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Disjoint Open Subsets of the Remaider of the Freudenthal Compactification

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Abstract

It is well-known that the Stone-Čech remainder $\beta\mathbb{N} - \mathbb{N}$ of the discrete space of cardinality $\aleph_0$ has a collection of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open subsets of $\beta\mathbb{N} - \mathbb{N}$. In this paper we consider an analogous theorem with respect to the Freudenthal compactification.

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1 Introduction

The maximal compactification with zero-dimensional remainder is called the Freudenthal compactification. Every rim-compact space has the Freudenthal compactification.

It is well-known [5] (or see [2])that the space $\beta\mathbb{N}-\mathbb{N}$, the remainder of the Stone-Čech compactification of the discrete space of cardinality $\aleph_0$, contains a collection of cardinality $\mathfrak{c}$ consisting of pairwise disjoint non-empty open subsets of $\beta\mathbb{N} - \mathbb{N}$. In [1] W. W. Comfort and H. Gordon generalized this theorem. Their main theorem asserts that for a Tychonoff space X and a cardinal number $\mathfrak{m}$, $\beta X - X$ admits a collection of $\mathfrak{m}$ pairwise disjoint non-empty open subsets of $\beta X - X$ if and only if X admits a collection of $\mathfrak{m}$ cozero-sets with certain properties.

In this paper we consider an analogous theorem with respect to the Freudenthal compactification. More precisely we shall characterize a rim-compact space of which the remainder of the Freudenthal compactification is the discrete space with cardinality $\mathfrak{m}$.

Throughout the rest of this paper all spaces considered are assumed to be rim-compact. We denote by γX the Freudenthal compactification of a space X.

2 Preliminaries

An open (resp. closed) set A of a space X is called *γ-open* (resp. *γ-closed*) if the boundary $\mathrm{Bd}_X A$ is compact. A finite open cover consisting of γ-open subsets of a space X is called a *γ-open cover* of X.

A space X is *rim-compact* if X has a base consisting of γ-open subsets of X.

Let αX be a compactification of a space X and U an open subset of X. Then we set $O_{\alpha X}(U) = \alpha X - \mathrm{Cl}_{\alpha X}(X - U)$.

The following proposition is a characterization of the Freudenthal compactifications which corresponds with those of the Stone-Čech compactifications.

Proposition 1. *Let αX be a compactification of a space X with zero-dimensional remainder. Then the following conditions are equivalent:*

(a) *αX is the Freudenthal compactification of X.*

(b) *$\mathrm{Cl}_{\alpha X} E \cap \mathrm{Cl}_{\alpha X} F = \mathrm{Cl}_{\alpha X}(E \cap F)$ for every pair E, F of γ-closed subsets of X.*

(c) *$O_{\alpha X}(U) \cup O_{\alpha X}(V) = O_{\alpha X}(U \cup V)$ for every pair U, V of γ-open subsets of X.*

(d) *$O_{\alpha X}(\mathcal{U}) = \{O_{\alpha X}(U) : U \in \mathcal{U}\}$ is an open cover of αX for every γ-open cover $\mathcal{U}$ of X.*

Proof. (a) $\Rightarrow$ (b). Obviously $X \cap \mathrm{Cl}_{\gamma X} E \cap \mathrm{Cl}_{\gamma X} F = X \cap \mathrm{Cl}_{\gamma X}(E \cap F)$. Since

$$\mathrm{Cl}_{\gamma X} H \cap (\gamma X - X) = (\gamma X - \mathrm{Cl}_{\gamma X}(X - H)) \cap (\gamma X - X)$$

for every γ-closed subset H of X (see e.g. [6]), we have

$$(\gamma X - X) \cap \mathrm{Cl}_{\gamma X} E \cap \mathrm{Cl}_{\gamma X} F = (\gamma X - X) \cap \mathrm{Cl}_{\gamma X}(E \cap F).$$

Hence we have $\mathrm{Cl}_{\gamma X} E \cap \mathrm{Cl}_{\gamma X} F = \mathrm{Cl}_{\gamma X}(E \cap F)$.

Implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). Assume that $\alpha X \neq \gamma X$. Since $\alpha X - X$ is zero-dimensional, there is a continuous mapping f of γX onto αX which leaves the points of X fixed. Since $\alpha X \neq \gamma X$, there are two points $x, y \in \gamma X - X$ such that $x \neq y$ and $f(x) = f(y)$.

We shall construct a γ-open cover $\mathcal{U}$ of X such that $O_{\alpha X}(\mathcal{U})$ is not an open cover of αX.

There are open subsets U, V of γX such that $x \in U \subset \mathrm{Cl}_{\gamma X} U \subset V$, $y \notin V$, $\mathrm{Bd}_{\gamma X} U \subset X$ and $\mathrm{Bd}_{\gamma X} V \subset X$. Let $U' = (\gamma X - \mathrm{Cl}_{\gamma X} U) \cap X$, $V' = V \cap X$ and $p = f(x) = f(y)$. Then $\{U', V'\}$ is a γ-open cover of X. It is easy to show that $x \in \mathrm{Cl}_{\gamma X}(X - U')$. From closedness of f it follows that $p = f(x) \in \mathrm{Cl}_{\gamma X}(X - U')$. Similarly, we have $p = f(y) \in \mathrm{Cl}_{\gamma X}(X - V')$. Hence we have $p \notin O_{\gamma X}(U') \cup O_{\gamma X}(V')$. This is a contradiction. This proves Proposition 1.

3 The main theorems

Let X be a space and $\mathfrak{m}$ a cardinal number. We say that X has $d(\mathfrak{m})$ provided that X admits a collection $\mathcal{U}$ of pairwise disjoint non-empty open subsets of X for which $|\mathcal{U}| = \mathfrak{m}$.

The smallest cardinal number $\mathfrak{m}$ such that every collection of pairwise disjoint open subsets of a space X has cardinality $\leq \mathfrak{m}$, is called the *Souslin number*, or *cellularity* of the space X.

A subset A of a space X is *relatively compact* if the closure $\mathrm{Cl}_X A$ is compact.

As for the Freudenthal compactification we can obtain an analogous theorem to the theorem by W. W. Comfort and H. Gordon.

Theorem 2. *Let X be a space and $\mathfrak{m}$ a cardinal number. Then the following conditions are equivalent:*

(a) *$\gamma X - X$ has $d(\mathfrak{m})$.*

(b) *X admits a collection $\{U_\lambda : \lambda \in \Lambda)$ of γ-open subsets of X, with $|\Lambda| = \mathfrak{m}$, for which*

(i) *each U_λ is not relatively compact,*

(ii) *$\mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu$ is compact for distinct $\lambda, \mu \in \Lambda$.*

Proof.

(a) $\Rightarrow$ (b).

Let $\{V_\lambda : \lambda \in \Lambda\}$ be a collection of pairwise disjoint non-empty open subsets of $\gamma X - X$, where $|\Lambda| = \mathfrak{m}$. For each $\lambda \in \Lambda$ there is an open subset O_λ of γX such that $V_\lambda = O_\lambda \cap (\gamma X - X)$. Since V_λ is not empty, we can choose a point $p_\lambda \in V_\lambda$. We take an open subset W_λ of γX such that $p_\lambda \in W_\lambda \subset \mathrm{Cl}_{\gamma X} W_\lambda \subset O_\lambda$ and $\mathrm{Bd}_{\gamma X} W_\lambda \subset X$. Now we set $U_\lambda = W_\lambda \cap X$ for each $\lambda \in \Lambda$. Then $\{U_\lambda : \lambda \in \Lambda\}$ is the desired collection.

Obviously U_λ is γ-open and non relatively compact. Next, we shall show that $\mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu$ is compact for distinct $\lambda, \mu \in \Lambda$. It suffices to show that $\mathrm{Cl}_{\gamma X} U_\lambda \cap \mathrm{Cl}_{\gamma X} U_\mu \subset X$. Since

$$\mathrm{Cl}_{\gamma X} U_\lambda \cap \mathrm{Cl}_{\gamma X} U_\mu \cap (\gamma X - X) \subset O_\lambda \cap O_\mu \cap (\gamma X - X) \subset V_\lambda \cap V_\mu = \emptyset,$$

the set $\mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu$ is compact.

