J. Saitama Univ. Fac. Educ., 61(2) : 155-172 (2012)

On Extinction of Measure-Valued Markov Processes

DÔKU, Isamu

Faculty of Education, Saitama University

Abstract

We begin with introducing superprocesses with branching rate functional and historical superprocesses. We consider the notions of recurrence, transience and extinction property of measure-valued Markov processes. Then we prove the finite time extinction for a class of measure-valued Markov processes.

Key Words : Extinction property, finite time extinction, measure-valued Markov process, superprocess.

1. Superdiffusion with Branching Rate Functional

In this section we shall introduce the superdiffusion with branching rate functional, which forms a general class of measure-valued Markov processes with diffusion as a underlying spatial motion. We write $\langle \mu, f\rangle = \int f d\mu$. $M_F = M_F(\mathbb{R}^d)$ is the space of finite measures on $\mathbb{R}^d$. Define a second order elliptic differential operator $L = \frac{1}{2}\nabla \cdot a\nabla + b\cdot\nabla$, and $a = (a_{ij})$ is positive definite and we assume that $a_{ij}, b_i \in C^{1,\varepsilon} = C^{1,\varepsilon}(\mathbb{R}^d)$. Here the space $C^{1,\varepsilon}$ indicates the totality of all Hölder continuous functions with index ε $(0 < \varepsilon \leqslant 1)$, allowing their first order derivatives to be locally Hölder continuous. $\Xi = \{\xi, \Pi_{s,a}, s \geq 0, a \in \mathbb{R}^d\}$ indicates a L-diffusion. CAF stands for contunuous additive functional in Probability Theory.

Definition 1. (A Locally Admissible Class of CAF; cf. Dynkin (1994), [24]) A continuous additive functional K is said to be in the Dynkin class with index q and we write $K \in \mathbb{K}^q$, (some $q > 0$) if (a)

$$\sup_{a\in\mathbb{R}^d} \Pi_{s,a}\int_s^t \phi_p(\xi_r)K(dr) \to 0, \quad (r_0 \geq 0) \quad \text{as} \quad s \to r_0, t \to r_0; \tag{1}$$

(b) each N, $\exists c_N > 0$:

$$\Pi_{s,a}\int_s^t \phi_p(\xi_r)K(dr) \leqslant c_N|t-s|^q \phi_p(a), \quad (\text{for} \quad 0 \leqslant s \leqslant t \leqslant N, \quad a \in \mathbb{R}^d). \tag{2}$$

□

When we write C_b as the set of bounded continuous functions on $\mathbb{R}^d$, then C_b^+ is the set of positive members g in C_b. The process $\mathbb{X} = \{X, \mathbb{P}_{s,\mu},\ s \geq 0, \mu \in M_F\}$ is said to be a *superdiffusion with branching rate functional* K or simply (L, K, μ)-superprocess if $X = \{X_t\}$ is a continous M_F-valued time-inhomogeneous Markov process with Laplace functional

$$\mathbb{P}_{s,\mu} e^{-\langle X_t, \varphi\rangle} = e^{-\langle \mu, v(s,t)\rangle}, \qquad 0 \leqslant s \leqslant t, \quad \mu \in M_F, \quad \varphi \in C_b^+. \tag{3}$$

Here the function v is uniquely determined by the log-Laplace equation

$$\Pi_{s,a}\varphi(\xi_t) = v(s,a) + \Pi_{s,a}\int_s^t v^2(r,\xi_r)K(dr), \qquad 0 \leqslant s \leqslant t, \quad a \in \mathbb{R}^d. \tag{4}$$

2. Historical Superprocess

The historical superprocess was initially studied by E.B. Dynkin (1991) [23], see also Dawson-Perkins (1991) [7]. $\mathbb{C} = C(\mathbb{R}_+, \mathbb{R}^d)$ denotes the space of continuous paths on $\mathbb{R}^d$ with topology of uniform convergence on compact subsets of $\mathbb{R}_+$. To each $w \in \mathbb{C}$ and $t > 0$, we write $w^t \in \mathbb{C}$ as the stopped path of w. We denote by $\mathbb{C}^t$ the totality of all these paths stopped at time t. To every $w \in \mathbb{C}$ we associate the corresponding stopped path trajectory $\tilde{w}$ defined by $\tilde{w}_t = w^t$ for $t \geq 0$. The image of L-diffusion w under the map : $w \mapsto \tilde{w}$ is called the *L-diffusion path process*. Moreover, we define

$$\mathbb{C}_R^{\times} \equiv \mathbb{R}_+ \hat{\times} \mathbb{C}^{\cdot} = \{(s,w) : s \in \mathbb{R}_+, w \in \mathbb{C}^s\} \tag{5}$$

and we denote by $M(\mathbb{C}_R^{\times}) \equiv M(\mathbb{R}_+ \hat{\times} \mathbb{C}^{\cdot})$ the set of measures η on $\mathbb{R}_+ \hat{\times} \mathbb{C}^{\cdot}$ which are finite, if restricted to a finite time interval. Let K be a positive CAF of ξ. $\tilde{\mathbb{X}} = \{\tilde{X}, \tilde{\mathbb{P}}_{s,\mu}, s \geq 0,\ \mu \in M_F(\mathbb{C}^s)\}$ is said to be a Dynkin's *historical superprocess* (cf. Dynkin (1991), [23]) if $\tilde{X} = \{\tilde{X}_t\}$ is a time-inhomogeneous Markov process with state $\tilde{X}_t \in M_F(\mathbb{C}^t)$, $t \geq s$, with Laplace functional

$$\tilde{\mathbb{P}}_{s,\mu} e^{-\langle \tilde{X}_t, \varphi\rangle} = e^{-\langle \mu, v(s,t)\rangle}, \quad 0 \leqslant s \leqslant t, \quad \mu \in M_F(\mathbb{C}^s), \quad \varphi \in C_b^+(\mathbb{C}), \tag{6}$$

where the function v is uniquely determined by the log-Laplace type equation

$$\tilde{\Pi}_{s,w_s}\varphi(\tilde{\xi}_t) = v(s,w_s) + \tilde{\Pi}_{s,w_s}\int_s^t v^2(r,\tilde{\xi}_r)K(dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \tag{7}$$

