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A Remark on Approximate Formula and Asymptotic Expansion for Pseudodifferential Operators of Kohn-Nirenberg Type

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Summary

In this paper we consider a class of pseudodifferential operators of Kohn-Nirenberg type, and make a remark on approximate formula for those pseudodifferential operators. Moreover, we also consider asymptotic expansion for pseudodifferential operators of the same kind. The explicit representation of the approximate formula can be given by the pseudodifferential operators with standard symbol replaced by an approximate sequence of symbols. While, the asymptotic expansion formula provides us with a tractable method of operations of seudodifferential operators, because the infinite series in the expansion may be convergent in the appropriate topology of symbol class.

Key Words: pseudodifferential operators of Kohn-Nirenberg type, symbol class, the Schwartz class, approximate formula, asymptotic expansion.

1. Introduction

In this paper we shall consider a class of pseudodifferential operators of Kohn-Nirenberg type, make a remark on a standard variant of approximate formula for pseudodifferential operators of the above-mentioned type, and also state a remark on a mathematical statement of asymptotic expansion formulation for pseudodifferential operators of the same kind. First of all, we shall treat an approximate formula for pseudodifferential operators of Kohn-Nirenberg type, where the explicit representation of the formula can be given by the pseudodifferential operators with symbol $a(x, \xi)$ replaced by an approximate sequence of symbols $a_\varepsilon(x, \xi)$. While, the symbol $a_\varepsilon(x, \xi)$ converges pointwise to $a(x, \xi)$ with all kinds of derivatives, and the corresponding pseudodifferential operator $a_\varepsilon(X, D)$ may converge to $a(X, D)$ in the sense that $a_\varepsilon(X, D)\, f(x)$ converges to $a(X, D)$ $f(x)$ in the topology of the Schwartz class $\mathcal{S}$. Secondly, we shall treat an asymptotic expansion for pseudodifferential operators of Kohn-Nirenberg type. On this account, this expansion theory provides us with a tractable method of operations of seudodifferential operators of Kohn-Nirenberg type, because the infinite series may be convergent in the sense of asymptotic expansion formula given.

2. Pseudodifferential operators of Kohn-Nirenberg type

In this section we shall see how the psedodifferential operators of Kohn-Nirenberg type are defined on the Schwartz class $\mathcal{S}$ that is the best class in functional analysis. Let $a \equiv a\,(x, \xi)$: $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ be a proper function. Then the pseudodifferential operator $(\Psi\mathrm{DO})$ $a(X, D)$ of Kohn-

Nirenberg type is defined by

$$a(X, D)f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(x, \xi)(\mathcal{F}f)(\xi)e^{ix\cdot\xi}d\xi, \quad f \in \mathcal{S}, \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x \cdot \xi = x_1\xi_1 + \cdots x_n\xi_n$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the Schwartz class, and $\mathcal{F}$ is the Fourier transform given by

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx, \quad f \in \mathcal{S}, \tag{2}$$

where

$$\int_{\mathbb{R}^n} g(x)dx = \iint \cdots (n) \cdots \int_{\mathbb{R}^n} g(x_1, x_2, \ldots, x_n)dx_1dx_2 \cdots dx_n \tag{3}$$

$$= \int_{\mathbb{R}} \cdots (n) \cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x_1, x_2, \ldots, x_n)dx_1 \right) dx_2 \cdots dx_n. \tag{4}$$

We call $a = a(x, \xi)$ a symbol. Note that we write the pseudodifferential operator by $a(X, D)$ instead of $a(x,D)$ which is used in most cases of usual textbooks. Because we would like to avoid misunderstanding the function $a(x, D)f$ as the value of function $a(x, D)\ f(x)$ when using the notation $a(x, D)$.

Next we shall define the symbol class $S^m_{\rho\delta}$ for $0 \leqslant \rho \leqslant 1$, $0 \leqslant \delta \leqslant 1$ and $m \in \mathbb{R}$. Let α, β be multi-indices, like $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_+$. For $a \in C^\infty(\mathbb{R}^{2n}) = C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, we define the norm $\|a\|_{s^m_{\rho\delta}(\alpha,\beta)}$ as

$$\|a\|_{s^m_{\rho\delta}(\alpha,\beta)} := \sup_{x,\xi\in\mathbb{R}^n} \langle\xi\rangle^{-(m+\delta|\beta|-\rho|\alpha|)}|\partial^\beta_x\partial^\alpha_\xi a(x, \xi)| \tag{5}$$

with $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $\langle x\rangle = (1 + |x|^2)^{1/2}$, $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$, $\partial^{\alpha_k}_{x_k} = \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}$ for $k = 1, 2, \ldots, n$, and

$$\partial^\alpha_x f(x) = \frac{\partial^{|\alpha|} f(x_1, \ldots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{6}$$

