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A Recursive Inequality of Empirical Measures Associated with EDM

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Summary

In this paper we consider a random model related to stochastic interacting systems, named an environment-dependent spatial model (EDM). As a matter of fact, this stochastic model is deeply connected with some Markov processes investigated by Liggett [18]. We shall show that rescaled processes of the empirical measures derived from EDMs satisfy some applicationally important recursive inequality.

Key Words: environment-dependent model, random model, stochastic interacting systems, empirical measures, rescaled process, recursive inequality.

1. Introduction

In this section we shall introduce an environment-dependent random model (EDM)[12]. Let $\mathbb{Z}^d$ be a d-dimensional lattice space, and we suppose that each site on $\mathbb{Z}^d$ is occupied by all means by either one of the two species. Just after a random time period, a particle dies out and is replaced by a new one, but the random time and the type chosen of the species are assumed to be determined by the environment conditions around the particle. The random function $\xi_t \equiv \xi_t(x)$: $\mathbb{Z}^d \to \{0, 1\}$ denotes the state at time t, and each number of $\{0, 1\}$ denotes the label of the type chosen of the two species. When we set $\|y\|_\infty := \max_i y_i$ for $y= (y_1, \ldots, y_d)$, then the R-neighborhood of x is defined by

$$\mathcal{N}_x := x + \{y : \ 0 < \|y\|_\infty \leqslant R\}, \tag{1}$$

where R is a positive constant given. For i = 0, 1, let $f_i(x, \xi)$ be a frequency of appearance of type i in $\mathcal{N}_x$ for ξ. More precisely, it can be expressed as

$$f_i(x) \equiv f_i(x,\xi) := \frac{\#\{y : \ \xi_t(y) = i \ ; \ y \in \mathcal{N}_x\}}{\#\mathcal{N}_x}. \tag{2}$$

For non-negative parameters $\alpha_{ij} \geq 0$, the dynamics of ξ_t is defined as follows. The state ξ makes transition $0 \to 1$ at rate

$$\frac{\lambda f_1(f_0 + \alpha_{01} f_1)}{\lambda f_1 + f_0}, \tag{3}$$

and it makes transition $1 \to 0$ at rate

$$\frac{f_0(f_1 + \alpha_{10} f_0)}{\lambda f_1 + f_0}. \tag{4}$$

The interpretation of the above rate is as follows. The particle of type i dies at rate $f_i + \alpha_{ij} f_j$, and is replaced instantaneously by either one of the two species chosen at random, according to the proliferation rate of type 0 and the interaction (= the competitive result) with the particle of type 1. We assume that competitive two species possess the same intensity of intraspecific interaction. The exchange of particles after death is described in the form being proportional to the weighted density between the two species, expressed by a parameter λ. Assume usually that $\lambda \geq 1$. The case of $\lambda = 1$ means that the contribution to a local appearance rate between the two competitive species is equivalent.

2. Scaling, rescaled process and empirical measure

For simplicity we shall treat a simple case $\lambda = 1$ only in what follows. For $N = 1, 2, \ldots$, let $M_N \in \mathbb{N}$, and we put $\ell_N := M_N \sqrt{N}$, and $\mathbb{S}_N := \mathbb{Z}^d/\ell_N$. And also $W_N = (W_N^1, \ldots, W_N^d) \in (\mathbb{Z}^d/M_N)\setminus\{0\}$ is defined as a random vector satisfying

$$\text{(i) } \mathcal{L}(W_N) = \mathcal{L}(-W_N); \qquad \text{(ii) } E(W_N^i W_N^j) \to \delta_{ij}\sigma^2 (\geq 0) \quad (\text{as } N \to \infty);$$
$$\text{(iii) } \{|W_N|^2\} \ (N \in \mathbb{N}) \text{ is uniformly integrable.}$$

Here $\mathcal{L}(Y)$ indicates the law of a random variable Y. For the kernel $p_N(x) := P(W_N/\sqrt{N}=x)$, $x \in \mathbb{S}_N$ and $\xi \in \{0, 1\}^{\mathbb{S}_N}$, we define the scaled frequency f_i^N as

$$f_i^N(x,\xi) = \sum_{y \in \mathbb{S}_N} p_N(y-x) 1_{\{\xi(y)=i\}}, \qquad (i = 0, 1). \tag{5}$$