3. Examples of Measure-Valued Processes

3.1 Dawson-Watanabe Superprocess

$X = \{X_t; t \geq 0\}$ is said to be the *Dawson-Watanabe superprocess* (cf. Watanabe (1968), [40], Dawson (1975), [1]) if $\{X_t\}$ is a Markov process taking values in the space $M_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$, satisfying the following martingale problem (MP): i.e., there exists a probability measure $\mathbb{P} \in \mathcal{P}(M_F(\mathbb{R}^d))$ on the sapce $M_F(\mathbb{R}^d)$ such that for all $\varphi \in \mathrm{Dom}(\Delta)$

$$M_t(\varphi) \equiv \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \int_0^t \langle X_s, \frac{1}{2}\Delta\varphi \rangle ds \tag{8}$$

is a $\mathbb{P}$-martingale and its quadratic variation process is given by

$$\langle M(\varphi) \rangle_t = \gamma \int_0^t \langle X_s, \varphi^2 \rangle ds. \tag{9}$$

Or equivalently, the Laplace functional of $\{X_t\}$ is given by

$$\mathbb{E}e^{-\langle X_t, \varphi \rangle} = e^{-\langle X_0, u(t) \rangle}, \qquad \text{for} \quad \varphi \in C_b^+(\mathbb{R}^d) \cap \mathrm{Dom}(\Delta), \tag{10}$$

where $u(t,x)$ is the unique positive solution of evolution equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \frac{1}{2}\gamma u^2, \qquad u(0,x) = \varphi(x). \tag{11}$$

The Dawson-Watanabe superprocess X_t inherits the branching property from the approximating branching Brownian motions (cf. Dawson (1993), [2]). Namely,

$$P_t(\cdot, \nu_1 + \nu_2) = P_t(\cdot, \nu_1) * P_t(\cdot, \nu_2), \tag{12}$$

where $P_t(\cdot, \mu)$ denotes the transition probability with the initial data μ. The branching property extends in the obvious way to initial measures of the form $\nu_1 + \cdots + \nu_n$. Conversely, for any integer n, the distribution of the superprocess started from initial measure μ is written as that of the sum of n independent copies of the superprocesses each started from μ/n. This implies that the Dawson-Watanabe superprocess is infinitely divisible.

The following proposition is well known. The statement is the analogue of the classical Lévy-Khintchine formula (cf. Sato (1999), [39]: Theorem 8.1, p.37), that characterizes the possible characteristic functions of infinitely divisible distributions on $\mathbb{R}^d$, in the measure-valued setting.

Theorem 2. (Canonical Representation Theorem) *Let $(E, \mathcal{E})$ be a Polish space, and X be infinitely divisible Random measure on $(E, \mathcal{E})$. Then there exist measures $X_d \in M_F(E)$ and $m \in M(M_F(E))$, $m \neq 0$ such that for $\forall \varphi \in C_b(E)$, $\int \{1 - e^{-\langle \nu, \varphi \rangle}\} m(d\nu) < \infty$ and*

$$-\log \mathbb{E}[e^{-\langle X, \varphi \rangle}] = \langle X_d, \varphi \rangle + \int \{1 - e^{-\langle \nu, \varphi \rangle}\} m(d\nu). \tag{13}$$

If $m(\{0\}) = 0$, then X_d and m are unique. (cf. Etheridge (2000), [28]: Theorem 1.28, p.18)

It is natural to ask whether we can construct other processes in $M_F(\mathbb{R}^d)$ (i.e. superprocesses) with infinitely divisible distributions. Using the above canonical representation formula as a starting point, we introduce the construction of general superprocess in a rather informal way. To keep the notation as simple as possible, we restrict our plan to time homogeneous Markov processes satisfying two conditions: (i) branching property; (ii) infinite divisibility. Let us denote by $Y = \{Y_t\}$ a time homogeneous Markov process. We have

$$\mathbb{E}_\mu[e^{-\langle Y_t, \varphi\rangle}] = e^{-\langle \mu, V_t\varphi\rangle}. \tag{14}$$

This operator V_t satisfies the property $V_{t+s} = V_t \circ V_s$. Comparing the formula (13) and Eq.(14), we can write with the uniqueness of the canonical representation,

$$\langle \mu, V_t\varphi\rangle = \int_{\mathbb{R}^d} \varphi(y) Y_d(\mu, t, dy) + \int_{M_F(\mathbb{R}^d)} (1 - e^{-\langle \nu, \varphi\rangle}) m(\mu, t, d\nu), \tag{15}$$

where we put for simplicity

$$Y_d(\mu, t, dy) = \int Y_d(x, t, dy)\mu(dx), \quad m(\mu.t, d\nu) = \int m(x, t, d\nu)\mu(dx).$$

Moreover, we can derive an important relation from (15)

$$V_t\varphi(x) = \mathbb{E}_{\delta_x}[\langle Y_t, \varphi\rangle] + \int (1 - e^{-\langle \nu, \varphi\rangle} - \langle \nu, \varphi\rangle) m(x, t, d\nu). \tag{16}$$

When we denote by P_t the linear semigroup associated with V_t, then we have

$$P_t\varphi(x) = \mathbb{E}_{\delta_x}[\langle Y_t, \varphi\rangle].$$

On the assumption that $V_t\varphi$ and $P_t\varphi$ are differentiable with respect to time, writing

$$\left.\frac{\partial P_t\varphi}{\partial t}(x)\right|_{x=0} = A\varphi(x),$$

we obtain

$$\left.\frac{\partial V_t\varphi}{\partial t}(x)\right|_{t=0} = A\varphi(x) + \lim_{t\to 0}\frac{1}{t}\int (1 - e^{-\langle \nu, \varphi\rangle} - \langle \nu, \varphi\rangle) m(x, t, d\nu). \tag{17}$$

Under the integrable condition on m

$$\int \langle \nu, 1\rangle \wedge \langle \nu, 1\rangle^2 \frac{1}{t} m(x, t, d\nu) \leqslant C, \quad (\exists C > 0), \quad \forall t < 1 \tag{18}$$

and some proper measurability on the kernel $n(x, d\theta)$, a measure on $(0, \infty)$ satisfying $\int_0^\infty \theta \wedge \theta\ 2n(x, d\theta) < \infty$, the compactness argument (e.g. Le Gall (1999)

[43]) allows us to pass to the limit as $t \to 0$. Finally, we obtain $v(t,x) = V_t\varphi(x)$ satisfying

$$\frac{\partial v}{\partial t}(t,x) = Av - b(x)v - c(x)v^2 + \int_0^\infty (1 - e^{-\theta v(t,x)} - \theta v(t,x))n(x,d\theta). \tag{19}$$

Here A is the generator of a Feller semigroup, $c \geq 0$ and b are bounded measurable functions, and $n: (0,\infty) \to (0,\infty)$ is a kernel satisfying the integrability condition, where required is uniformity with respect to the parameter x. Such a superprocess can indeed be constructed, and the corresponding martingale problem has a unique solution. For the rigorous treatment, see e.g. Fitzsimmons (1988) [29] and ElKaroui-Roelly (1991) [26]. Moreover, the time inhomogeneous case is treated by Dynkin, Kuznetsov and Skorokhod (1994) [25].