Then a set $S^m_{\rho\delta}$ of $C^\infty(\mathbb{R}^{2n})$-class functions is defined by

$$S^m_{\rho\delta} := \bigcap_{\alpha,\beta\in\mathbb{Z}^n_+} \left\{ a \in C^\infty(\mathbb{R}^{2n});\ \|a\|_{S^m_{\rho\delta}(\alpha,\beta)} < \infty \right\}. \tag{7}$$

We call $S^m_{\rho\delta}$ a symbol class and call its element or component $a \in S^m_{\rho\delta}$ a symbol simply. In addition, we use the notation S^m for S^m_{10}. Notice that $a(x, \xi) = \xi_j \in S^1 \equiv S^1_{10}$ for $j = 1, 2, \ldots, n$.

In what follows we shall investigate some conditions in order that the expression given in (1) makes sense.

Theorem 1. *Assume that $a \in S^m_{\rho\delta}$. Then it follows that $a(X, D)f \in \mathcal{S}$ for every $f \in \mathcal{S}$. Moreover, the correspondence*

$$\mathcal{S} \ni f \mapsto a(X, D)f \in \mathcal{S} \tag{8}$$

is a continuous mapping.

Proof. Since f belongs to the Schwartz class $\mathcal{S}$, the integral in (1) is absolutely convergent. On this account, it follows immediately that $a(X, D)f \in \mathcal{S}$. For $\xi \in \mathbb{R}^n$, we define a differential operator L_ξ as

$$L_\xi := \langle x \rangle^{-2}(I - \Delta_\xi) = \frac{I - \Delta_\xi}{\langle x \rangle^2} \tag{9}$$

with an identity I. This is nothing but a pseudodifferential operator generated by the symbol

$$a(x, \xi) := \frac{1 + |\xi|^2}{1 + |x|^2}. \tag{10}$$

That is to say, it proves to be that

$$\frac{I - \Delta_\xi}{\langle x \rangle^2} f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{1 + |\xi|^2}{1 + |x|^2} (\mathcal{F}f)(\xi) e^{ix\cdot\xi} d\xi. \tag{11}$$

Then, an easy calculation yields to a useful equation $L_\xi\left[e^{ix\cdot\xi}\right] = e^{ix\cdot\xi}$. When we denote by $(L_\xi)^N$ an n-times composite operation, we shall make an estimate of the term $\partial^\alpha a(X, D)f(x)$ below by employing the integration by parts. The following computation is essentially due to a typical elaborate technique in the theory of pseudodifferential operators. An application of the integration by parts formula leads to

$$\begin{aligned}
a(X, D)f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(x, \xi)(\mathcal{F}f)(\xi) e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(x, \xi)(\mathcal{F}f)(\xi)(L_\xi)^N e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} L_\xi(a(x, \xi)(\mathcal{F}f)(\xi)) \cdot (L_\xi)^{N-1} e^{ix\cdot\xi} d\xi \\
&= \cdots\cdots \text{(by mathematical induction)} \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (L_\xi)^k(a(x, \xi)(\mathcal{F}f)(\xi)) \cdot (L_\xi)^{N-k} e^{ix\cdot\xi} d\xi \\
&= \cdots\cdots (k = 0, 1, 2, \ldots, N) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (L_\xi)^{N-1}(a(x, \xi)(\mathcal{F}f)(\xi)) \cdot L_\xi e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (L_\xi)^N(a(x, \xi)(\mathcal{F}f)(\xi)) e^{ix\cdot\xi} d\xi.
\end{aligned} \tag{12}$$

Notice that

$$|\partial_x^\alpha L_\xi^N(a(x, \xi)(\mathcal{F}f)(\xi))| \leqslant C_{M,N,\alpha} \langle \xi \rangle^{-M} \tag{13}$$

holds by virtue of the Leibniz formula. Consequently, it follows that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta a(X,D)f(x)| < \infty, \tag{14}$$

where $D^\beta = D_{x_1}^{\beta_1}\cdots D_{x_N}^{\beta_N}$ and $D_{x_k} = -i\frac{\partial}{\partial x_k}$. In other words, this implies that $a(X, D)f \in \mathcal{S}$. The aforementioned computation yields lucidly to the continuity of the mapping $a(X, D)(\cdot)$. This finishes the proof. $\square$

