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Compactifications of Product Spaces

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Abstract

Let X be a compact metric space and let Y be a non-compact, locally compact metric space. In this paper we give conditions on X and Y which characterize the product space $X \times Y$ having all compact metric spaces as remainders.

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1. Introduction

Throughout this paper all spaces are assumed to be completely regular and T_1 unless otherwise stated.

A space Y is a remainder of another space X if Y is homeomorphic to $\alpha X - X$ for some compactification αX of X.

In the theory of compactifications one of the major problems has been that of characterizing when all members of a certain class of spaces can serve as remainders for each member of another class of spaces (cf. [2], [3], [6], [7], [8], [12] etc.).

Hatzenbuhler and Mattson [6] characterized spaces having all compact metric spaces as remainders. In this paper we consider this problem on product spaces.

In [7] Hatzenbuhler and Mattson gave conditions (see Theore 3.1 below) on X and Y which characterize when all compact metric spaces are continuous images of $\beta X \times \beta Y - X \times Y$, where βX is the Stone-Čech compactification of X. By Magill's theorem, their conditions are sufficient in order that the product space $X \times Y$ have all compact metric spaces as remainders. However, in general, $\beta X \times \beta Y \neq \beta(X \times Y)$, therefore their conditions need not be necessary. In [7] they asked whether their condition are not only sufficient but also necessary.

In this paper we give necessary and sufficient conditions on a compact metric space X and a metric space Y which characterize the product space $X \times Y$ having all compact metric spaces as remainders.

2. Preliminaries

Since every space having a compactification with compact remainder is locally compact, we only consider locally compact spaces. The following theorem, which was proved by Magill [10], is a basic result on remainders of compactifications of locally compact spaces.

2.1. Theorem (Magill [10]). *For every locally compact space X and any compact space K the following conditions are equivalent;*
(a) *K is a remainder of X*,
(b) *K is a continuous image of $\beta X - X$*,
(c) *K is a continuous image of a remainder of X.*

Since every compact metric space is a continuous image of the Cantor set, we obtain the following corollary.

2.2. Corollary. *For every locally compact space X the following conditions are equivalent;*
(a) *X has all compact metric spaces as remainders,*
(b) *X has a compactification with the Cantor set as remainder,*
(c) *X has a compactification αX such that $\alpha X - X$ is a continuous preimage of the Cantor set.*

Recall that a space X is *scattered* if every non-empty closed subset of X has an isolated point. The following lemma is easily, so we omit the proof.

2.3. Lemma. *Let X be a compact, non-scattered, zero-dimensional space. Then there exists a continuous mapping from X to the Cantor set.*

Telgársky ([14], p.64 Remark) proved the following lemma.

2.4. Lemma (Telgarsky [14]). *Let f be a perfect mapping from a space X onto a space Y.*
(a) *If X is scattered, then so is Y.*
(b) *If Y is scattered and if $f^{-1}(y)$ is scattered for every $y \in Y$, then X is scattered.*

Let Y be a subspace of another space X. Then *Y is zero-dimensionally embedded in* X if there exists a collection $\mathcal{U}$ of open subsets of X satisfying the following conditions;

(i) $\{U \cap Y : U \in \mathcal{U}\}$ is a base for Y, and
(ii) $\mathrm{Bd}_X\, U \cap Y = \emptyset$ for every $U \in \mathcal{U}$.

The maximal compactification of a space X with zero-dimensionally embedded remainder is called the *Freudenthal* comapctification of X and denoted by γX. Every locally compact space X has the Freudenthal compactification γX. In the case when X is locally compact it is easy to see that for every compactification αX of X, $\alpha X - X$ is zero-dimensionally embedded in αX if and only if $\alpha X - X$ is zero-dimensional (see [1] p.273). In [8] Hatzenbuhler and Mattson pointed out the following theorem without the proof. They stated this theorem follows their theorem [6] which characterizes a space having all compact metric space as remainders. However, it is easy to show that this follows Lemmas 2.3 and 2.4.

2.5. Theorem (Hatzenbuhler and Mattson [8]). *A locally compact space X has all compact metric spaces as remainders if and only if the remainder $\gamma X - X$ of the Freudenthal compactification of X is not scattered.*

Proof. Suppose that X has all compact metric spaces as remainders. Then X has a compactification αX with the Cantor set as remainder. Since the Cantor set is not scattered, by Lema 2.4(a), $\gamma X - X$ is not scattered.

Conversely, if $\gamma X - X$ is not scattered, then, by Lemma 2.3, the Cantor set is a continuous image of $\gamma X - X$. Hence, by Corollary 2.2, X has all compact metric spaces as remainders.