Actually, ξ_t^N is given by $\xi_t^N = \xi_{Nt}(x\sqrt{N})$. As a matter of fact, the rescaled process $\xi_t^N : \mathbb{S}_N \ni x \mapsto \xi_t^N(x) \in \{0, 1\}$ is determined by the following state transition law, nemaly, it makes transition $0 \to 1$ at rate $Nf_1^N(f_0^N + \alpha_0^N f_1^N)$, or else it makes transition $1 \to 0$ at rate $Nf_0^N(f_1^N + \alpha_1^N f_0^N)$. We also denote the rescaled process ξ_t^N by the symbol $Res(p_N, \alpha_i^N)$. On this account, we may define the associated measure-valued process (or its corresponding empirical measure) as

$$X_t^N := \frac{1}{N} \sum_{x \in \mathbb{S}_N} \xi_t^N(x) \delta_x. \tag{6}$$

For the initial value X_0^N, we assume that

$$\sup_N \langle X_0^N, 1 \rangle < \infty, \qquad X_0^N \to X_0 \quad \text{in} \quad M_F(\mathbb{R}^d) \quad (N \to \infty), \tag{7}$$

where $M_F(\mathbb{R}^d)$ is the totality of all the finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. For a finite measure $\mu \in M_F(E)$ with a topological space E, we use the notation $\langle \mu, \varphi \rangle = \int_E \varphi(x)\mu(dx)$ for integral of a measurable function φ over E with respect to a measure μ on E. Note that the convergence in (7) is that in the sense of weak convergence for measures [17].

3. Main theorem : recursive inequality

In this section we shall introduce the principal result on an estimate of the maximum of the moment of total mass process for the empirical measure. To prove it we need some precise estimate of the quantity in question, and in fact, that can be realized by a certain recursive type inequality for the empirical measures.

THEOREM 1. (Main Result) *Let $F(N)$ be a function of N that satisfies $1 \leqslant F(N) \leqslant N$ and $\lim_{N\to\infty} F(N)/N=0$. If the condition $N^{5/7}/F(N) \to 0$ holds as $N \to \infty$, then for any $p > 1$ and $T > 0$, there exists a finite constant $c(p, T) > 0$ such that*

$$E[\sup_{t\leqslant T}\langle X_t^N, 1\rangle^p] \leqslant c(p,T)\left(\frac{N}{F(N)}\right)^{p-1/2}(\langle X_0^N, 1\rangle^p + 1). \tag{8}$$

4. Sketch of proof of main result

Step 1. In this section we shall introduce a sketch of proof of our main result Theorem 1. First of all, we begin with showing a useful equality.

LEMMA 2. *The following equality holds for every $t > 0$:*

$$E[\langle X_t^N, 1\rangle] = \langle X_0^N, 1\rangle. \tag{9}$$

Proof. First we consider a bounded function $\psi : \mathbb{S}_N \to \mathbb{R}$. For a continuous time random walk $B_t^{x,N}$ with rate N and step distribution p_N starting at x,

$$\phi_s(x) \equiv \phi(s,x) = P_{t-s}^N\psi(x) := E[\psi(B_{t-s}^{x,N})] \tag{10}$$

defines a semigroup. Indeed, this newly defined function ϕ satisfies a differential equation $\partial_s\phi(s) + \mathcal{A}_N\phi(s) = 0$ by virtue of the backward equation argument for continuous time Markov chains. Recall that $\mathcal{A}_N$ is its generator, and is given by

$$\mathcal{A}_N\phi(x) := N\sum_y p_N(y-x)(\phi(y)-\phi(x)). \tag{11}$$

According to the theory of semimartingales [21], we may apply Itô's formula in stochastic calculus [14] to a relationship of rescaled EDMs to obtain

$$\langle X_t^N, \phi_t\rangle = \langle X_0^N, \phi_0\rangle + \int_0^t X_s^N(\partial_s\phi(s) + \mathcal{A}_N\phi(s))ds + M_t^N(\phi), \tag{12}$$

for $0 \leqslant t \leqslant T$, where $M_t^N(\phi)$ is a martingale term and $X_t(\phi)$ denotes an integral of the test function ϕ relative to the measure-valued process dX_t. Taking the expectation operation $E[\cdot]$ at the both sides of (12), we can get the equality $E[X_t^N(\phi_t)] = E[X_0^N(\phi_0)] = E[X_0^N(P_t^N\psi)]$, and besides we readily obtain the desired expression (9) with $E[X_t^N(\psi)] = E[X_0^N(P_t^N\psi)]$ by changing the function $\phi(t)$ to a general one ψ, where we have substituted $\phi\cdot(x) \equiv 1$ instead of ψ and we also have made use of

$$\phi(s,x) = \phi_s(x) = P_{t-s}^N 1(x) = E[1(B_{t-s}^{x,N})] = E[1(x)] = 1. \tag{13}$$