Theorem 2. *Let* $0 \leqslant \rho \leqslant 1,\ 0 \leqslant \delta \leqslant 1$ *and* $m \in \mathbb{R}$. *We assume that a sequence* $\{a_\varepsilon\}_{\varepsilon\in[0,1]} \subset S^m_{\rho\delta}$ *satisfies the inequality*

$$(i) \qquad |\partial_x^\beta \partial_\xi^\alpha a_\varepsilon(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}, \tag{15}$$

and also that

$$(ii) \qquad \lim_{\varepsilon\downarrow 0} \partial_x^\beta\partial_\xi^\alpha a_\varepsilon(x,\xi) = \partial_x^\beta\partial_\xi^\alpha a(x,\xi), \qquad \text{(pointwiae convergence)} \qquad \forall x,\xi\in\mathbb{R}^n. \tag{16}$$

Then, it follows that the equality

$$\lim_{\varepsilon\downarrow 0} a_\varepsilon(X,D)f = a(X,D)f \tag{17}$$

holds in the topology of the Schwartz class $\mathcal{S}$.

Proof. It goes almost similarly as in the proof of the previous theorem. As a matter of fact, if we resort to the same technique in the above computation in (12), and if we apply the Lebesgue convergence theorem, then the conclusion yields from the standard argument. $\square$

3. Approximate formula for pseudodifferential operators

In this section we shall introduce the first main result, namely, the approximate formula for pseudodifferential operators of Kohn-Nirenberg type. Before stating the principal statement, we will provide with a concrete useful example of the symbol $a_\varepsilon(x, \xi)$, which has been discussed in Theorem 2.

Example 3. Let ρ, δ and m be the same as in the previous discussion. Assume that $a \in S^m_{\rho,\delta}$. We may choose a smooth function $\gamma \in C^\infty \equiv C^\infty(\mathbb{R}^n)$ satisfying the condition: $1_{Q(1)} \leqslant \gamma \leqslant 1_{Q(2)}$. When we put

$$a_\varepsilon(x,\xi) := a(x,\xi)\gamma(\varepsilon x)\gamma(\varepsilon\xi), \quad \forall\varepsilon : 0\leqslant\varepsilon\leqslant 1, \tag{18}$$

then this sequence $\{a_\varepsilon\}$, $\varepsilon \in [0, 1]$, satisfies the conditions (i) and (ii) in Theorem 2 uniformly with respect to the parameter $\varepsilon > 0$.

Theorem 4. (Approximate Formula for ΨDOs) *Assume that* $a \in S^m_{\rho,\delta}$. *For such a symbol* $a = a(x, \xi)$, *we are supposed to take a sequence* $a_\varepsilon(x, \xi)$ *as in the above example with* $0 \leqslant \varepsilon \leqslant 1$. *Then for* $f \in \mathcal{S}$, *the approximate formula* $a(X, D)$ *for* ΨDO

$$a(X,D)f(x) = \frac{1}{(2\pi)^n}\lim_{\varepsilon\downarrow 0}\iint_{\mathbb{R}^{2n}} a_\varepsilon(x,\xi)f(y)e^{i(x-y)}d\xi dy \tag{19}$$

holds in the topology of the Schwartz class $\mathcal{S}$.

Proof. We are going to approximate the pseudodifferential operator $a(X, D)$ by making use of Theorem 2. In fact, by virtue of Theorem 2, it follows immediately that

$$a(X, D)f(x) = \lim_{\varepsilon \downarrow 0} a_\varepsilon(X, D)f(x), \quad \forall f \in \mathcal{S}. \tag{20}$$

Since the symbol $a_\varepsilon \equiv a_\varepsilon(x, \xi)$ has a compact support, a simple computation with Fubini theorem reads

$$\begin{aligned} a(X, D)f(x) &= \frac{1}{(2\pi)^{n/2}} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} a_\varepsilon(x, \xi)(\mathcal{F}f)(\xi)e^{ix\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \iint_{\mathbb{R}^{2n}} a_\varepsilon(x, \xi) f(y) e^{i(x-y)\cdot\xi} d\xi dy. \end{aligned} \tag{21}$$

