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On A Class of Historical Superprocesses in The Dynkin Sense

DÔKU, Isamu

Faculty of Education, Saitama University

Abstract

We study a class of historical superprocesses in the Dynkin sense. This class is closely related to a group of superprocesses, namely, measure-valued branching Markov processes, which are associated with stable random measure. Then we prove the existence theorem for the class of historical superprocesses in Dynkin's sense.

Key Words : Dynkin's historical superprocess, measure-valued Markov process, superprocess, stable random measure, existence.

1. Introduction

The purpose of this paper is to investigate a class of historical superprocesses in Dynkin's sense. This class is closely related to another class of superprocesses, namely, measure-valued branching Markov processes, which are associated with stable random measure. Then it is proven that the class of historical superprocesses in Dynkin's sense exists under some suitable conditions. Now we shall begin with introducing a fundamental relationship between superprocess and its associated historical superprocess.

1.1 Superprocess with Branching Rate Functional

In this subsection we shall introduce the so-called superprocess with branching rate functional, which forms a general class of measure-valued branching Markov processes with diffusion as a underlying spatial motion. We write the integral of measurable function f with respect to measure μ as $\langle \mu, f\rangle = \int f d\mu$. For simplicity, $M_F = M_F(\mathbb{R}^d)$ is the space of finite measures on $\mathbb{R}^d$. Define a second order elliptic differential operator $L = \frac{1}{2}\nabla \cdot a\nabla + b\cdot\nabla$, and $a = (a_{ij})$ is positive definite and we assume that $a_{ij}, b_i \in C^{1,\varepsilon} = C^{1,\varepsilon}(\mathbb{R}^d)$. Here the space $C^{1,\varepsilon}$ indicates the totality of all Hölder continuous functions with index ε $(0 < \varepsilon \leqslant 1)$, allowing their first order derivatives to be locally Hölder continuous. $\Xi = \{\xi, \Pi_{s,a}, s \geq 0, a \in \mathbb{R}^d\}$ indicates a corresponding L-diffusion. Moreover, CAF stands for continuous additive functional in Probability Theory.

Definition 1. (A Locally Admissible Class of CAF; cf. Dynkin (1994), [17]) A continuous additive functional K is said to be in the Dynkin class with index q and we write $K \in \mathbb{K}^q$, (some $q > 0$) if (a)

$$\sup_{a \in \mathbb{R}^d} \Pi_{s,a} \int_s^t \phi_p(\xi_r) K(dr) \to 0, \quad (r_0 \geq 0) \quad \text{as} \quad s \to r_0, t \to r_0; \tag{1}$$

(b) each N, $\exists c_N > 0$:

$$\Pi_{s,a} \int_s^t \phi_p(\xi_r) K(dr) \leqslant c_N |t-s|^q \phi_p(a), \quad (\text{for} \quad 0 \leqslant s \leqslant t \leqslant N, \quad a \in \mathbb{R}^d). \tag{2}$$

□

When we write C_b as the set of bounded continuous functions on $\mathbb{R}^d$, then C_b^+ is the set of positive members g in C_b. The process $\mathbb{X} = \{X, \mathbb{P}_{s,\mu}, \, s \geq 0, \mu \in M_F\}$ is said to be a *superprocess (or superdiffusion) with branching rate functional* K or simply (L, K, μ)-*superprocess* if $X = \{X_t\}$ is a continous M_F-valued time-inhomogeneous Markov process with Laplace transition functional

$$\mathbb{P}_{s,\mu} e^{-\langle X_t, \varphi \rangle} = e^{-\langle \mu, v(s,t) \rangle}, \qquad 0 \leqslant s \leqslant t, \quad \mu \in M_F, \quad \varphi \in C_b^+. \tag{3}$$

Here the function v is uniquely determined by the following log-Laplace equation

$$\Pi_{s,a} \varphi(\xi_t) = v(s,a) + \Pi_{s,a} \int_s^t v^2(r, \xi_r) K(dr), \qquad 0 \leqslant s \leqslant t, \quad a \in \mathbb{R}^d. \tag{4}$$

1.2 Associated Historical Superprocess

The historical superprocess was initially studied by E.B. Dynkin (1991) [15],[16], see also Dawson-Perkins (1991) [5]. $\mathbb{C} = C(\mathbb{R}_+, \mathbb{R}^d)$ denotes the space of continuous paths on $\mathbb{R}^d$ with topology of uniform convergence on compact subsets of $\mathbb{R}_+$. To each $w \in \mathbb{C}$ and $t > 0$, we write $w^t \in \mathbb{C}$ as the stopped path of w. We denote by $\mathbb{C}^t$ the totality of all these paths stopped at time t. To every $w \in \mathbb{C}$ we associate the corresponding stopped path trajectory $\tilde{w}$ defined by $\tilde{w}_t = w^t$ for $t \geq 0$. The image of L-diffusion w under the map : $w \mapsto \tilde{w}$ is called the *L-diffusion path process.* Moreover, we define

$$\mathbb{C}_R^{\times} \equiv \mathbb{R}_+ \hat{\times} \mathbb{C}^{\cdot} = \{(s,w) : s \in \mathbb{R}_+, w \in \mathbb{C}^s\} \tag{5}$$

and we denote by $M(\mathbb{C}_R^{\times}) \equiv M(\mathbb{R}_+ \hat{\times} \mathbb{C}^{\cdot})$ the set of measures η on $\mathbb{R}_+ \hat{\times} \mathbb{C}^{\cdot}$ which are finite, if restricted to a finite time interval. Let K be a positive CAF of ξ. $\tilde{\mathbb{X}} = \{\tilde{X}, \tilde{\mathbb{P}}_{s,\mu}, s \geq 0, \, \mu \in M_F(\mathbb{C}^s)\}$ is said to be a Dynkin's *historical superprocess* (cf.

Dynkin (1991), [16]) if $\tilde{X} = \{\tilde{X}_t\}$ is a time-inhomogeneous Markov process with state $\tilde{X}_t \in M_F(\mathbb{C}^t)$, $t \geq s$, with Laplace transition functional

$$\tilde{\mathbb{P}}_{s,\mu} e^{-\langle \tilde{X}_t, \varphi \rangle} = e^{-\langle \mu, v(s,t) \rangle}, \quad 0 \leqslant s \leqslant t, \quad \mu \in M_F(\mathbb{C}^s), \quad \varphi \in C_b^+(\mathbb{C}), \tag{6}$$

where the function v is uniquely determined by the log-Laplace type equation

$$\tilde{\Pi}_{s,w_s} \varphi(\tilde{\xi}_t) = v(s, w_s) + \tilde{\Pi}_{s,w_s} \int_s^t v^2(r, \tilde{\xi}_r) \tilde{K}(dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \tag{7}$$

We call this class of process an associted historical superprocess in Dynkin's sense in this article.