3.2 Stable Superprocess

Let α be a parameter such that $(0 < \alpha \leqslant 2)$. $X = \{X_t; t \geq 0\}$ is called an *α-stable superprocess* on $\mathbb{R}^d$ with branching of index $1+\beta \in (1,2)$ (cf. Fleischmann (1988), [30]) if X is a finite measure-valued stochastic process and the log-Laplace equation appearing in the characterization of X is given by

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + au - bu^{1+\beta}, \tag{20}$$

where $a \in \mathbb{R}$, $b > 0$ are any fixed constants, and $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ is fractional Laplacian. The underlying spatial motion of superprocess X is described by a symmetric α-stable motion in $\mathbb{R}^d$ with index $\alpha \in (0,2]$. Especially when $\alpha = 2$, then it just corresponds to the Brownian motion. While, its continuous-state branching mechanism desribed by

$$v \mapsto \Psi(v) = -av + bv^{1+\beta}, \quad v \geq 0$$

belongs to the domain of attraction of a stable law of index $1+\beta \in (1,2]$. The branching is critical if $a = 0$.

It is well known that in dimensions $d < \frac{\alpha}{\beta}$ at any fixed time $t > 0$, the measure $X_t = X_t(dx)$ is absolutely continuous with probability one. That is, there is a density function $\tilde{X}_t(x)$, $x \in \mathbb{R}^d$, such that

$$X_t(dx) = \tilde{X}_t(x)dx.$$

For the case $d < \frac{\alpha}{\beta}$, $\beta \in (0,1)$ and $\alpha = 2$, if $a = 0$ (critical branching), it is proven that a version of the density $\tilde{X}_t(x)$ of the measure $X_t(dx)$ exists and satisfies, in a weak sense, the SPDE

$$\frac{\partial}{\partial t}X_t(x) = \Delta X_t(x) + (bX_{t-}(x))^{1/(1+\beta)}\dot{L}(t,x), \tag{21}$$

where $\dot{L}$ is a $(1+\beta)$-stable noise without negative jumps, cf. Mytnik-Perkins (2003), [36].

4. Measure-Valued Diffusions

The operator L is defined by

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} \quad \text{on} \quad \mathbb{R}^d. \tag{22}$$

The coefficient $a(x) = \{a_{ij}(x)\}$ is positive definite, and $a_{ij}, b_i \in C^{\varepsilon}(\mathbb{R}^d)$ with $\varepsilon \in (0,1]$. We suppose the following assumptions:

(A.1) the martingale problem (MP) for L is well-posed;
(A.2) the diffusion process $\{Y_t\}$ on $\mathbb{R}^d$ corresponding to L is conservative;
(A.3) $\{T_t\}$ is C_0-preserving.

Here $C_0(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \lim_{|x|\to\infty} f(x) = 0\}$, and $\{T_t\}$ is the semigroup corresponding to the L-diffusion Y_t.

We shall explain briefly below the construction of measure-valued diffusions. Each $n \in \mathbb{N}$, consider N_n-particles with each of mass $\frac{1}{n}$, starting at points $x_i^{(n)} \in \mathbb{R}^d$ $(i = 1, 2, \ldots, N_n)$. They are performing independent branching diffusions according to the operator L with branching rate cn, $(c > 0)$ and branching distribution $\{p_k^{(n)}\}_{k=1}^{\infty}$, where

$$\sum_{k=0}^{\infty} k \cdot p_k^{(n)} = 1 + \frac{\gamma}{n}, \quad (\gamma > 0)$$

and

$$\sum_{k=0}^{\infty}(k-1)^2 p_k^{(n)} = m + o(1), (m > 0), (n \to \infty).$$

Let $N_n(t)$ be the number of particles alive at time t, and $\{x_i^{(n)}\}_{i=1}^{N_n(t)}$ be their positions. Define an $M_F(\mathbb{R}^d)$-valued process $X_n(t)$ by

$$X_n(t) = \frac{1}{n}\sum_{i=1}^{N_n(t)} \delta_{x_i^{(n)}(t)}. \tag{23}$$

$\langle \mu, f\rangle$ means the integration of f relative to measure μ, i.e., $\displaystyle\int_{\mathbb{R}^d} f(x)\mu(dx)$. We put $\alpha = cm$ and $\beta = c\gamma$.

Theorem 3. (Roelly-Coppoletta (1986), [42]) *If*

$$X_n(0) = \frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i^{(n)}} \quad \Rightarrow \quad \mu \in M_F(\mathbb{R}^d), \tag{24}$$

then $X_n(\cdot)$ converges weakly to an $M_F(\mathbb{R}^d)$-valued process, which can be uniquely characterized as the solution to the following martingale problem (MP): *the process $X_t \in M_F(\mathbb{R}^d)$ satisfies $X_0 = \mu$,* a.s. *for $\forall f \in C_c^2(\mathbb{R}^d)$*

$$M_t(f) \equiv \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_s, Lf\rangle ds - \beta \int_0^t \langle X_s, f\rangle ds \tag{25}$$

is a martingale with increasing process

$$\langle M(f)\rangle_t = 2\alpha \int_0^t \langle X_s, f^2\rangle ds. \tag{26}$$