(b) $\Rightarrow$ (a).

We set $V_\lambda = \mathrm{Cl}_{\gamma X} U_\lambda \cap (\gamma X - X)$ for each $\lambda \in \Lambda$. Then $\{V_\lambda : \lambda \in \Lambda\}$ is a collection of pairwise disjoint non-empty open subsets of $\gamma X - X$, with $|\Lambda| = \mathfrak{m}$.

Indeed, since $\mathrm{Cl}_{\gamma X} U_\lambda \cap (\gamma X - X) = (\gamma X - \mathrm{Cl}_{\gamma X}(X - U_\lambda)) \cap (\gamma X - X)$, V_λ is open in $\gamma X - X$. From non relative compactness of U_λ it follows that V_λ is non-empty. By Proposition1, (a) $\Leftrightarrow$ (b) and compactness of $\mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu$, we have

$$V_\lambda \cap V_\mu = \mathrm{Cl}_{\gamma X} U_\lambda \cap \mathrm{Cl}_{\gamma X} U_\mu \cap (\gamma X - X) = \mathrm{Cl}_{\gamma X}(\mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu) \cap (\gamma X - X)$$

$$= \mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu \cap (\gamma X - X) = \emptyset.$$

Hence $\gamma X - X$ has $d(\mathfrak{m})$. This completes the proof.

Foe a space X $R(X)$ denotes the set of all points having no compact neighborhood. A collection $\mathcal{G} = \{G_1, G_2, \cdots, G_n\}$ of pairwise disjoint open subsets of a space X is called an *n-star* of X if (1) $X - G_1 \cup G_2 \cup \cdots \cup G_n$ is compact and (2) $X - G_1 \cup G_2 \cup \cdots \cup G_{i-1} \cup G_{i+1} \cup \cdots \cup G_n$ is not compact for each $1 \leq i \leq n$. K. D. Magill [3] proved that a locally compact space has an n-point compactification if and only if X has an n-star. It is easy to see that for a locally compact space X, the Freudenthal compactification is an n-point compactification if and only if X has an n-star and X has no n+1-star.

The following lemma is due to K. Morita [4].

Lemma 3. *For a γ-closed subset F of a space X γF is homeomorphic to the closure of F in the space γX.*

T. Terada [6] gave a characterization of a metrizable space having a compactification with countable discrete remainder. We characterize a space of which the remainder of the Freudenthal compactification is the discrete space with cardinality $\mathfrak{m}$.

Theorem 4. *The remainder of the Freudenthal compactification of a space X is the discrete space with cardinality $\mathfrak{m}$ if and only if X admits a collection $\{U_\lambda : \lambda \in \Lambda\}$ of γ-open subsets of X with $|\Lambda| = \mathfrak{m}$, for which*

(i) $\mathrm{Cl}_X U_\lambda$ *is non-compact, locally compact and has no 2-star for each $\lambda \in \Lambda$,*

(ii) $\mathrm{Cl}_X U_\lambda \cap \mathrm{Cl}_X U_\mu$ *is compact for distinct $\lambda, \mu \in \Lambda$,*

(iii) $X - \bigcup\{U_\lambda : \lambda \in \Lambda\}$ *is compact, and*

(iv) *for every γ-open subset U of X containing $R(X)$ there is a finite subset Λ' of Λ such that $X - U \cup \bigcup\{U_\lambda : \lambda \in \Lambda'\}$ is compact.*

Proof.

Suppose that $\gamma X - X$ is the discrete space with cardinality $\mathfrak{m}$. For each $x \in \gamma X - X$ there are open subsets V_x, W_x of γX such that $x \in V_x \subset \mathrm{Cl}_{\gamma X} V_x \subset W_x$ and $W_x \cap (\gamma X - X) = \{x\}$. Now we set $U_x = V_x \cap X$ for each $x \in \gamma X - X$. We shall prove that $\mathcal{U} = \{U_x : x \in \gamma X - X\}$ is the desired collection.