2. Examples of Measure-Valued Processes

2.1 Dawson-Watanabe Superprocess

$X = \{X_t; t \geq 0\}$ is said to be the *Dawson-Watanabe superprocess* (cf. Watanabe (1968), [29], Dawson (1975), [1]) if $\{X_t\}$ is a Markov process taking values in the space $M_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$, satisfying the following martingale problem (MP): i.e., there exists a probability measure $\mathbb{P} \in \mathcal{P}(M_F(\mathbb{R}^d))$ on the sapce $M_F(\mathbb{R}^d)$ such that for all $\varphi \in \mathrm{Dom}(\Delta)$

$$M_t(\varphi) \equiv \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \int_0^t \langle X_s, \frac{1}{2} \Delta \varphi \rangle ds \tag{8}$$

is a $\mathbb{P}$-martingale and its quadratic variation process is given by

$$\langle M(\varphi) \rangle_t = \gamma \int_0^t \langle X_s, \varphi^2 \rangle ds. \tag{9}$$

Or equivalently, the Laplace transition functional of $\{X_t\}$ is given by

$$\mathbb{E} e^{-\langle X_t, \varphi \rangle} = e^{-\langle X_0, u(t) \rangle}, \qquad \text{for} \quad \varphi \in C_b^+(\mathbb{R}^d) \cap \mathrm{Dom}(\Delta), \tag{10}$$

where $u(t,x)$ is the unique positive solution of evolution equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u - \frac{1}{2} \gamma u^2, \qquad u(0,x) = \varphi(x). \tag{11}$$

The Dawson-Watanabe superprocess X_t inherits the branching property from the approximating branching Brownian motions (cf. Dawson (1993), [2]). Namely,

$$P_t(\cdot, \nu_1 + \nu_2) = P_t(\cdot, \nu_1) * P_t(\cdot, \nu_2), \tag{12}$$

where $P_t(\cdot, \mu)$ denotes the transition probability with the initial data μ. The branching property extends in the obvious way to initial measures of the form $\nu_1 + \cdots + \nu_n$. Conversely, for any integer n, the distribution of the superprocess started from initial measure μ is written as that of the sum of n independent copies of the superprocesses each started from μ/n. This implies that the Dawson-Watanabe superprocess is infinitely divisible.

The following proposition is well known. The statement is the analogue of the classical Lévy-Khintchine formula (cf. Sato (1999), [28]: Theorem 8.1, p.37), that characterizes the possible characteristic functions of infinitely divisible distributions on $\mathbb{R}^d$, in the measure-valued setting.

THEOREM 2. (Canonical Representation Theorem) *Let $(E,\mathcal{E})$ be a Polish space, and X be infinitely divisible Random measure on $(E,\mathcal{E})$. Then there exist measures $X_d \in M_F(E)$ and $m \in M(M_F(E))$, $m \neq 0$ such that for $\forall\varphi \in C_b(E)$, $\int\{1 - e^{-\langle\nu,\varphi\rangle}\}m(d\nu) < \infty$ and*

$$-\log \mathbb{E}[e^{-\langle X,\varphi\rangle}] = \langle X_d, \varphi\rangle + \int\{1 - e^{-\langle\nu,\varphi\rangle}\}m(d\nu). \tag{13}$$

If $m(\{0\}) = 0$, then X_d and m are unique. (cf. Etheridge (2000), [20]: Theorem 1.28, p.18)

It is natural to ask whether we can construct other processes in $M_F(\mathbb{R}^d)$ (i.e. superprocesses) with infinitely divisible distributions. As a matter of fact, using the above canonical representation formula as a starting point, we can construct a more general superprocess. To keep the notation as simple as possible, we restrict our plan to time homogeneous Markov processes satisfying two conditions: (i) branching property; (ii) infinite divisibility. Let us denote by $Y = \{Y_t\}$ a time homogeneous Markov process. We have

$$\mathbb{E}_\mu[e^{-\langle Y_t,\varphi\rangle}] = e^{-\langle\mu,V_t\varphi\rangle}. \tag{14}$$

This operator V_t satisfies the property $V_{t+s} = V_t \circ V_s$. Comparing the formula (13) and Eq.(14), we can write with the uniqueness of the canonical representation,

$$\langle\mu, V_t\varphi\rangle = \int_{\mathbb{R}^d} \varphi(y)Y_d(\mu,t,dy) + \int_{M_F(\mathbb{R}^d)} (1 - e^{-\langle\nu,\varphi\rangle})m(\mu,t,d\nu), \tag{15}$$

where we put for simplicity

$$Y_d(\mu,t,dy) = \int Y_d(x,t,dy)\mu(dx), \quad m(\mu.t,d\nu) = \int m(x,t,d\nu)\mu(dx).$$

Moreover, we can derive an important relation from (15)

$$V_t\varphi(x) = \mathbb{E}_{\delta_x}[\langle Y_t,\varphi\rangle] + \int (1 - e^{-\langle\nu,\varphi\rangle} - \langle\nu,\varphi\rangle)m(x,t,d\nu). \tag{16}$$

When we denote by P_t the linear semigroup associated with V_t, then we have

$$P_t\varphi(x) = \mathbb{E}_{\delta_x}[\langle Y_t, \varphi\rangle], \qquad \left.\frac{\partial P_t\varphi}{\partial t}(x)\right|_{x=0} = A\varphi(x). \tag{17}$$

Under the integrable condition on m

$$\int \langle \nu, 1\rangle \wedge \langle \nu, 1\rangle^2 \frac{1}{t} m(x, t, d\nu) \leqslant C, \quad (\exists C > 0), \quad \forall t < 1 \tag{18}$$

and some proper measurability on the kernel $n(x, d\theta)$, a measure on $(0, \infty)$ satisfying $\int_0^\infty \theta \wedge \theta^2 n(x, d\theta) < \infty$, the compactness argument (e.g. Le Gall (1999) [31]) allows us to obtain $v(t, x) = V_t\varphi(x)$ satisfying

$$\begin{aligned}\frac{\partial v}{\partial t}(t, x) &= Av - b(x)v - c(x)v^2 \\ &\quad + \int_0^\infty (1 - e^{-\theta v(t,x)} - \theta v(t, x))n(x, d\theta).\end{aligned} \tag{19}$$

Here A is the generator of a Feller semigroup, $c \geq 0$ and b are bounded measurable functions, and $n: (0, \infty) \to (0, \infty)$ is a kernel satisfying the integrability condition, where required is uniformity with respect to the parameter x. Note that the corresponding martingale problem has a unique solution. For the rigorous treatment, see e.g. Fitzsimmons (1988) [21] and ElKaroui-Roelly (1991) [19]. Moreover, the time inhomogeneous case is treated by Dynkin, Kuznetsov and Skorokhod (1994) [18].

