# Star-Product Functional and Related Integral Equations

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**Abstract**

In this paper we consider a class of deterministic nonlinear integral equations. While, we begin with constructing a branching model, define a star-product and construct a tree-based star-product functional. Finally we study the mathematical structure of the functional and prove that the expectation of the functional with respect to a time-reversed law of the branching process satisfies the original integral equations.
**Key Words**: Nonlinear integral equation, branching model, tree structure, star-product, branching process, star-product functional.

## 1. Notations

For simplicity, let $D_0 := \mathbb{R}^3 \setminus \{0\}$, and we put $\mathbb{R}_+ := [0, \infty)$. For every $\alpha, \beta \in \mathbb{C}^3$, we use the symbol $\alpha \cdot \beta$ for the inner product, and we define $e_x := x/|x|$ for every $x \in D_0$. In this article we consider the following deterministic nonlinear integral equation:

$$e^{\lambda t|x|^2} u(t,x) = u_0(x) + \frac{\lambda}{2} \int_0^t ds \; e^{\lambda s|x|^2} \int p(s,x,y;u) n(x,y) dy + \frac{\lambda}{2} \int_0^t e^{\lambda s|x|^2} f(s,x) ds, \qquad \text{for} \quad \forall (t,x) \in \mathbb{R}_+ \times D_0. \tag{1}$$

Here $u \equiv u(t,x)$ is an unknown function : $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, $\lambda > 0$, and $u_0 : D_0 \to \mathbb{C}^3$ is the initial data such that $u(t,x)|_{t=0} = u_0(x)$. Moreover, $f(t,x) : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3$ is a given function satisfying $f(t,x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$. The integrand $p$ in (1) is given by

$$p(t,x,y;u) = u(x,y) \cdot e_x \{ u(t, x-y) - e_x (u(t,x-y) \cdot e_x) \}. \tag{2}$$

On the other hand, we consider a Markov kernel $K : D_0 \to D_0 \times D_0$. Actually, for every $z \in D_0$, $K_z(dx, dy)$ lies in the space $\mathcal{P}(D_0 \times D_0)$ of all probability measures on a product space $D_0 \times D_0$. When the kernel $k$ is given by $k(x,y) = i|x|^{-2} n(x,y)$, then we define $K_z$ as a Markov kernel satisfying that for any positive measurable function $h = h(x,y)$ on $D_0 \times D_0$,

$$\iint h(x,y) K_z(dx,dy) = \int h(x, z-x) k(x,z) dx. \tag{3}$$

Moreover, we assume that for every measurable functions $f, g > 0$ on $\mathbb{R}^+$,

$$\int h(|z|) \nu(dz) \int g(|x|) K_z(dx,dy) = \int g(|z|) \nu(dz) \int h(|y|) K_z(dx,dy) \tag{4}$$

holds, where the measure $\nu$ is given by $\nu(dz) = |z|^{-3} dz$.

## 2. Principal theorem

In this section we shall state our main result, which asserts the existence and uniqueness of solutions to the nonlinear integral equation (1). As a matter of fact, the solution $u(t, x)$ can be expressed as the expectation of a star-product functional, which is nothing but a probabilistic solution constructed by making use of the below-mentioned branching particle systems and branching models. Let

$$M_{\bigstar}^{\langle u_0, f\rangle}(\omega) = \prod \bigstar_{[x_{\tilde{m}}]} \Xi_{m_2.m_3}^{m_1}[u_0, f](\omega), \tag{5}$$

be a probabilistic representation in terms of tree-based star-product functional with weight $(u_0, f)$. For the details of the definition, see the succeeding sections. On the other hand, $M_*^{\langle U,F\rangle}(\omega)$ denotes the corresponding $*$-product functional with weight $(U, F)$. In fact, as to be seen in what follows, in a similar manner as the case of a star-product functional we can construct a $(U, F)$-weighted tree-based $*$-product functional $M_*^{\langle U,F\rangle}(\omega)$, which is indexed by the nodes $(x_m)$ of a binary tree. Here we suppose that $U = U(x)$ (resp. $F = F(t, x)$) is a non-negative measurable function on $D_0$ (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each $x$. Indeed, in construction of the $*$-product functional, the product in question is taken as ordinary multiplication $*$ instead of the star-product $\bigstar$ in the definition of star-product functional.