This implies the conclusion (19). □

4. Pseudodifferential operators on the tempered distributions

In this section we shall extend the definition of the pseudodifferential operators of Kohn-Nirenberg type, and would like to define them on the space $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. In order to define $a(X, D)f$ for $f \in \mathcal{S}'$, we need to consider the adjoint $a(X, D)^*$ of $a(X, D)$. By employing the same approximate symbol $a_\varepsilon \equiv a_\varepsilon(x, \xi)$, the adjoint operator $a(X, D)^*$ can be defined by

$$a(X, D)^* g(y) := \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} a_\varepsilon(x, \xi) e^{i\xi\cdot(y-x)} g(y) dx d\xi, \quad \forall g \in \mathcal{S}(\mathbb{R}^n). \tag{22}$$

Indeed, it is interesting to note that almost the same properties as to the operator $a(X, D)$ are valid even for $a(X, D)^*$. The following result indicated that the adjoint $a(X, D)^*$ is equivalent to the transposed linear trnasformation $a(X, D)$, that is,

Theorem 5. (Duality Formula) *For every $f, g \in \mathcal{S}$, we admit the following dual relation*

$$\langle a(X, D)f, g\rangle = \langle f, a(X, D)^* g\rangle. \tag{23}$$

Proof. We may rewrite the term $\langle a(X, D)f, g\rangle$ into another form by making use of the limit procedure, that is to say,

$$\begin{aligned} \langle a(X, D)f, g\rangle &= \int_{\mathbb{R}^n} a(X, D)f(x) \cdot g(x) dx \\ &= \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^{2n}} a_\varepsilon(x, \xi) g(x) f(y) e^{i(x-y)\cdot\xi} d\xi dy \right) dx \\ &= \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^{2n}} a_\varepsilon(y, \xi) g(y) f(x) e^{i(y-x)\cdot\xi} d\xi dx \right) dy \\ &= \int_{\mathbb{R}^n} f(x) \cdot a(X, D)^* g(x) dx = \langle f, a(X, D)^* g\rangle, \end{aligned} \tag{24}$$

where we have used the Fubini theorem in the third equality because the integral has been truncated by the approximation. □

Definition 6. ($a(X, D)f$ for $f \in \mathcal{S}'$) Let $0 \leqslant \rho \leqslant 1$, $0 \leqslant \delta \leqslant 1$ and $m \in \mathbb{R}$. Suppose that $a \in$

$S^{m}_{\rho\delta}$. We define $a(X, D) f \in \mathcal{S}'$(for all $f \in \mathcal{S}'$) as

$$\langle a(X,D)f, g\rangle = \langle f, a(X,D)^{*}g\rangle, \quad \forall g \in \mathcal{S} \tag{25}$$

by making use of $a(X, D)^{*}$. Then the mapping $a(X, D) : \mathcal{S}' \to \mathcal{S}'$proves to be continuous by virtue of the duality property. □

5. Asymptotic expansion for pseudodifferential operators

This section treats the second principal result, Theorem 7, which is about the asymptotic expansion for pseudodifferential operators of Kohn-Nirenberg type.

THEOREM 7. (Asymptotic Expansion for ΨDO) *Let* $0 \leqslant \rho \leqslant 1$ *and* $0 \leqslant \delta \leqslant 1$. *Suppose that the sequence of numbers* $\{m_j\}_{j=0}^{\infty}$ *satisfies the condition*

$$\sup_{j\in\mathbb{N}_0}(m_{j+1} - m_j) < 0. \tag{26}$$

If the symbols $a_j \in S^{mj}_{\rho\delta}$ *for each* $j = 0, 1, 2, ...$ *are given, then there exists a certain proper symbol* $\exists\, a \in S^{m0}_{\rho\delta}$ *such that*

$$a(x,\xi) - \sum_{k=0}^{j-1} a_k(x,\xi) \in S^{m_j}_{\rho\delta} \tag{27}$$

holds for every natural number j.

Proof. Take a proper smooth function $\varphi \in C^{\infty} = C^{\infty}(\mathbb{R}^n)$ satisfying $1_{B(1)} \leqslant \varphi \leqslant 1_{B(2)}$. Then, define the symbol $a = a(x, \xi)$ as

$$a(x,\xi) := \sum_{k=0}^{\infty}\left(1 - \varphi\left(\frac{\xi}{2^k}\right)\right) a_k(x,\xi) \tag{28}$$

by employing the sequence of symbols $\{a_k\}$ given. Notice that the above-defined symbol $a(x, \xi)$ is freely termwise differentiable, since the infinite sum in $a(x, \xi)$ turns out to be a finite sum on an arbitrary compact set in $\mathbb{R}^{2n}$. We have only to show that

$$a(x,\xi) - \sum_{k=0}^{j-1} a_k(x,\xi) \in S^{m_j}_{\rho\delta} \tag{29}$$

for each $j = 0, 1, 2, \ldots$ for this newly defined symbol $a(x, \xi)$. By the definition of symbol class, it suffices to show the following lemma in order to verify (29).