Let $Q(X)$ be the set of all quasi-components of a space X and let $p : X \to Q(X)$ be the natural projection of X onto $Q(X)$. We give $Q(X)$ the topology generated by

$$\{\mathcal{U} : \mathcal{U} \subset Q(X) \text{ and } p^{-1}(\mathcal{U}) \text{ is open-and-closed in X}\}$$

as a base for open sets. We call the space $Q(X)$ with this topology the *quasi-component* space of X. It is easy to see that the quasi-component space $Q(X)$ is zero-dimensional. For more detailed information about the Freudenthal compactification and the quasi-component space, the reader is referred to Aarts and Nishiura [1] and Dickson and McCoy [4].

3. Sufficient conditions

In [7] Hatzenbuhler and Mattson proved the following theorem

3.1. Theorem (Hatzenbuhler and Mattson [7]). *All compact metric spaces are continuous image of $\beta X \times \beta Y - X \times Y$ if and only if*

(i) *βX or βY has all compact metric spaces as continuous images, or*

(ii) *one factor of $X \times Y$ has a compact quasi-component and the other has all compact metric spaces as remainders.*

By Corollary 2.2, each of the above conditions (i) and (ii) is sufficient in order that the product space $X \times Y$ have all compact metric spaces as remainders.

In this section we shall give some sufficient conditions on metric spaces X and Y in order that the product space $X \times Y$ have all compact metric spaces as remainders.

Let $\mathcal{D}$ be a pairwise disjoint collection of closed subsets of a space X. If the collection $\mathcal{D}' = \mathcal{D} \cup \{\{x\} : x \in X - \cup\mathcal{D}\}$ is an upper semi-continuous decomposition of X, then we denote by $X/\mathcal{D}$ the quotient space $X/\mathcal{D}'$.

3.2. Theorem. *Let X be a compact space. If a space Y has all compact metric spaces as remainders, then so does the product space $X \times Y$.*

Proof. By Corollary 2.2, Y has a compactification αY with the Cantor set as remainder. Let us set

$$\mathcal{D} = \{X \times \{y\} : y \in \alpha Y - Y\} \text{ and } \alpha(X \times Y) = (X \times \alpha Y)/\mathcal{D}.$$

Then $\alpha(X\times Y)$ is a compactification of $X\times Y$. The remainder $\alpha(X\times Y)-X\times Y$ is homeomorphic to the Cantor set. Thus, by Corollary 2.2, the product space $X\times Y$ has all compact metric spaces as remainders.

3.3. Theorem. *Let X be a space for which the quasi-component space $Q(X)$ is compact and non-scattered. Then for every non-compact, locally compact space Y the product space $X\times Y$ has all compact metric spaces as remainders.*

Proof. By Lemma 2.3, there exists a continuous mapping f from $Q(X)$ onto the Cantor set C. Let $\omega Y=Y\cup\{\infty\}$ be the one-point compactification of Y. Then, obviously, $\beta X\times\omega Y$ is a compactification of $X\times Y$. Let φ be the mapping from $\beta X\times\omega Y-X\times Y$ to C defined by $\varphi(x, y)=\beta(f\circ p)(x)$ for every $(x, y)\in\beta X\times\omega Y-X\times Y$, where $p : X\to Q(X)$ is the natural projection and $\beta(f\circ p)$ is the Čech extension of $f\circ p$. Then φ is a continuous surjection. Hence, by Corollary 2.2, the product space $X\times Y$ has all compact metric spaces as remainders.

Steiner and Steiner [13] proved the following thereom.

3.4. Theorem (Steiner and Steiner [13], Corollary 3)**.** *Let X be an infinite discrete space and let K be a compact space with a dense subset of cardinality less than or equal to that of X. Then X has a compactification with K as remainder.*

Using the same technique of the proof of Theorem 3.4, it is easy to show that the gollowing theorem, so we omit the proof.

3.5. Theorem. *Let X be a locally compact space which can be represented as an infinite topological sum. Then X has the Cantor set as rmainder, therefore X has all compact metric spaces as remainders.*

It is well-known that every locally compact, non-separable metric space can be represented as an infinite disjoint topological sum. Thus we obtain the following proposition.

3.6. Proposition. *Let X and Y be locally compact metric spaces. If X or Y is not separable, then the product space $X\times Y$ has all compact metric spaces as remainders.*

Because the purpose of this paper is to give conditions on metric spaces X and Y which characterize the product space $X\times Y$ having all compact metric spaces as remainders, we only consider locally compact metric spaces X and Y.