Theorem 1. *Suppose that* $|u_0(x)| \leqslant U(x)$ *for* $\forall x$ *and* $|\tilde{f}(t, x)| \leqslant F(t, x)$ *for* $\forall t,\ x$*, and also that for some* $T > 0$ $(T >> 1$ *sufficiently large*$)$,

$$E_{T,x}[M_*^{\langle U,F\rangle}] < \infty, \quad \text{a.e.} - x \tag{6}$$

*Then there exists a* $(u_0, f)$*-weighted tree-based star* $\bigstar$*-product functional* $M_{\bigstar}^{\langle u_0, f\rangle}(\omega)$*, indexed by a set of node labels accordingly to the tree structure which a binary critical branching process* $Z^{K_x}(t)$ *determines. Furthermore, the function*

$$u(t, x) = E_{t,x}[M_{\bigstar}^{\langle u_0, f\rangle}] \tag{7}$$

*gives a unique solution to the integral equation (1). Here* $E_{t,x}$ *denotes the expectation with respect to a probability measure* $P_{t,x}$ *as the time-reversed law of* $Z^{K_x}(t)$.

## 3. Branching model and tree-like structure

In this section we consider a continuous time binary critical branching process $Z^{K_x}(t)$ on $D_0$, whose branching rate is given by a parameter $\lambda |x|^2$, whose branching mechanism is binary with equi-probability, and whose descendant branching particle behavior (or distribution) is determined by the kernel $K_x$. Next, taking notice of the tree structure which the process $Z^{K_x}(t)$ determines, we denote the space of marked trees

$$\omega = (t, (t_m), (x_m), (\eta_m), m \in \mathcal{V}) \tag{8}$$

by $\Omega$. Furthermore, we write the time-reversed law of $Z^{K_x}(t)$ being a probability measure on $\Omega$ as

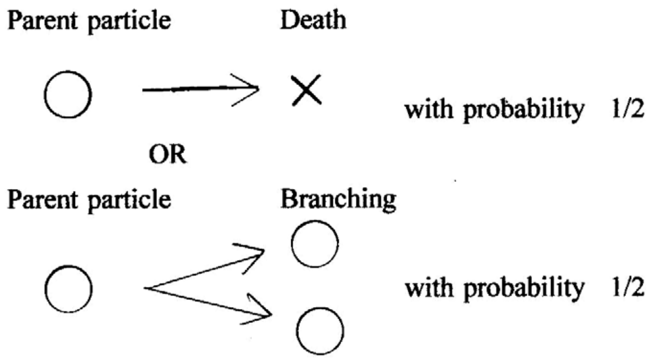


Figure 1: Binary Branching

$P_{t,x}\in\mathcal{P}(\Omega)$. Here $t$ denotes the birth time of common ancestor, and the particle $x_m$ dies when $\eta_m=0$, while it generates two descendants $x_{m1}, x_{m2}$ when $\eta_m=1$. On the other hand,

$$\mathcal{V}=\bigcup_{\ell\geq 0}\{1,2\}^{\ell}$$

is a set of all labels, namely, finite sequences of symbols with length $\ell$, which describe the whole tree structure given. For $\omega\in\Omega$ we denote by $\mathcal{N}(\omega)$ the totality of nodes being branching points of tree, and let $N_+(\omega)$ be the set of all nodes $m$ being a member of $\mathcal{V}\backslash\mathcal{N}(\omega)$, whose direct predecessor lies in $\mathcal{N}(\omega)$ and which satisfies the condition $t_m(\omega)>0$, and let $N_-(\omega)$ be the same set as described above, but satisfying $t_m(\omega)\leqslant 0$. Finally we put

$$N(\omega)=N_+(\omega)\cup N_-(\omega). \tag{9}$$

## 4. Star-product functional and $*$-product functional

In what follows we shall introduce a tree-based star-product functional in order to construct a probabilistic solution to the class of integral equations (1). First of all, we denote by the symbol $\mathrm{Proj}^z(\cdot)$ a projection of the objective element onto its orthogonal part of the $z$ component in $\mathbb{C}^3$, and we define a $\bigstar$-product of $\beta,\gamma$ for $z\in D_0$ as

$$\beta\bigstar_{[z]}\gamma=-i(\beta\cdot e_z)\mathrm{Proj}^z(\gamma). \tag{10}$$

We shall define $\Theta^m(\omega)$ for each $\omega\in\Omega$ realized as follows. When $m\in N_+(\omega)$, then $\Theta^m(\omega)=\tilde{f}(t_m(\omega),x_m(\omega))$, while $\Theta^m(\omega)=u_0(x_m(\omega))$ if $m\in N_-(\omega)$. Then we define

$$\Xi^{m_1}_{m_2.m_3}(\omega)\equiv\Xi^{m_1}_{m_2,m_3}[u_0,f](\omega):=\Theta^{m_2}(\omega)\bigstar_{[x_{m_1}]}\Theta^{m_3}(\omega), \tag{11}$$

where as for the product order in the star-product $\bigstar$, when we write $m\prec m'$ lexicographically with respect to the natural order $\prec$, the term $\Theta^m$ labelled by $m$ necessarily occupies the left-hand side and the other $\Theta^{m'}$ labelled by $m'$ occupies the right-hand side by all means. And besides, as abuse of notation we write

