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The Limit Function and Characterization Equation for Fluctuation in The Tumour Angiogenic SDE Model

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Abstract

In this paper we study a tumour angiogenic SDE model, which describes the vessel dynamics of tips in tumour angiogenesis. We derive an explicit expression of the limit function in mean principle and an explicit representation of the characterization equation in fluctuation for the tumour angiogenic SDE model.

Key Words : Tumour angiogenesis, tip dynamics, stochastic differential equation, random model, mean principle, fluctuation analysis.

1. Introduction

In Dôku-Misawa (2013) [1] we studied mean principle and fluctuation of SDE model for tumour angiogenesis, see also Dôku (2011) [2] and Misawa (2013) [3]. In this paper we propose a new mathematical model which is a generalization of the previous tumour angiogenic SDE model in Dôku-Misawa (2013) [1], and derive an explicit expression of the limit function in the mean principle of the model, as well as an explicit representation of the characterization equation in the fluctuation.

We shall introduce below some notations, terminology and modelling of blood vessel networks in angiogenesis. Let N_0 be the initial number of tips, $N(t)$ be the total number of tips at time t, $X^i(t) \in \mathbb{R}^d$ be the position of the i-th tip at time t with $d = 3$, and $v^i(t)$ be the moving velocity of the i-th tip at time t. Then the network of endothelial cells is expressed as the union of the trajectories of the tips, namely,

$$X(t) \equiv X(t, \omega) := \bigcup_{i=1}^{N(t)} \{X^i(s),\, T_i \le s \le t\}, \tag{1}$$

where T_i denotes the birth time of the i-th tip, that is to say, the time when an existing vessel branches and the i-th trajectory springs up. As is well known, the tip generating process is described by a marked point process. However, in the standpoint of its analysis and applications, it is more convenient to give it as a probability measure on the product space between time space and position space. Hence, the corresponding process is given as a probability measure $G \equiv G(dt \times dx)$, i.e.,

$$G(dt \times dx) = \sum_n \delta_{\{(T^n, Y^n)\}}, \tag{2}$$

where T^n is the birth time of the n-th tip and Y^n is the spatial position of the n-th tip that has been newly born. For each i we write

$$\tilde{X}_t^i \equiv \tilde{X}^i(t) = (X_1^i(t), X_2^i(t), X_3^i(t)) \in \mathbb{R}^3, \quad v_t^i \equiv v^i(t) = (v_1^i(t), v_2^i(t), v_3^i(t)) \in \mathbb{R}^3, \tag{3}$$

and for each j (j = 1, 2, 3) we have $X_j^i(t) \in \mathbb{R}$ and $v_j^i(t) \in \mathbb{R}$. Next we shall propose a new stochastic differential equation (SDE) model which describes the blood vessel dynamics. Under these circumstances the formulation via a random model (i.e., an SDE model) on the vessel motion is given by the following simultaneous equations. As a matter of fact, for each i,

$$\begin{cases} d\tilde{X}^i(t) = \Xi(t, \tilde{X}) v_t^i dt, \\ dv^i(t) = a(t, \tilde{X}^i, v^i)dt + \sigma v_t^i dW_t^i, \end{cases} \qquad (t > T_i) \tag{4}$$

where $W_t^i \equiv W^i(t) = (W_1^i(t), W_2^i(t), W_3^i(t)) \in \mathbb{R}^3$ is a three-dimensional Brownian motion (or Wiener process). Next we refer to the concrete components of the afore-mentioned equations. Namely, $C(t, x)$ denotes the concentration rate of TAF (tumour angiogenic factors), and $f\,(t, x)$ is the fibronectin and/or their gradients. The positive constant $\sigma > 0$ is a diffusion coefficient, and the term Ξ is given by $\Xi(t, \tilde{X}) := 1 - p_a \mathbb{I}_{\tilde{X}_t}\{\tilde{X}_t^k\}$, where p_a is a switching parameter, and the parameter p_a takes only the 0 and 1 values. Actually, the state $p_a = 0$ indicates that no impingement is considered, while $p_a = 1$ means that the phenomenon of anastosis is taken into account. $\mathbb{I}_{(\cdot)}\{\cdot\}$ is the indicator or characteristic function associated with the existing blood network status. According to several system biological or molecular biological observations, the coefficient term (or the drift term) $a\,(t, x, v)$ of (4) is thought to be a function of $C(t, x)$ and $f(t, x)$. Here we suppose that it is given by

$$a(t, \tilde{X}^i, v^i) := -kv_t^i + \Phi(C(t, \tilde{X}_t^i), f(t, \tilde{X}_t^i)), \quad k > 0. \tag{5}$$

There are surely various discussions for the term Φ to be described. Suggested by considerations of the bias depending on TAF and the fivronectin field of Plank- Sleeman (2004) [4], and also inspired by the argument on the magnitude of the chemotactic and haptotactic gradient for the reorientation of the cell increase of Stéphanou et al. (2006) [5], we adopt the function Φ of the following form:

$$\begin{aligned} \Phi(C, f) &\equiv \Phi(C(t, \tilde{X}_t^i), f(t, \tilde{X}_t^i)) \\ &= d_C \cdot \nabla C(t, \tilde{X}_t^i) + d_f \cdot \nabla f(t, \tilde{X}_t^i) \end{aligned} \tag{6}$$

with $d_1 > 0,\ d_2 > 0,\ \gamma > 0,\ q > 0,$

$$d_C = d_1 \frac{|\nabla C(t, \tilde{X}_t^i)|}{(1 + \gamma C(t, \tilde{X}_t^i))^q}, \quad \text{and} \quad d_f = d_2 |\nabla f(t, \tilde{X}_t^i)|. \tag{7}$$

Note that

$$\nabla C = (\partial_1 C, \partial_2 C, \partial_3 C) = (\frac{\partial C}{\partial x_1}, \frac{\partial C}{\partial x_2}, \frac{\partial C}{\partial x_3}) \quad \text{and}$$
$$\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) \quad \text{with} \quad \partial_i g = \frac{\partial g}{\partial x_i} \ (i = 1, 2, 3), \tag{8}$$

and also that the term $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ is considered to have a form

$$\Phi_j \equiv \Phi_j(C(t, x), f(t, x)) = K_1 \frac{\partial C}{\partial x_j} + K_2 \frac{\partial f}{\partial x_j}, \quad \text{with } (j = 1, 2, 3). \tag{9}$$