2.2 Stable Superprocess

Let α be a parameter such that $(0 < \alpha \leqslant 2)$. $X = \{X_t; t \geq 0\}$ is called an *α-stable superprocess* on $\mathbb{R}^d$ with branching of index $1+\beta \in (1, 2)$ (cf. Fleischmann (1988), [22]) if X is a finite measure-valued stochastic process and the log-Laplace equation appearing in the characterization of X is given by

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + au - bu^{1+\beta}, \tag{20}$$

where $a \in \mathbb{R}$, $b > 0$ are any fixed constants, and $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ is fractional Laplacian. The underlying spatial motion of superprocess X is described by a symmetric α-stable motion in $\mathbb{R}^d$ with index $\alpha \in (0, 2]$. Especially when $\alpha = 2$, then it just corresponds to the Brownian motion. While, its continuous-state branching mechanism desribed by

$$v \mapsto \Psi(v) = -av + bv^{1+\beta}, \quad v \geq 0$$

belongs to the domain of attraction of a stable law of index $1 + \beta \in (1, 2]$. The branching is critical if $a = 0$. See also Mytnik-Perkins (2003), [26].

2.3 Measure-Valued Diffusions

The operator L is defined by

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} \quad \text{on} \quad \mathbb{R}^d. \tag{21}$$

The coefficient $a(x) = \{a_{ij}(x)\}$ is positive definite, and $a_{ij}, b_i \in C^{\varepsilon}(\mathbb{R}^d)$ with $\varepsilon \in (0,1]$. We suppose the following assumptions:

(A.1) the martingale problem (MP) for L is well-posed;
(A.2) the diffusion process $\{Y_t\}$ on $\mathbb{R}^d$ corresponding to L is conservative;
(A.3) $\{T_t\}$ is C_0-preserving.

Here $C_0(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \lim_{|x|\to\infty} f(x) = 0\}$, and $\{T_t\}$ is the semigroup corresponding to the L-diffusion Y_t.

We shall explain briefly below the construction of measure-valued diffusions. Each $n \in \mathbb{N}$, consider N_n-particles with each of mass $\frac{1}{n}$, starting at points $x_i^{(n)} \in \mathbb{R}^d$ $(i = 1, 2, \ldots, N_n)$. They are performing independent branching diffusions according to the operator L with branching rate cn, $(c > 0)$ and branching distribution $\{p_k^{(n)}\}_{k=1}^{\infty}$, where

$$\sum_{k=0}^{\infty} k \cdot p_k^{(n)} = 1 + \frac{\gamma}{n}, \quad (\gamma > 0)$$

and

$$\sum_{k=0}^{\infty} (k-1)^2 p_k^{(n)} = m + o(1), (m > 0), (n \to \infty).$$

Let $N_n(t)$ be the number of particles alive at time t, and $\{x_i^{(n)}\}_{i=1}^{N_n(t)}$ be their positions. Define an $M_F(\mathbb{R}^d)$-valued process $X_n(t)$ by

$$X_n(t) = \frac{1}{n} \sum_{i=1}^{N_n(t)} \delta_{x_i^{(n)}(t)}. \tag{22}$$

$\langle \mu, f \rangle$ means the integration of f relative to measure μ, i.e., $\displaystyle\int_{\mathbb{R}^d} f(x)\mu(dx)$. We put $\alpha = cm$ and $\beta = c\gamma$.

THEOREM 3. (Roelly-Coppoletta (1986), [30]) *If*

$$X_n(0) = \frac{1}{n} \sum_{i=1}^{N_n} \delta_{x_i^{(n)}} \quad \Rightarrow \quad \mu \in M_F(\mathbb{R}^d), \tag{23}$$

then $X_n(\cdot)$ converges weakly to an $M_F(\mathbb{R}^d)$-valued process, which can be uniquely characterized as the solution to the following martingale problem (MP): *the process*

$X_t \in M_F(\mathbb{R}^d)$ *satisfies* $X_0 = \mu$, a.s. *for* $\forall f \in C_c^2(\mathbb{R}^d)$

$$M_t(f) \equiv \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_s, Lf\rangle ds - \beta \int_0^t \langle X_s, f\rangle ds \tag{24}$$

is a martingale with increasing process

$$\langle M(f)\rangle_t = 2\alpha \int_0^t \langle X_s, f^2\rangle ds. \tag{25}$$

Such a process $X = \{X_t; t \geq 0\} = \{X_t, t \geq 0; \mathbb{P}_\mu^{\alpha,\beta}, \mu \in M_F(\mathbb{R}^d)\}$ is called a measure-valued diffusion with papameters $\{\alpha, \beta, L\}$, or $\{\alpha, \beta, L\}$-superprocess. The next is the alternative characterization of measure-valued diffusion via the log-Laplace equation. The $M_F(\mathbb{R}^d)$-valued diffusion $X = \{X_t\}$ with parameters (α, β, L) is characterized by the log-Laplace equation:

$$\mathbb{E}_\mu \exp\left\{-\langle X_t, g\rangle - \int_0^t \langle X_s, \psi\rangle ds\right\} = e^{-\langle \mu, u(t)\rangle} \quad \text{for} \quad \forall g \geq 0, \psi \in C_c^2(\mathbb{R}^d) \tag{26}$$

where $u \equiv u(x,t) \in C^{2,1}(\mathbb{R}^d \times [0,\infty))$ is the unique positive solution of the evolution equation :

$$\begin{aligned} \partial_t u = Lu + \beta u - \alpha u^2 + \psi, \quad (x,t) \in \mathbb{R}^d \times [0,\infty) \\ u(\cdot, 0) = g(\cdot), \quad u(\cdot, t) \in C_0(\mathbb{R}^d). \end{aligned} \tag{27}$$

Remark 4. (a) The existence of a classical solution u to the log-Laplace equation follows from the method of semigroups by Pazy (1983) [27]. (b) For the non-negativity of the solution u, the type of argument provided by Iscoe (1986) [24] is used. (c) The uniqueness yields from the parabolic maximum principle in a standard way, see e.g. Lieberman (1996) [25].