3.7. Lemma. *Let X be a locally compact separable metric space. If the quasi-component space $Q(X)$ is not compact, then X can be represented as an infinite disjoint topological sum.*

Proof. Since X is locally compact and second-countable, X is σ-compact, therefore so is $Q(X)$. Let $Q(X)=\cup\{Y_i : i < \omega\}$, where Y_i is compact. Since $Q(X)$ is zero-dimensional and Lindelöf,

$Q(X)$ is strongly zero-dimensional (see [5], Theorem 1.6.5). Thus there exists a zero-dimensional compactification Y of $Q(X)$. Take a point $y \in Y - Q(X)$. For every $i < \omega$ we take an open-and-closed subset U_i of Y such that $y \in U_i$, $U_i \cap Y_i = \emptyset$, and $U_i \subset U_{i-1}$, where $U_{-1} = Y$. Let us set $Z_i = U_{i-1} - U_i$ for every $i < \omega$. Since $Z_i \neq \emptyset$ for infinitely meny $i < \omega$, we can assume that $Z_i \neq \emptyset$ for every $i < \omega$. Thus we have $Q(X) = \oplus\{Z_i : i < \omega\}$, where $Z_i \neq \emptyset$. Let us set $X_i = p^{-1}(Z_i)$ for every $i < \omega$, where $p : X \to Q(X)$ is the natural projection. Then, obviously, we have $X = \oplus\{X_i : i < \omega\}$, where $X_i \neq \emptyset$. This completes the proof of Lemma 3.7.

3.8. Theorem. *Let X be a locally compact separable metric space for which the quasi-component space $Q(X)$ is not compact. Then for every locally compact space Y the product space $X \times Y$ has all compact metric spaces as remainders.*

Proof. By Lemma 3.7, X can be represented as an infinite disjoint topological sum, therefore so can $X \times Y$. Hense, by Theorem 3.5, the product space $X \times Y$ has all compact metric spaces as remainders.

4. A characterization of the Freudenthal compactification

In this section we shall give a necessary and sufficient condition in order that a compactification of a locally compact space with scattered remainder be equivalent to the Freudenthal compactification.

Let $\alpha_1 X$ and $\alpha_2 X$ be compactifications of a space X. We say that $\alpha_1 X$ is equivalent to $\alpha_2 X$ if there exists a homeomorphism $f : \alpha_1 X \to \alpha_2 X$ such that $f(x) = x$ for every $x \in X$, and write $\alpha_1 X = \alpha_2 X$. If there exists a continuous mapping $f : \alpha_1 X \to \alpha_2 X$ such that $f(x) = x$ for every $x \in X$, then we write $\alpha_1 X \geq \alpha_2 X$. We write $\alpha_1 X > \alpha_2 X$ if $\alpha_1 X = \alpha_2 X$ and $\alpha_1 X \neq \alpha_2 X$.

A compactification αX of a space X is called an n-point compactification if $\alpha X - X$ consists of n points. Following Magill [9] we say that a pairwise disjoint collection $\{G_1, G_2, \cdots, G_n\}$ of open subsets of a space X is an *n-star* of X provided;

(1) $K = X - (G_1 \cup G_2 \cup \cdots \cup G_n)$ is compact and

(2) $K \cup G_i$ is not compact for every $i = 1, 2, \cdots, n$.

Magill [9] characterized a locally compact space having an n-point compactification as foloes.

4.1 Theorem (Magill [9]). *A locally compact space X has an n-point compactification if and only if X has an n-star.*

For every closed subset F of a space X we denote by F^d the set of all accumulation points of F. For a space X, inductively, we can define the closed subset $X^{(\alpha)}$ for every ordinal α as follows;

$X^{(0)} = X$,
$X^{(\alpha+1)} = (X^{(\alpha)})^d$ and
$X^{(\lambda)} = \cap\{X^{(\alpha)} : \alpha < \lambda\}$ for a limit ordinal λ.

A space X is scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. For every point x of a scattered space X *the rank* of x in X, denoted by rank $(x; X)$, is the maxmail ordinal α with $x \in X^{(\alpha)}$. It is easy to see that for every point x of a scattered space X there exists a neighborhood U of x in X such that rank $(y; X) <$ rank $(x; X)$ for every $y \in U$ with $y \neq x$.