For brevity's sake, we abbreviate its individual tag number i in what follows. We also use the following notations.

$$X_t = (\tilde{X}_t, v_t) = (X_t^1, X_t^2, X_t^3, v_t^1, v_t^2, v_t^3) \quad \text{and}$$
$$B_t = (\tilde{B}_t, W_t) = (\tilde{B}_t^1, \tilde{B}_t^2, \tilde{B}_t^3, W_t^1, W_t^2, W_t^3)$$

where $\tilde{B}_t = (\tilde{B}_t^i, i = 1, 2, 3)$ is a three-dimensional Brownian motion independent of W_t. Then our newly proposed tumour angiogenic SDE model for vessel tip dynamics (4) is equivalent to

$$d \begin{pmatrix} \tilde{X}_t \\ v_t \end{pmatrix} = \begin{pmatrix} (1 - p_a \mathbb{I}_{\tilde{X}_t}(\tilde{X}_t^k)) v_t \\ a(t, \tilde{X}, v) \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \tilde{X}_t \\ v_t \end{pmatrix} d \begin{pmatrix} \tilde{B}_t \\ W_t \end{pmatrix}, \tag{10}$$

and furthermore, for simplicity, we shall write it as follows:

$$\begin{cases} dX_t = F(t, X_t)dt + G(X_t)dB_t, \\ X_0 = Z, \end{cases} \tag{11}$$

with, for $T > 0$,

$$\begin{cases} F(t, x) : [0, T] \times \mathbb{R}^6 \to \mathbb{R}^6, \\ G(x) : \mathbb{R}^6 \to (\mathbb{R}^6 \otimes \mathbb{R}^6) \times \mathbb{R}^6 \cong \mathbb{R}^6. \end{cases}$$

This is nothing but an Itô type stochastic differential equation with respect to a Brownian motion, to which the usual stochastic calculus (or Iô calculus) can be applied.

2. Main Results

According to the general theory on stochastic differential equations (cf. Øksendal (1998) [6] or Ikeda-Watanabe (1989) [7]), in order to obtain the existence and uniqueness result for solutions to the stochastic differential equation (SDE) of Itô type (11), we have only to assume the following conditions. For the function $G(t, x) = G(x)$ by convention, we assume:

ASSUMPTION. (A.1) (Restriction on growth) There exists a proper positive constant $C > 0$ such that for $\forall t \in [0, T]$ and $\forall x \in \mathbb{R}^6$

$$|F(t, x)| + \|G(t, x)\| \leq C(1 + |x|). \tag{12}$$

(A.2) (Lipschitz continuity) There exists a proper positive constant $D > 0$ such that for $\forall t \in [0, T]$ and $\forall x, y \in \mathbb{R}^6$

$$|F(t, x) - F(t, y)| + \|G(t, x) - G(t, y)\| \leq D|x - y|. \tag{13}$$

Here note that $G\,(t, x) = (G_{ij}(t, x)) \in M(6 \times 6)$ and $\|G(t, x)\| = \sum_{i,j} |G_{ij}(t, x)|^2$ $= \sum_{i=1}^{6} \sum_{j=1}^{6} G_{ij}^2(t, x)$, where $M\,(6 \times 6)$ denotes the totality of (6, 6)-type square matrices.
(A.3) (Initial value) The initial value Z is a random variable and is independent of the σ-algebra $\mathcal{F}_\infty^B = \sigma(B_s : s \geq 0)$, and satisfies the integrability condition

$$\mathbb{E}\,|\,Z\,|^2 < +\infty. \tag{14}$$

Then it is well known as the theorem on existence and uniqueness of solutions to SDEs that under the assumptions (A.1), (A.2) and (A.3), the SDE (11) possesses the unique solution which is t-continuous and satisfies (i) X_t is $\mathcal{F}_t^Z$-adapted where $\mathcal{F}_t^Z \equiv \sigma(Z) \vee \sigma(B_s : s \leq t)$; and (ii) $\mathbb{E} \int_0^T |X_t|^2 \, dt < \infty$. On this account, we prove the following first main result. For simplicity we set

$$\Upsilon^{[f]}(t, x, y) := f(t, x) - f(t, y) \quad \text{and} \quad \mathcal{F}_t^Z := \sigma(Z) \vee \sigma(B_s : s \leqslant t). \tag{15}$$

THEOREM 1. (Existence and uniqueness of solution to SDE) *Assume* (A.3). *We also suppose that*

$$|\nabla C(t, x)| + |\nabla f(t, x)| \leqslant C_1(1 + |x|), \quad \text{for} \quad \exists C_1 > 0, \forall t \geq 0, \forall x \tag{16}$$

$$|\Upsilon^{[\nabla C]}(t, x, y)| + |\Upsilon^{[\nabla f]}(t, x, y)| \leqslant C_2|x - y|, \quad \text{for} \quad \exists C_2 > 0, \forall t > 0, \forall x, y. \tag{17}$$

Then SDE (11) *possesses the unique solution* $X = (X_t) \in \mathbb{R}^6$ *such that* (a) X_t *is t-continuous*, (b) X_t *is* $\mathcal{F}_t^Z$*-adapted, and* (c) X_t *satisfies the integrability condition*

$$\mathbb{E} \int_0^T |X_t|^2 dt < +\infty. \tag{18}$$

We use the scaling to the model relative to $\varepsilon > 0$, and consider a scaled process

$X_t^\varepsilon(\omega) \equiv X^\varepsilon(t,\omega) := X(\frac{t}{\varepsilon},\omega)$. In this stage we are very concerned on the asymptotic behavior of $X^\varepsilon(t,\omega)$ as $\varepsilon \to 0$. In order to analyze the asymptotic behaviors and derive the mean principle for our SDE model, we need the following conditions.

$$\sup_{t>0} |v_t(\omega)| < +\infty, \qquad \mathbb{P}-a.s. \tag{19}$$

$$\sup_{t>0} |\nabla C(t,x)| < +\infty \quad \text{uniformly in} \quad x, \tag{20}$$

$$\sup_{t>0} |\nabla f(t,x)| < +\infty \quad \text{uniformly in} \quad x. \tag{21}$$

For $\forall s, u$ such that $0 < s < u$,

$$\lim_{\varepsilon\to 0} \int_s^u F^\varepsilon(t,y)dt = \int_s^u F^0(t,y)dt \tag{22}$$

where $F^\varepsilon(t,y)$ is defined by $F(\frac{t}{\varepsilon},y)$. Then we call F^ε is integrally continuous at $\varepsilon = 0$ with respect to (t,y).