3. Superprocess Related to Random Measure

Suppose that $p > d$, and let $\phi_p(x) = (1 + |x|^2)^{-p/2}$ be the reference function. $C = C(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$, and define

$$C_p = \{f \in C : |f| \leqslant C_f \cdot \phi_p, \exists C_f > 0\}. \tag{28}$$

We denote by $M_p = M_p(\mathbb{R}^d)$ the set of non-negative measures μ on $\mathbb{R}^d$, satisfying

$$\langle \mu, \phi_p\rangle = \int \phi_p(x)\mu(dx) < \infty. \tag{29}$$

It is called the space of p-tempered measures. When $\Xi = \{\xi_t, \Pi_{s,a},\ s \geq 0, a \in \mathbb{R}^d\}$ is an L-diffusion, then we define the continuous additive functional K_η of ξ by

$$K_\eta = \langle \eta, \delta_x(\xi_r)\rangle dr = \left(\int \delta_X(\xi_r)\eta(dx) \right) dr \quad \text{for} \quad \eta \in M_p \tag{30}$$

or equivalently, we define

$$K[s,t] \equiv K_\eta[s,t] = \int_s^t dr \int \eta(dx)\delta_x(\xi_r), \quad \eta \in M_p. \tag{31}$$

Then $X^\eta = \{X_t^\eta; t \geq 0\}$ is said to be a measure-valued diffusion with branching rate functional K_η if for the initial measure $\mu \in M_F$, X satisfies the Laplace functional of the form

$$\mathbb{P}^\eta_{s,\mu} e^{-\langle X_t^\eta, \varphi\rangle} = e^{-\langle \mu, v(s)\rangle}, \quad (\varphi \in C_b^+), \tag{32}$$

where the function $v \geq 0$ is uniquely determined by

$$\Pi_{s,a}\varphi(\xi_t) = v(s,a) + \Pi_{s,a}\int_s^t v^2(r,\xi_r)K_\eta(dr), \quad (0 < s \leqslant t, a \in \mathbb{R}^d). \tag{33}$$

Assume that $d = 1$ and $0 < \nu < 1$. Let $\lambda \equiv \lambda(dx)$ be the Lebesgue measure on $\mathbb{R}$, and let $(\gamma, \mathbb{P})$ be the stable random measure on $\mathbb{R}$ with Laplace functional

$$\mathbb{P}e^{-\langle \gamma, \varphi\rangle} = \exp\left\{ -\int \varphi^\nu(x)\lambda(dx) \right\}, \quad \varphi \in C_b^+. \tag{34}$$

Note that $\mathbb{P}$-a.a ω realization, $\gamma(\omega) \in M_p$ under the condition $p > \nu^{-1}$. We consider a positive CAF K_γ of ξ for $\mathbb{P}$-a.a. ω. So that, thanks to Dynkin's general formalism for superprocess with branching rate functional (see Section 1), there exists an (L, K_γ, μ)-superprocess X^γ when we adopt a p-tempered measure γ for CAF K_η in (30) instead of η, as far as $K_\gamma = K_\gamma(\omega; dr)$ may lie in the class $\mathbb{K}^q$ for some $q > 0$. Namely, we can get:

Theorem 5. *Let $K_\gamma \in \mathbb{K}^q$. For $\mu \in M_F$ with compact support, there exists an (L, K_γ, μ)-superprocess with branching rate functional K_γ*, i.e.,

$$\mathbb{P} - \text{a.a.}\omega, \qquad \exists\ \mathbb{X}^\gamma = \{X^\gamma, \mathbb{P}^\gamma_{s,\mu}, s \geq 0, \mu \in M_F\}.$$

Example 6. For $d = 1, a = 1$ and $b = 0$, X^γ is a stable catalytic SBM. This was initially constructed and investigated by Dawson-Fleischmann-Mueller (2000) [3].

4. Historical Superprocess Related to Random Measure

Since the initial measure μ has compact support, according to Dawson-Li-Mueller : Ann. Probab. **23** (1995) [4], X^γ has the compact support property, with the result that the range $\mathcal{R}(X)$ of X^γ is compact. We are going to work with historical superprocesses. As a matter of fact, for the sake of convenient criterion, we put the superprocess X^γ lifted up to the historical superprocess setting $\tilde{X}^\gamma_t(dw)$. For a path $w \in \mathbb{C} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, we consider the stopped path $w^t_\cdot \in \mathbb{C}$ defined by $w^t_s = w_{t\wedge s}$, $(s \geq 0)$, cf. E.B. Dynkin: Probab. Theory Relat. Fields **90** (1991) [16]. Then we can show the existence of the corresponding historical superprocess in the Dynkin sense, namely, we can prove that $\exists\ \{\tilde{X}^\gamma, \tilde{\mathbb{P}}^\gamma_{s,\mu}, s \geq 0, \mu \in M_F(\mathbb{C}^s)\}$.

Theorem 7. (Main Result) *Let K_γ be a positive CAF of ξ lying in the Dynkin class $\mathbb{K}^q$. Then there exists a historical superprocess in the Dynkin sense*

$$\tilde{\mathbb{X}}^\gamma = \{\tilde{X}^\gamma, \tilde{\mathbb{P}}^\gamma_{s,\mu}, s \geq 0, \mu \in M_F(\mathbb{C}^s)\}.$$

