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A construction of a generalized infinite-dimensional Cantor-manifold

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Abstract

We construct a separable metrizable space T satisfying the following conditions;

(1) T is weakly infinite-dimensional, and

(2) T can not be separated by any hereditarily weakly infinite-dimensional subspace of T.

This is a negative answer to a problem of Krasinkiewicz.

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1 Introduction

In this paper we assume that all spaces are separable and metrizable. A collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of a space X is *essential* in X if there is no partition L_i in X between A_i and B_i such that $\bigcap\{L_i : i < \omega\} = \emptyset$. A space X is said to be *strongly infinite-dimensional* (we abbrev. *s.i.d.*) provided that there exists an essential collection in X. If X is not *s.i.d.*, then we call X *weakly infinite-dimensional* (we abbrev. *w.i.d.*). For a closed subset F of a space X we say that X is *separated* by F (or F is a *separator* of X) if $X - F$ is not connected. A compact space X is *infinite-dimensional Cantor-manifold* if X can not be separated by any w.i.d. subspace of X. The existence of infinite-dimensional Cantor-manifolds is obvious. Indeed, the Hilbert cube is such one. Furthermore, Sklyarenko [5, p.165] proved that every s.i.d. compact space contains an infinite-dimensional Cantor-manifold.

Krasinkiewicz [7] asked whether the existence of another type of like infinite-dimensional Cantor-manifolds. Namely, he asked whether for every w.i.d. space X it is possible to separate X by a hereditarily w.i.d. subspace of X. Here a space X is *hereditarily w.i.d.* if every subspace of X is w.i.d. E. Pol [9] gave a negative answer to this problem. Namely, she cnstructed a w.i.d. compact space which can not be separated by any hereditarily w.i.d. subspace of X. We may regard such a space as a generalized infinite-dimensional Cantor-manifold.

The purpose of this paper is to construct another counterexample to this problem.

2 Construction

In this section we shall construct a w.i.d. space T which can not be separated by any hereditarily w.i.d. subspace of T.

we begin with the following example.

2.1. Example ([10]). *There exists a space S satisfying the following conditions;*

(1) *S is s.i.d.,*

(2) *S is Čech-complete, and*

(3) *S is totally disconnected.*

In this section we denote by S the above space. By (1), there exists an essential collection $\{(A_i, B_i) : i < \omega\}$ in S. The following fact and lemma are easily proved, so we omit the proofs.

2.2. Fact. *Every partition in S between A_0 and B_0 is s.i.d.*

2.3. Lemma. *Let Y be a space and X a subspace of Y. If the following two conditions hold, then Y is w.i.d.*

(1) *$Y - X$ is w.i.d., and*

(2) *for every closed subset F of Y, if $F \subset X$, then F is w.i.d.*

Let Y be a compactification of S and $X = S$. Then, by Example 2.1 (3), the condition (2) in Lemma 2.3 holds. Thus if the remainder $Y - S$ is w.i.d., then so is Y.

Schurle [12] proved that every Čech-complete space has a compactification with strongly countable-dimensional remainder. Engelking and Pol [4] gave a simple proof of Schurle's theorem above. Since the space S in Example 2.1 is Čech-complete, there exists a compactification Y of S such that $Y - S$ is strongly countable-dimensional. Thus, by Lemma 2.3, Y is w.i.d. However, the equality $\mathrm{Cl}_Y A_0 \cap \mathrm{Cl}_Y B_0 = \emptyset$ need not hold. We need a compactification Y of S with this property.

2.4. Lemma. *There exists a compactification Y of S satisfying the following conditions;*

(1) *Y is w.i.d.,*

(2) *$\mathrm{Cl}_Y A_0 \cap \mathrm{Cl}_Y B_0 = \emptyset$, and*

(3) *$Y - S$ is countable-dimensional.*

Proof. The proof of this lemma is essentially due to Engelking and Pol [4]. To satisfy the condition (2) we slightly improve the last part of their proof.