We are now in a position to state the second main result in this paper, which supplies with an explicit expression of the limit function in mean principle. Although our SDE model (4) (or (10), (11)) is an extension of the tumour angiogenic model treated in Dôku-Misawa (2013) [1] and Misawa (2013) [3], this result sharpens the previous mean principle theorem (cf. Theorem 22, §4.2 in [1]).

THEOREM 2. *Suppose the same conditions* (16) *and* (17) *as in Theorem 1. In addition, we assume* (19), (20), (21) *and* (22).
(a) *Under the hypothesis that* $X_s(\omega) = y = (\tilde{y},\hat{y}) \in \mathbb{R}^6 \cong \mathbb{R}^3 \times \mathbb{R}^3$, $\mathbb{P}$-a.s., *there exists a proper function* $\tilde{F}(y) : \mathbb{R}^6 \to \mathbb{R}^6$ *such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(s,y)ds = \tilde{F}(y) \tag{23}$$

holds uniformly in y.
(b) *Moreover, if* $u \equiv u(t)$ *is a solution of the Cauchy problem for deterministic dynamic differential equation*

$$\frac{du}{dt}(t) = \tilde{F}(u(t)) \qquad \text{with} \quad u(t)|_{t=0} = z, \tag{24}$$

then the convergence in law $X(\frac{t}{\varepsilon}) \Longrightarrow u(t)$ *holds as* ε *approaches to zero.*
(c) *The limit function* $\tilde{F}$ *is given concretely by*

$$\tilde{F} := \begin{pmatrix} c_0 \cdot \hat{y} \\ -k\hat{y} + \alpha(\infty)\cdot\nabla C(\infty,\tilde{y}) + \beta(\infty)\cdot\nabla f(\infty,\tilde{y}) \end{pmatrix} \tag{25}$$

with $\alpha(\infty)=d_C(\infty,\tilde{y})$, $\beta(\infty)=d_f(\infty,\tilde{y})$ *for* $d_C\equiv d_C(t,\tilde{X}_t)$ *and* $d_f\equiv d_f(t,\tilde{X}_t)$*. Here we set* $c_0=1-p_a\mathbb{I}_{\tilde{X}}\{\tilde{X}^k\}$.

Next we shall introduce the third main result in this paper, which provides with an explicit representation of the characterization equation for the fluctuation of the rescaled tumour angiogenic SDE model. Before stating the theorem, we define the fluctuation quantity based upon the fundamental results in Lemma 6 and in the proof of Theorem 2 (see below): i.e., (i) vanishing of the Itô type stochastic integral of rescaled function

$$\sqrt{\varepsilon}\int_0^t G^\varepsilon(y)dB_s^\varepsilon \Longrightarrow 0 \quad (\text{as} \quad \varepsilon\to 0); \tag{26}$$

(ii) the limiting equality of the SDE model

$$\lim_{\varepsilon\searrow 0} X\left(\frac{t}{\varepsilon}\right)=z+\int_0^t \tilde{F}(\lim_{\varepsilon\searrow 0} X\left(\frac{s}{\varepsilon}\right))ds. \tag{27}$$

As a matter of fact, we define the fluctuation as

$$V_t^\varepsilon \equiv V^\varepsilon(t,\omega):=\frac{1}{\sqrt{\varepsilon}}\{X\left(\frac{t}{\varepsilon},\omega\right)-u(t)\}, \quad \mathbb{P}-\text{a.s.} \tag{28}$$

for $t>0$ and $\varepsilon>0$.

THEOREM 3. *We assume* (16), (17), (19), (20), (21) *and* (22).

(a) *There exist some proper functions* $\xi(t,\omega)\in L^1(0,T)$*,* $\mathbb{P}$*-a.s., and* $\Psi(t,\omega)\in L^2(0,T)$*,* $\mathbb{P}$*-a. s. such that*

$$\lim_{\varepsilon\to 0}\int_0^t (\nabla\cdot F(\frac{s}{\varepsilon},z))\cdot y ds=\int_0^t \xi(s)\cdot y ds, \tag{29}$$

$$\text{and} \qquad \lim_{\varepsilon\to 0}\int_0^t G^\varepsilon(X^\varepsilon(s))ds=\int_0^t \Psi(s)ds. \tag{30}$$

(b) *The fluctuation* V_t^ε *converges in law to some process* Z_t *as* $\varepsilon\to 0$*.*

(c) *The limit process* Z_t *satisfies the following SDE :*

$$dZ_t=\xi(t)Z_t dt+\Psi(t)dB_t. \tag{31}$$

(d) *Actually, the limit functions in* (29) *and* (30) *which determine the characterization equation of the fluctuation, are explicitly presented as*

$$\xi(s,\omega)=\nabla\cdot F(\infty,u(s)) \ \ \text{and} \tag{32}$$

$$\Psi(s,\omega)=G(\infty,u(s))=\begin{pmatrix}0&0\\0&\sigma\end{pmatrix}u(s)=\begin{pmatrix}0&0\\0&\sigma\end{pmatrix}\begin{pmatrix}\tilde{u}(s)\\\hat{u}(s)\end{pmatrix}=\begin{pmatrix}0\\\hat{u}(s)\end{pmatrix}. \tag{33}$$

If we rewrite the definition (28) of fluctuation, then we immediately obtain

$$X\left(\frac{t}{\varepsilon}\right) = u(t) + \sqrt{\varepsilon}V^{\varepsilon}(t). \tag{34}$$

Here $u(t)$ is the solution of the ordinary differential equation like $\frac{d}{dt}u(t) = \tilde{F}(u(t))$, so that, the solution curve (parametrized by time t) is a smooth curve with respect to t. The expression (34) suggests that the rescaled process $X(\frac{t}{\varepsilon})$ (which satisfies a SDE (42) below) is obtained by adding a random quantity (fluctuation) $\sqrt{\varepsilon}V^{\varepsilon}(t)$ to the curve $u(t)$ additively for each t. In other words, the random quantity $X(\frac{t}{\varepsilon})$ (controlled by our SDE model) can be regarded as the sum structure being decomposed as the deterministic term $u(t)$ and randomly perturbed term.

