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# A dimension-lowering mappings theorem for $C_f$-spaces

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**Abstract**

In this paper we prove the following theorem: If $f : X \longrightarrow Y$ is an open mapping of a paracompact space $X$ onto a $C_f$-space $Y$ such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $X$ is a $C_f$-space.

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## 1 Introduction

The present paper is a continuation of [2]. In this paper we assume that all spaces are normal unless otherwise stated. We refer the readers to [1] for dimension theory.

In [3] the second author introduced the notion of $C_f$-spaces, which is a generalization of $C$-spaces. A space $X$ is a *$C_f$-space* if for every countable collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of

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finite open covers of $X$ there exists a countable collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$ of collections of pairwise disjoint open subsets of $X$ such that $\mathcal{H}_i$ is a refinement of $\mathcal{G}_i$ for every $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ covers $X$.

# 2 The main theorem

Polkowski [4] proved the following theorem.

**2.1 Theorem** [4]. *If $f : X \longrightarrow Y$ is an open mapping of a metacompact space $X$ onto an $A$-weakly infinite-dimensional space $Y$ such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $X$ is $A$-weakly infinite-dimensional.*

It is known [1, 6.3.G] that the above theorem remains true if we replace 'metacompact' by 'countably paracompact'.

On the other hand, we [2, Theorem 2.10] proved the following theorem.

**2.2. Theorem** [2] *If $f : X \longrightarrow Y$ is an open mapping of a $C_f$-space $X$ onto a countably paracompact space $Y$ such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $Y$ is a $C_f$-space.*

This is a dimension-raising theorem for open mappings. In this section we shall prove a dimension-lowering theorem for open mappings, which is analogous to the above theorem of Polkowski.

**2.3. Theorem** *If $f : X \longrightarrow Y$ is an open mapping of a paracompact space $X$ onto a $C_f$-space $Y$ such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $X$ is a $C_f$-space.*

To prove our main theorem we need the following lemma. For the sake of completeness we give a proof.

**2.4. Lemma.** *If $f : X \longrightarrow Y$ is an open mapping of a space $X$ to a space $Y$ and there exists an integer $n \geq 1$ such that $|f^{-1}(y)| = n$ for every $y \in Y$, then $f$ is a local homeomorphism.*

**Proof.** For every $x_0 \in X$ let $f^{-1}(f(x_0)) = \{x_0, x_1, \dots, x_{n-1}\}$. Take a collection $\{V_i : 0 \le i \le n-1\}$ of pairwise disjoint open subsets of $X$ with $x_i \in V_i$. Assume that for every neighborhood $U$ of $x_0$ with $U \subset V_0$ the restriction $f|_U : U \longrightarrow Y$ is not injective. We can take distinct points $a_U$ and $b_U$ in $U$ such that $f(a_U) = f(b_U)$. Letting $c_U = f(a_U)$ and $W = \bigcap_{i=1}^{n-1} f(V_i)$, we have $c_U \notin W$. Indeed, assume that $c_U \in W$. Since $c_U \in f(V_i)$ for every $1 \le i \le n-1$, there exists a point $z_i \in V_i$ such that $f(z_i) = c_U$. It is easy to see that the set $\{a_U, b_U, z_1, z_2, \dots, z_{n-1}\}$ consists of exactly $n+1$ points. This is a contradiction, because $|f^{-1}(c_U)| = n$. Thus we have $c_U \notin W$. On the other hand, we have $a_{f^{-1}(W)\cap V_0} \in f^{-1}(W) \cap V_0$, therefore $c_{f^{-1}(W)\cap V_0} = f(a_{f^{-1}(W)\cap V_0}) \in W$. This is a contradiction. Hence there exists a neighborhood $U$ of $x_0$ such that the restriction $f|_U : U \longrightarrow Y$ is injective. Obviously, the restriction $f|_U$ is an embedding. Lemma 2.4 has been proved.