It is easy to see that (see [2] or [6]) there exists a compactification X of S such that $\mathrm{Cl}_X A_0 \cap \mathrm{Cl}_X B_0 = \emptyset$. Since S is Čech-complete, there is a collection $\{U_i : i < \omega\}$ of open subsets of X such that $\bigcap\{U_i : i < \omega\} = S$ and $U_{i+1} \subset U_I$ for every $i < \omega$.

Let d be a metric on X such that $d(x, y) < 1$ for every $x, y \in X$. From the compactness of X it follows that for every $i < \omega$ there is a finite collection $\mathcal{U}_i = \{U_{ij} : j < n(i)\}$ of open subsets of X such that

$$S \subset \bigcup \mathcal{U}_i \subset U_i \text{ and mesh } \mathcal{U}_i < \frac{1}{i}.$$

Let $f_{ij} : X \to I = [0,1]$ be the mapping defined by

$$f_{ij}(x) = d(x, X - U_{ij})$$

for every $i, j; i < \omega$ and $j < n(i)$. Since $\mathrm{Cl}_X A_0 \cap \mathrm{Cl}_X B_0 = \emptyset$, there is a continuous mapping $f_{00} : X \to I$ such that

$(*)$ $\qquad f_{00}(\mathrm{Cl}_X A_0) = \{0\}$ and $f_{00}(\mathrm{Cl}_X B_0) = \{1\}$.

Let f be the diagonal of the mappings $\{f_{ij} : i < \omega$ and $j < n(i)\}$, where $n(0) = 1$, that is,

$$f = \triangle\{f_{ij} : i < \omega \text{ and } j < n(i)\} : X \to I^\omega$$

defined by

$$f(x) = (f_{ij}(x)) \in \prod_{i<\omega}(\prod_{j<n(i)} I_{ij}) = I^\omega,$$

where I_{ij} is a copy of I. Then the restriction $f|S$ of f to S is an embedding and $f(X - S) \subset K_\omega$, where $K_\omega = \{(t_i) \in I^\omega : \{i : t_i \neq 0\}$ is finite $\}$. The closure Y of $f(S)$ in I^ω is a compactification of S and we have $Y - S \subset f(X - S) \subset K_\omega$. Since K_ω is countable-dimensional, so is $Y - S$. Thus, by Lemma 2.3, Y is w.i.d. By $(*)$, we have $\mathrm{Cl}_Y A_0 \cap \mathrm{Cl}_Y B_0 = \emptyset$. Hence Y has all the required properties. Lemma 2.4 has been proved.

Let Y be as in Lemma 2.4, $Z' = Y/\{\mathrm{Cl}_Y A_0, \mathrm{Cl}_Y B_0\}$ the quotient space and $q : Y \to Z'$ the quotient mapping. Let us set

$\{a'\} = q(\mathrm{Cl}_Y A_0)$,

$\{b'\} = q(\mathrm{Cl}_Y B_0)$,

$Z = Z' \times C$,

$a = (a', 0) \in Z$, and

$b = (b', 0) \in Z$,

where C is the standard Cantor set in I. We regard $q : Y \to Z'$ as $q : Y \to Z' \simeq Z' \times \{0\} \subset Z$.

2.5. Lemma. *Let Z be as above. Then*

(1) *Z has no isolated point,*

(2) *Z is w.i.d., and*

(3) *for every open neighborhood U of a in Z with $b \notin \mathrm{Cl}_Z U$ the boundary $\mathrm{Bd}_Z U$ of U is not hereditarily w.i.d.*

Proof. Since C has no isolated point, the condition (1) holds. Let us set

$$S' = S - (A_0 \cup B_0) \text{ and } W = Z' - (S' \cup \{a', b'\}).$$

Then we have $Z' - S' = W \cup \{a', b'\}$. By Example 2.1(3), S' is totally disconnected, therefore so is $S' \times C$. We can regard W as the natural subspace of $Y - S$. By Lemma 2.4(3), W is countable-dimensional, therefore so is $Z' - S'$. Thus $(Z' - S') \times C$ is also countable-dimensional. By Lemma 2.3, Z is w.i.d. Hence the condition (2) holds.