3. Proof of Theorem 1

In order to prove the existence and uniqueness of solutions to SDE (11), it suffices to show the restriction on growth (A.1) and the Lipschitz continuity (A.2) under the conditions (16), (17). In what follows we shall verify it when $d_C(\omega) = \alpha$, $\mathbb{P}$-a.s and $d_f(\omega) = \beta$, $\mathbb{P}$-a.s. for simplicity. Moreover, it is sufficient to show it for a simpler case

$$\tilde{F}(t, X_t) = \begin{pmatrix} c_0 v_t \\ -kv_t + \alpha\partial_i C + \beta\partial_i f \end{pmatrix} \tag{35}$$

instead of $F(t, X_t) = \begin{pmatrix} \Xi(t, \tilde{X})v_t \\ a(t, \tilde{X}, v) \end{pmatrix} = \begin{pmatrix} \Xi(t, \tilde{X}v_t \\ -kv_t + \Phi \end{pmatrix}$. In fact, we have

$$|\tilde{F}(t, X_t)| = \sqrt{|c_0 v_t|^2 + |-kv_t + \alpha\partial_i C + \beta\partial_i f|^2},$$

$$|c_0 v_t|^2 = c_0^2|v_t|^2, \quad \text{and} \quad |X_t| = \sqrt{|\tilde{X}_t|^2 + |v_t|^2}.$$

Finally we would like to show the estimate results just similar to a type of $|\tilde{F}(t, X_t)| \leqslant K(1 + |X_t|)$ for some constant $K > 0$. It follows from the condition (16) that

$$\begin{aligned}
&|-kv_t + \alpha\partial_i C + \beta\partial_i f|^2 \\
&\leqslant 3(k^2|v_t|^2 + \alpha^2|\partial_i C(t, \tilde{X}_t)|^2 + \beta^2|\partial_i f(t, \tilde{X}_t)|^2) \\
&\leqslant 3(k^2|v_t|^2 + \alpha^2|\nabla C|^2 + \beta^2|\nabla f|^2) \\
&\leqslant 3\{k^2|v_t|^2 + \alpha^2 K^2(1 + |\tilde{X}_t|)^2 + \beta^2 K^2(1 + |\tilde{X}_t|)^2\} \\
&\leqslant 3\{k^2|v_t|^2 + 2\alpha^2 K^2(1 + |\tilde{X}_t|^2) + 2\beta^2 K^2(1 + |\tilde{X}_t|^2)\} \\
&\leqslant M_0^2(1 + |v_t|^2 + |\tilde{X}_t|^2) = M_0^2(1 + |X_t|^2). \quad (\exists M_0 > 0)
\end{aligned} \tag{36}$$

Similarly, using (20) we obtain

$$
\begin{aligned}
|\tilde{F}(t,X_t)| &\leqslant \sqrt{|c_0 v_t|^2 + |-kv_t + \alpha\partial_i C + \beta\partial_i f|^2} \\
&\leqslant M_0\sqrt{1+|v_t|^2+|\tilde{X}_t|^2} \\
&\leqslant M_0\sqrt{1+|X_t|^2} \leqslant M_0(1+|X_t|), \qquad (37)
\end{aligned}
$$

where we have employed an elementary inequality $1+a^1 \leqslant 1+2a+a^2=(1+a)^2$ with $a>0$. While, when $X_t=(\tilde{X}_t, v_t)$ and $Y_t=(\tilde{Y}_t,\ w_t)$, then we get

$$
|X_t - Y_t|^2 = \left|\begin{pmatrix}\tilde{X}_t - \tilde{Y}_t \\ v_t - w_t\end{pmatrix}\right|^2 = |\tilde{X}_t - \tilde{Y}_t|^2 + |v_t - w_t|^2.
$$

Since we have

$$
\begin{aligned}
&\tilde{F}(t,X_t) - \tilde{F}(t,Y_t) \\
&= \begin{pmatrix} c_0 v_t - c_0 w_t \\ -kv_t + \alpha\partial_i C(t,\tilde{X}_t) + \beta\partial_i f(t,\tilde{X}_t) + kw_t - \alpha\partial_i C(t,\tilde{Y}_t) - \beta\partial_i f(t,\tilde{Y}_t)\end{pmatrix},
\end{aligned}
$$

by using a simple inequality $(\alpha+\beta+\gamma)^2 \leqslant 3(\alpha^2+\beta^2+\gamma^2)$, we can obtain easily together with (17)

$$
\begin{aligned}
&|\tilde{F}(t,X_t) - \tilde{F}(t,Y_t)|^2 \\
&= |c_0 v_t - c_0 w_t|^2 \\
&\quad + |-kv_t + \alpha\partial_i C(t,\tilde{X}_t) + \beta\partial_i f(t,\tilde{X}_t) + kw_t - \alpha\partial_i C(t,\tilde{Y}_t) - \beta\partial_i f(t,\tilde{Y}_t)|^2 \\
&\leqslant c_0|v_t - w_t|^2 \\
&\quad + |k(w_t - v_t) + \alpha(\partial_i C(t,\tilde{X}_t) - \partial_i C(t,\tilde{Y}_t)) + \beta(\partial_i f(t,\tilde{X}_t) - \partial_i f(t,\tilde{Y}_t))|^2 \\
&\leqslant c_0|v_t - w_t|^2 + 3k^2|v_t - w_t|^2 \\
&\quad + 3\alpha^2|\partial_i C(t,\tilde{X}_t) - \partial_i C(t,\tilde{Y}_t)|^2 + 3\beta^2|\partial_i f(t,\tilde{X}_t) - \partial_i f(t,\tilde{Y}_t)|^2 \\
&\leqslant (c_0^2 + 3k^2)|v_t - w_t|^2 + 3\alpha^2 K_2^2|\tilde{X}_t - \tilde{Y}_t|^2 + 3\beta^2 K_2^2|\tilde{X}_t - \tilde{Y}_t|^2 \\
&= (c_0^2 + 3k^2)|v_t - w_t|^2 + 3K_2^2(\alpha^2+\beta^2)|\tilde{X}_t - \tilde{Y}_t|^2 \\
&\leqslant \max\{c_0^2 + 3k^2, 3K_2^2(\alpha^2+\beta^2)\}(|v_t - w_t|^2 + |\tilde{X}_t - \tilde{Y}_t|^2) \\
&= M_1^2 \cdot |X_t - Y_t|^2,
\end{aligned}
$$

where we have put $M_1 := \sqrt{\max\{c_0^2 + 3k^2, 3K_2^2(\alpha^2+\beta^2)\}} > 0$. Hence, we have verified that

$$
|\tilde{F}(t,X_t) - \tilde{F}(t,Y_t)| \leqslant M_1 \cdot |X_t - Y_t|.
$$

Thus we attain the establishment of the restriction on growth and Lipschitz continuity from the conditions (16) and (17). □