4.2. Theorem. *Let αX be a compactification of a locally compact space X such that $\alpha X - X$ is scattered. Then αX is equivalent to the Freudenthal compactification γX if and only if $\alpha X - \{x\}$ has no 2-point compactification for every $x \in \alpha X - X$.*

Proof. Necessity. We shall prove that $\gamma X - \{x\}$ has no 2-point compactification for every $x \in \gamma X - X$. Assume that $\gamma X - \{x\}$ has a 2-point compactification $Y = (\gamma X - \{x\}) \cup \{a, b\}$ for some $x \in \gamma X - X$. Then Y is a compactification of X. It is easy to see that $\gamma X < Y$ and $Y - X$ is zero-dimensional. This contradicts the maximality of the Freudenthal compactificatiom. Hence $\gamma X - \{x\}$ has no 2-point compactification for every $x \in \gamma X - X$.

Sufficiency. Since X is locally compact, $\alpha X - X$ is compact. Thus $\alpha X - X$ is zero-dimensional, because $\alpha X - X$ is scattered. Hence we have $\gamma X \geq \alpha X$; let $f : \gamma X \to \alpha X$ be the continuous mapping such that $f(x) = x$ for every $x \in X$. Assume that $\gamma X > \alpha X$. Then there exists a point $x \in \alpha X - X$ such that $|f^{-1}(x)| \geq 2$. Let us set

$$\lambda = \min\{\text{rank } (x; \alpha X - X) : x \in \alpha X - X \text{ and } |f^{-1}(x)| \geq 2\}.$$

Take a point $x \in \alpha X - X$ such that $\lambda =$ rank $(x; \alpha X - X)$. We shall prove that $\alpha X - \{x\}$ has a 2-point compactification. Since $f^{-1}(x)$ is zero-dimensional and $|f^{-1}(x)| \geq 2$, there exists open-and-closed subsets A and B in $f^{-1}(x)$ such that $f^{-1}(x) = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset$ and $B \neq \emptyset$. Let us set

$$\mathcal{D} = \{f^{-1}(y) : y \in \alpha X \text{ with } y \neq x\} \cup \{A, B\}.$$

Then we shall prove that $\mathcal{D}$ is an upper semi-continuous decomposition of γX. To this end, it suffices to show that for every open subset O in γX with $A \subset O$ there exists an open subset W in γX such that $A \subset W \subset O$ and $D \subset W$ for every $D \in \mathcal{D}$ with $D \cap W \neq \emptyset$. Take an open neighborhood U of x in αX such that rank $(y; \alpha X - X) < \lambda$ for every $y \in U \cap (\alpha X - X)$ with $y \neq x$. Let us set $W = f^{-1}(U) \cap O \cap (\alpha X - B)$. Then, obviously, we have $A \subset W \subset O$. Let $D \in \mathcal{D}$ with $D \cap W \neq \emptyset$ and $D \neq A$. Then we have $D \cap f^{-1}(U) \neq \emptyset$. Therefore $f(D) \cap U \neq \emptyset$. Take a point $y \in f(D) \cap U$. Since $y \in U$, we have rank $(y; \alpha X - X) < \lambda$. This implies that $f^{-1}(y)$ is a singleton. Thus we have $D = f^{-1}(y) = \{y\} \subset W$. Hence $\mathcal{D}$ is an upper semi-continuous decomposition of γX. Let Y be the quotient space $\gamma X/\mathcal{D}$. Then Y is a 2-point compactification of $\alpha X - \{x\}$. This is a contradiction. Hence αX is equivalent to the Freudenthal compactification.

We should point out that the assumption that $\alpha X - X$ is scattered can not be replaced by the assumption that $\alpha X - X$ is zero-dimensional.

4.3. Example. *There exists a compactification αX of a locally comapct, separable metric space X satisfying the following conditions;*
(1) $\alpha X - X$ *is zero-dimensional,*
(2) $\alpha X \neq \gamma X$, *and*
(3) $\alpha X - \{x\}$ *has no 2-point compactification for every* $x \in \alpha X - X$.

Let S^1 be the circle. Fix a point $a \in S^1$. Let us set $X = (S^1 - \{a\}) \times C$, where C is the Cantor set. Then the space $\alpha X = S^1 \times C$ is a compactification of X. Obviously, $\alpha X - X = \{a\} \times C$ is zero- dimensional. Since $S^1 - \{a\} \approx (-1, 1)$, we have $X = (S^1 - \{a\}) \times C \approx (-1, 1) \times C$. Thus the space $\alpha' X = [-1, 1] \times C$ is a comapctification of X. Obviously, $\alpha' X - X = \{-1, 1\} \times C$ is zero-dimensional and $\alpha X < \alpha' X$. Hense we have $\alpha X \neq \gamma X$.