Let U be an open neighborhood of a in Z with $b \notin \mathrm{Cl}_Z U$. Let us set

$$L = q^{-1}(\mathrm{Bd}_Z U \cap (Z' \times \{0\}) \cap S.$$

Then L is a partition in S between A_0 and B_0, therefore, by Fact 2.2, L is s.i.d. On the other hand, we can regard L as a subspace of $\mathrm{Bd}_Z U$. Thus $\mathrm{Bd}_Z U$ is not hereditarily w.i.d. Hence the condition (3) holds. Lemma 2.5 has been proved.

2.6. Construction. Let d be a metric on Z with $d(a,b)=1$ and $\mathcal{B}$ a countable base for Z. For every $n<\omega$ let us set

$$Z_n = Z, a_n = a, b_n = b, d_n = d \text{ and } \mathcal{B}_n = \mathcal{B},$$

$$\Lambda_n = \{(B,B') : B, B' \in \mathcal{B}_n \text{ with } B \cap B' = \emptyset \text{ or } B \in \mathcal{B}_n, B' \in \mathcal{B}_i \text{ for some } i<n\}$$

and

$$\Lambda = \bigcup\{\Lambda_n : n<\omega\}.$$

Since Λ is countable, it is easy to see that there is a bijection $\psi : \omega - \{0\} \to \Lambda$ such that $\psi(n) \in \bigcup\{\Lambda_i : i<n\}$. By Lemma 2.5(1), for every $\psi(n) = (B_n, B'_n) \in \Lambda$, inductively, we can take points $x_n \in B_n$ and $y_n \in B'_n$ such that

$$x_n, y_n \notin \{x_i, y_i, a_i, b_i : i<n\}.$$

Let us set

$$Z" = \oplus\{Z_n : n<\omega\} \text{ and } \mathcal{D} = \{\{a_n, x_n\}, \{b_n, y_n\} : 0<n<\omega\}.$$

Let $T = Z"/\mathcal{D}$ be the quotient set and $p : Z" \to T$ the quotient mapping.

2.7. Definition. Let $x, y \in T$ and $\{z_i : 0 \le i \le n\}$ a finite subset of T. Then $\{z_i : 0 \le i \le n\}$ is a *chain* from x to y if the following conditions are satisfied;

(1) $x = z_0, y = z_n$ and

(2) for every $i; 0 \le i \le n-1$, there is $\ell(i) < \omega$ such that $p^{-1}(z_i) \cap Z_{\ell(i)} \neq \emptyset$ and $p^{-1}(z_{i+1}) \cap Z_{\ell(i)} \neq \emptyset$.

Then both $p^{-1}(z_i) \cap Z_{\ell(i)}$ and $p^{-1}(z_{i+1}) \cap Z_{\ell(i)}$ are singletons, so we set

$$\{r_i\} = p^{-1}(z_i) \cap Z_{\ell(i)} \text{ and } \{s_i\} = p^{-1}(z_{i+1}) \cap Z_{\ell(i)}.$$

2.8. Definition. For every $x, y \in T$ we set

$$\rho(x,y) = \inf\{\sum_{i=0}^{n-1} d_{\ell(i)}(r_i, s_i) : \{z_i : 0 \le i \le n\} \text{ is a chain from } x \text{ to } y\}.$$

2.9. Lemma. *Let ρ be as in Definition 2.8. Then ρ is a metric on T.*

Proof. By the construction of ρ, it is obvious that ρ is a pseudometric on T. Thus it suffices to prove that $\rho(x,y)=0$ implies $x=y$.