4. Proof of Theorem 2

Let Λ be an index set. For convention, we use the following notations. Let us consider the Itô type stochastic differential equation with parameter $\lambda \in \Lambda$

$$dX_t^\lambda(\omega) = F^\lambda\left(t, X_t^\lambda(\omega)\right) dt + G^\lambda\left(t, X_t^\lambda(\omega)\right) dB_t^\lambda(\omega),$$

and we write its solution as $X^\lambda = \{X_t^\lambda; t \geq 0\}$ with $\lambda \in \Lambda$. We assume that the drift term F^λ and the diffusion term G^λ satisfy the same conditions as those stated in the existence and uniqueness theorem (Theorem 1) for the previous SDE (11). For $\varepsilon > 0$ we consider a scaled process $X^\varepsilon(t,\omega) = X(t/\varepsilon, \omega)$, and we are very concerned on the asymptotic behaviour of $X^\varepsilon(t,\omega)$ as $\varepsilon \to 0$. We need the following two technical lemmas.

LEMMA 4. *We assume that the term F^λ is integrally continuous at $\lambda = \lambda_0$. If for $\forall t > 0$,*

$$\int_0^t G^\lambda(X_s^\lambda)dB_s^\lambda \Rightarrow \int_0^t G^{\lambda_0}(X_s^{\lambda_0})dB_s^{\lambda_0}, \quad \text{as} \quad \lambda \to \lambda_0, \tag{38}$$

then the convergence in law $X_t^\lambda \Rightarrow X_t^{\lambda_0}$ holds as λ tends to λ_0.

Proof. For λ in the index set Λ, we consider the parametrized SDE

$$dX_t^\lambda(\omega) = F^\lambda\left(t, X_t^\lambda(\omega)\right) dt + G^\lambda\left(X_t^\lambda(\omega)\right) dB_t^\lambda(\omega). \tag{39}$$

Then we have an integral form of (39)

$$X_t^\lambda(\omega) = X_0^\lambda(\omega) + \int_0^t F^\lambda\left(s, X_s^\lambda(\omega)\right) ds + \int_0^t G^\lambda\left(X_s^\lambda(\omega)\right) dB_s^\lambda(\omega). \tag{40}$$

The assertion yields from the limit procedure $\lambda \to \lambda_0$ in (40), because the proof goes almost similarly as in the proof of Theorem 20 in [1]. □

LEMMA 5. *For the parameter $\varepsilon > 0$, let $X_t^\varepsilon(\omega)$ be the solution to the initial value problem for the scaled Itô type SDE*

$$dX_t^\varepsilon = \varepsilon^{-1} F^\varepsilon(t, X_t^\varepsilon)dt + \varepsilon^{-\frac{1}{2}} G^\varepsilon(X_t^\varepsilon)dB_t^\varepsilon(\omega), \quad \text{with} \quad X_0^\varepsilon = z. \tag{41}$$

Then the rescaled process $X\left(\frac{t}{\varepsilon}, \omega\right)$ satisfies the following integral equation :

$$X\left(\frac{t}{\varepsilon}\right) = z + \int_0^t F\left(\frac{s}{\varepsilon}, X\left(\frac{s}{\varepsilon}\right)\right) ds + \sqrt{\varepsilon}\int_0^t G\left(X\left(\frac{s}{\varepsilon}\right)\right) dB_s^\varepsilon(\omega), \tag{42}$$

where we put $B^\varepsilon(s) = \sqrt{\varepsilon} B\left(\frac{s}{\varepsilon}\right)$ in the above expression.

Proof. In the case of $X^\varepsilon \equiv X^\varepsilon(t) := X\left(\frac{t}{\varepsilon}\right)$, it is necessary to think of what will happen in the stochastic differential term after we change $dX^\varepsilon(t)$ into $\frac{d}{dt}X^\varepsilon(t)$. Actually, we readily obtain

$$\frac{d}{dt}X^{\varepsilon}(t)=\frac{d}{dt}\left\{X\left(\frac{t}{\varepsilon}\right)\right\}=\frac{1}{\varepsilon}\frac{dX}{dt}\left(\frac{t}{\varepsilon}\right)=\frac{1}{\varepsilon}\frac{dX^{\varepsilon}}{dt}(t). \tag{43}$$

On the other hand, when B_t is a one-dimensional Brownian motion, then the scaling property of the Brownian motion (cf. Durrett (1996) [8]) yields immediately to the equivalence in law: $\mathcal{L}(B_t)=\mathcal{L}\left(\frac{1}{\sqrt{a}}B_{at}\right)$. Hence, a similar rule remains valid even for the term $B_t=(\tilde{B}_t,\ W_t)\in\mathbb{R}^6$ with three-dimensional Brownian motion for each component. That is to say, it follows that

$$\begin{aligned}&\mathcal{L}(\tilde{B}_t^1,\tilde{B}_t^2,\tilde{B}_t^3,W_t^1,W_t^2,W_t^3)\\&=\mathcal{L}\left(\frac{1}{\sqrt{a}}\tilde{B}_{at}^1,\frac{1}{\sqrt{a}}\tilde{B}_{at}^2,\frac{1}{\sqrt{a}}\tilde{B}_{at}^3,\frac{1}{\sqrt{a}}W_{at}^1,\frac{1}{\sqrt{a}}W_{at}^2,\frac{1}{\sqrt{a}}W_{at}^3\right)\end{aligned}$$