In fact, $\tilde{\mathbb{X}}^\gamma = \{\tilde{X}^\gamma, \tilde{\mathbb{P}}^\gamma_{s,\mu}, s \geq 0, \mu \in M_F(\mathbb{C}^s)\}$ is a time-inhomogeneous Markov process with state $\tilde{X}^\gamma_t \in M_F(\mathbb{C}^t)$, $t \geq s$, with Laplace transition functional

$$\tilde{\mathbb{P}}^\gamma_{s,\mu} \exp\left\{-\langle \tilde{X}^\gamma_t, \varphi\rangle\right\} = e^{-\langle \mu, v(s,t)\rangle}, \quad 0 \leqslant s \leqslant t, \quad \mu \in M_F(\mathbb{C}^s), \quad \varphi \in C^+_b(\mathbb{C}), \tag{35}$$

where the function v is uniquely determined by the log-Laplace type equation

$$\tilde{\Pi}_{s,w_s}\varphi(\tilde{\xi}_t) = v(s, w_s) + \tilde{\Pi}_{s,w_s}\int_s^t v^2(r, \tilde{\xi}_r)\tilde{K}_\gamma(\omega; dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \tag{36}$$

Remark 8. There is another important key point, i.e., decomposition of initial measures. Suppose that the initial measure has a finite deconposition $\mu = \sum_i \mu_i$. If we can show finite time extinction for each initial measure $X^\gamma_0 = \mu_i$, then the branching property implies finite time extinction for $X^\gamma_0 = \mu$. Therefore it is very useful that the stable random measure γ admits a representation of sum of discrete points.

5. Proof of Main Result

Roughly speaking, in order to prove our principal result Theorem 7, we need to apply Theorem 5 (the existence theorem for superprocess related to stable random measure) to historical process.

5.1 Reformulation

In this subsection for convenience we shall adopt some notation and terminology from Dynkin (1991) : Ann. Probab. **19** (1991) [15]. Let $(E_t, \mathcal{B}_t)$ be a measurable space that describes the state space of the underlying process ξ at time t (which can usually be imbedded isomorphically into a compact metrizable space $\mathcal{C}$), and $\hat{E}$ be the global state space given by the set of pairs $t \in \mathbb{R}_+$ and $x \in E_t$. The symbol $\mathcal{B}(\hat{E})$ denotes the σ-algebra in $\hat{E}$, generated by functions $f : \hat{E} \to \mathbb{R}$. Note that

$$\hat{E}(I) = \{(r, x) : r \in I, x \in E_r\} \in \mathcal{B}(\hat{E})$$

for every interval I. The sample space W is a set of paths (or trajectories) $\xi_t(w) = w_t$ for each $w \in W$. Furthermore, $\mathcal{F}(I)$ is the σ-algebra generated by $\xi_t(w)$ for $t \in I$. Let $w(I)$ denote the restriction of $w \in W$ to I, and $W(I)$ be the image of W under this mapping. Especially when $\mathcal{F}^\circ(I)$ is the σ-algebra in $W(I)$ generated by ξ_t, $t \in I$, then $B \in \mathcal{F}^\circ(I)$ is equivalent to the inverse image of B under the mapping $w \mapsto w(I)$ belongs to $\mathcal{F}(I)$. Let $r < t < u$. Suppose that $w^1 \in W[r, t]$ and $w^2 \in W[t, u]$. Then we write $w = w^1 \vee w^2$ if $w_s = w^1_s$ for $s \in [r, t]$ and if $w_s = w^2_s$ for $s \in [t, u]$. Moreover,

$$\tilde{\Xi} = (\xi_{\leqslant t}, \mathcal{F}_{\tilde{\Xi}}(I), \tilde{\Pi}_{r,w(\leqslant r)}) = (\xi(-\infty, t], \mathcal{F}_{\tilde{\Xi}}(I), \tilde{\Pi}_{r,w(-\infty,r]})$$

is the historical process for $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,a})$. Under those circumstances, it suffices to prove the following assertion Theorem 9 in order to prove the main result Theorem 7.

THOREM 9. *Let $\tilde{\Xi}$ be a historical process, $\tilde{K}_\gamma = \tilde{K}_\gamma(\omega)$ be its CAF associated to stable random measure γ with properties*:
(a) *For every $q > 0$, $r < t$ and $x \in E_r$,*

$$\tilde{\Pi}_{r,x(\leqslant r)} e^{q\tilde{K}_\gamma(\omega;(r,t))} < \infty.$$

(b) *For every $t_0 < t$, there exists a positive constant C such that*

$$\tilde{\Pi}_{r,x(\leqslant r)} \tilde{K}_\gamma(\omega; (r, t)) \leqslant C$$

holds for $r \in [t_0, t)$, $x \in E_r$. Put $\psi^t(x, z) = b^t(x)z^2 = 1 \times z^2$. Then there exists a Markov process

$$M^\gamma = (M^\gamma_t, \mathcal{G}(I), P^\gamma_{r,\mu})$$

on the space $\mathcal{M}_{\leqslant t} = M_F(\mathbb{C}^t)$ of all finite measures on $(W, \mathcal{F}^*_{\leqslant t}) = (W, \mathcal{F}^*(-\infty, t])$ with the universal completion $\mathcal{F}^*_{\leqslant t}$ of *σ-algebra, such that for every $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{F}^*_{\leqslant t}$,*

$$P^\gamma_{r,\mu} \exp\{-\langle M^\gamma_t, \varphi\rangle\} = e^{-\langle \mu, v(r,\cdot)\rangle}, \quad 0 \leqslant r \leqslant t, \quad \mu \in \mathcal{M}_{\leqslant r}, \tag{37}$$

where $v^r(w_{\leqslant r}) = v(r, w(-\infty, r])$ is a progressive function determined uniquely by the equations

$$v^r(x_{\leqslant r}) + \tilde{\Pi}_{r,x(\leqslant r)} \int_r^t \psi^s(\xi_{\leqslant s}, v^s(\xi_{\leqslant s}))\tilde{K}_\gamma(\omega; ds) = \tilde{\Pi}_{r,x(\leqslant r)}\varphi(\xi_{\leqslant t}) \quad \text{for} \quad r \leqslant t \tag{38}$$

$$v^r(x_{\leqslant r}) = 0 \quad \text{for} \quad r > t.$$

5.2 Key Lemma

First of all we define some spaces which are used later. Let $\mathbb{H}$ be the cone of all bounded functions $f \in \mathcal{B}$ with the topology of bounded convergence, where we say that a sequence $\{f_n\}$ of $\mathcal{B}$-measurable functions converges boundedly to f as n tends to infinity, if $f_n \to f$ pointwise and if f_n are uniformly bounded. In addition, we define $\mathbb{H}_c = \mathbb{H} \cap \{f : 0 \leqslant f \leqslant c\}$. In order to prove Theorem 5 we need the following key lemma (Lemma 10), which plays an important role in the succeeding subsection. Based upon the discussion on approximation in terms of branching particle systems (cf. §2, Chapter 3, pp.45–52 in Dynkin (1994) [17]), we suppose that the function $v_t^r(\beta, x)$ satisfies

$$v_t^r(\beta, x) + \Pi_{r,x} \int_r^t \psi_\beta^s(\xi_s, v_t^s(\beta, \xi_s)) K_\gamma(\omega; ds) = \Pi_{r,x} F_\beta(\xi_t) \tag{39}$$

with $F_\beta(x) = \frac{1}{\beta}(1 - e^{-\beta f(x)})$.