This finishes the proof of lemma. □

Step 2. Recall standard results for stochastic integrals with respect to Poisson processes N_s with the intensity $E[N_s] = \lambda_s$. Since $\hat{N}_s = N_s - \lambda_s$ is a martingale, the stochastic integral $M_s = \int_0^t \Psi(s, \omega)\, d\hat{N}_s$ becomes a martingale. Furthermore, it follows that

$$E|M_t|^2 = E\left|\int_0^t \Psi(s,\omega)d\hat{N}_s\right|^2 = E\int_0^t \Psi^2(s,\omega)d\lambda_s. \tag{14}$$

Let $\{\Lambda_t^N(x, y) : x, y \in \mathbb{S}_N\}$ be a family of independent Poisson processes with rate $N \cdot p_N(y-x)$ defined on a complete probability space. Note that its compensated process

$$\hat{\Lambda}_t^N(x,y) = \Lambda_t^N(x,y) - N \cdot p_N(y-x)t \tag{15}$$

are $(\mathcal{F}_t)$-martingale. On this account, for every test function $\psi \equiv \psi(s, x) \in M_b([0, T] \times \mathbb{S}_N)$ with $T < \infty$,

$$M_t^N(\psi) := \frac{1}{F(N)}\sum_s\sum_y\int_0^t \psi_s(x)(\xi_{s-}(y) - \xi_{s-}(x))d\hat{\Lambda}_s(x,y) \tag{16}$$

is a cadlag L^2 $(\mathcal{F}_t)$-martingale, and its predictable square function is given by

$$\langle M^N(\psi)\rangle_t = \frac{N}{F(N)^2}\int_0^t \sum_x\sum_y \psi_s(x)^2(\xi_s(y) - \xi_s(x))^2 p_N(y-x)ds, \quad t \in [0,T] \tag{17}$$

where $M_b(D)$ is the totality of all bounded measurable functions defined on a proper space D, and the summation $\sum_x$ is taken over the whole space $\mathbb{S}_N$. In particular, the equality

$$\langle M^N(1)\rangle_t = 2\int_0^t \langle X_s^N, \frac{N}{F(N)}V_N(s,x)\rangle ds \tag{18}$$

holds, where $V_N(t, x) = \sum_y p_N(y-x)1\{\xi_t(y)=0\}$. An application of the results (16), (17) and (18) with the expression (12) leads to $X_t^N(1) = X_0^N(1) + \mathrm{M}_t^N(1)$.

LEMMA 3. *The random quantity* $\langle X_t^N, 1\rangle$ *is an* L^2*-martingale such that*

$$\langle X^N(1)\rangle_t = \frac{2N}{F(N)}\int_0^t \frac{1}{F(N)}\sum_x \xi_s(x)V_N(s,x)ds \leqslant \frac{2N}{F(N)}\int_0^t \langle X_s^N, 1\rangle ds \tag{19}$$

holds.

Proof. The expression (16) implies that $\mathrm{M}_t^N(1)$ is also a martingale as a special case. While, since $\langle X_0^N, 1\rangle = const.$, it is easy to see that for $0 < \forall s < t$

$$\begin{aligned} E[X_t^N(1)|\mathcal{F}_s] &= E[X_0^N(1)|\mathcal{F}_s] + E[M_t^N(1)|\mathcal{F}_s] \\ &= X_0^N(1) + M_s^N(1) = X_s^N(1). \end{aligned} \tag{20}$$

Moreover, we can deduce that $\langle X_t^N, 1\rangle$ is an L^2 martingale because $M_t^N(\psi)$ is an L^2 martingale. For a proper sequence of stopping times (T_n), the quadratic variation satisfies

$$Var(\langle X_0^N, 1\rangle) := \sum_k \left|\langle X^N(0\wedge T_{k+1}), 1\rangle - \langle X^N(0\wedge T_k), 1\rangle\right|^2 = 0, \tag{21}$$

hence it follows immediately that $\langle X^N(1)\rangle_t = \langle M_t^N(1)\rangle_t$ holds for every $t > 0$. Elementary results for Poisson process and stochastic integral with respect to Poisson process reads

$$\begin{aligned} E|M_t^N(1)|^2 &= E\left|\frac{1}{F(N)}\sum_x\sum_y\int_0^t(\xi_{s-}(y)-\xi_{s-}(x))d\hat{\Lambda}_s(x,y)\right|^2 \\ &= E[\frac{1}{F(N)^2}\sum_x\sum_y\int_0^t(\xi_s(y)-\xi_s(x))^2 N\cdot p_N(y-x)ds]. \end{aligned} \tag{22}$$