Such a process $X = \{X_t; t \geq 0\} = \{X_t, t \geq 0; \mathbb{P}_\mu^{\alpha,\beta}, \mu \in M_F(\mathbb{R}^d)\}$ is called a measure-valued diffusion with papameters $\{\alpha, \beta, L\}$, or $\{\alpha, \beta, L\}$-superprocess. The next is the alternative characterization of measure-valued diffusion via the log-Laplace equation. The $M_F(\mathbb{R}^d)$-valued diffusion $X = \{X_t\}$ with parameters (α, β, L) is characterized by the log-Laplace equation:

$$\mathbb{E}_\mu \exp\left\{-\langle X_t, g\rangle - \int_0^t \langle X_s, \psi\rangle ds\right\} = e^{-\langle \mu, u(t)\rangle} \quad \text{for} \quad \forall g \geq 0, \psi \in C_c^2(\mathbb{R}^d) \tag{27}$$

where $u \equiv u(x,t) \in C^{2,1}(\mathbb{R}^d \times [0,\infty))$ is the unique positive solution of the evolution equation :

$$\begin{aligned} \partial_t u &= Lu + \beta u - \alpha u^2 + \psi, \quad (x,t) \in \mathbb{R}^d \times [0,\infty) \\ u(\cdot, 0) &= g(\cdot), \quad u(\cdot, t) \in C_0(\mathbb{R}^d). \end{aligned} \tag{28}$$

Remark 4. (a) The existence of a classical solution u to the log-Laplace equation follows from the method of semigroups by Pazy (1983) [37]. (b) For the non-negativity of the solution u, the type of argument provided by Iscoe (1986) [33] is used. (c) The uniqueness yields from the parabolic maximum principle in a standard way, see e.g. Lieberman (1996) [35].

4.1 Fundamental Properties

We denote by $Z_t = \langle X_t, 1\rangle$ the total mass process. Under the probability measure $\mathbb{P}_\mu$, Z_t is a 1-dim diffusion process on $[0,\infty)$, corresponding to the operator

$$\mathcal{L} = \alpha x \frac{\partial^2}{\partial x^2} + \beta x \frac{\partial}{\partial x} \tag{29}$$

satisfying $Z_0 = \mu(\mathbb{R}^d)$, ($\mu \in M_F(\mathbb{R}^d)$). Standard techniques from the theory of one-dimensional diffusion processes show that

$$\mathbb{P}_\mu(Z_t > 0, \forall t \geq 0, \lim_{t\to\infty} Z_t = \infty) = 1 - \exp\{-\frac{\beta}{\alpha}\mu(\mathbb{R}^d)\} \tag{30}$$

and also

$$\mathbb{P}_\mu(Z_t = 0, \forall t \gg 1 : \text{large}) = \exp\{-\frac{\beta}{\alpha}\mu(\mathbb{R}^d)\}. \tag{31}$$

DEFINITION 5. (a) The path $X(\cdot)$ *survives*
$\Leftrightarrow$ if $Z_t > 0$ for $\forall t \geq 0$.
(b) The path $X(\cdot)$ becomes *extinct*
$\Leftrightarrow$ if $Z_t = 0$ for $\forall t \gg 1$: large.

The *critical* measure-valued diffusion is obtained by choosing $\gamma = 0$

$$\text{i.e.} \quad \sum_{k=0}^{\infty} k \cdot p_k^{(n)} = 1. \tag{32}$$

In that case, it follows that $\beta \equiv c\gamma = 0$. So that, we obtain

$$\mathbb{P}_\mu(Z_t = 0, \forall t \gg 1 : \text{large}) = e^{-\frac{\beta}{\alpha}\mu(\mathbb{R}^d)} = 1, \tag{33}$$

which implies that $X(\cdot)$ dies out with probability one.

4.2 Transience and Recurrence

Let $X = \{X_t; t \geq 0\}$ be a supercritical $M_F(\mathbb{R}^d)$-valued diffusion, and we denote by $\text{supp}(X)$ the support of the process $X = \{X_t\}$.

DEFINITION 6. (a) $\text{supp}(X)$ is *recurrent* if, for $\forall \mu \in M_F(\mathbb{R}^d), \forall B \subset \mathbb{R}^d$: open set,
$\mathbb{P}_\mu(X_t(B) > 0, \exists t \geq 0 | X(\cdot) \quad \text{survives}) = 1$.
(b) $\text{supp}(X)$ is *transient* if, for $\forall \mu \in M_F(\mathbb{R}^d), B \subset \mathbb{R}^d$: bounded such that $\text{supp}(\mu) \cap \bar{B} = \emptyset$,
$\mathbb{P}_\mu(X_t(B) > 0, \exists t \geq 0 | X(\cdot) \quad \text{suvives}) < 1$.

For $x_0 \in \mathbb{R}^d$ and $R > 0$ fixed, we choose a positive function ϕ such that $\phi \equiv \phi(x) \in C^{2,\eta}(\mathbb{R}^d \setminus \bar{B}_R(x_0))$ is the minimal positive solution to the equation

$$Lu + \beta u - \alpha u^2 = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \bar{B}_R(x_0), \quad \lim_{|x-x_0|\to R} u(x) = \infty. \tag{34}$$

THEOREM 7. *For each $\mu \in M_F(\mathbb{R}^d)$, we have the following expression*

$$\mathbb{P}_\mu(X_t(B_R(x_0)) = 0, \quad \text{for} \quad \forall t \geq 0) = e^{-\langle \mu,\phi\rangle}. \tag{35}$$

Theorem 8. (Criterion)

(a) *If* $\inf_{x \in \mathbb{R}^d \setminus \bar{B}_R(x_0)} \phi(x) \geq \frac{\beta}{\alpha}$, *then* $\mathrm{supp}(X)$ *is recurrent.*

(b) *If* $\liminf_{|x| \to \infty} \phi(x) = 0$, *then* $\mathrm{supp}(X)$ *is transient.*

This result (Theorem 8) is used to obtain the criteria which depend more explicitly on the operator L.

Theorem 9. *If the underlying L-diffusion $Y = \{Y_t\}$ is recurrent, then* $\mathrm{supp}(X)$ *is also recurrent.*

In order to treat the case of transience, we need to define the motions of generalized principal eigenvalues.

4.3 Criticality Theory for Second Order Elliptic Operator

Let $D \subset \mathbb{R}^d$ be a domain, and $\lambda \in \mathbb{R}$. We define

$$C_{L-\lambda}(D) = \{u \in C^2(D) : (L - \lambda)u = 0, u > 0 \quad \text{in} \quad D\}. \tag{36}$$

Definition 10. (a) The operator $L - \lambda$ on D is *subcritical* if it possesses a positive Green's function: in this case, $C_{L-\lambda}(D) \neq \emptyset$.
(b) The operator $L - \lambda$ on D is *critical* if the operator $L - \lambda$ on D does not possess a positive Green's function, but $C_{L-\lambda}(D) \neq \emptyset$.
(c) The operator $L - \lambda$ on D is *supercritical* if the operator $L - \lambda$ on D is neither subcritical nor critical: i.e., $C_{L-\lambda}(D) = \emptyset$.