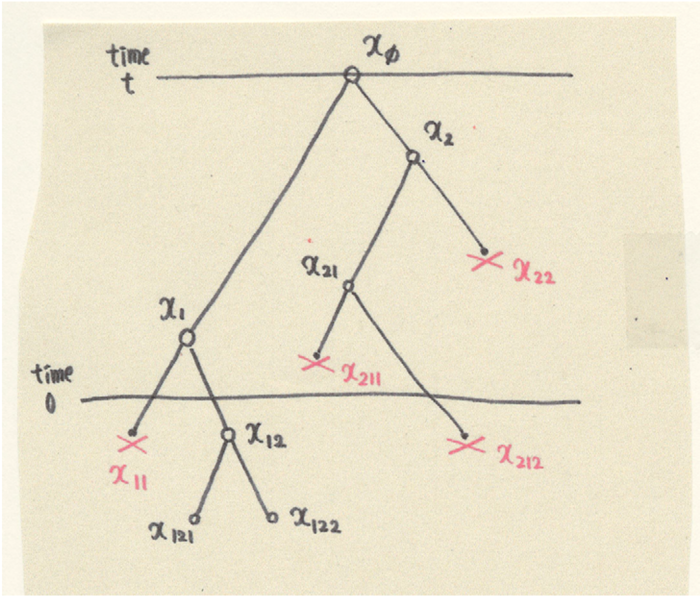

Figure 2: Example: A realized Tree

$$\Xi^{\emptyset}_{m,\emptyset}(\omega) \equiv \Xi^{\emptyset}_{m,\emptyset}[u_0, f](\omega) := \Theta^m(\omega), \tag{12}$$

especially when $m \in \mathcal{V}$ is a label of single terminal point in the restricted tree structure in question.

Under these circumstances, we consider a random quantity which obtained by executing the star-product $\bigstar$ inductively at each node in $\mathcal{N}(\omega)$, and we call it a tree-based $\bigstar$-product functional, and we express it symbolically as

$$M^{\langle u_0, f\rangle}_{\bigstar}(\omega) = \prod \bigstar_{[x_{\tilde{m}}]} \Xi^{m_1}_{m_2.m_3}[u_0, f](\omega), \tag{13}$$

where $m_1 \in \mathcal{N}(\omega)$ and $m_2, m_3 \in N(\omega)$, and by the symbol $\prod \bigstar$ (as a product relative to the star-product) we mean that the star-products $\bigstar$'s should be succeedingly executed in a lexicographical manner with respect to $x_{\tilde{m}}$ such that $\tilde{m} \in \mathcal{N}(\omega) \cap \{|\tilde{m}| = \ell - 1\}$ when $|m_1| = \ell$.

Example 1. Now let us suppose that a tree structure $\omega_1$ ($\in \Omega$) has been realized here (see Figure 2). Next we shall classify those nodes in the realized tree $\omega_1$. As a matter of fact, as to those two particles located at $x_{11}$ and $x_{12}$ with nodes of the level $|m| = \ell = 2$ accompanied by the pivoting node $x_1$, we can construct

$$\Xi^1_{11,12}(\omega_1) = \Theta^{11}(\omega_1) \bigstar_{[x_1]} \Theta^{12}(\omega_1)$$

by a star-product $u_0(x_{11}(\omega_1)) \bigstar_{[x_1]} u_0(x_{12}(\omega_1))$ in accordance with the rule, because both $m_1 = 11$ and $m_2 = 12$ lie in $N_-(\omega)$. As to the node $x_{21}$, how to construct $\Xi(\omega_1)$ is the almost same thing as described above. In fact, it goes similarly because $x_{211}$ lies in $N_+(\omega_1)$ and $x_{212}$ lies in $N_-(\omega_1)$. According to the rule, it follows that

