-space.*

**Proof.** Assume that the space $L$ *is a* $\mathcal{C}_f$*-space.* By Lemma 2.2, there exists a finite discrete $\mathcal{C}_f$-refinement $\{\mathcal{U}_{2i+1}: i\in\mathbb{N}\}$ of $\{\mathcal{G}_{2i+1}|L: i\in\mathbb{N}\}$. By Lemma 2.3, there exists a collection $\{\mathcal{H}_{2i+1}: i\in\mathbb{N}\}$ of discrete collections of open subsets of $X$ such that $\mathcal{H}_{2i+1}|L = \mathcal{U}_{2i+1}$ and $\mathcal{H}_{2i+1} < \mathcal{G}_{2i+1}$. We have $\bigcup_{i\in\mathbb{N}}\bigcup\mathcal{H}_{2i+1} \supset \bigcup_{i\in\mathbb{N}}\bigcup\mathcal{H}_{2i+1}|L = \bigcup_{i\in\mathbb{N}}\bigcup\mathcal{U}_{2i+1} = L$. This contradicts that $\{\mathcal{G}_i : i \in \mathbb{N}\}$ is essential, thus the space $L$ is not a $\mathcal{C}_f$-space.

**2.7. Lemma** *Let* $\{\mathcal{G}_i : i \in \mathbb{N}\}$ *be an essential collection of finite open covers of a hereditarily normal space X. For every collection* $\{\mathcal{H}_{2i} : i \in \mathbb{N}\}$ *of pairwise disjoint open subsets of X such that* $H_\lambda \subset G_\lambda$, *the space* $L = X - \bigcup_{i\in\mathbb{N}}\bigcup\mathcal{H}_{2i}$ *is not a* $\mathcal{C}_f$*-space.*

**Proof.** Assume that the space $L$ is a $\mathcal{C}_f$-space. There exists a $\mathcal{C}_f$-refinement $\{\mathcal{U}_{2i+1}: i\in\mathbb{N}\}$ of $\{\mathcal{G}_{2i+1}|L: i\in\mathbb{N}\}$. We may assume that every $\mathcal{U}_{2i+1}$ is finite. By Lemma 2.4, there exists a collection $\{\mathcal{H}_{2i+1}: i\in\mathbb{N}\}$ of finite collections of pairwise disjoint open subsets of $X$ such that $\mathcal{H}_{2i+1}|L = \mathcal{U}_{2i+1}$ and $\mathcal{H}_{2i+1} < \mathcal{G}_{2i+1}$. We have $\bigcup_{i\in\mathbb{N}}\bigcup\mathcal{U}_{2i+1} \supset \bigcup_{i\in\mathbb{N}}\bigcup\mathcal{H}_{2i+1}|L = \bigcup_{i\in\mathbb{N}}\bigcup\mathcal{U}_{2i+1} = L$. This contradicts that $\{\mathcal{G}_i : i \in \mathbb{N}\}$ is essential, thus the space $L$ is not a $\mathcal{C}_f$-space.

**2.8 Proof of Lemma 2.1.** As $Y$ is a not $\mathcal{C}_f$-space, we take an essential collection $\{\mathcal{U}_i : i\in\mathbb{N}\}$ of finite open covers of $Y$, where $\mathcal{U}_i = \{U_\lambda : \lambda\in\Lambda_i\}$. Since $X$ is a countably paracompact $\mathcal{C}_f$-space, by lemma 2.2, there exists a discrete $\mathcal{C}_f$-refinement $\{\{V_\lambda:\lambda\in\Lambda_{2i}\}: i\in\mathbb{N}\}$ of $\{\{f^{-1}(U_\lambda):\lambda\in\Lambda_{2i}\}: i\in\mathbb{N}\}$ such that $V_\lambda \subset f^{-1}(U_\lambda)$ and $V_\lambda$ is a $F_\sigma$-set. Let us set $V_\lambda = \bigcup_{n\in\mathbb{N}} F_{\lambda n}$, where $F_{\lambda n}$ is closed. For every, $\lambda \in \Lambda_{2i}$ we set