Assume that there exist $x, y \in T$ such that $\rho(x,y)=0$ and $x \neq y$. Since $p^{-1}(z)$ consists of at most two points for every $z \in T$, we set

$$p^{-1}(x) = \{x', \alpha_{\ell(x)}\} \text{ or } \{x'\},$$

where $x' \in Z_{m(x)}, \alpha_{\ell(x)} = a_{\ell(x)}$ or $b_{\ell(x)}$, and $m(x) \neq \ell(x)$, and

$$p^{-1}(y) = \{y', \alpha_{\ell(y)}\} \text{ or } \{y'\},$$

where $y' \in Z_{m(y)}, \alpha_{\ell(y)} = a_{\ell(y)}$ or $b_{\ell(y)}$, and $m(y) \neq \ell(y)$.

Let us set

$$\alpha'_{\ell(x)} = \begin{cases} b_{\ell(x)} & \text{if } \alpha_{\ell(x)} = a_{\ell(x)} \\ a_{\ell(x)} & \text{if } \alpha_{\ell(x)} = b_{\ell(x)} \end{cases}$$

and

$$\beta'_{\ell(y)} = \begin{cases} b_{\ell(y)} & \text{if } \beta_{\ell(y)} = a_{\ell(y)} \\ a_{\ell(y)} & \text{if } \beta_{\ell(y)} = b_{\ell(y)}. \end{cases}$$

We distinguish three cases.

Case 1. $x \neq p(a_0), p(b_0)$

Let us set

$$\varepsilon_x = \min\{d_{m(x)}(x', a_{m(x)}), d_{m(x)}(x', b_{m(x)})\}$$

and

$$\varepsilon_y = \begin{cases} 1 & \text{if } y = p(a_0) \text{ or } p(b_0) \\ \min\{d_{m(y)}(y', a_{m(y)}), d_{m(y)}(y', b_{m(y)})\} & \text{otherwise.} \end{cases}$$

Then, obviously, we have $\varepsilon_x > 0$ and $\varepsilon_y > 0$. We fix ε with

$$0 < \varepsilon < \min\{\varepsilon_x, \varepsilon_y, 1\}.$$

Since $\rho(x, y) = 0$, there exists a chain $\{z_i : 0 \leq i \leq n\}$ from x to y such that

$$\sum_{i=0}^{n-1} d_{\ell(i)}(r_i, s_i) < \varepsilon,$$

where r_i and s_i are as in Definition 2.7. Without loss of generality we can assume that

$$r_i \neq s_i \text{ for every } i; 0 \leq i \leq n-1,$$

and

$$\ell(i) \neq \ell(i+1) \text{ for every } i; 0 \leq i \leq n-2.$$

By the assumption that $\ell(i) \neq \ell(i+1)$, if $s_i \neq a_{\ell(i)}, b_{\ell(i)}$, then $r_{i+1} = a_{\ell(i+1)}$, or $b_{\ell(i+1)}$.

Claim. $r_i = a_{\ell(i)}$ or $b_{\ell(i)}$ for every $i; 1 \leq i \leq n-1$.

We shall prove this claim by the induction on $i = 1, 2, \cdots, n-1$.

First, we shall show that $r_1 = a_{\ell(1)}$ or $b_{\ell(1)}$. Suppose that $r_0 = x'$. Then we have $m(x) = \ell(0)$, therefore we have

$$d_{m(x)}(x', s_0) = d_{\ell(0)}(r_0, s_0) < \varepsilon < \varepsilon_x.$$

Thus we have $s_0 \neq a_{\ell(0)}, b_{\ell(0)}$. This implies $r_1 = a_{\ell(1)}$ or $b_{\ell(1)}$. Suppose that $r_0 \neq x'$. Then we have $r_0 = \alpha_{\ell(x)}$ and $\ell(x) = \ell(0)$, therefore we have

$$d_{\ell(x)}(\alpha_{\ell(x)}, s_0) = d_{\ell(0)}(r_0, s_0) < \varepsilon < 1.$$

Thus we have $s_0 \neq \alpha'_{\ell(0)}$. Since $r_0 \neq s_0$, we have $s_0 \neq \alpha_{\ell(0)}$. Thus we have $s_0 \neq a_{\ell(0)}, b_{\ell(0)}$. This implies $r_1 = a_{\ell(1)}$ or $b_{\ell(1)}$.