We shall prove that $\alpha X - \{x\}$ has no 2-point compactification for every $x \in \alpha X - X$. Assume that $\alpha X - \{x\}$ has a 2-point compactification for some $x = (a, c) \in \{a\} \times C = \alpha X - X$. Then, by Theorem 4.1, there exists a 2-star $\{G_1, G_2\}$ of $\alpha X - \{x\}$. Since $K = (\alpha X - \{a\}) - G_1 \cup G_2$ is compact, we can take an open subset U in S^1 and an open subset V in C such that $x = (a, c) \in U \times V \subset \alpha X - K = G_1 \cup G_2 \cup \{x\}$. We may assume that $x = (a, c) = (0, c) \in (-1, 1) \times C \approx U \times V$. Then we have $(-1, 1) \times C - \{(0, c)\} \subset G_1 \cup G_2$. Since G_1 and G_2 are disjoint open subsets of $\alpha X - \{x\}$, we have $(-1, 1) \times \{t\} \subset G_1$ or $(-1, 1) \times \{t\} \subset G_2$ for every $t \in C - \{c\}$. Thus it is easy to see that

$$(-1, 1) \times U(c; \varepsilon) - \{(0, c)\} \subset G_1 \text{ or } (-1, 1) \times U(c; \varepsilon) - \{(0, c)\} \subset G_2$$

for some $\varepsilon > 0$, where $U(c; \varepsilon)$ is the ε-neighborhood of c in C. Suppose that $(-1, 1) \times U(c; \varepsilon) - \{(0, c)\} \subset G_1$. Then we have $K \cup G_2 \subset \alpha X - V$ for some open neighborhood V of x in αX. Thus $K \cup G_2$ is compact. This is a contradiction. Hence $\alpha X - \{x\}$ has no 2-point compactification for every $x \in \alpha X - X$.

5. The Freudenthal compactification of $X \times Y$

Let X be a compact metric space and let Y be a non-compact, locally compact metric space. In section 3 we proved that the following condition is sufficient in order that the product space $X \times Y$ have all compact metric spaces as remainders;
(1) the quasi-component space $Q(X)$ is not scattered, or
(2) the space Y has all compact metric spaces as remainders.

In this section we shall prove that the above condition is not only sufficient but also necessary.

We begin with the following thereom.

5.1. Theorem. *Let X be a compact connected space and let Y be a locally compact space which does not have all compact metric spaces as remainders. Then the equlity*

$$\gamma(X \times Y) = (X \times \gamma Y)/\mathcal{D}$$

holds, where $\mathcal{D} = \{X \times \{y\} : y \in \gamma Y - Y\}$.

Proof. Let $\alpha(X\times Y)=(X\times\gamma Y)/\mathcal{D}$ and let $\pi : X\times\gamma Y\to\alpha(X\times Y)$ be the projection. Since $\alpha(X\times Y)-X\times Y\approx\gamma Y-Y$, by Theorem 2.5, $\alpha(X\times Y)-X\times Y$ is scattered. Thus, by Theorem 4.2, it suffices to prove that $\alpha(X\times Y)-\{y^*\}$ has no 2-point compactification for every $y^*\in\alpha(X\times Y)-X\times Y$. Here we set $\{y^*\}=\pi(X\times\{y\})$ for every $y\in\gamma Y-Y$. Assume that there exists a point $y_0\in\gamma Y-Y$ such that the space $Z=\alpha(X\times Y)-\{y_0^*\}$ has a 2-point compactification. By Theorem 4.1, we can take a 2-star $\{U, V\}$ of Z, Then the set $U\cup V\cup\{y_0^*\}=\alpha(X\times Y)-(Z-U\cup V)$ is open in $\alpha(X\times Y)$, because $Z-U\cup V$ is compact. Since X is compact and since $X\times\{y_0\}=\pi^{-1}(y_0^*)\subset\pi^{-1}(U\cup V\cup\{y_0^*\})$, we can take an open subset W in γY such that $X\times\{y_0\}\subset X\times W\subset\pi^{-1}(U\cup V\cup\{y_0^*\})$. For every $y\in W-\{y_0\}$ we have $X\times\{y\}\subset X\times(W-\{y_0\})\subset\pi^{-1}(U\cup V)=\pi^{-1}(U)\cup\pi^{-1}(V)$. On the other hand, $X\times\{y\}$ is connected and $\pi^{-1}(U)\cap\pi^{-1}(V)=\emptyset$. This implies that $X\times\{y\}\subset\pi^{-1}(U)$ or $X\times\{y\}\subset\pi^{-1}(V)$ for every $y\in W-\{y_0\}$. Let us set $U'=\{y\in\gamma Y-\{y_0\} : X\times\{y\}\subset\pi^{-1}(U)\}$ and $V'=\{y\in\gamma Y-\{y_0\} : X\times\{y\}\subset\pi^{-1}(V)\}$. Then we have $W-\{y_0\}\subset U'\cup V'$. We shall prove that $\{U, V\}$ is a 2-star of $\gamma Y-\{y_0\}$. Obviously, U' and V' are disjoint open subsets in $\gamma Y-\{y_0\}$. Since $(\gamma Y-\{y_0\})-U'\cup V'\subset(\gamma Y-\{y_0\})-(W-\{y_0\})=\gamma Y-W$, the space $K=(\gamma Y-\{y_0\})-U'\cup V'$ is compact. Next we shall show that $K\cup U'$ is not compact. Let $q : \alpha(X\times Y)\to\gamma Y$ be the natural mapping defined by $q(z)=y$, where $z=y^*\in\alpha(X\times Y)-X\times Y$ or $z=(x, y)\in X\times Y$. Since $q(Z-V)\subset(\gamma Y-\{y_0\})-V'$, we have

