An Estimate of Survival Probability for Superprocesses

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Summary

In this paper we consider a branching particle system that is governed by diffusion. When we construct its rescaled branching diffusion particle system with three parameters, then the measure-valued diffusion process or superdiffusion is obtained from the rescaled branching particle systems via limiting procedure. We discuss the survival probability for the limiting superprocess, and derive an estimate of the survival probability for the superprocesses.

Key Words: branching particle system, diffusion, rescaled branching particle system, limiting procedure, measure-valued diffusion, superprocess, survival probability.

1. Introduction

In this paper we consider a branching particle system whose motion is described by a diffusion process. Actually, the diffusion is governed by the second order partial differential operator on the assumption that the operator is uniformly elliptic. Next we define a rescaled branching diffusion particle system with three parameters, and recall the well known limiting procedure. Then the measure-valued diffusion process or superdiffusion is obtained from the rescaled branching particle systems by virtue of the limit proposition. Lastly we discuss the survival probability for the limiting superprocess, and derive an estimate of the survival probability for the superprocesses. Our result is a generalization of the result obtained by Sheu (1997) [15], where an estimate of survival probability for super-Brownian motion was derived.

2. Rescaled branching diffusion particle system

Let L be a second order partial differential operator of the form

$$L := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} \tag{1}$$

where a_{ij}, $b_i \in C_b(\mathbb{R}^d)$ for every i, j. We suppose that L is uniformly elliptic. Then $\xi=(\xi_t, \Pi_x)$ denotes an L-diffusion, and $Z=\{Z_t;\ t\geq 0\}$ is a branching diffusion particle system. Namely, Z is a probabilistic model of a system of particles living in the space $\mathbb{R}^d$, dying and producing random number of offspring in accordance with exponentially distributed lifetime at their death time and death location, and moving in such a way as governed by the law of the L-diffusion. Actually, Z_t is a point measure on $\mathbb{R}^d$ and $Z_t(B)$ indicates the number of particles at time t in a set $B\subset \mathbb{R}^d$. Moreover, the distribution of Z_t is determined by the three parameters q, k and φ. Here, q is a random measure on $\mathbb{R}^d$describing the initial distribution of the particle system, k is the killing rate for each particle, and φ is a generating function describing the distribution of number of offspring.

Then for every β > 0, we can define a rescaled branching diffusion particle system Z^β=($Z^\beta{}_t$, $P_{q\ (\mu/\beta)}$) with parameters ($q_{(\mu/\beta)}$, k_β , φ_β), where, when $b_1 \in \mathbb{R}$, $b_2 \geq 0$, m is a measure on [1, ∞) such that

$$\int_1^\infty u \cdot m(du) < \infty. \tag{2}$$

and n is a measure on (0, 1) such that

$$\int_0^1 u^2 \cdot n(du) < \infty, \tag{3}$$

we put

$$c_1 := |b_1|, \quad c_2 := 2b_2 + \int_0^1 u^2 \cdot n(du), \quad c_3 := m([1, \infty)), \tag{4}$$

q_μ is the Poisson random measure with intensity $\mu \in M_F\,(\mathbb{R}^d)$,

$$k_\beta := \frac{c_1 + c_2}{\beta} + c_3\beta \tag{5}$$

and $\varphi_\beta := \varphi_1(\beta) + \varphi_2(\beta) + \varphi_3(\beta) + \varphi_4(\beta)$ with

$$\varphi_1(\beta) := \frac{1}{k_\beta}\left\{c_3\beta + \frac{c_1 + c_2}{\beta}z + b_1(1-z)\right\}, \tag{6}$$

$$\varphi_2(\beta) := \frac{1}{k_\beta}\left(c_1 + \frac{b_2}{\beta}\right)(1-z)^2, \tag{7}$$

$$\varphi_3(\beta) := \beta\int_1^\infty (e^{-u(1-z)/\beta} - 1)m(du), \quad \text{and} \tag{8}$$

$$\varphi_4(\beta) := \beta\int_0^1 \left\{e^{-u(1-z)/\beta} - 1 + \frac{u(1-z)}{\beta}\right\} n(du). \tag{9}$$

Then note that our Z_β admits the following characterization via Laplace transition functional. That is to say, it is well known that the Laplace transition functional is given by

$$P_{q_{(\mu/\beta)}} e^{-\langle f, Z_t^\beta\rangle} = e^{-\langle u_t, \mu/\beta\rangle} \tag{10}$$

for every bounded function $f > 0$, where $u \equiv u_t(x) = u(t, x)$ satisfies

$$\Pi_x[1 - e^{-f(\xi_t)}] - u(t,x) \tag{11}$$

$$= \Pi_x \int_0^t k\{\varphi(1 - u(t-s, \xi_s)) - 1 + u(t-s, \xi_s)\}ds \tag{12}$$