Then there exists a number $\lambda_c \equiv \lambda_c(D) \in (-\infty, 0]$ such that $L - \lambda$ on D is subcritical for $\lambda > \lambda_c(D)$, and $L - \lambda$ on D is supercritical for $\lambda < \lambda_c(D)$. However, it is either subcritical or critical for $\lambda = \lambda_c(D)$.

Definition 11. Such a number $\lambda_c(D)$ is called the *generalized principal eigenvalue* (GPE) for L on D.

It is monotone non-decreasing as a function of D. Note that

$$\lambda_C(D) = \inf\{\lambda \in \mathbb{R} : C_{L-\lambda}(D) \neq \emptyset\}.$$

Remark 12. If D is bounded and its boundary ∂D is smooth, and if coefficients a_{ij}'s and b_i's of L are smooth up to the boundary ∂D, then $\lambda_c(D) = \lambda_0$, where λ_0 is the classical principal eigenvalue, namely, $\lambda_0 = \sup \mathrm{Re}\{\sigma((L, \mathcal{D}_\alpha))\}$. Let

$\{D_n\}_{n=1}^{\infty}$ be an increasing sequence of bounded domains such that $\mathbb{R}^d = \cup_{n=1}^{\infty} D_n$. Furthermore, we define $\lambda_{c,\infty} = \lim_{n\to\infty} \lambda_c(\mathbb{R}^d \setminus \bar{D}_n)$ $(\leqslant \lambda_c)$, and it is called the generalized principal eigenvalue at ∞.

4.4 Local Extinction

THEOREM 13. (one-dimensional case)

(i) *If* $\beta < -\lambda_{c,\infty}$, *then* $\mathrm{supp}(X)$ *is transient.*

(ii) *If* $\beta > -\lambda_{c,\infty}$, *then* $\mathrm{supp}(X)$ *is recurrent.*

(iii) *If* $\beta = -\lambda_{c,\infty} = -\lambda_c$, *then* $\mathrm{supp}(X)$ *is transient.*

THEOREM 14. (Multidimensional case)
If $\beta < -\lambda_{c,\infty}$ *or if* $\beta = -\lambda_{c,\infty} = -\lambda_c$, *then* $\mathrm{supp}(X)$ *is transient.*

Remark 15. There is an example which illustrates the assertion that it is possible to obtain "transience" in multidimensional case even if $\beta > -\lambda_{c,\infty}$, cf. Pinsky (1996), [38].

DEFINITION 16. The support $\mathrm{supp}(X)$ exhibits *local extinction* if for each $\mu \in M_F(\mathbb{R}^d)$ and each bounded set $B \subset \mathbb{R}^d$, there exists a finite random time $\zeta_B < \infty$, $\mathbb{P}_\mu$-a.s. such that $X_t(B) = 0$ for $\forall t \geq \zeta_B$.

As a matter of fact, the notion of "local extinction" is not equivalent to "transience". It is a rather stronger condition, compared to transience.

THEOREM 17. *The local extinction of* $\mathrm{supp}(X)$ *occurs if and only if* $\beta \leqslant -\lambda_c$.

Remark 18. Thus, if $\lambda_c \neq \lambda_{c,\infty}$ and $\beta \in (-\lambda_c, -\lambda_{c,\infty})$, then $\mathrm{supp}(X)$ is transient. But $\mathrm{supp}(X)$ does not exhibit local extinction. When we denote by $\lambda_c^{(\beta)}$ the GPE for $L+\beta$ on $\mathbb{R}^d$, then Theorem 17 implies that local extinction occurs if and only if $\lambda_c^{(\beta)} \leqslant 0$ according to the terminology of Pinsky (1996) [38].

EXAMPLE 19. (Case study) We consider the operator $L = \frac{1}{2}\frac{d^2}{dx^2} + b_0\frac{d}{dx}$ on $\mathbb{R}$, where $b_0 \neq 0$ is a constant. Then L just corresponds to a transient diffusion $Y = \{Y_t\}$. Since $d = 1$, we define $\lambda_{c,\infty} = \lim_{n\to\infty} \lambda_c((n,\infty))$ and $\lambda_{c,-\infty} = \lambda_c((-\infty,-n))$. Then we obtain $\lambda_c = \lambda_{c,\infty} = \lambda_{c,-\infty} = -\frac{b_0^2}{2}$. Note that $L - \lambda_c$ is critical. If $\beta < \frac{b_0^2}{2}$, then the Grenn's function for $L + \beta = L - (-\beta)$ is given by

$$G_{-\beta}(x,y) = \frac{2\pi}{\sqrt{b_0^2 - 2\beta}} \exp\{-(b_0^2 - 2\beta)^{1/2}|y-x| - b_0(x-y)\}. \tag{37}$$

(a) If $\beta \in (0, \frac{b_0^2}{2})$, then $\mathrm{supp}(X)$ is transient, and also $\mathrm{supp}(X)$ exhibits local extinction.
(b) If $\beta = \frac{b_0^2}{2}$, then $\mathrm{supp}(X)$ is transient, and $\mathrm{supp}(X)$ exhibits local extinction.
(c) If $\beta > \frac{b_0^2}{2}$, then $\mathrm{supp}(X)$ is no more transient, but it is recurrent.

5. Principal Results on Extinction Properties

Case I is the result proved by D.A. Dawson and K. Fleischmann in Z. Wahrsch. Verw. Geb. **70** (1985), [3], where the super-Brownian motion (SBM) X starting from Lebesgue measure λ satisfies *weak local extinction* in lower dimensions. Precisely,

(a) When $d = 1, 2$, for any compact set A,

$$X_t(A) \to 0 \quad \text{in probability as} \quad t \to \infty.$$

(b) However, when $d \geq 3$, then X is persistent, i.e., the limiting random measure X_∞ satisfies

$$\mathbb{P}_\lambda[X_\infty] = \lambda.$$

Case II is about the result that a $(1 + \beta)$-SBM X (with $0 < \beta \leqslant 1$) admits *weak local extinction.* In fact,

(a) X is persistent if and only if $d\beta > 2$.