LEMMA 8. (Reduction) *For any multi-indices* $\alpha, \beta \in \mathbb{Z}^n_+$, *the following estimate*

$$\sup_{x,\xi\in\mathbb{R}^n} \langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \left|\partial_x^{\beta}\partial_{\xi}^{\alpha}\left(a - \sum_{k=1}^{j-1} a_k\right)(x,\xi)\right| < \infty \tag{30}$$

holds.

Proof. The assertion of Lemma 8 can be verified naturally by showing the suceeding series of technical lemmas. Firs to fall, recall that the sequence of numbers $\{m_k\}_{k=0}^{\infty}$ is monotone decreasing and it goes to minus infinity, namely, $m_k > m_{k+1} > \cdots \searrow -\infty$, by definition. Taking this fact into account, we put

$$K(j;\alpha) := \min\{k \in \mathbb{N}_0 : m_k - m_j + \rho|\alpha| < 0\} \geq j. \tag{31}$$

An easy calculation leads to

$$a(x,\xi) - \sum_{k=0}^{j-1} a_k(x,\xi) = \sum_{k=j}^{\infty} \left(1 - \varphi\left(\frac{\xi}{2^k}\right)\right) a_k(x,\xi) - \sum_{k=0}^{j-1} \varphi\left(\frac{\xi}{2^k}\right) a_k(x,\xi). \tag{32}$$

Hence it is easy to see that for any α, $\beta \in \mathbb{Z}_+^n$,

$$\begin{aligned} \partial_x^\beta \partial_\xi^\alpha \left(a(x,\xi) - \sum_{k=0}^{j-1} a_k(x,\xi) \right) &= \sum_{k=j}^{\infty} \partial_x^\beta \partial_\xi^\alpha \left\{ (1 - \varphi\left(\frac{\xi}{2^k}\right)) a_k(x,\xi) \right\} \\ &\quad - \sum_{k=0}^{j-1} \varphi\left(\frac{\xi}{2^k}\right) \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi) \\ &\quad - \sum_{k=0}^{\infty} \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ \alpha \geq \gamma \neq 0}} \binom{\alpha}{\gamma} \partial_\xi^\gamma \varphi\left(\frac{\xi}{2^k}\right) \cdot \partial_x^\beta \partial_\xi^{\alpha-\gamma} a_k(x,\xi) \\ &=: J_1 - J_2 - J_3. \end{aligned} \tag{33}$$

The following lemmas just correspond to each term J_k $(k = 1, 2, 3)$ respectively.

LEMMS 9. (As for the term J_1) *We have the following estimate*

$$\begin{aligned} &\sup_{\substack{x \in \mathbb{R}^n \\ \xi \in \mathbb{R}^n}} \langle \xi \rangle^{-(m_j + \delta|\beta| - \rho|\alpha|)} \left| \sum_{k=j}^{\infty} \partial_x^\alpha \partial_\xi^\alpha (1 - \varphi\left(\frac{\xi}{2^k}\right)) a_k(x,\xi) \right| \\ &\leqslant \sup_{\xi \in \mathbb{R}^n} \sum_{k=j}^{\infty} 2^{-m_k + m_j} < \infty. \end{aligned} \tag{34}$$

LEMMA 10. (As for the term J_2) *For any* α, β ∈ $\mathbb{Z}_+^n$,

$$\sup_{\substack{x \in \mathbb{R}^n \\ \xi \in \mathbb{R}^n}} \langle \xi \rangle^{-(m_j + \delta|\beta| - \rho|\alpha|)} \left| \sum_{k=0}^{j-1} \varphi\left(\frac{\xi}{2^k}\right) \partial_x^\alpha \partial_\xi^\alpha a_k(x,\xi) \right| < \infty \tag{35}$$

holds.