Next, suppose that $r_i = a_{\ell(i)}$ or $b_{\ell(i)}; 1 \leq i \leq n-2$. Since $d_{\ell(i)}(r_i, s_i) < \varepsilon < 1$ and $r_i \neq s_i$, we have $s_i \neq a_{\ell(i)}, b_{\ell(i)}$. Hence we have $r_{i+1} = a_{\ell(i+1)}$ or $b_{\ell(i+1)}$. Claim has been proved.

Case 1-1. $y \neq p(a_0), p(b_0)$.

Suppose that $s_{n-1} = y'$. Then we have $m(y) = \ell(n-1)$ and

$$\varepsilon > d_{\ell(n-1)}(r_{n-1}, s_{n-1}) \geq \min\{d_{m(y)}(y', a_{m(y)}), d_{m(y)}(y', b_{m(y)}) = \varepsilon_y.$$

This is a contradiction.

Suppose that $s_{n-1} = \beta_{\ell(n-1)}$. Then we have

$$\varepsilon > d_{\ell(n-1)}(r_{n-1}, s_{n-1}) = d_{\ell(n-1))}(a_{\ell(n-1)}, b_{\ell(n-1)}) = 1.$$

This is a contradiction.

Case 1-2. $y = p(a_0)$.

This case implies $\ell(n-1) = 0$. Thus we have $s_{n-1} = a_0$ and $r_{n-1} = b_0$. Hence we have

$$\varepsilon > d_{\ell(n-1)}(r_{n-1}, s_{n-1}) = d_0(a_0, b_0) = 1.$$

This is a contradiction.

Case 1-3. $y = p(b_0)$.

Similarly as in Case 1-2.

Case 2. $x = p(a_0)$

If $y \neq p(b_0)$ then $y \neq p(a_0), p(b_0)$. Thus we get a contradiction similarly as in Case 1. Suppose that $y = p(b_0)$. Then there exists a chain $\{z_i : 0 \leq i \leq n\}$ from x to y such that

$$\sum_{i=0}^{n-1} d_{\ell(i)}(r_i, s_i) < 1 \text{ and } \ell(i) \neq \ell(i+1) \text{ for every } i; 0 \leq i \leq n-2.$$

If $n = 1$ then we have $r_0 = a_0$ and $s_0 = b_0$. Thus we have

$$1 > d_{\ell(0)}(r_0, s_0) = d_0(a_0, b_0) = 1.$$

Suppose that $n \geq 2$. Since $\ell(1) \neq \ell(0)$, we have $r_1 \notin Z_0$. Similarly as in Case 1, we can show that $r_i = a_{\ell(i)}$ or $b_{\ell(i)}$ for every $i; 0 \leq i \leq n-1$. By the construction of T, $\ell(i) < \ell(i+1)$ for every $i; 0 \leq i \leq n-2$. Thus we have $\ell(n-1) \neq 0$. Hence we have $s_{n-1} \in Z_0$. On the other hand, by the assumption that $y = p(b_0)$, we have $s_{n-1} = b_0$. This is a contradiction.

Case 3. $x = p(b_0)$

Similarly as in Case 2.

In any case we obtain a contradiction. Hence $\rho(x, y) = 0$ implies $x = y$. Lemma 2.9 has been proved.

3 Main result

Throughout the rest of this paper we regard T as the metric space with the metric ρ defined in Definition 2.8.

In this section we shall prove that the metric space T is a counterexample to a Krasinkiewicz's problem.

3.1. Lemma. *Let p, Z_n and T be as in Section 2. Then $p|Z_n : Z_n \to T$ is an embedding for every $n < \omega$.*

Proof. To prove this lemma it suffices to prove the following claim.

Claim. For every $x \in Z_n$ there exists $\varepsilon > 0$ satisfying the following condition (*);

(*) for every $y \in Z_n$ with $d_n(x, y) < \varepsilon$ the equality $\rho(p(x), p(y)) = d_n(x, y)$ holds.

