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On Regular Supercompact Spaces

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Abstract

In this paper we prove that every compact tree-like space is regular supercompact. This is a positive answer to a question of J. van Mill. As an application we obtain that the Stone-Čech compactification and the Freudenthal compactification of a rim-compact tree-like space are regular supercompact.

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1. Introduction

The notion of supercompactness was introduced by J. de Groot in [5]. A collection of sets is *linked* if every two members have a non-empty intersection. A collection of sets is *binary* if every linked subcollection has a non- empty intersection. A space is *supercompact* if it has a binary subbase for its closed sets. By Alexander's lemma, every supercompact space is compact. Many compact spaces are supercompact. Examples of supercompact spaces are compact ordered spaces, compact metrizable spaces([13] or see [9]) and compact tree-like spaces([4] and [15], or see [17]). However, not all compact spaces are supercompact. M. G. Bell [2] proved that if the Stone-Čech compactification βX of a space X is supercompact, then X is pseudocompact.

J. van Mill [17] introduced the notion of regular supercompact spaces in analogy with regular Wallman spaces defined by E. F. Steiner [12]. A space is regular *supercompact* if it has a binary subbase $\mathcal{F}$ for its closed sets such that the ring generated by $\mathcal{F}$ consists of regular closed sets. A compact space is *regular Wallman* if it has a subbase $\mathcal{F}$ for its closed sets such that the ring generated by $\mathcal{F}$ consists of regular closed sets. Every regular supercompact space is supercompact as well as regular Wallman. Every compact ordered space is regular supercompact. E. K. van Douwen [14] proved that every compact metrizable space is regular supercompact. J. van Mill [17] proved that a compact tree-like space X is regular supercompact in case X has the weight at most 2^ω and asked whether all compact tree-like spaces are regular supercompact. In [6] the author announced that every compact tree-like space is regular supercompact. The purpose of this paper is to give a proof of this result. J. Nikiel [11] also obtained this result, independently. As a corollary it follows that the Stone-Čech compactification and the Freudenthal compactification of a rim-compact tree-like space are regular supercompact. Some of the results and notation are taken from [7].

2. Lemmas

When $A, B \subset X$, $A \cap B = \emptyset$ and both A and B are open in $A \cup B$, we frequently write $A + B$

instead of $A \cup B$. We often write $X - x$ instead of $X - \{x\}$.

A space is *tree-like* if it is connected and every two distinct points can be separated by a third point.

Let X be a space and x, y, $z \in X$. Then we say that z *separates* x and y if there exist open subsets U and V of X such that $x \in U$, $y \in V$, $U \cap V = \emptyset$ and $X - \{z\} = U \cup V$. In such a case we simply write $X - z = U\langle x\rangle + V\langle y\rangle$.

For $a, b \in X$ we set

$$E(a, b) = \{x \in X : x \text{ separates } a \text{ and } b\} \text{ and } S(a, b) = E(a, b) \cup \{a, b\}.$$

Throughout the rest of this paper, the letter X will always denote a given non-empty compact tree-like space.

Since X is not empty, we take a point $x^* \in X$. For $x, y \in X$ we define the relation $x \leq_{x^*} y$ if and only if $x \in S(x^*, y)$.

Throughout the rest of this paper, we shall fix a point $x^* \in X$ and give a partial order $\leq$ on X by $\leq_{x^*}$. We always X as a partially ordered set with $\leq$.

A subset C of X is called a *segment* if C is a component of $X - x$ for some $x \in X$. In particular, a component C of $X - x$ containing y is called *a segment of y in $X - x$*. A point x of X is called an *end-point* if $X - x$ is connected. We denote by $E(X)$ the set of all end-points of X.

To prove our main theorem we need several lemmas.

2. 1. Lemma $X - x = \bigcup\{S(x^*, x) : x \in E(X)\}$.

Proof. Suppose not, i.e. $y \in X - \bigcup\{S(x^*, x) : x \in E(X)\}$. By Zorn's lemma, there is a maximal chain A' containing y. We put $A = A' \cap \{x \in X : y \leq x\}$. Then each point of A is not an end-point. For each $x \in A$, let $X - x = A_x\langle x^*\rangle + B_x$, where A_x is a segment. By [7, Lemma 9], note that $B_x = \{z \in X : x < z\}$.

First we shall prove the following claim.

Claim. $\{A_z : z \in A\}$ is an open cover of X.