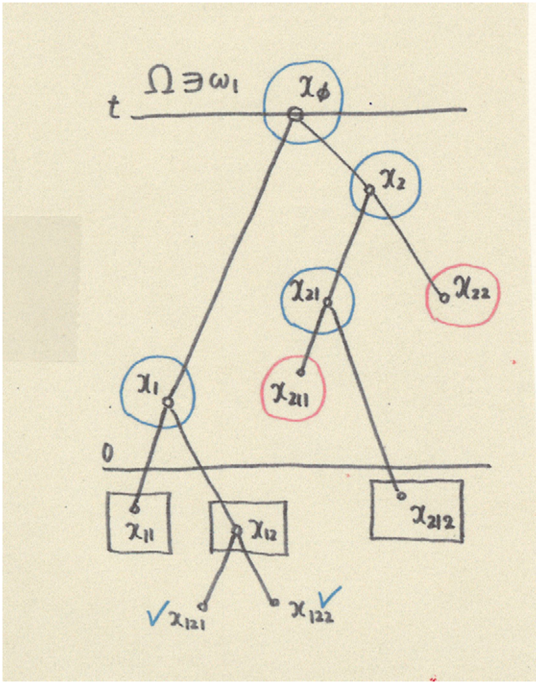


Figure 3: Classification of Nodes

$$\Theta^{211}(\omega_1) = \tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \quad \text{and} \quad \Theta^{212}(\omega_1) = u_0(x_{212}(\omega_1)),$$

hence $\Xi^{21}_{211,212}(\omega_1)$ is given by $\tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \bigstar_{[x_{21}]} u_0(x_{212}(\omega_1))$, see Figure 3. Consequently, we obtain finally

$$M_{\bigstar}^{\langle u_0, f\rangle}(\omega_1) = \big(u_0(x_{11}) \bigstar_{[x_1]} u_0(x_{12})\big) \bigstar_{[x_\phi]} \left\{ \left( \tilde{f}(t_{211}, x_{211}) \bigstar_{[x_{21}]} u_0(x_{212}) \right) \bigstar_{[x_2]} \tilde{f}(t_{22}, x_{22}) \right\}. \tag{14}$$

□

## 5. Sketch of proof

In this section we are first going to construct a $(U, F)$-weighted tree-based $*$-product functional $M_*^{\langle U,F\rangle}(\omega)$, which is indexed by the nodes $(x_m)$ of a binary tree. Here recall that $U = U(x)$ (resp $F = F(t\ x)$) is a non-negative measurable function on $D_0$ (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each $x$. Moreover, in construction of the functional, the product is taken as ordinary multiplication $*$ instead of the star-product $\bigstar$.

In what follows we shall give an outline of the proof of Theorem 1. We need the following technical lemma, which plays an essential role in the proof.

Lemma 2. *For $0 \leqslant t \leqslant T$ and $x \in D_0$, the function $V(t, x) = E_{t,x}[M_*^{\langle U,F\rangle}(\omega)]$ satisfies*

$$e^{\lambda t|x|^2} V(t, x) = U(x) + \int^t ds \frac{\lambda |x|^2}{2} e^{\lambda s |x|^2} \left\{ F(s, x) + \int V(s, y) V(s, z) K_x(dy, dz) \right\}. \tag{15}$$

*Proof of lemma 2.* By making use of the conditional expectation we can decompose the function $V(t,x)$ as follows:

$$\begin{aligned} V(t,x) &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega)] \\ &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi \leqslant 0] + E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0] \\ &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi \leqslant 0] + E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0,\ \eta_\phi = 0] \\ &\qquad + E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0,\ \eta_\phi = 1]. \end{aligned} \tag{16}$$

Next we are going to take into consideration an equivalence between the event $t_\phi \leqslant 0$ and $T \notin [0,t]$. Indeed, as to the first term in the third line of (16), since the condition $t_\phi \leqslant 0$ means that $T$ never lies in an interval $[0,t]$, and since $m=\phi \in N_-(\omega)$ leads to a non-random functional expression

$$M_* = \Theta^\phi = U(x),$$

the tree-based $*$-product functional is allowed to have a simple representation:

$$\begin{aligned} E_{t,x}[M_*^{\langle U,F\rangle}, t_\phi \leqslant 0] &= E_{t,x}[M_*^{\langle U,F\rangle} \cdot 1_{\{t_\phi \leqslant \leqslant 0\}}] = U(x) \cdot P_{t,x}(t_\phi \leqslant 0) \\ &= U(x) \cdot P(T \notin [0,t]) = U(x) \cdot P(T \in (t,\infty)) \\ &= U(x) \int_t^\infty f_T(s)\mathrm{d}s = U(x) \int_t^\infty \lambda |x|^2 e^{-\lambda s |x|^2} \mathrm{d}s \\ &= U(x) \cdot \exp\{-\lambda t |x|^2\}. \end{aligned} \tag{17}$$