Lemma 10. (Key Lemma) *Let K_γ be the CAF of the underlying L-diffusion ξ as stated in Theorem 5. For $\beta > 0$, we assume that $\psi_\beta^t(x, z)$ converges to $\psi^t(x, z)$ uniformly on the set $(t, x) \in \hat{E}$, $z \in [0, c]$ for every $c \in (0, \infty)$. Then the function $v_t^r(\beta, x)$ given by (39) converges uniformly on every set $r \in [t_0, t)$ and $f \in \mathbb{H}_c$ to the unique solution $v^r(x)$ of the following integral equation*

$$\begin{aligned} &v^r(x) + \Pi_{r,x} \int_r^t \psi^s(\xi_s, v^s(\xi_s)) K_\gamma(\omega; ds) = \Pi_{r,x} f(\xi_t) \quad \text{for} \quad r \leqslant t \\ &v^r(x) = 0 \quad \text{for} \quad r > t. \end{aligned} \tag{40}$$

Example 11. We give a typical example for $\psi_\beta^t(x, z)$ to converge to $\psi^t(x, z)$ uniformly on the set.

$$\begin{aligned} \psi_\beta^t(x, z) = \frac{b^t(x)}{k(k-1)} \left\{(1 - \beta z)^k - 1 + k\beta z\right\} \beta^{-2} \\ + \int_0^{1/\beta} (e^{-uz} - 1 + zu) n^t(x, du) \end{aligned}$$

and

$$\psi^t(x, z) = \frac{1}{2} b^t(x) z^2 + \int_0^\infty (e^{-uz} - 1 + zu) n^t(x, du)$$

where $b^t(x)$ is a bounded progressive function and $n^t(x, du)$ is a kernel from $(\hat{E}, \mathcal{B}^*(\hat{E}))$ to $(0, +\infty)$ such that

$$\int_0^1 u^2 \cdot n^t(x, du) \qquad \text{and} \qquad \int_1^\infty u \cdot n^t(x, du)$$

are bounded functions on $\hat{E}$ and $\int_N^\infty u{\cdot}n^t(x,du) \to 0$ uniformly in (t,x) as $N \to \infty$. $\square$

Proof of Lemma 10. We have

$$0 \leqslant v_t^r(\beta,x) \leqslant \Pi_{r,x} f(\xi_t) \leqslant c \tag{41}$$

for all $f \in \mathbb{H}_c$. By virtue of the assumption, for every $\varepsilon > 0$ there exists a positive constant β_0 such that

$$|\psi_\beta^s(x,z) - \psi^s(x,z)| \leqslant \varepsilon \tag{42}$$

holds for all $\beta \in I_0 = (0,\beta_0)$, $(s,x) \in \hat{E}$ and $z \in [0,c]$. Since the function $\psi^t(x,z)$ is locally Lipschitz in z uniformly in (t,x), it is obvious to see that there exists a constant $K_c > 0$ such that

$$|\psi^t(x,z_1) - \psi^t(x,z_2)| \leqslant K_c|z_1 - z_2| \tag{43}$$

for all $z_1, z_2 \in [0,c]$ and $(t,x) \in \hat{E}$. On this account, a simple computation with (42) and (43) leads to

$$\begin{aligned} &|\psi_\beta^s(x, v_t^s(\beta,y)) - \psi_{\beta_1}^s(x, v_t^s(\beta_1,y))| \\ &\leqslant |\psi_\beta^s(x, v_t^s(\beta,y)) - \psi^s(x, v_t^s(\beta,y))| + |\psi^s(x, v_t^s(\beta,y)) - \psi^s(x, v_t^s(\beta_1,y))| \\ &\qquad + |\psi^s(x, v_t^s(\beta_1,y)) - \psi_{\beta_1}^s(x, v_t^s(\beta_1,y))| \\ &\leqslant 2\varepsilon + K_c|v_t^s(\beta,y) - v_t^s(\beta_1,y)| \end{aligned} \tag{44}$$

for all $\beta, \beta_1 \in (0,\beta_0)$, $f \in \mathbb{H}_c$ and all $(s,x),(s,y) \in \hat{E}$. We deduce from (39), (41) and properties of K_γ that

$$\begin{aligned} |v_t^r(\beta,x) - v_t^r(\beta_1,x)| &\leqslant \|F_\beta - F_{\beta_1}\| + 2\varepsilon C_1 \\ &\quad + C_2 \cdot \Pi_{r,x} \int_r^t |v_t^s(\beta,x) - v_t^s(\beta_1,x)| \cdot K_\gamma(\omega; ds) \end{aligned} \tag{45}$$

holds for $\exists C_1, C_2 > 0$. So that, the generalized Gronwall inequality applied to (45) allows us to obtain

$$|v_t^r(\beta,x) - v_t^r(\beta_1,x)| \leqslant (\|F_\beta - F_{\beta_1}\| + 2\varepsilon C_1)\Pi_{r,x} e^{C_2 \cdot K_\gamma(\omega;(r,t))}. \tag{46}$$

We may apply an elementary inequality to get

$$\|F_\beta - F_{\beta_1}\| \leqslant \|F_\beta - f\| + \|f - F_{\beta_1}\| \leqslant (\beta + \beta_1)\|f\|^2/2, \tag{47}$$

paying attention to the fact that $|F_\beta(x) - f(x)| \leqslant \frac{1}{2}\beta c^2$ with $0 \leqslant f(x) \leqslant c$ for all x. Since $\psi_\beta^s(x, v_t^s(\beta,x))$ converges to $\psi^s(x, v^s(x))$ uniformly on the set of $s \in [r,t]$

and $x \in E_s$, passage to the limit $\beta \to 0$ in (39) is legitimate so as to derive the integral equation (40). Lastly the uniqueness of the solution $v^r(x)$ for (40) yields again from the generalized Gronwall inequality. $\square$