Proof of Claim. By [8,Theorem IV.4, 3, Proposition III.2 or 18, Lemma 2.1]), A_z is open in X for each $z \in A$. Hence it suffices to prove that $\{A_z : z \in A\}$ is a cover of X. We distinguish tree cases.

Case 1. $x \in A_y$.

Since $y \in A$, $x \in \bigcup\{A_z : z \in A\}$.

Case 2. $x \notin A_y$ and $x \in A$

Since $x \in A$, x is not an end-point of X. Thus $z \in B_x$ for some $z \in X$. Assume that $B_x \cap A = \emptyset$. Then $z \notin A$ and, by [7, Lemma 9], $x < z$. This implies that $z \notin A'$, therefore A' is properly contained in $A' \cup \{z\}$. Let u be any point of A'. If $u \notin A$, then $u < z$, because $u < y$. If $u \in A$, then $u \notin B_x$, because $B_x \cap A = \emptyset$.

Thus we have $u < x$, therefore $u < z$. Hence $A' \cup \{z\}$ is a chain, which contradicts the maximality of A'. We can take a point $v \in B_x \cap A$. Then $x \in A_v$, because $x < v$. Hence $x \in \bigcup\{A_z : z \in A\}$.

Case 3. $x \notin A_y$ and $x \notin A$.

This case implies that $x \notin A'$. Since A' is maximal, there is a point $z \in A'$ such that $x \not< z$ and $z \not< x$. Then we see $y \leq z$, that is, $z \in A$. Since $z \not< x$, we have $x \in A_z$. Hence $x \in \bigcup\{A_z : z \in A\}$.

This completes the proof of Claim.

Since X is compact, we can now take a finite subset $\{z_1, z_2, \dots , z_n\}$ from A such that $\{A_{z_i} : i = 1, 2, \dots , n\}$ covers X. Let $z = \max\{z_1, z_2, \dots , z_n\}$. Then, by [7, Lemma 10], $A_{z_i} \subset A_z$ for each i. Thus $X = A_z$, this contradicts $z \notin A_z$. Lemma 2.1 has been proved.

Let $B \subset A \subset X$. Then B is <-*dense* in A if for each $x, y \in A$ with $x < y$ there is a point $z \in B$ such that $x < z < y$.

2.2. Lemma *For each a* $\in X$ *with a* $\neq x^*$*, there are disjoint subsets P and Q of* $S(x^*, a)$ *such that both P and Q are <-dense in* $S(x^*, a)$.

Proof. Let $\mathcal{S} = \{(x, y) : x, y \in S(x^*, a)$ and $x < y\}$, and enumerate $\mathcal{S}$ as $\mathcal{S} = \{(x_\alpha, y_\alpha) : \alpha < \tau \}$, where τ is an ordinal number. Suppose that for each $\beta < \alpha$, p_β and q_β are taken and satisfy the following conditions (i) - (iv).

(i) $p_\beta, q_\beta \in E(x_\beta , y_\beta)$,

(ii) $P_\alpha \cap Q_\alpha = \emptyset$, where $P_\alpha = \{p_\beta : \beta < \alpha\}$ and $Q_\alpha = \{q_\beta : \beta < \alpha\}$,

(iii) if $P_\beta \cap E(x_\beta, y_\beta) \neq \emptyset$, then $p_\beta = p_\gamma$, where $\gamma = \min\{\delta : p_\delta \in E(x_\beta, y_\beta)\}$,

(iv) if $Q_\beta \cap E(x_\beta, y_\beta) \neq \emptyset$, then $q_\beta = q_\gamma$, where $\gamma = \min\{\delta : q_\delta \in E(x_\beta, y_\beta)\}$.

We take p_α and q_α in ech of the following cases.

Case 1. $P_\alpha \cap E(x_\alpha, y_\alpha) \neq \emptyset$ and $Q_\alpha \cap E(x_\alpha, y_\alpha) \neq \emptyset$.

Let $\beta = \min\{\delta : p_\delta \in E(x_\alpha, y_\alpha)\}$ and $p_\alpha = p_\beta$. Let $\gamma = \min\{\delta : q_\delta \in E(x_\alpha, y_\alpha)\}$ and $q_\alpha = q_\gamma$.

Case 2. $P_\alpha \cap E(x_\alpha, y_\alpha) = \emptyset$ and $Q_\alpha \cap E(x_\alpha, y_\alpha) = \emptyset$.