However,

(b) if $d\beta = 2$, then

$$X_t(A) \to 0 \quad \text{in probability} \quad (t \to \infty).$$

This was proved in D.A. Dawson and K. Fleischmann: Stoch. Proc. Appl. **30** (1988), [4].

Case III is the result by I. Iscoe: Ann. Probab. **16** (1988), [34], where a 1-dimensional (1+1)-SBM X satisfies *alomost sure local extinction.* That is,

$$\mathbb{P}_\lambda - \text{a.s.} \quad X_t(A) = 0 \quad \text{for} \quad t \quad \text{large enough.}$$

It is interesting to note that the Laplace functional of the weighted occupation time for for SBM can also be expressed in terms of the solution to a nonlinear PDE of similar type, cf. I. Iscoe: Probab. Th. Relat. Fields **71** (1986), [33].

Case IV treats the case of a measure-valued diffusion X with constant parameters α, β. Let λ_c be the generalized principal eigenvalue for

$$L = \frac{1}{2} \sum_{i,j} a_{ij} \partial_i \partial_j + \sum_i b_i \partial_i \quad \text{on} \quad \mathbb{R}^d.$$

Assume that X_t is a supercritical measure-valued diffusion. The support supp(X) of the process exhibits *local extinction* if and only if

$$\beta \leqslant -\lambda_c.$$

This result was obtained by R.G. Pinsky in Ann. Probab. **24** (1996), [38].

Case V is about an (L, β, α, D)-superprocess X. The support $\text{supp}(X)$ of the process X exhibits *local extinction* if and only if there exists a positive solution $u > 0$ to the equation

$$(L + \beta)u = 0 \quad \text{on} \quad D.$$

This was proved in J. Engländer and R.G. Pinsky: Ann. Probab. **27** (1999), [27].

Case VI is devoted to a new type of result, where a (1+1)-SBM X exhibits a *stronger version of a.s. local extinction*. Indeed, for $\forall a$ $(0 < a < 1)$,

$$X_t([-t^a, t^a]) = 0 \quad \text{holds} \quad \mathbb{P}_\mu - \text{a.s. for} \quad t: \quad \text{large enough.}$$

This is the result shown by K. Fleischmann, A. Klenke and J. Xiong: J. Theoret. Probab. **19** (2006), [31].

The last Case VII is also about the result of a *stronger version of a.s. local extinction*. Let X be a $(1+\beta)$-SBM with Lebesgue initial measure λ. Suppose that $d < \frac{2}{\beta}$, and let $g(t)$ be a nondecreasing and right-continuous radius and let $B_{g(t)}$ denote a closed ball in $\mathbb{R}^d$ with radius $g(t)$. X. Zhou proved in Stoch. Proc. Appl. **118** (2008) [41] that

$$X_t(B_{g(t)}) = 0 \quad \text{holds} \quad \mathbb{P}_\lambda - \text{a.s. for} \quad \forall t: \quad \text{large enough,}$$

provided that the integrability condition

$$\int_1^\infty g(y)^d y^{-1-1/\beta} dy < \infty$$

is satisfied.

6. Finite Time Extinction

Suppose that $p > d$, and let $\phi_p(x) = (1 + |x|^2)^{-p/2}$ be the reference function. $C = C(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$, and define

$$C_p = \{f \in C : |f| \leqslant C_f \cdot \phi_p, \exists C_f > 0\}. \tag{38}$$

We denote by $M_p = M_p(\mathbb{R}^d)$ the set of non-negative measures μ on $\mathbb{R}^d$, satisfying

$$\langle \mu, \phi_p \rangle = \int \phi_p(x)\mu(dx) < \infty. \tag{39}$$

It is called the space of p-tempered measures. When $\xi = \{\xi_t, P_{s,a},\ s \geq 0, a \in \mathbb{R}^d\}$ is an L-diffusion, then we define the continuous additive functional K_η of ξ by

$$K_\eta = \langle \eta, \delta_x(\xi_r) \rangle dr = \left(\int \delta_X(\xi_r)\eta(dx) \right) dr \quad \text{for} \quad \eta \in M_p \tag{40}$$

or equivalently, we define

$$K[s,t] \equiv K_\eta[s,t] = \int_s^t dr \int \eta(dx)\delta_x(\xi_r), \quad \eta \in M_p. \tag{41}$$

Then $X^\eta = \{X_t^\eta; t \geq 0\}$ is said to be a measure-valued diffusion with branching rate functional K_η if for the initial measure $\mu \in M_F$, X satisfies the Laplace functional of the form

$$\mathbb{P}_{s,\mu}^\eta e^{-\langle X_t^\eta, \varphi\rangle} = e^{-\langle \mu, v(s)\rangle}, \quad (\varphi \in C_b^+), \tag{42}$$

where the function $v \geq 0$ is uniquely determined by

$$P_{s,a}\varphi(\xi_t) = v(s,a) + P_{s,a}\int_s^t v^2(r,\xi_r)K_\eta(dr), \quad (0 < s \leqslant t, a \in \mathbb{R}^d). \tag{43}$$

Assume that $d = 1$ and $0 < \nu < 1$. Let $\lambda \equiv \lambda(dx)$ be the Lebesgue measure on $\mathbb{R}$, and let $(\gamma, \mathbb{P})$ be the stable random measure on $\mathbb{R}$ with Laplace functional

$$\mathbb{P}e^{-\langle \gamma, \varphi\rangle} = \exp\left\{-\int \varphi^\nu(x)\lambda(dx)\right\}, \quad \varphi \in C_b^+. \tag{44}$$

Note that $\mathbb{P}$-a.a ω realization, $\gamma(\omega) \in M_p$ under the condition $p > \nu^{-1}$. We consider a positive CAF K_γ of ξ for $\mathbb{P}$-a.a. ω. So that, thanks to Dynkin's general formalism for superprocess with branching rate functional (see Section 1), there exists an (L, K_γ, μ)-superprocess X^γ when we adopt a p-tempered measure γ for CAF K_η in (40) instead of η, as far as K_γ may lie in the class $\mathbb{K}^q$ for some $q > 0$. Namely, we can get:

THEOREM 20. *Let $K_\gamma \in \mathbb{K}^q$. For $\mu \in M_F$ with compact support, there exists an (L, K_γ, μ)-superprocess with branching rate functional K_γ*, i.e.,

$$\mathbb{P} - \text{a.a.}\omega, \qquad \exists \mathbb{X}^\gamma = \{X^\gamma, \mathbb{P}_{s,\mu}^\gamma, s \geq 0, \mu \in M_F\}.$$

THEOREM 21. (Main Result) *Suppose that $p > 1/\nu$. Let $\mu \in M_F$ with compact support. Then the superprocess X^γ with branching rate functional K_γ dies out for finite time with pwobability one. That is to say,*

$$\mathbb{P} - \text{a.a.} \quad \gamma, \quad \mathbb{P}_{0,\mu}^\gamma(X_t^\gamma = 0, \quad \exists t > 0) = 1 \tag{45}$$

holds.