$$\begin{aligned}(Z-U\cup V)\cup U&=Z-V\\&\subset q^{-1}((\gamma Y-\{y_0\})-V')\\&=q^{-1}((\gamma Y-\{y_0\})-U'\cup V')\cup U')\\&=q^{-1}(K\cup U').\end{aligned}$$

On the other hand, $\{U, V\}$ is a 2-star of Z. Thus $(Z-U\cup V)\cup U$ is not compact. This implies that $K\cup U'$ is not compact. Similarly, $K\cup V'$ is not compact. Thus $\{U', V'\}$ is a 2-star of $\gamma Y-\{y_0\}$. Hence, by Theorem 4.1, $\gamma Y-\{y_0\}$ has a 2-point compactification. However, by Theorem 4.2, this is a contradiction. This completes the proof of Theorem 5.1.

Nowinski [11] proved that $\gamma(X\times Y)=(X\times\gamma Y)/\mathcal{D}$, where $\mathcal{D}=\{X\times\{y\} : y\in\gamma Y-Y\}$, in the case when X is a compact conected space and Y is a locally compact weakly paracompact space. Hence in Theorem 5.1, the assumption that Y does not have all compact metric spces as remainders can be replaced by the assumption that Y is weakly paracompact.

5.2. Question. *Let X be a compact connected space and let Y be a locally compact space. Let $\mathcal{D}=\{X\times\{y\} : y\in\gamma Y-Y\}$. Does the equality $\gamma(X\times Y)=(X\times\gamma Y)/\mathcal{D}$ hold?*

5.3. Theorem. *Let X be a compact metric space for which the quasi-component space $Q(X)$ is scattered and let Y be a locally compact space which does not have all compact metric spaces as remainders. Then the equality*

$$\gamma(X\times Y)=(X\times\gamma Y)/\mathcal{D}$$

holds, where $\mathcal{D}=\{F\times\{y\} : F\in Q(X)$ and $y\in\gamma Y-Y\}$.

Therefore, the product space $X\times Y$ does not have all compact metric spaces as remainders.

Proof. Let $\alpha(X\times Y)=(X\times\gamma Y)/\mathcal{D}$ and $\pi : X\times\gamma Y\to \alpha(X\times Y)$ be the projection. Since $\alpha(X\times Y)-X\times Y\approx Q(X)\times(\gamma Y-Y)$, by Theorem 2.5, $\alpha(X\times Y)-X\times Y$ is scattered. Thus, by Theorem 4.2, it suffices to prove that $\alpha(X\times Y)-\{(F, y)^*\}$ has no 2-point compactification for every point $(F, y)^*$ of $\alpha(X\times Y)-X\times Y$. Here we set $\{(F, y)^*\}=\pi(F\times\{y\})$ for every $F\in Q(X)$ and for any $y\in \gamma Y-Y$. Assume that there exists a point $(F, y)^*\in \alpha(X\times Y)-X\times Y$ such that the space $Z=\alpha(X\times Y)-\{(F, y)^*\}$ has a 2-point compactification $\alpha Z=Z\cup\{a, b\}$. Let us set $\tilde{F}=\pi(F\times\gamma Y)$. Since F is compact and connected, by Theorem 5.1, we have $\tilde{F}=\gamma(X\times Y)$. By Theorem 4.2, $\tilde{F}-\{(F, y)^*\}$ has no 2-point compactification. Thus we may assume that $\mathrm{Cl}_{\alpha Z}(\tilde{F}-\{(F, y)^*\})=(\tilde{F}-\{(F, y)^*\})\cup\{a\}$, because $\tilde{F}-\{(F, y)^*\}$ is a non-compact closed subset of Z. We distinguish two cases.