Therefore we can deduce from (43) that

$$\frac{1}{\varepsilon}\cdot dX\left(\frac{t}{\varepsilon},\omega\right)=\frac{1}{\varepsilon}F\left(\frac{t}{\varepsilon},X_t^{\varepsilon}\right)dt+\frac{1}{\sqrt{\varepsilon}}G\left(X_t^{\varepsilon}\right)dB_t^{\varepsilon}. \tag{44}$$

Immediately, (44) reads equivalently

$$dX\left(\frac{t}{\varepsilon},\omega\right)=F\left(\frac{t}{\varepsilon},X_t^{\varepsilon}\right)dt+\sqrt{\varepsilon}\cdot G\left(X_t^{\varepsilon}\right)dB_t^{\varepsilon}. \tag{45}$$

When we rewrite the above (45) into an integral form, then the required expression (42) can be obtained. □

By virtue of the assumption on integral continuity of the drift term F, the passage to the limit $\varepsilon\to 0$ allows us to get

$$\lim_{\varepsilon\to 0}\int_{s_1}^{s_2}F^{\varepsilon}(s,y)ds=\int_{s_1}^{s_2}\lim_{\varepsilon\to 0}F^{\varepsilon}(s,y)ds=\int_{s_1}^{s_2}F^{0}(s,y)ds \tag{46}$$

for arbitrary pairs $s_1,s_2>0$ such that $s_1<s_2$. On the other hand, by the scaling we have

$$\lim_{\varepsilon\to 0}\int_{s_1}^{s_2}F\left(\frac{s}{\varepsilon},y\right)ds=\int_{s_1}^{s_2}\lim_{\varepsilon\to 0}F\left(\frac{s}{\varepsilon},y\right)ds=\int_{s_1}^{s_2}F(\infty,y)ds, \tag{47}$$

since we have

$$\begin{aligned}F(s,y)&=\begin{pmatrix}(1-p_a\mathbb{I}_{\tilde{X}_s}(\tilde{X}_s^{[k]}))v_s\\ a(s,\tilde{X},v)\end{pmatrix}\\&=\begin{pmatrix}(1-p_a\mathbb{I}_{\tilde{y}}(\tilde{y}^*))\hat{y}\\ a(s,\tilde{y},\hat{y})\end{pmatrix}=\begin{pmatrix}c_0\hat{y}\\ -k\hat{y}+\Phi(C(s,\tilde{y}),f(s,\tilde{y}))\end{pmatrix}\\&=\begin{pmatrix}c_0\hat{y}\\ -k\hat{y}+d_C(s,\tilde{y})\cdot\nabla C(s,\tilde{y})+d_f(s,\tilde{y})\cdot\nabla f(s,\tilde{y})\end{pmatrix}\end{aligned}$$

with $y = (\tilde{y}, \hat{y})$ and $F(s, y)$ is a continuous function in s. Moreover, we can calculate a little bit further by making use of the hypothesis (23):

$$\begin{aligned}\tilde{F}(y) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T F(s, y) ds = \lim_{T\to\infty} \frac{1}{T} \int_0^1 F(T\beta, y) \cdot T \cdot d\beta \\ &= \int_0^1 F(\infty, y) d\beta = F(\infty, y).\end{aligned} \tag{48}$$

This implies that the real body of the limit function $\tilde{F}$ in mean principle is given by $F(\infty, y)$, where we have employed in the above a transformation of variables $s = T\beta$.

LEMMA 6. *We have the following convergence in law*

$$\sqrt{\varepsilon} \int_0^t G(y) dB_s^{\varepsilon} \quad \Rightarrow \quad 0 \qquad (\text{as} \quad \varepsilon \downarrow 0) \tag{49}$$

for every $t > 0$.

Proof. By the definition of G, since $G(y) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma\hat{y} \end{pmatrix}$, we readily get

$$\begin{aligned}|G(y)| &= |(0, \sigma\hat{y})| = \sqrt{\sigma^2 |\hat{y}|^2} = \sigma\sqrt{|\hat{y}|^2} \\ &\leqslant \sigma\sqrt{|\tilde{y}|^2 + |\hat{y}|^2} = \sigma\sqrt{|y|^2} = \sigma|y| \\ &\leqslant C_1(1 + |y|). \qquad (\exists C > 0).\end{aligned} \tag{50}$$

This means that the restriction on growth condition for the diffusion term is satisfied. Consequently, we can have an estimate

$$\begin{aligned}&\mathbb{E}\left|\sqrt{\varepsilon} \int_0^t G^{\varepsilon}(y) dB_s^{\varepsilon}\right|^2 = \mathbb{E}[\varepsilon \int_0^t \|G^{\varepsilon}(y)\|^2 ds] \\ &\leq \mathbb{E}\left[\varepsilon \int_0^t C_1^2 (1 + |y|)^2 ds\right] \leq \varepsilon t \cdot C_1^2 (1 + |y|)^2 \to 0, \qquad (\text{as} \quad \varepsilon \downarrow 0).\end{aligned} \tag{51}$$

The first equality in the above computation is due to the Itô isometry, cf. Ikeda- Watanabe (1989) [7]. □

Next we observe from (42) in Lemma 5 that

$$\lim_{\varepsilon} X\left(\frac{t}{\varepsilon}\right) = z + \lim_{\varepsilon\downarrow 0} \int_0^t F\left(\frac{s}{\varepsilon}, X\left(\frac{s}{\varepsilon}\right)\right) ds + \lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int_0^t G^{\varepsilon}\left(X\left(\frac{s}{\varepsilon}\right)\right) dB_s^{\varepsilon}(s). \tag{52}$$

Therefore, an application of Lemma 6 to (52) yields to

$$\lim_{\varepsilon\to 0} X\left(\frac{t}{\varepsilon}\right) = z + \lim_{\varepsilon\to 0} \int_0^t F\left(\frac{s}{\varepsilon}, X\left(\frac{s}{\varepsilon}\right)\right) ds$$

$$
\begin{aligned}
&= z + \int_0^t \lim_{\varepsilon\to 0} F\left(\frac{s}{\varepsilon}, X\left(\frac{s}{\varepsilon}\right)\right) ds = z + \int_0^t F\left(\infty, \lim_{\varepsilon} X\left(\frac{s}{\varepsilon}\right)\right) ds \\
&= z + \int_0^t \tilde{F}\left(\lim_{\varepsilon}\left(\frac{s}{\varepsilon}\right)\right) ds.
\end{aligned} \tag{53}
$$