3. Superprocess and survival probability

As β goes to zero, then our rescaled process $Z^\beta{}_t$ converges to ameasure-valued diffusion or superdiffusion X=(X_t), $t \geq 0$. Henceforth we shall call it a superprocess X. Moreover, for every bounded function $f > 0$ on $\mathbb{R}^d$, the function

$$v(t,x) \equiv v^{[f]}(t,x) \equiv v(t,x;f) := -\log P_x e^{-\langle f, X_t \rangle} \tag{13}$$

satisfies the integral equation (i.e. the log-Laplace type equation in Dynkin sense)

$$v(t,x) + \Pi_x \int_0^t \psi(v(t-s,\xi_s))ds = \Pi_x f(\xi t), \tag{14}$$

where $P_x$denotes the law $P_{\delta x}$ of the superprocess X with the initial measure δ_x, and δ_x denotes the Dirac delta function at the position x. In addition, we have put

$$\psi(z) := \left\{ b_1 - \int_1^\infty u \cdot m(du) \right\} z + \phi(z) \tag{15}$$

$$\phi(z) := b_2 z^2 + \int_0^\infty (e^{-uz} - 1 + uz)(m(du) + n(du)). \tag{16}$$

Let X be the superprocess obtained in the above argument. The quantity

$$\langle 1, X_t \rangle = \int_{\mathbb{R}^d} 1 X_t(dx) \tag{17}$$

is called the total mass process.For every positive constant $c > 0$, the function

$$v_c(t) := -\log P_x e^{-\langle c, X_t \rangle} \tag{18}$$

satisfies the ordinary differential equation

$$\frac{d}{dt} v_c(t) + \psi(v_c(t)) = 0, \qquad \forall t > 0, \tag{19}$$

with the initial value $v_c(0) = c$. Under these circumstances, we put

$$z_0 := \max\{z > 0; \psi(z) = 0\}. \tag{20}$$

Furthermore, if $\{z > 0;\ \psi(z) > 0\} = \emptyset$, then it follows that $z_0 = \infty$. It is obvious that the function $v_c(t)$ is increasing in c, and there exists the limit

$$\exists \quad v_\infty(t) := \lim_{c \to \infty} v_c(t). \tag{21}$$

Now we are in a position to define the lifetime ℓ of the superprocess $X = (X_t, P_\mu)$ as

$$\ell := \sup\{t \geq 0; X_t \neq 0\}. \tag{22}$$

Then we say that X survives if $\ell = \infty$.

Theorem 1. (Survival condition) *Let $\mu \in M_F(\mathbb{R}^d)$. Assume that $z_0 \in [0, \infty)$. If the function $\phi(z)$ satisfies the condition*

$$\int_z^\infty \frac{1}{\phi(z)} dz = \infty \quad \text{for} \quad \forall z > 0, \tag{23}$$

then the superprocess X survives, P_μ-a.s.

Theorem 2. (Estimate of survival probability) *If the condition in Theorem 1 is not necessarily satisfied, then we have the following estimate of the survival probability for superprocess $X = (X_t, P_\mu)$:*

$$P_\mu(X \quad survives) = 1 - e^{z_0\langle 1,\mu\rangle}, \tag{24}$$

where $\mu \in M_F(\mathbb{R}^d)$ *and* $z \in [0, \infty)$.

4. Sketch of proof of main results

First of all we get

$$P_\mu(X_t \neq 0) = 1 - P_\mu(X_t = 0) \tag{25}$$

$$= 1 - e^{-v_\infty(t)\langle 1,\mu\rangle}, \tag{26}$$

and we can also obtain the expression

$$P_\mu(X \quad \text{survives}) = \lim_{t\to\infty} P_\mu(X_t \neq 0). \tag{27}$$

We need the following lemmas.

Lemma 3. (cf. Sheu [15], Proposition 2.1) If $z_0 \in [0, \infty)$, then $v_\infty(t) = \infty$ for every $t > 0$ if and only if the function ψ satisfies

$$\int_z^\infty \frac{1}{\psi(u)} du = \infty, \qquad \forall z > z_0. \tag{28}$$

Lemma 4. (cf. Sheu [15], Proposition 2.2) If ψ does not satisfy the condition stated in the above lemma 3, then $v_\infty(t)$ converges to z_0 as $t \to \infty$.

Lemma 5. (cf Sheu [15], Proposition 2.4) Assume that $z_0 \in [0, \infty)$. The function ϕ satisfies the condition (23) stated in Theorem 1 if and only if the function ψ satisfies the condition (28).

The assertions yields immediately from the above lemmas 3, 4 and 5. □

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