From this fact, we can get easily that

$$\begin{aligned} \langle M_t^N(1)\rangle_t &= \frac{N}{F(N)^2}\sum_x\sum_y\int_0^t(\xi_s(y)-\xi_s(x))^2 p_N(y-x)ds \\ &= \frac{N}{F(N)^2}\sum_x\int_0^t\xi_s(x)\left(\sum_y p_N(y-x)\cdot 1\{\xi_s(x)=0\}\right)ds \\ &\qquad + \frac{N}{F(N)^2}\sum_y\int_0^t\xi_s(y)\left(\sum_x p_N(x-y)\cdot 1\{\xi_s(x)=0\}\right)ds \\ &= \frac{2N}{F(N)}\int_0^t\left(\frac{1}{F(N)}\sum_x\xi_s(x)V_N(s,x)\right)ds \\ &\leqslant \frac{2N}{F(N)}\int_0^t\left(\frac{1}{F(N)}\sum_x\xi_s(x)1(x)\right)ds = \frac{2N}{F(N)}\int_0^t\langle X_s^N,1\rangle ds \end{aligned} \tag{23}$$

where we made use of a simple inequality $V_N(s,x)\leqslant 1$. □

Step 3. To complete the proof of Theorem 1, we need the following lemmas.

LEMMA 4. *There exists a positive constant $K(p) > 0$ depending on $\forall p > 1$ such that*

$$E[\sup_{t\leqslant T}\langle X_t^N,1\rangle^p]\leqslant K(p)\left\{X_0^N(1)^p + E[\langle X^N(1)\rangle_t^{p/2} + E[\sup_{t\leqslant T}|\varDelta X_t^N(1)|^p]\right\} \tag{24}$$

holds.

LEMMA 5. *There exists a positive constant $K_1(p) > 0$ such that*

$$E[\sup_{t\leqslant T}\langle X_t^N,1\rangle^p]\leqslant K_1(p)\left\{X_0^N(1)^p + \left(\frac{2NT}{F(N)}\right)^{p/2}E[\sup_{t\leqslant T}X_t^N(1)^{p/2}]+1\right\} \tag{25}$$

holds.

LEMMA 6. *There exists a positive constant $K_2 > 0$ such that*

$$E[\sup_{t\leqslant T}X_t^N(1)^2]\leqslant K_2\left(\frac{NT}{F(N)}+1\right)(X_0^N(1)^2+1) \tag{26}$$

holds.

LEMME 7. *There exist certain positive constants $K_3(p) > 0$ and $c(p,T) > 0$ such that*

$$E[\sup_{t\leqslant T}\langle X_t^N,1\rangle^p]\leqslant K_3(p)[X_0^N(1)^p+\left(\frac{2Nt}{F(N)}\right)^{p/2}\left\{c(p,T)\left(\frac{N}{F(N)}\right)^{(p-1)/2}(X_0^N(1)+1)\right\}+1] \tag{27}$$

holds.

The above-mentioned Lemma 4 is a direct result of Burkholder-Davis-Gundy inequality [21] together with a trivial inequality $(a+b)^p\leqslant c_p(a^p+b^p)$ and $\langle X\rangle_t=\langle X\rangle_t^c+\sum_{0\leqslant s\leqslant t}(\Delta X_s)^2$ and $\Delta X_s=\Delta M_s$ for $\forall_s$. Lemma 5 yields from a standard technique for estimation, the inequality (19) in Lemma 3, and an estimate for jump term:

$$E[\sup_{s\leqslant T}|\Delta X_s^N(1)|^p]\leqslant\left(\frac{1}{F(N)}\right)^p\leqslant 1. \tag{28}$$

Another trivial inequality $ab\leqslant\frac{1}{2}(a^2+b^2)$ yields to Lemma 6. Moreover, Lemma 7 can be verified by induction hypothesis. Combining those results, we can deduce that (8) is valid for $\forall p>1$ by taking advantage of mathematical induction method. This completes the proof of Theorem 1. □

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