**2.5 Proof of Theorem 2.3.** For every $n \in \mathbb{N}$ we set

$$Y_n = \{y \in Y : |f^{-1}(y)| = n\} \text{ and } X_n = f^{-1}(Y_n).$$

It is easy to see that the union $Y'_n = \bigcup_{k \le n} Y_k$ is closed in $Y$ for every $n \in \mathbb{N}$, therefore the union $X'_n = \bigcup_{k \le n} X_k$ is also closed in $X$. Since $X$ is the union of countable collection $\{X'_n : n \in \mathbb{N}\}$ of closed subsets of $X$, by the countable sum thereom for $C_f$-spaces, we only prove that $X'_n$ is a $C_f$-space for every $n \in \mathbb{N}$. Let $f_n : X_n \longrightarrow Y_n$ be the mapping defined by $f_n(x) = f(x)$ for every $x \in X_n$.

Obviously, $X'_1$ is a $C_f$-space, because $f_1$ is a homeomorphism. Assume that $X'_{n-1}$ is a $C_f$-space. To prove that $X'_n$ is a $C_f$-space, it suffices to show that every closed subset $Z$ of $X'_n$ contained in $X_n$ is a $C_f$-space.

By Lemma 2.4, the mapping $f_n$ is a local homeomorphism. Thus for every $x \in X_n$ we can take a neighborhood $U_x$ of $x$ in $X_n$ such that the restriction $f_n|_{U_x} : U_x \longrightarrow Y_n$ is an embedding. Since $X_n$ is open in $X'_n$, $U_x$ is open in $X'_n$. We may assume that $U_x$ is an $F_\sigma$-set of $X'_n$. Let $U_x = \cup\{A(x,m) : m \in \mathbb{N}\}$, where $A(x,m)$ is closed in $X'_n$. For every $y \in Y_n$ let us set $f^{-1}(y) = \{x(y,1), x(y,2), \ldots, x(y,n)\}$. Then the intersection $\bigcap_{i=1}^{n} f(U_{x(y,i)})$ is a neighborhood of $y$ in $Y'_n$. Take an open $F_\sigma$-set $V_y$ of $y$ in $Y'_n$ such that $y \in V_y \subset \bigcap_{i=1}^{n} f(U_{x(y,i)})$. Let $V_y = \cup\{B(y,\ell) : \ell \in \mathbb{N}\}$, where $B(y,\ell)$ is closed in $Y'_n$. The set $W(y,i) = U_{x(y,i)} \cap f^{-1}(V_y)$ is homeomorphic to $f(W(y,i))$. We have

$$W(y,i) = \bigcup\{A(x(y,i),m) \cap f^{-1}(B(y,\ell)) : m, \ell \in \mathbb{N}\}.$$

We shall prove that $A(x(y,i),m) \cap f^{-1}(B(y,\ell))$ is a $C_f$-space. Since $f_n|_{U_x(y,i)}$ is an embedding, $A(x(y,i),m) \cap f^{-1}(B(y,\ell))$ is homeomorphic to $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$.

By [1, Lemma 6.3.12], $f_n$ is closed, therefore $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$ is closed in $Y_n$. Since

$$f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell))) \subset B(y,\ell) \subset Y_n,$$

$f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$ is closed in $B(y,\ell)$. As $B(y,\ell)$ is a $C_f$-space, $f_n(A(x(y,i),m) \cap f^{-1}(b(y,\ell)))$ is a $C_f$-space. Thus $A(x(y,i),m) \cap f^{-1}(b(y,\ell))$ is a $C_f$-space. By the countable sum theorem for $C_f$-spaces, $W(y,i)$ is a $C_f$-space. Since $Z$ is paracompact, the open cover $\mathcal{W} = \{W(y,i) \cap Z : y \in Y_n, 1 \leq i \leq n\}$ of $Z$ has a locally-finite closed refinement $\mathcal{F}$. Since every member of $\mathcal{F}$ is a $C_f$-space, by the locally finite sum theorem for $C_f$-spaces, $Z$ is a $C_f$-space. Theorem 2.3 has been proved.

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