EXAMPLE 22. For $d = 1, a = 1$ and $b = 0$, X^γ is a stable catalytic SBM. This was initially constructed and investigated by Dawson-Fleischmann-Mueller (2000).

7. Sketch of the Proof

Since the initial measure μ has compact support, according to Dawson-Li-Mueller : Ann. Probab. **23** (1995) [6], X^γ has the compact support property, with the result that the range $\mathcal{R}(X)$ of X^γ is compact. We are going to work with historical superprocesses. As a matter of fact, for the sake of convenient criterion, we put the superprocess X^γ lifted up to the historical superprocess setting $\tilde{X}_t^\gamma(dw)$. For a path $w \in \mathbb{C} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, we consider the stopped path $w^t_\cdot \in \mathbb{C}$ defined by $w^t_s = w_{t\wedge s}$, $(s \geq 0)$, cf. E.B. Dynkin: Probab. Theory Relat. Fields **90** (1991) [23]. Then we can show the existence of the corresponding historical superprocess in Dynkin sense, namely, we can prove that $\exists\ \{\tilde{X}^\gamma, \tilde{\mathbb{P}}_{s,\mu}, s \geq 0, \mu \in M_F(\mathbb{C}^s)\}$.

Theorem 23. *There exists a Dynkin's historical superprocess* $\tilde{\mathbb{X}}^\gamma = \{\tilde{X}^\gamma, \tilde{\mathbb{P}}^\gamma_{s,\mu}, s \geq 0, \mu \in M_F(\mathbb{C}^2)\}$.

We want to show that

$$\lim_{t\to\infty} \tilde{\mathbb{P}}^\gamma_{0,\mu}(\tilde{X}^\gamma_t \neq 0) = 0, \quad \mathbb{P}-\text{a.s.} \tag{46}$$

Moreover, we define

$$\mathbb{C}_K = \{w \in \mathbb{C} : |w_s| < K, \forall s \geq 0\} \quad \text{for} \quad K \geq 1.$$

By the compact support property, we have

$$\lim_{K\to\infty} \inf_{t\geq 0} \tilde{\mathbb{P}}^\gamma_{0,\mu}\left(\text{supp}(\tilde{X}^\gamma_t) \subseteq \mathbb{C}_K\right) = 1, \quad \mathbb{P}-\text{a.a.}\omega. \tag{47}$$

The goal is to show that, $\mathbb{P}$-a.s., $\tilde{\mathbb{P}}^\gamma_{0,\mu}(\tilde{X}^\gamma_t \neq 0)$ vanishes for large t. Hence it suffices to show that, for $\forall K$: large

$$\lim_{t\to\infty} \tilde{\mathbb{P}}^\gamma_{0,\mu}(\tilde{X}^\gamma_t \neq 0 \quad \text{and} \quad \text{supp}(\tilde{X}^\gamma_t) \subset \mathbb{C}_K) = 0. \tag{48}$$

By emplying the periodic extension technique $\gamma \to \gamma^K$, it suffices to show finite time extinction of the historical Dynkin-superprocess $\tilde{X}^{\gamma^K}$ with fixed periodic extension γ^K: i.e.

$$\lim_{t\to\infty} \tilde{\mathbb{P}}^{\gamma^K}_{0,\mu}(\tilde{X}^{\gamma^K}_t \neq 0) = 0, \quad \text{each fixed} \quad K > 1.$$

As a matter of fact, we can show the above expression by using the comparison of extinction probabilities of Dawson-Fleischmann-Mueller: Ann. Probab. **28** (2000) [5] and also by a similar technique on finite time extinction of catalytic branching process of Fleischmann-Mueller (2000) [32]. There is another important key point, i.e., decomposition of initial measures. Suppose that the initial measure has a finite deconposition $\mu = \sum_i \mu_i$. If we can show finite time extinction for each initial measure $X_0^\gamma = \mu_i$, then the branching property implies finite time extinction for $X_0^\gamma = \mu$. Therefore it is very useful that the stable random measure γ admits a representation of sum of discrete points. After all, we obtain

PROPOSITION 24. *For a fixed sample* $\gamma(\omega)$

$$\lim_{t\to\infty} \tilde{\mathbb{P}}_{0,\mu}^\gamma(\tilde{X}_t^\gamma \neq 0 \quad \text{and} \quad \mathrm{supp}(\tilde{X}_t^\gamma) \subseteq \mathbb{C}_K) = 0.$$

Finally, through the projection technique (cf. Dawson-Perkins (1991) [7]; Dôku (2003) [9]), we obtain the following result. For $\mathbb{P}$-a.a. ω,

$$\mathbb{P}_{0,\mu}^\gamma(X_t^\gamma = 0 \quad \text{for some} \quad t > 0) = 1$$

which means that the process X^γ exhibits finite time extinction.

Acknowledgements
This work is supported in part by Japan MEXT Grant-in Aids SR(C) 20540106 and also by ISM Coop. Res. 23-CR-5006.

References

1. Dawson, D.A. : Stochastic evolution equations and related measure processes. J. Multivariate Anal. **5** (1975), 1–52.

2. Dawson, D.A. : *Measure-valued Markov processes.* In "Ecole d'Eté de Probabilité de Saint-Flour, XXI- 1991", pp.1-260. LN Math. **1541** (1993), Springer, Berlin.

3. Dawson, D.A. and Fleischmann, K. : Critical dimension for a model of branching in a random medium. Z. Wahrsch. Verw. Geb. **70** (1985), 315–334.

4. Dawson, D.A. and Fleischmann, K. : Strong clumping of critical sapce-time branching models in subcritical dimensions. Stoch. Proc. Appl. **30** (1988), 193–208.

5. Dawson, D.A., Fleischmann, K. and Mueller, C. : Finite time extinction of superprocesses with catalysts. Ann. Probab. **28** (2000), no.2, 603–642.