By the construction of ρ, it is obvious that the inequality $\rho(p(x), p(y)) \leq d_n(x, y)$ holds.

Case 1. $x \neq a_n, b_n$.

Let us set $\varepsilon = \min\{d_n(x, a_n), d_n(x, b_n)\}$. Assume that there exists $y \in Z_n$ such that $\rho(p(x), p(y)) < d_n(x, y) < \varepsilon$. We take a chain $\{z_i : 0 \leq i \leq m\}$ from $p(x)$ to $p(y)$ such that $\sum_{i=0}^{m-1} d_{\ell(i)}(r_i, s_i) < \varepsilon$, $r_i \neq s_i$ for every $i; 0 \leq i \leq m-1$, and $\ell(i) \neq \ell(i+1)$ for every $i; 0 \leq i \leq m-2$.

Similarly as the proof of Lemma 2.9, we have

$$r_i = a_{\ell(i)} \text{ or } b_{\ell(i)} \text{ for every } i; 0 \leq i \leq m-1, \text{ and } n \leq \ell(0) < \ell(1) < \cdots < \ell(m-1).$$

On the other hand, since $p(s_{m-1}) = y \in Z_n$ and $s_{m-1} \neq r_{m-1}$, we have

$$s_{m-1} = \begin{cases} a_{\ell(m-1)} & \text{if } r_{m-1} = b_{\ell(m-1)} \\ b_{\ell(m-1)} & \text{if } r_{m-1} = a_{\ell(m-1)}. \end{cases}$$

Thus we have

$$1 > \varepsilon > d_{\ell(m-1)}(r_{m-1}, s_{m-1}) = d_{\ell(m-1)}(a_{\ell(m-1)}, b_{\ell(m-1)}) = 1.$$

This is a contradiction.

Case 2. $x = a_n$ and $n \geq 1$.

Let us set

$p^{-1}p(b_n) = Z_k = \{c\}$,

$p^{-1}p(a_n) = Z_{k'} = \{c'\}$, and

$\varepsilon = \min\{d_k(a_k, c), d_k(b_k, c), d_{k'}(a_{k'}, c'), d_{k'}(b_{k'}, c')$.

Assume that there exists $y \in Z_n$ such that $\rho(p(x), p(y)) < d_n(x, y) < \varepsilon$. We take a chain $\{z_i : 0 \le i \le m\}$ from $p(x)$ to $p(y)$ such that $\sum_{i=0}^{m-1} d_{\ell(i)}(r_i, s_i) < \varepsilon$, $r_i \neq s_i$ for every $i; 0 \le i \le m-1$, and $\ell(i) \neq \ell(i+1)$ for every $i; 0 \le i \le m-2$.

Similarly as the proof of Lemma 2.9, we have $r_i = a_{\ell(i)}$ or $b_{\ell(i)}$ for every $i; 0 \le i \le m-1$. Assume that $c = s_i$ for some $i; 0 \le i \le m-1$. Then we have $\ell(i) = k$. Since $r_i = a_{\ell(i)}$ or $b_{\ell(i)}$, we have $\varepsilon > d_{\ell(i)}(r_i, s_i) = d_k(r_i, c) \ge \varepsilon$. This is a contradiction. Thus we have $c \neq s_i$ for every $i; 0 \le i \le m-1$. This implies that $b_n \neq r_i$ for every $i; 0 \le i \le m-1$. Since $r_{m-1} = a_{\ell(m-1)}$ or $b_{\ell(m-1)}$, $s_{m-1} = \beta_{\ell(m-1)} = a_{\ell(m-1)}$ or $b_{\ell(m-1)}$, $\ell(y) = \ell(m-1)$ and $r_{m-1} \neq s_{m-1}$, we have

$$1 > d_{\ell(m-1)}(r_{m-1}, s_{m-1}) = d_{\ell(y)}(a_{\ell(y)}, b_{\ell(y)}) = 1$$

This is a contradiction.

Case 3. $x = b_n$ and $n \ge 1$.

Similarly as in Case 2, we obtain a contradiction.