Case 1. F is open in X.

In this case $Z-\tilde{F}$ is compact. Thus we have

$$\begin{aligned}\alpha Z&=\mathrm{Cl}_{\alpha Z}Z\\ &=\mathrm{Cl}_{\alpha Z}(Z-\tilde{F})\cup\mathrm{Cl}_{\alpha Z}(\tilde{F}-\{(F, y)^*\})\\ &=(Z-\tilde{F})\cup(\tilde{F}-\{(F, y)^*\})\cup\{a\}\\ &=Z\cup\{a\}\\ &=\alpha Z-\{b\}.\end{aligned}$$

This is a contradiction.

Case 2. F is not open in X.

In this case F is not an isolated point of $Q(X)$. Since $Q(X)$ is scattered, the set of all isolated points of $Q(X)$ is dense in $Q(X)$. Since X is a compact metric space, so is $Q(X)$. Thus we can take a sequence $\{E_i : i<\omega\}$ of isolated points of $Q(X)$ which converges to F. Let us set $\tilde{E}_i=\pi(E_i\times\gamma Y)$ for every $i<\omega$. Since $b\notin \mathrm{Cl}_{\alpha Z}(\tilde{F}-\{(F, y)^*\})$, we take an open subset U in αZ such that $b\in U\subset \mathrm{Cl}_{\alpha Z}U\subset \alpha Z-\mathrm{Cl}_{\alpha Z}(\tilde{F}-\{(F, y)^*\})$. Then we have $a, b\notin \mathrm{Bd}_{\alpha Z}U$, therefore we have $\mathrm{Bd}_{\alpha Z}U\subset Z$. Let us set $U'=U\cap Z$. Since $\mathrm{Bd}_Z U'\subset \mathrm{Bd}_{\alpha Z}U$, $\mathrm{Bd}_Z U'$ is compact. On the other hand, since $b\in U$, we have $b\in \mathrm{Cl}_{\alpha Z}U'$. This implies that $\mathrm{Cl}_Z U'$ is not compact. Since $\mathrm{Cl}_Z U'\subset \mathrm{Cl}_{\alpha Z}U\subset \alpha Z-\mathrm{Cl}_{\alpha Z}(\tilde{F}-\{(F, y)^*\})$, we have $\mathrm{Cl}_Z U'\cap(\tilde{F}-\{(F, y)^*\})=\emptyset$. This imples that

(*) $\quad\mathrm{Cl}_{\alpha(X\times Y)}U'\cap\tilde{F}=\{(F, y)^*\}$.

For every $i<\omega$ let us set $U_i=U'\cap\tilde{E}_i$.

Claim. $\{\mathrm{Bd}_{\tilde{E}_i}U_i : i<\omega\}$ is discrete in Z.

Take a point $z\in Z$. Suppose that $z\in\tilde{F}$. Then the set $W=Z-\mathrm{Cl}_Z U'$ is an open neighborhood of z in Z such that $W\cap \mathrm{Bd}_{\tilde{E}_i}U_i=\emptyset$ for every $i<\omega$.

Suppose that $z\notin\tilde{F}$. Then we have

$z=(x, y')\in X\times Y$ with $x\notin F$, or

$z=(G, y')^*\in \alpha(X\times Y)-X\times Y\approx Q(X)\times(\gamma Y-Y)$ with $F\neq G$.

Since $\{E_i : i<\omega\}$ converges to F, in any case there exists an open subset V of X such that $x\in V$ (resp. $G\subset V$), $V\cap F=\emptyset$ and $V\cap E_i=\emptyset$ for all but finitely many $i<\omega$. Since $\{\tilde{E}_i : i<\omega\}$ is pairwise disjoint, we can take an open neighborhood W of z in Z such that $W \cap \mathrm{Bd}_{\tilde{E}_i} Ui = \emptyset$ for all but at most one $i < \omega$. Hence $\{\mathrm{Bd}_{\tilde{E}_i} Ui : i < \omega\}$ is discrete in Z.