6. Dawson, D.A., Li, Y. and Mueller, C. : The support of measure-valued branching processes in a random environment. Ann. Probab. **23** (1995), no.4, 1692–1718.

7. Dawson, D.A. and Perkins, E.A. : *Historical Processes.* Mem. Amer. Math. Soc. **93**, no.454, Providence, 1991.

8. Dôku, I. : Exponential moments of solutions for nonlinear equations with catalytic noise and large deviation. Acta Appl. Math. **63** (2000), 101–117.

9. Dôku, I. : Weighted additive functionals and a class of measure-valued Markov processes with singular branching rate. Far East J. Theo. Stat. **9** (2003), 1–80.

10. Dôku, I. : A certain class of immigration superprocesses and its limit theorem. Adv. Appl. Stat. **6** (2006), 145–205.

11. Dôku, I. : A limit theorem of superprocesses with non-vanishing deterministic immigration. Sci. Math. Jpn. **64** (2006), 563–579.

12. Dôku, I. : Limit theorems for rescaled immigration superprocesses. RIMS Kôkyûroku Bessatsu B6 (2008), 55–69.

13. Dôku, I. : A limit theorem of homogeneous superprocesses with spatially dependent parameters. Far East J. Math. Sci. **38** (2010), 1–38.

14. Dôku, I. : A mathematical model for immune response to the cancer cells. The twentieth conference of JSMB, Abstract Collection, Disease I, O-8 (2010), p.66.

15. Dôku, I. : An application of random model to mathematical medicine. ISM Cop. Res. Rept. **262** (2011), 108–118.

16. Dôku, I. : On mathematical modelling for immune response to the cancer cells. J. Saitama Univ. Fac. Educ. Math. Nat. Sci. **60** (2011), no.1, 137–148.

17. Dôku, I. : On a random model for immune response: toward a modelling of antitumor immune responses. RIMS Kôkyûroku, **1751** (2011), 18–24.

18. Dôku, I. : A remark on tumor-induced angiogenesis from the viewpoint of mathematical cell biology: mathematical medical approach via stochastic modelling. J. Saitama Univ. Fac. Educ. Math. Nat. Sci. **60** (2011), no.2, 205–217.

19. Dôku, I. : On extinction property of superprocesses. ISM Cop. Res. Rept. **275** (2012), 34–42.

20. Dôku, I. : A stochastic model for tumor immune response from the standpoint of mathematical medicine: immunomechanism against cancer cells and extinction property of superprocesses. J. Saitama Univ. Fac. Educ. Math. Nat. Sci. **61** (2012), no.1, 223–232.

21. Dôku, I. : A random model for tumor immunobiomechanism arising in mathematical physiology. preprint.

22. Dôku, I. : A random model for tumor immunobiomechanism: theoretical implication for host-defense mechanism. preprint.

23. Dynkin, E.B. : Path processes and historical superprocesses. Probab. Theory Relat. Fields **90** (1991), 1–36.

24. Dynkin, E.B. : *An Introduction to Branching Measure-Valued Processes.* AMS, Providence, 1994.

25. Dynkin, E.B., Kuznetsov, S.E. and Skorokhod, A.V. : Branching measure-valued processes. Prob. Theory Relat. Fields **99** (1994), no.1, 55–96.

26. ElKaroui, N. and Roelly, S. : Proprietés de martingales, explosion et representation de Lévy-Khinchine d'une classe du processus de branchement à valeurs mesures. Stoch. Proc. Appl. **38** (1991), no.2, 239–266.

27. Engländer, J. and Pinsky, R.G. : On the construction and support properties of measure-valued diffusions on $D \subseteq \mathbb{R}^d$ with spatially dependent branching. Ann. Probab. **27** (1999), no.2, 684–730.

28. Etheridge, A.M. : *An Introduction to Superprocesses.* Univ. Lecture Series, volume 20, Amer. Math. Soc., Providence, 2000.

29. Fitzsimmons, P.J. : Construction and regularity of measure-valued branching processes. Israel J. Math. **64** (1988), 337–361; & **73** (1991), 127.

30. Fleischmann, K. : Critical behavior of some measure-valued processes. Math. Nachr. **135** (1988), 131–147.

31. Fleischmann, K., Klenke, A. and Xiong, J. : Pathwise convergence of a rescaled super-Brownian catalyst reactant process. J. Theoret. Probab. **19** (2006), no.3, 557–588.

32. Fleischmann, K. and Mueller, C. : Finite time extinction of catalytic branching processes. In *Stochastic Models.* A Conference in Honour of Donald A. Dawson (eds. Gorostiza, L.G. and Ivanoff, B.G.), pp.125–139. Amer. Math. Soc., Providence, 2000.

33. Iscoe, I. : A weighted occupation time for a class of measure-valued branching processes. Probab. Theory Relat. Fields **71** (1986), 85–116.

34. Iscoe, I. : On the supports of measure-valued critical branching Brownian motion. Ann. Probab. **16** (1988), 200–221.

35. Lieberman, G.M. : *Second Order Parabolic Differential Equations.* World Scientific, Singapore, 1996.

36. Mytnik, L. and Perkins, E.A. : Regularity and irregularity of $(1+\beta)$-stable super-Brownian motion. Ann. Probab. **31** (2003), 1413–1440.

37. Pazy, A. : *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, Berlin, 1983.

38. Pinsky, R.G. : Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions. Ann. Probab. **24** (1996), 237–267.

39. Sato, K. : *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Univ. Press, Cambridge, 1999.

40. Watanabe, S. : A limit theorem of branching processes and continuous state branching processes. J. Math. Kyoto Univ. **8** (1968), 141–167.

41. Zhou, X. : A zero-one law of almost sure local extinction for $(1+\beta)$-super-Brownina motion. Stoch. Proc. Appl. **118** (2008), 1982–1996.

42. Roelly-Coppoletta, S. : A criterion of convergence of measure-valued processes: application to measure branching processes. Stochastics **17** (1986), 43–65.

43. Le Gall, J.-F. : *Spatial Branching processes, Random Snakes and Partial Differential Equations.* Lectures in Math. ETH Zürich, Birkhäuser, Basel, 1999.

Department of Mathematics
Faculty of Education, Saitama University
Saitama, 338-8570 Japan
e-mail : idoku@mail.saitama-u.ac.jp

(Received March 19, 2012)
(Accepted May 18, 2012)