By Lemma 4, when we write the limit of $\lim_\varepsilon X(\frac{t}{\varepsilon})$ as $u(t)$, then $X(\frac{t}{\varepsilon})$ converges in law to $u(t)$ for every $t > 0$ as $\varepsilon \to 0$, and from (53) we get

$$
u(t) = z + \int_0^t \tilde{F}(u(s))ds, \tag{54}
$$

which implies that the limit $u(t)$ is a solution of the Cauchy problem for the ordinary differential equation $\frac{du}{dt}(t) = \tilde{F}(u(t))$ with the initial value $u(t)|_{t=0} = z$. This completes the proof of Theorem 2. □

5. Proof of Theorem 3

For the initial data of SDE model, we can assume without loss of generality that $Z(\omega) = x$, $\mathbb{P}$-a.s. ($x \in \mathbb{R}^6$), cf. see (11). Then we have

$$
\begin{aligned}
V_t^\varepsilon(\omega) &= \frac{1}{\sqrt{\varepsilon}}\left\{x + \int_0^t F\left(\frac{s}{\varepsilon}, X\left(\frac{s}{\varepsilon}\right)\right) ds + \sqrt{\varepsilon}\int_0^t G^\varepsilon\left(X\left(\frac{s}{\varepsilon}\right)\right) dB_s^\varepsilon \right. \\
&\qquad\qquad \left. - \left(x + \int_0^t \tilde{F}(u(s))ds\right)\right\} \\
&= \frac{1}{\sqrt{\varepsilon}}\left\{\int_0^t (F\left(\frac{s}{\varepsilon}, X\left(\frac{s}{\varepsilon}\right)\right) - \tilde{F}(u(s)))ds + \sqrt{\varepsilon}\int_0^t G^\varepsilon\left(X\left(\frac{s}{\varepsilon}\right)\right) dB_s^\varepsilon\right\} \\
&= \frac{1}{\sqrt{\varepsilon}}\int_0^t \left\{F\left(\frac{s}{\varepsilon}, u(s) + \sqrt{\varepsilon}V_s^\varepsilon\right) - \tilde{F}(u(s))\right\} ds + \int_0^t G^\varepsilon\left(u(s) + \sqrt{\varepsilon}V_s^\varepsilon\right) dB_s^\varepsilon \\
&=: I_1 + I_2,
\end{aligned} \tag{55}
$$

where we made use of the expression (34).

(As to I_1) : We decompose the term I_1 into three distinct terms, and investigate each component one by one by taking the limit procedure $\varepsilon \to 0$. As a matter of fact, we get

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{\varepsilon}}\int_0^t \left\{F^\varepsilon(s, u(s) + \sqrt{\varepsilon}y) - F^\varepsilon(u(s)) - \sqrt{\varepsilon}(\nabla \cdot F^\varepsilon(s,z)) \cdot y\right\} ds \\
&\quad + \frac{1}{\sqrt{\varepsilon}}\int_0^t \{F^\varepsilon(s, u(s)) - \tilde{F}(u(s))\}ds + \int_0^t (\nabla \cdot F^\varepsilon(s,z)) \cdot y ds \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{56}
$$

As to the term J_1, paying attention to the expansion

$$
A(s, z + \sqrt{\varepsilon}y) - A(s,z) - (\nabla \cdot A(s,z)) \cdot \sqrt{\varepsilon}y = o(\sqrt{\varepsilon}y), \tag{57}
$$

we readily obtain

$$J_1 = \frac{1}{\sqrt{\varepsilon}} \int_0^t o(\sqrt{\varepsilon} y) ds = \int_0^t \frac{o(\sqrt{\varepsilon} y)}{\sqrt{\varepsilon} y} \times y ds, \tag{58}$$

where we applied the expansion formula (57) to the three integrand terms in J_1. Hence it follows by definition of infinitesimal of higher order that

$$\lim_{\varepsilon \searrow 0} J_1 = \int_0^t \lim_{\varepsilon \searrow 0} \frac{o(\sqrt{\varepsilon} y)}{\sqrt{\varepsilon} y} \times y ds = \int_0^t \{0 \times y\} ds = 0. \tag{59}$$

Next, as to J_2, by transformation of variables $s = \varepsilon\sigma$ we have

$$\lim_{\varepsilon \searrow 0} J_2 = \lim_{\varepsilon \searrow 0} \frac{1}{\sqrt{\varepsilon}} \int_0^{t/\varepsilon} \{F(\sigma, u(\varepsilon\sigma)) - \tilde{F}(u(\varepsilon\sigma))\varepsilon d\sigma. \tag{60}$$

Moreover, employing another transformation of variables $t = \varepsilon T$, noting the equivalence between limit $\varepsilon \to 0$ and limit $T \to \infty$, the expression (60) can be reduced to

$$\lim_{\varepsilon \searrow 0} \frac{t}{\sqrt{\varepsilon}} \lim_{T \to \infty} \frac{1}{T} \int_0^T \{F(\sigma, u(\varepsilon\sigma)) - \tilde{F}(u(\varepsilon\sigma))\} d\sigma. \tag{61}$$

And besides, taking the uniformness in y of the limit procedure (23) in Theorem 2 into account, we observe easily that (61) vanishes and $\lim_{\varepsilon \to 0} J_2 = 0$, because we applied the expression

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(s, y) ds = \tilde{F}(y) = F(\infty, y) \quad \text{uniformly in} \quad y.$$

Lastly, for the term J_3, we may apply the transformation of variables $s = ut$ together with integral continuity of F to obtain

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} J_3 &= \lim_{\varepsilon \searrow 0} \int_0^t (\nabla \cdot F\left(\frac{s}{\varepsilon}, z\right)) \cdot y ds \\
&= \lim_{\varepsilon \searrow 0} \int_0^1 (\nabla \cdot F\left(\frac{ut}{\varepsilon}, z\right)) \cdot ytdu \\
&= \int_0^1 \lim_{\varepsilon \searrow 0} (\nabla \cdot F\left(\frac{ut}{\varepsilon}, z\right)) \cdot ytdu \\
&= \int_0 (\nabla \cdot F(\infty, z)) \cdot ytdu \\
&= \int_0^t (\nabla \cdot F(\infty, u(s))) \cdot y ds (= \int_0^t \xi(s) \cdot y ds).
\end{aligned} \tag{62}$$

This implies that there exists a function $\xi(s) \equiv \nabla \cdot F(\infty, u(s)) \in L^1(0, T)$ such that (29) holds; i.e.,

$$\lim_{\varepsilon\searrow 0}\int_0^t (\nabla\cdot F\left(\frac{s}{\varepsilon},z\right))\cdot y ds=\int_0^t \xi(s)\cdot y ds. \tag{63}$$

Here we regard $u(s)$ as z in the above-mentioned calculation. And also note that we are very concerned on the convergence in law of the fluctuation $V_s^\varepsilon(\omega)\to Z_s(\omega)$ as the parameter ε approaches to zero.