Similarly as the proof of Theorem 2, $\mathrm{Cl}_X U_x$ is non-compact and $\mathrm{Cl}_X U_x \cap \mathrm{Cl}_X U_y$ is compact for distinct $x, y \in \gamma X - X$. Obviously $X - \bigcup\{U_x : x \in \gamma X - X\}$ is compact. Since $\mathrm{Cl}_X U_x = \mathrm{Cl}_{\gamma X} U_x - \{x\}$, $\mathrm{Cl}_X U_x$ is locally compact. It is easy te see that $\mathrm{Cl}_X U_x$ is γ-closed. By Lemma 3, we have $\gamma(\mathrm{Cl}_X U_x) \approx \mathrm{Cl}_X U_x \cup \{x\}$. Thus $\mathrm{Cl}_X U_x$ has no 2-star. Since $\mathrm{Cl}_{\gamma X}(\gamma X - X) = (\gamma X - X) \cup R(X)$, $(\gamma X - X) \cup R(X)$ is compact. Let U be a γ-open subset of X containing $R(X)$. Then we have $x \in O_{\gamma X}(U)$ for all but finitely many $x \in \gamma X - X$. Let Λ' be the set of all points $x \in \gamma X - X$ such that $x \notin O_{\gamma X}(U)$. Since $\gamma X - O_{\gamma X}(U) \cup \bigcup\{O_{\gamma X}(U_x) : x \in \Lambda'\} \subset X$, we have

$$X - U \cup \bigcup\{U_x : x \in \Lambda'\} = \gamma X - O_{\gamma X}(U) \cup \bigcup\{O_\gamma X(U_x) : x \in \Lambda'\},$$

therefore $X - U \cup \bigcup\{U_x : x \in \Lambda'\}$ is compact. Hence $\mathcal{U}$ satisfies the conditions (i) - (iv).

Conversely, since $\mathrm{Cl}_X U_\lambda$ is γ-closed, $\gamma(\mathrm{Cl}_X U_\lambda) \approx \mathrm{Cl}_{\gamma X} U_\lambda$. Since $\mathrm{Cl}_X U_\lambda$ is non-compact, locally compact and has no 2-star, $\mathrm{Cl}_{\gamma X} U_\lambda \cap (\gamma X - X)$ is exactly one-point set. Thus we set $\mathrm{Cl}_{\gamma X} U_\lambda \cap (\gamma X - X) = \{x_\lambda\}$ for each $\lambda \in \Lambda$. Let $Y = X \cup \{x_\lambda : \lambda \in \Lambda\}$. Similarly as the proof of Theorem 2 $\{x_\lambda\}$ is open in $\gamma X - X$. Thus $Y - X$ is the discrete space with cardinality $\mathfrak{m}$. Hence it suffices to show that $Y = \gamma X$. Assume that there is a point $x \in \gamma X - Y$. Since every point $x \in \bigcup\{U_\lambda : \lambda \in \Lambda\}$ has a compact neighborhood, we have $R(X) \subset X - \bigcup\{U_\lambda : \lambda \in \Lambda\}$, therefore $R(X)$ is compact. From compactness of $R(X)$ it follows that there is an open subset V of γX such that $R(X) \subset V$, $\mathrm{Bd}_{\gamma X} V \subset X$ and $x \notin \mathrm{Cl}_{\gamma X} V$. Let $U = V \cap X$. Then U is a γ-open subset of X containing $R(X)$. Thus there is a finite subset Λ' of Λ such that $X - U \cup \bigcup\{U_\lambda : \lambda \in \Lambda'\}$

is compact. Since $\gamma X - X \subset \mathrm{Cl}_{\gamma X} U \cup \bigcup\{\mathrm{Cl}_{\gamma X} U_\lambda : \lambda \in \Lambda'\}$ and $x \notin \mathrm{Cl}_{\gamma X} U$, there is an element $\lambda \in \Lambda$ such that $x \in \mathrm{Cl}_{\gamma X} U_\lambda$. However, since $x \in \gamma X - Y$, this is a contradiction. Thus we have $Y = \gamma X$. Hence $\gamma X - X$ is the discrete space with cardinality $\mathfrak{m}$.

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