As to the third term, we need to note the following things. A particle generates two offsprings or descendants $x_1, x_2$ with probability $\frac{1}{2}$ under the condition $\eta_\phi = 1$; since $t_\phi > 0$, when the branching occurs at $t_\phi = s$, then, under the conditioning operation at $t_\phi$, the Markov property guarantees that the lower tree structure below the first generation branching node point (or lo-cation) $x_1$ is independent of that below the location $x_2$ with realized $\omega \in \Omega$, hence a tree-based $*$-product functional branched after time $s$ is also probabilistically independent of the other tree-based $*$-product functional branched after time $s$; and besides, the distributions of $x_1$ and $x_2$ are totally controlled by the Markov kernel $K_x$. Therefore, an easy computation provides with an impressive expression

$$\begin{aligned} E_{t,x}[M_*^{\langle U,F\rangle}, t_\phi > 0, \eta_\phi = 1] &= \frac{1}{2} \int_0^t \mathrm{d}s \lambda |x|^2 e^{-\lambda |x|^2 (t-s)}. \\ &\times \iint E_{s,x_1}[M_*] \cdot E_{s,x_2}[M_*] K_x(\mathrm{d}x_1, \mathrm{d}x_2). \end{aligned}$$

Note that as for the second term, it goes almost similarly as the computation of the second one. Finally, summing up we obtain

$$\begin{aligned} V(t,x) &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega)] \\ &= U(x) r^{-\lambda t |x|^2} + \int_0^t \frac{\lambda |x|^2}{2} e^{-\lambda |x|^2 (t-s)} F(s,x) \mathrm{d}s \\ &+ \int_0^t \frac{\lambda |x|^2}{2} e^{-\lambda |x|^2 (t-s)} \iint V(s,y) V(s,z) K_x(\mathrm{d}y, \mathrm{d}z) \mathrm{d}s. \end{aligned} \tag{18}$$

On this account, if we multiply both sides of (18) by $\exp\{\lambda t |x|^2\}$, then the required expression

(15) in Lemma 2 can be derived, which completes the proof. □

As a matter of fact, the mapping : $[0,T] \ni t \mapsto e^{\lambda|x|^2 t}V(t,x) \in \bar{\mathbb{R}}_+$ is non-decreasing, so that, it proves to be that

$$|M_\bigstar^{\langle u_0,f\rangle}(\omega)| \leqslant |M_*^{\langle U,F\rangle}(\omega)| \tag{19}$$

holds for $\forall t \in [0,T]$ and $x \in E_c$, where $E_c$ is a measurable set on which the validity of $E_{t,x}[M_*^{\langle U,F\rangle}] < \infty$ may be kept. Another important aspect for the proof consists in establishment of the following $M_*$-control inequality. That is to say, we have

$$|M_\bigstar^{\langle u_0,f\rangle}(\omega)| \leqslant |M_*^{\langle U,F\rangle}(\omega)| \tag{20}$$

because of the validity of a simple inequality

$$|w \bigstar_{[x]} v| \leqslant |w| \cdot |v| \ \text{ for } \ w, v \in \mathbb{C}^3 \ \text{ and } \ x \in D_0.$$

On the other hand, it is derived that the space of solutions to (1) is formed by the condition

$$\int_0^T ds \int |u(s,y)| \cdot |u(s,z)| K_x(dy,dz) < \infty \quad \text{for } \ x \in E_c.$$

A similar discussion as above leads to

$$\begin{aligned} u(t,x) = E_{t,x}[M_\bigstar^{\langle u_0,f\rangle}(\omega)] &= e^{-\lambda t|x|^2} u_0(x) + \int_0^t ds\ \lambda|x|^2 e^{-\lambda(t-s)|x|^2} \times \\ &\times \frac{1}{2}\left\{ \tilde{f}(s,x) + \iint E_{s,x_1}[M_\bigstar] \bigstar_{[x]} E_{s,x_2}[M_\bigstar] K_x(dx_1,dx_2) \right\}. \end{aligned} \tag{21}$$

Finally we can deduce that $u(t,x) = E_{t,x}[M_\bigstar^{\langle u_0,f\rangle}(\omega)]$ satisfies the integral equation (1), and this $u(t,x)$ is a solution lying in the space $\mathcal{D}$. Actually, $\mathcal{D}$ is a space of all functions $\varphi : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, being continuous in $t$ and measurable such that

$$\int_0^\infty ds \int |p(s,x,y;\varphi)| K_x(dy,dz) < \infty, \quad \text{a.e.} - x.$$

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