5.3 Proof of Theorem 5

For every probability measure M on $M_F(E)$ (i.e. $M \in \mathcal{P}(M_F(E))$), the formula

$$L_M(f) = \int_{M_F(E)} e^{-\langle \nu, f \rangle} M(d\nu), \qquad f \in \mathbb{H} \tag{48}$$

defines a continuous functional on $\mathbb{H}$, which is called the Laplace functional of the measure M. We denote by π_μ the Poisson random measure on $(\hat{E}, \mathcal{B}(\hat{E}))$ with intensity μ, and notice that

$$\int e^{\langle \nu, f \rangle} \pi_\mu(d\nu) = \exp\{\langle \mu, e^f - 1 \rangle\}$$

holds, where $\langle \eta, v \rangle = \int_{\hat{E}} v(r,x)\eta(dr,dx)$. Based upon the existence argument for superprocess (cf. §4, Chapter 3 of Dynkin (1994) [17]), the formula

$$Q^\beta_{\pi_{\eta/\beta}} \exp\{-\langle \beta Y_t, f \rangle\} = e^{-\langle \eta, v^t(\beta) \rangle} \tag{49}$$

with $\beta > 0$ and a counting measure Y_t, means that

$$L_{M_\beta}(f) = \exp\{-\langle \eta, v^t(\beta) \rangle\} \tag{50}$$

where $\mathcal{M}_t = M_F(E_t)$, and $M_\beta(\cdot)$ is a probability measure on $(\mathcal{M}_t, \mathcal{B}(\mathcal{M}_t)) = (M_F(E_t), \mathcal{B}(M_F(E_t)))$, that is to say, $M_\beta \in \mathcal{P}(\mathcal{M}_t)$, which is defined by

$$M_\beta(C) = Q^\beta_{\pi_{\eta/\beta}}(\{\beta Y_t \in C\}), \qquad \forall C \in \mathcal{B}(\mathcal{M}_t). \tag{51}$$

Let η be an admissible measure on $\hat{E}$, concentrated on $\hat{E}_{\geq t_0}$ and satisfying that $\eta(\hat{E}_{\leqslant t}) < \infty$ for all $t > 0$. We write such a measure as $\eta \in \mathfrak{M}(\hat{E})$. Then by the key lemma (Lemma 10) a uniform convergence

$$\langle \eta, v^t(\beta) \rangle \to \langle \eta, v^t \rangle$$

on each set $\mathbb{H}_c$ is induced naturally. On the other hand, a general theory on Laplace functionals (cf. e.g. §3.4, Chapter 3, pp.50–51 in Dynkin (1994) [17]) guarantees that, if $L_{M_n}(f) \to L(f)$ uniformly on each set $\mathbb{H}_c$, then the limit L is the Laplace functional of a probability measure. According to this argument, there exists a probability measure $\mathcal{P}(\eta; t, \cdot)$ on $(\mathcal{M}_t, \mathcal{B}(\mathcal{M}_t))$ such that

$$\int e^{-\langle \nu, f \rangle} \mathcal{P}(\eta; t, d\nu) = e^{-\langle \eta, v \rangle}. \tag{52}$$

For an arbitrary $\eta \in \mathfrak{M}(\hat{E})$, we consider restrictions η_n of η to $\hat{E}[n, n+1)$, namely, $\eta_n = \eta \restriction \hat{E}[n, n+1)$, and we write $\mathcal{P}(\eta; t, \cdot)$ for $\forall\ \eta \in \mathfrak{M}(\hat{E})$ as the convolution of measures $\mathcal{P}(\eta_n; t, \cdot)$. Since the formula (52) is valid for η_n, it is also true for $\eta \in \mathfrak{M}(\hat{E})$. Hence it follows by Fatou's lemma that

$$\begin{aligned} \int \langle \nu, 1\rangle \mathcal{P}(\eta; t, d\nu) &\leqslant \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int (1 - e^{-\lambda\langle \nu, 1\rangle}) \mathcal{P}(\eta; t, d\nu) \\ &\leqslant \eta(\hat{E}_{\leqslant t}) < \infty \end{aligned} \tag{53}$$

and therefore it turns out that the measure $\mathcal{P}(\eta; t, \cdot)$ is concentrated on $\mathcal{M}_t$. Take a measure $\mu \in \mathcal{M}_r$, and let η_r be the image of μ under the mapping $\tau : x \mapsto (r, x)$ from E_r to $(\hat{E}, \mathcal{B}(\hat{E}))$. While we write the transition probability as

$$\hat{\mathcal{P}}(r, \mu; t, \cdot) = \mathbb{P}(X_t^\gamma \in (\cdot) | X_r^\gamma = \mu).$$

On this account, the formula

$$\hat{\mathcal{P}}(r, \mu; t, \cdot) = \hat{\mathcal{P}}(r, \tau^{-1}(\eta_r); t, \cdot) = \mathcal{P}(\eta_r; t, \cdot) \tag{54}$$

determines a Markov transition function $\hat{\mathcal{P}}$ by virtue of the well-known argument seen e.g. in the proof of Theorem 3.1 in Dynkin, E.B. : Trans. Amer. Math. Soc. **314** (1989), pp.255–282. Thus we attain that Theorem 5 (insisting the existence theorem for superprocess $X^\gamma = \{X_t^\gamma\}$ related to stable random measure $\gamma(\omega)$) holds for any Markov process X^γ with this transition function $\hat{\mathcal{P}}$. □