Since $|E(x_\alpha, y_\alpha)| \geq 2$, we take $p_\alpha, q_\alpha \in E(x_\alpha, y_\alpha)$ with $p_\alpha \neq q_\alpha$.

Case 3. $P_\alpha \cap E(x_\alpha, y_\alpha) \neq \emptyset$ and $Q_\alpha \cap E(x_\alpha, y_\alpha) = \emptyset$.

Let $\beta = \min\{\delta : p_\delta \in E(x_\alpha, y_\alpha)\}$ and $p_\alpha = p_\beta$. Assume that $E(x_\alpha, y_\alpha) \subset P_\alpha$. Let $\gamma = \min\{\delta : p_\delta \in E(x_\alpha, y_\alpha)$ and $p_\delta \neq p_\beta\}$. Obviously, $\beta < \gamma$. Then, by [7, Lemma 7], $p_\beta < p_\gamma$ or $p_\gamma < p_\beta$. Suppose that $p_\beta < p_\gamma$. Since $E(p_\beta, p_\gamma) \subset E(x_\alpha, y_\alpha) \subset P_\alpha$, we take $p_\xi \in E(p_\beta, p_\gamma)$. Obviously, $\gamma < \xi$. Then $q_\xi < p_\xi$ or $p_\xi < q_\xi$. If $q_\xi < p_\xi$, then $q_\xi < x_\alpha$, because $q_\xi \notin E(x_\alpha, y_\alpha)$. Thus $x_\xi < p_\beta < y_\xi$, this contradicts (iii). If $p_\xi < q_\xi$, then $y_\alpha < q_\xi$, because $q_\xi \notin E(x_\alpha, y_\alpha)$. Thus $x_\xi < p_\gamma < y_\xi$, this contradicts (iii). Similarly, if $p_\gamma < p_\beta$, then we have a contradiction. Hence $E(x_\alpha, y_\alpha) \not\subset P_\alpha$. Thus we can take a point $q_\alpha \in E(x_\alpha, y_\alpha) - P_\alpha$.

Case 4. $P_\alpha \cap E(x_\alpha, y_\alpha) = \emptyset$ and $Q_\alpha \cap E(x_\alpha, y_\alpha) \neq \emptyset$.

Similarly as in Case 3.

In any case it is easy to see that p_α and q_α satisfy the conditions (i) - (iv).

We set $P = \{p_\alpha : \alpha < \tau \}$ and $Q = \{q_\alpha : \alpha < \tau \}$. Then P and Q have all the required properties. Lemma 2.2 has been proved.

Let $\mathcal{U}(X)$ be the collection of all segments of X. Then $\mathcal{U}(X)$ is a subbase for its open sets, because X is compact.

2.3. Lemma *There is a closure-distributive subcollection $\mathcal{B}$ of $\mathcal{U}(X)$ such that $\mathcal{B}$ is a subbase for its closed sets.*

Proof. Enumerate $E(X)$ as $E(X) = \{x_\alpha : \alpha < \tau\}$, where τ is an ordinal number. Let $X_\alpha = S(x^*, x_\alpha) - \bigcup\{S(x^*, x_\beta) : \beta < \alpha\}$. Obviously, $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$, and, by Lemma 2.1, $X = \bigcup\{X_\alpha : \alpha < \tau\}$. By Lemma 2.2, we can take two disjoint <-dense subsets P'_α and Q'_α in $S(x^*, x_\alpha)$ and set $P_\alpha = P'_\alpha \cap X_\alpha$, $Q_\alpha = Q'_\alpha \cap X_\alpha$, $P = \cup\{P_\alpha : \alpha < \tau\}$ and $Q = \bigcup\{Q_\alpha : \alpha < \tau\}$. For each $p \in P_\alpha$, let B_p be a segment of x_α in $X - p$. For each $q \in Q_\alpha$, let A_q be a segment of x^* in $X - q$.

Let us set $\mathcal{B} = \{A_q : q \in Q\} \cup \{B_p : p \in P\}$. We shall prove that $\mathcal{B}$ is a closure-distributive subbase for its open sets.

Since for each $B \in \mathcal{B}$, Bd$B = \{q\}$ or $\{p\}$ according $B = A_q$ or B_p. Since $P \cap Q = \emptyset$, we have Bd$B_0 \cap$ Bd$B_1 = \emptyset$ for $B_0, B_1 \in \mathcal{B}$ with $B_0 \neq B_1$. Hence $\mathcal{B}$ is closure-distributive.