Case 4. $x = a_0$.

Assume that there exists $y \in Z_0$ such that $\rho(p(x), p(y)) < d_0(x, y) < 1$. We take a chain $\{z_i : 0 \le i \le m\}$ from $p(x)$ to $p(y)$ such that $\sum_{i=0}^{m-1} d_{\ell(i)}(r_i, s_i) < 1$, $r_i \neq s_i$ for every $i; 0 \le i \le m-1$, and $\ell(i) \neq \ell(i+1)$ for every $i; 0 \le i \le m-2$. Similarly as the proof of Lemma 2.9, we have $r_i = a_{\ell(i)}$ or $b_{\ell(i)}$ for every $i; 0 \le i \le m-1$. In particular, we have $r_{m-1} = a_{\ell(m-1)}$ or $b_{\ell(m-1)}$. Since $p(s_{m-1}) = y \in Z_0$, we have

$$s_{m-1} = \begin{cases} a_{\ell(m-1)} & \text{if } r_{m-1} = b_{\ell(m-1)} \\ b_{\ell(m-1)} & \text{if } r_{m-1} = a_{\ell(m-1)}. \end{cases}$$

Then we have

$$1 > d_{\ell(m-1)}(r_{m-1}, s_{m-1}) = d_{\ell(m-1)}(a_{\ell(m-1)}, b_{\ell(m-1)}) = 1.$$

This is a contradiction.

Case 5. $x = b_0$.

Similarly as in Case 4, we obtain a contradiction.

In any case we get a contradiction. Hence the condition (*) holds. Lemma 3.1 has been proved.

3.2. Lemma. *The metric space T is w.i.d.*

Proof. By Lemmas 2.5(2) and 3.1, T is the union of countable w.i.d. subspaces $p(Z_n)$. Hence, by [8] (or see [11]), T is w.i.d. Lemma 3.2 has been proved.

3.3. Lemma. *Every separator L of T is not hereditarily w.i.d.*

Proof. Since L is a separator, there exist two non-empty open subsets U and V of T such that $T - L = U \cup V$ and $U \cap U = \emptyset$. Then we can take $\psi(n) = (B, B') \in \Lambda$ such that $B \subset p^{-1}(U)$ and $B' \subset p^{-1}(V)$. Let us set $U' = p^{-1}(U) \cap Z_n$. Then we can assume that $a_n \in U'$ and $b_n \notin \mathrm{Cl}_{Z_n} U'$. By Lemma 2.5(3), $\mathrm{Bd}_{Z_n} U'$ is not hereditarily w.i.d. Since $p^{-1}(\mathrm{Bd}_{Z_n} U') \subset L$ and , by Lemma 3.1, $p^{-1}(\mathrm{Bd}_{Z_n} U')$ is homeomorphic to $\mathrm{Bd}_{Z_n} U'$, L is not hereditarily w.i.d. Lemma 3.3 has been proved.

By Lemmas 3.2 and 3.3, we obtain the following example, which is a negative answer to a problem of Krsinkiewicz [7].

3.4. Example. There exists a space T satisfying the following conditions (1) and (2);

(1) T is w.i.d., and

(2) T can not be separated by any hereditarily w.i.d. subspace of T.

The space T in Example 3.4 is not compact. If this space T has a w.i.d. compactification X, then X satisfies the conditions (1) and (2) in Example 3.4. Borst [1] introduced the concept of small weakly infinite-dimensionality, and he proved that a space X has a w.i.d. compactification if and only if X is small weakly infinite-dimensional.

3.5. Problem. *Is the space T in Example 3.4 small weakly infinite-dimensional ?*

It is well-known [3, 1.9.8] that every n-dimensional compact space contains a n-dimensional Cantor-manifold. Furthermore, every s.i.d. compact space contains an infinite dimensional Cantor-manifold.

3.5. Problem. *Let X be a w.i.d. compact space which is not hereditarily w.i.d. Does X contains a compact subspace satisfying the conditions (1) and (2) in Example 3.4 ?*

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