(As to I2) : Resorting to Itô's isometry for the Itô type stochastic integral with respect to a Brownian motion, we obtain

$$\begin{aligned}\mathbb{E}|I_2|^2&=\mathbb{E}\left|\int_0^t G^\varepsilon(u(s)+\sqrt{\varepsilon}V_s^\varepsilon)dB_s^\varepsilon\right|^2\\&=\mathbb{E}\int_0^t\{G^\varepsilon(u(s)+\sqrt{\varepsilon}V_s^\varepsilon)\}^2ds.\end{aligned} \tag{64}$$

Hence, it follows immediately from (64) that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\mathbb{E}|I_2|^2&=\lim_{\varepsilon\searrow 0}\mathbb{E}\int_0^t\{G^\varepsilon(u(s)+\sqrt{\varepsilon}V_s^\varepsilon)\}^2ds\\&=\mathbb{E}\int_0^t\lim_{\varepsilon\searrow 0}\{G^\varepsilon(u(s)+\sqrt{\varepsilon}V_s^\varepsilon)\}^2ds.\end{aligned} \tag{65}$$

In fact, we have $G=\begin{pmatrix}0&0\\0&\sigma\end{pmatrix}\begin{pmatrix}\tilde{X}_t\\v_t\end{pmatrix}=\begin{pmatrix}0\\\sigma v_t\end{pmatrix}$, and when we put $u(s)=(\tilde{u}(s),\hat{u}(s))$ and $V_s^\varepsilon=(\tilde{V}_s^\varepsilon,\hat{V}_s^\varepsilon)$, then the scaled term G^ε can be rewritten into

$$G^\varepsilon(u(s)+\sqrt{\varepsilon}V_s^\varepsilon)=\begin{pmatrix}0\\\sigma(\hat{u}(s)+\sqrt{\varepsilon}\hat{V}_s^\varepsilon)\end{pmatrix},$$

so that, we finally get $G(\infty,u(s))=\begin{pmatrix}0\\\sigma\hat{u}(s)\end{pmatrix}$. On this account, we can deduce from (65) that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\mathbb{E}|I_2|^2&=\mathbb{E}\int_0^t G(\infty,u(s))^2ds\\&=\mathbb{E}\int_0^t\Psi^2(s,\omega)ds=\mathbb{E}\left|\int_0^t\Psi(s,\omega)dB_s\right|^2,\end{aligned} \tag{66}$$

because we made use of the Itô isometry again in the above computation, but this time we applied it for the above term in the reverse direction. Thus we attain at last that

$$\lim_{\varepsilon\searrow 0}\int_0^t G^\varepsilon(X_s^\varepsilon)^2ds=\int_0^t G(\infty,u(s))^2ds=\int_0^t\Psi(s)^2ds, \tag{67}$$

with the result that, for $u(s)=(\tilde{u}(s),\hat{u}(s))$,

$$G(\infty, u(s)) \equiv \begin{pmatrix} 0 \\ \sigma\hat{u}(s) \end{pmatrix} = \Psi(s) = (\tilde{\Psi}(s), \hat{\Psi}(s))$$
$$= (0, \sigma\hat{u}(s)) = (0, \sigma \cdot Proj_L(u(s))) = (0, \sigma \cdot u(s)|_L),$$

where $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3 = U \oplus L$ and we define the notation $Proj_L$ as $Proj_L(h) = \hat{h}$ for $h = (\tilde{h}, \hat{h}) \in U \oplus L$. Then from the definition of fluctuation V_s^ε and its decomposition

$$V_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\{X(\frac{t}{\varepsilon}, \omega) - u(t)\}$$
$$= \frac{1}{\sqrt{\varepsilon}}\int_0^t (F^\varepsilon(X\left(\frac{s}{\varepsilon}\right)) - \tilde{F}(u(s)))ds + \frac{1}{\sqrt{\varepsilon}} \cdot \sqrt{\varepsilon}\int_0^t G^\varepsilon\left(X\left(\frac{s}{\varepsilon}\right)\right) dB_s^\varepsilon,$$

we observe that the last stochastic integral in the above vanishes $(\Longrightarrow 0)$ as ε tends to zero. When we write the limit process of V_s^ε as Z_s, then under the circumstances $V_s^\varepsilon(\omega) = y$, $\mathbb{P}$-a.s., from (55) the aforementioned discussion on convergence in law yields to

$$V_t^\varepsilon = I_1 + I_2 = (J_1 + J_2 + J_3) + I_2$$
$$\longrightarrow (0 + 0 + \lim_\varepsilon J_3) + \lim_\varepsilon I_2$$
$$= \int_0^t (\nabla \cdot F(\infty, u(s))) \cdot \lim_\varepsilon V_s^\varepsilon ds + \int_0^t \Psi(s, \omega)dB_s \tag{68}$$

as $\varepsilon \to 0$. Since $V_t^\varepsilon \Longrightarrow Z_t(\omega)$, summing up, we thus attain the derivation of the stochastic integral equation that should satisfy the limit process $Z_t \equiv Z_t(\omega)$ appearing in the limiting procedure $\varepsilon \to 0$ for the fluctuation V_t^ε. That is to say,

$$Z_t = \int_0^t \xi(s)Z_s ds + \int_0^t \Psi(s)dB_s. \tag{69}$$

When we rewrite it into a differential form, then we observe that Z_t is a stochastic process which is characterized by the following Itô type SDE :

$$dZ_t = \xi(t)Z_t dt + \Psi(t)dB_t$$
$$= \nabla \cdot F(\infty, u(t))dt + G(\infty, u(t))dB_t. \tag{70}$$

□

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