5.4 Proof of Theorem 9

As stated in the assertion of Theorem 9, set $\psi = \psi^t(x, z)$ as special branching mechanism. The historical superprocess $M^\gamma = (M_t^\gamma,\ \mathcal{G}(I),\ P_{r,\mu}^\gamma)$ with parameters $(\tilde{\Xi},\ \tilde{K}_\gamma,\ \psi)$ can be obtained from the superprocess X^γ with the almost same parameters (Ξ, K_γ, ψ) by the direct construction. First of all we define the finite-dimensional distributions of the random measure M_t^γ as

$$\mu_{t_1 t_2 \cdots t_n}(A_1 \times A_2 \times \cdots \times A_n) = M_t^\gamma(\{w(t_1) \in A_1, w(t_2) \in A_2, \ldots, w(t_n) \in A_n\}) \tag{55}$$

for time partition $\mathit{\Delta} = \{t_k\}$ with $t_1 < t_2 < \cdots < t_n \leqslant t$ and $A_1 \in \mathcal{B}_{t_1}, A_2 \in \mathcal{B}_{t_2}, \ldots,$ $A_n \in \mathcal{B}_{t_n}$. Actually, this $\mu_{t_1 \cdots t_n}$ determines uniquely the probability distribution on $\mathcal{B}(E_{t_1} \times \cdots \times E_{t_n})$. To this end we replace $X_{t_1}^\gamma$ by its restriction $\hat{X}_{t_1}^\gamma$ $(= X_{t_1}^\gamma \restriction A_1)$ to A_1 and run the superprocess during the time interval $[t_1, t_2]$ starting from $\hat{X}_{t_1}^\gamma$. Moreover we can proceed analogously until getting a $Z \in \mathcal{M}_t$ and then take $Z(E_t)$ as the value for (55). Then we construct a measure M_t^γ on $\mathcal{M}_{\leqslant t}$ by applying the Kolmogorov extension theorem to the family (55) $\{\mu_{t_1 \cdots t_n}\}$. Indeed, if $\{\mu_{t_1 \cdots t_n}\}$ satisfies the consistency condition:

$$\begin{aligned} &\mu_{t_1 \cdots t_{k-1} t_{k+1} \cdots t_n}(A_1 \times \cdots \times A_{k-1} \times \check{A}_k \times A_{k+1} \times \cdots \times A_n) \\ &\qquad = \mu_{t_1 \cdots t_{k-1} t_k t_{k+1} \cdots t_n}(A_1 \times \cdots \times A_{k-1} \times E_{t_k} \times A_{k+1} \times \cdots \times A_n) \end{aligned} \tag{56}$$

for $k = 1, 2, \ldots, n$ and $A_k \in \mathcal{B}_{t_k}$ $(k = 1, 2, \ldots, n)$, where the symbol $\vee$ means exclusion of the number or item crowned with $\vee$ from the set $N = \{1, 2, \ldots, n\}$, then the Kolmogorov extension theorem guarantees that there exists a unique probability measure P on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ such that the finite-dimensional distribution of $M_t^\gamma \in \mathcal{M}_{\leqslant t}$ is equal to $\{\mu_{t_1 \cdots t_n}\}$. Here $\hat{\Omega}$ is given by

$$\hat{\Omega} = (\mathcal{M}_{\leqslant t})^{[0,\infty)} = \{\omega | \omega(\cdot) : [0, \infty) \to \mathcal{M}_{\leqslant t}\}. \tag{57}$$

It is said that the historical superprocess can be obtained from branching particle systems (= BPS), however it cannot be obtained from BPS by the limit procedure (cf. e.g. §1.2 in Dynkin (1991) [15]) applied to the Markov process $Y = \{Y_t(\cdot)\}$ indicating the number of particles alive at time t in a set $(\cdot)$. As a matter of fact, it can be obtained from BPS by the limit procedure applied to the special process $\mathcal{Y} = \{\mathcal{Y}_t\}$. In fact, as a function of t, $\mathcal{Y}_t$ is a measure-valued process in functional spaces $W_{\leqslant t} = W(-\infty, t]$ (called historical path space). Moreover, note that the complete picture of a branching particle system is given not by the process Y_t but by the random tree composed of the paths of all particles. We sgall give below a rough sketch about construction of $\mathcal{Y}_t$. Now let us pick up a particle $< P >$ at time t at a point z. Its genealogy can be represented by a scheme

$$(r, x) \to (s_1, y_1) \to (s_2, y_2) \to \cdots \to (s_k, y_k) \to (t, z). \tag{58}$$

The rabels (s_i, y_i) indicates the birth time and birth place of the particle $< P >$ and its ancestors, and the label (r, x) refers to the immigration time and place of the first member of the family. An arrow a from (s, y) to (s', y') corresponds to a path $w \in W_{\leqslant t}$ which we call the historical path for $< P >$, and note that we usually set $w_t = \partial$ for $t < r$ with an extra state ∂. The historical paths of all particles which are alive at time t form a configuration in $W_{\leqslant t}$ which can also be described by an integer-valued measure $\mathcal{Y}_t$ on $W_{\leqslant t}$. In this way, as a function of t, $\mathcal{Y}_t$ is constructed as a measure-valued process in functional space $W_{\leqslant t}$. Lastly some comments on progressivity of transition probablity should be mentinoed. Indeed, a natural question is to ask whether that kind of progressivity for the underlying Markov process $\Xi = \{\xi\}$ implies an analogous condition for the historical process $\tilde{\Xi}$. Here the condition in question is as follows.

Condition. (TPP) The transition probabilities are progressive, i.e. the function

$$f^t(x) = 1_{\{t<u\}} \Pi_{t,x}(\xi_u \in B)$$

is progressive for every $u \in \mathbb{R}_+$ and $B \in \mathcal{B}_u$. $\square$

Take a set $B = \{w : w(t_1) \in A_1, \ldots, w(t_n) \in A_n\}$ with $t_1 < t_2 < \cdots < t_n = u$. Then it is easy to see from the progressivity of ξ that

$$
\begin{aligned}
& 1_{\{t<u\}}\tilde{\Pi}_{t,x(\leqslant t)}(\xi_{\leqslant u} \in B) = \Pi_{t,x_t}(w(t_1) \in A_1, \cdots, w(t_n) \in A_n), \quad t < t_1 \qquad (59) \\
& = 1_{A_1}(x_{t_1}) \cdot \Pi_{t_1,x_{t_1}}(w(t_2) \in A_2, \cdots, w(t_n) \in A_n), \qquad t \in [t_1, t_2) \\
& = \cdots\cdots\cdots \\
& = 1_{A_1}(x_{t_1}) \cdots 1_{A_i}(x_{t_i}) \cdot \Pi_{t_i,x_{t_i}}(w(t_{i+1}) \in A_{i+1}, \cdots, w(t_n) \in A_n), \quad t \in [t_i, t_{i+1}) \\
& = \cdots\cdots\cdots \\
& = 0, \qquad t \geq u.
\end{aligned}
$$

Therfore, the condition (TPP) is satisfied even for the historical process $\tilde{\Xi}$ as far as it may be valid for the underlying process ξ. $\square$

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Department of Mathematics
Faculty of Education, Saitama University
Saitama, 338-8570 Japan
e-mail : idoku@mail.saitama-u.ac.jp

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