On the other hand, since $\mathrm{Bd}_{\tilde{E}_i} Ui \subset \mathrm{Bd}_Z U'$ and $\mathrm{Bd}_Z U'$ is compact, $\mathrm{Bd}_{\tilde{E}_i} Ui = \emptyset$ for all but finitely many $i < \omega$. Thus we can assume that $\mathrm{Bd}_{\tilde{E}_i} Ui = \emptyset$ for all $i < \omega$. Thus U_i is open-and-closed in $\tilde{E}_i$ for every $i < \omega$. Let $q : \alpha(X \times Y) \to \gamma Y$ be the mapping defined by

$$q(z) = y, \text{ where } \pi^{-1}(z) = \{(x, y)\} \text{ or } F \times \{y\},$$

for every $z \in \alpha(X \times Y)$. Let us set $V_i = q(Ui)$ for every $i < \omega$. Then it is easy to see that V_i is open-and-closed in γY. By (*),

(**) for every neighborhood O of y in γY there exists $n < \omega$ such that $V_i \subset O$ for every i with $n \leq i$. Let $i_1 = 0$. Take a point $x_1 \in V_{i\ell}$ with $x_1 \neq y$ and a neighborhood O_1 of y in γY with $x_1 \notin O_1$. By (**), for every n with $0 < n < \omega$, inductively, we can take a point $x_{i_n} \in \gamma Y - \{y\}$, a neighborhood O_n of y in γY and $i_n < \omega$ such that $i_n < i_{n+1}$, $x_n \in V_{i_n}$, $x_n \notin O_n$, $O_{n+1} \subset O_n$ and $V_{i_{n+1}} \subset O_n$. Let us set $D_n = (V_{i_n} - \cup\{V_{i_m} : n < m\}) - \{y\}$, for every $n < \omega$, where $V_{i_0} = \gamma Y$. Then we have $x_n \in D_n$ for every n with $0 < n < \omega$, therefore $D_n \neq \emptyset$. Since V_i is open-and-closed in γY, D_n is closed in $\gamma Y - \{y\}$. We shall show that D_n is open in $\gamma Y - \{y\}$. To this end it suffices to show that $\cup\{V_{i_m} : n < m\} - \{y\}$ is closed in $\gamma Y - \{y\}$. For a point z $\in \mathrm{Cl}_{\gamma Y} (\cup\{V_{i_m} : n < m\}) - \{y\}$ we take a neighborhood O of z in γY with $y \notin \mathrm{Cl}_{\gamma Y} O$. Then, by (**), there exists $k < \omega$ such that $V_i \subset \gamma Y - \mathrm{Cl}_{\gamma Y} O$ for every $i > k$. This implies that $z \in \mathrm{Cl}_{\gamma Y} V_{i_\ell}$ for some ℓ with $n < \ell$ and $i_\ell \leq k$. Thus D_n is open-and-closed in $\gamma Y - \{y\}$. Hence we have $\gamma Y - \{y\} = \oplus\{D_n : n < \omega\}$. By Theorem 3.5, $\gamma Y - \{y\}$ has a 2-point compactification. However, by Theorem 4.2, this is a contradiction. Hence the equality $\gamma(X \times Y) = (X \times \gamma Y)/\mathcal{D}$ holds.

Since $\gamma(X \times Y) - X \times Y \approx Q(X) \times (\gamma Y - Y)$, the remainder of the Freudenthal compactification is scattered. Hence by Theorem 2.5, the product space $X \times Y$ does not have all compact metric spaces as remainders.

We now come to the main result in this paper.

5.4. Corollary. *Let X be a compact metric space and let Y be a non-compact, locally compact space. Then the product space $X \times Y$ has all compact metric spaces as remainders if and only if one of the following conditions is satisfied;*

(1) *the quasi-component space $Q(X)$ is not scattered,*

(2) *the space Y has all compact metric spaces as remainders.*

5.5. Remark. *A compact metric space X is not scattered if and only if X has cardinality 2^ω. If X is a compact metric space, then so is the quasi-component space $Q(X)$.* Hence the condition (1) in Corollary 5.4 is equivalent to the following condition (1');

(1') *the quasi-component space $Q(X)$ has cardinality 2^ω.*

For the sake of this paper we need not weaken the assumption in Theorem 5.3. However, it is natural to ask the following question.

5.6. Question. *Let X be a compact space and let Y be a non-compact, locally compact space. Is the Freudenthal compactification of the product space $X \times Y$ equivalent to $(X \times \gamma Y)/\mathcal{D}$, where D $= \{F \times \{y\} : F \in Q(X)$ and $y \in \gamma Y - Y\})$?*

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