$$W_\lambda = Y - f(X - V_\lambda).$$

Then $\mathcal{W}_i = \{W_\lambda : \lambda \in \Lambda_{2i}\}$ is a collection of pairwise disjoint open subsets of $Y$ such that $W_\lambda \subset U_\lambda$. Let us set

$$L = X - \bigcup_{i\in\mathbb{N}}\bigcup\mathcal{W}_i.$$

By Lemma 2.6 (resp. Lemma 2.7), $L$ is a not $\mathcal{C}_f$-space. we have

$$L = L\cap Y = L\cap f(X)$$

$$= L\cap f(\bigcup_{i\in\mathbb{N}}\bigcup_{\lambda\in\Lambda_{2i}} V_\lambda)$$

$$= \bigcup_{i\in\mathbb{N}}\bigcup_{\lambda\in\Lambda_{2i}} L\cap f(V_\lambda)$$

$$= \bigcup_{i\in\mathbb{N}}\bigcup_{\lambda\in\Lambda_{2i}} (L\cap f(\bigcup_{n\in\mathbb{N}} F_{\lambda n}))$$

$$= \bigcup_{i\in\mathbb{N}}\ \bigcup_{\lambda\in\Lambda_{2i}}\ \bigcup_{n\in\mathbb{N}} (L\cap f(F_{\lambda n}))$$

Assume that $L\cap f(F_{\lambda n})$ is a $\mathcal{C}_f$-space for every $i \in \mathbb{N}$, $\lambda \in \Lambda_{2i}$ and $n \in \mathbb{N}$. From Lemma 2.5, $L$ is a $\mathcal{C}_f$-space. This contradicts that $L$ is not a $\mathcal{C}_f$-space. Thus $L\cap f(F_{\lambda n})$ is not a $\mathcal{C}_f$-space for some $i \in \mathbb{N}$, $\lambda \in \Lambda_{2i}$ and $n \in \mathbb{N}$. Let us set

$$Y_0 = L \cap f(F_{\lambda n}),\ X_0 = F_{\lambda n} \cap f^{-1}(L) \text{ and } X_1 = (X - V_\lambda) \cap f^{-1}(L \cap f(F_{\lambda n})).$$

It is easy to show that these subsets $X_0$, $X_1$ and $Y_0$ have all the required properties.

We now come to our main theorems.

**2.9. Theorem** *If $f : X \to Y$ is a closed mapping of a countably paracompact $C_f$-space $X$ onto either countably paracompact or hereditarily normal space $Y$ and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $Y$ is a $C_f$-space.*
**Proof.** Assume that $Y$ is not a $C_f$-space. By applying $k$ times Lemma 2.1, we can find a point $y \in Y$ with $|f^{-1}(y)| \geq k+1$. This is a contradiction.

**2.10. Theorem** *$f : X \to Y$ is an open mapping of a $C_f$-space $X$ onto a countably paracompact space $Y$ such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $Y$ is a $C_f$-space.*
**Proof.** Let $K_j = \{y \in Y : |f^{-1}(y)| = j\}$ for every $j \in \mathbb{N}$. It is easy to see that the union $\bigcup_{j \leq i} K_j$ is closed in $Y$ for every $i \in \mathbb{N}$. Inductively, we show that the union $\bigcup_{j \leq i} K_j$ is a $C_f$-space for every $i \in \mathbb{N}$. To this end, it suffices to show that every closed subspace $Z$ of $Y$ contained in $K_i$ is a $C_f$-space. By [2, Lemma 6.3.12], the restriction $f|_Z : f^{-1}(Z) \to Z$ is perfect. Since the inverse image of a countably paracompact space under a perfect mapping is countably paracompact, by Theorem 2.9, $Z$ is a $C_f$-space. Thus the union $\bigcup_{j \leq i} K_j$ is a closed $C_f$-space for every $i \in \mathbb{N}$. By Lemma 2.5, $Y$ is a $C_f$-space.

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