We need the following claim to prove that $\mathcal{B}$ is a subbase for its open sets.

Claim. For $x, y \in X$ with $x < y$ there are a point $z \in E(x, y)$ and an ordinal number $\alpha < \tau$ such that $E(x, z) \subset X_\alpha$.

Proof of Claim. Let $\alpha = \min\{\beta : E(x, y) \cap S(x^*, x_\beta) \neq \emptyset\}$. By [7, Lemma 6], $S(x, y) \cap S(y, x_\alpha) \cap S(x_\alpha, x)$ is one point set $\{z\}$. Assume that there is a point $w \in E(x, z) - X_\alpha$. Since $X = \{X_\beta : \beta < \tau\}$, we take a β with $w \in X_\beta$. Then $E(x, y) \cap S(x^*, x_\beta) \neq \emptyset$. From the minimality of α it follows that $\alpha < \beta$. Since $w \in E(x, z)$, $x^* < x < w < z < x_\alpha$. Thus $w \in S(x^*, x_\alpha)$. This contradicts $w \in X_\beta$. Hence $E(x, z) \subset X_\alpha$. This completes the proof of Claim.

Next, we shall prove that $\mathcal{B}$ is a subbase for its open sets. Since $\mathcal{U}(X)$ is a subbase for its open sets, it suffices to prove that for each $x \in X$ and $U \in \mathcal{U}(X)$ with $x \in U$ there is a $B \in \mathcal{B}$ such that $x \in B \subset U$.

Let U be a segment of x in $X - z$. There are two cases to consider.

Case 1. $x^* \in U$.

Let $S(x, z) \cap S(z, x^*) \cap S(x^*, x) = \{y\}$. By Claim, $E(y, w) \subset X_\alpha$ for some $w \in E(y, z)$ and some $\alpha < \tau$. We take a point $q \in Q_\alpha$ such that $y < q < w$. Then, by [7, Lemmas 3 and 10], $x \in A_q \subset U$.

Case 2. $x^* \notin U$.

Since $x \in U$, by [7, Lemma 9], this case implies that $z < x$. Hence, by Claim, $E(z, w) \subset X_\alpha$ for some $w \in E(z, x)$ and some $\alpha < \tau$. We take a point $p \in P$ such that $z < p < w$. Similarly, we have $x \in B_p \subset U$.

In any case we can take an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Hence $\mathcal{B}$ is a subbase for its open sets. Lemma 2.3 has been proved.

3. The main result

We are now in a position to establish our main theorem.

3.1. Theorem *Every compact tree-like space is regular supercompact.*

Proof. By Lemma 2.3, there is a closure-distributive subbase $\mathcal{B}$ for its open sets such that $\mathcal{B} \subset$

$\mathcal{U}(X)$. Let $\mathcal{F} = \{\mathrm{Cl}(\cap\mathcal{B}') : \mathcal{B}'$ is a finite subcollection of $\mathcal{B}\}$. Then $\mathcal{F}$ is a subbase for its closed sets, since X is compact. Since $\mathcal{B}$ is closure-distributive, by [7, Lemma 12], $\mathcal{F}$ is binary, moreover the ring generated by $\mathcal{F}$ consists of regular closed sets. Hence X is regular supercompact. Theorem 3.1 has been proved.

Since every regular supercompact space is regular Wallman, we have

3.2. Corollary *Every compact tree-like space is regular Wallman.*

This is also a positive answer to a question of J. van Mill [16], who proved that every compact tree-like space of weight at most 2^{ω} is regular Wallman.

K. R. Allen [1] proved that the Freudenthal compactification of a rim-compact tree-like space is tree-like. Our next corollary is a direct consequence of this result and Theorem 3.1.

3.3. Corollary *The Freudenthal compactification of a rim-compact tree-like space is regular supercompact (hence it is regular Wallman).*

In [16] J. van Mill proved that if a rim-compact tree-like space Y has at most 2^{ω} closed subsets, then βX is regular Wallman. K. Misra [10] extended this result to the case that Y is a rim-compact tree-like space of weight at most 2^{ω} , Moreover, he proved that for a rim-compact tree-like space Y, βY is regular Wallman if and only if Y has a compactification which is regular Wallman. Thus we obtain the following corollary.

3.4. Corollary *The Stone-Čech compactification of a rim-compact tree-like space is regular Wallman.*

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