LEMMA 11. (As for the term J_3) *For any* α, β ∈ Z_+^n, *we have*

$$\sup_{\substack{x \in \mathbb{R}^n \\ \xi \in \mathbb{R}^n}} \langle \xi \rangle^{-(m_j + \delta|\beta| - \rho|\alpha|)} \left| \sum_{k=0}^{\infty} \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ \alpha \geq \gamma \neq 0}} \binom{\alpha}{\gamma} \partial_\xi^\gamma \varphi\left(\frac{\xi}{2^k}\right) \cdot \partial_x^\beta \partial_\xi^{\alpha-\gamma} a_k(x,\xi) \right| < \infty. \tag{36}$$

Thus, summing up the above three lemmas, we conclude the statement of Lemma 8. This finishes the proof of Theorem 7. □

6. Proofs of key lemmas

Note that the quantity $K(j;\,\alpha)$ is directly related to a verification of Lemma 11 only. Recall here that $\alpha \leqslant \beta$ means $\alpha_j \leqslant \beta_j$ for any $j = 1, 2, \ldots, n$, and by definition we have

$$\binom{\alpha}{\beta} := \prod_{j=1}^{n} {}_{\alpha_j}C_{\beta_j} = \prod_{j=1}^{n} \frac{\alpha_j!}{\beta_j!(\alpha_j - \beta_j)!}. \tag{37}$$

Proof of Lemma 9. A direct computation with estimation order in the symbol class enables us to make an estimate of the J_1-term, and the following inequalities are derived with ease:

$$\begin{aligned} &\sup_{\substack{x\in\mathbb{R}^n\\ \xi\in\mathbb{R}^n}} \langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \left|\sum_{k=j}^{\infty} \partial_x^\beta \partial_\xi^\alpha \{(1-\varphi\left(\frac{\xi}{2^k}\right))a_k(x,\xi)\}\right| \\ &\leqslant \sup_{\substack{x\in\mathbb{R}^n\\ \xi\in\mathbb{R}^n}} \langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \sum_{k=j}^{\infty} \left|1_{B(2^k)^c}(\xi)\partial_x^\alpha \partial_\xi^\alpha \{(1-\varphi\left(\frac{\xi}{2^k}\right))a_k(x,\xi)\}\right| \end{aligned} \tag{38}$$

$$\leqslant \sup_{\xi\in\mathbb{R}^n} \sum_{k=j}^{\infty} \langle\xi\rangle^{m_k-m_j} 1_{B(2^k)^c}(\xi) \leqslant \sup_{\xi\in\mathbb{R}^n} \sum_{k=j}^{\infty} 2^{-m_k+m_j} < \infty \tag{39}$$

holds, where we made use of the assumption $\sup_{j\in\mathbb{N}0} (m_{j+1} - m_j) < 0$ in derivation of the last inequality. This finishes the proof. □

Proof of Lemma10. If we take the fact $\varphi \in C_c^\infty$ into consideration, then it is obvious to see the establishment of the following estimate, namely,

$$\sup_{\substack{x\in\mathbb{R}^n\\ \xi\in\mathbb{R}^n}} \langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \left|\sum_{k=0}^{j-1} \varphi\left(\frac{\xi}{2^k}\right) \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)\right| < \infty. \tag{40}$$

This concludes the assertion of Lemma 10. □

Proof of Lemma 11. Finally we consider the third term J_3. When we set $\alpha \geq \gamma \neq 0$, then we readily obtain

$$\begin{aligned} &\langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \left|\partial_\xi^\gamma \varphi\left(\frac{\xi}{2^k}\right) \cdot \partial_x^\beta \partial_\xi^{\alpha-\gamma} a_k(x,\xi)\right| \\ &\leqslant 2^{-|\gamma|} \langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \cdot \langle\xi\rangle^{m_k+\delta|\beta|-\rho|\alpha-\gamma|} \cdot 1_{B(2)\setminus B(1)}\left(\frac{\xi}{2^k}\right) \\ &\leqslant 2^{-|\gamma|} \langle\xi\rangle^{m_k-m_j+\rho|\gamma|} \cdot 1_{B(2)\setminus B(1)}\left(\frac{\xi}{2^k}\right), \end{aligned} \tag{41}$$

where we have used the fact $a_k(x, \xi) \in S^{mk}_{\rho\delta}$ for $k = 0, 1, 2, \ldots$. On the other hand, when $k > K(j; \alpha)$, then noting that a simple inequality

$$\langle\xi\rangle^{m_k-m_j+\rho|\gamma|} \leqslant |\xi|^{m_k-m_j+\rho|\gamma|} \tag{42}$$

is valid, we can derive the following inequalities in a similar manner as above. Indeed, we can get

$$
\begin{aligned}
&\left|\partial_\xi^\gamma \varphi\left(\frac{\xi}{2^k}\right) \cdot \partial_x^\beta \partial_\xi^{\alpha-\gamma} a_k(x,\xi)\right| \\
&\leqslant \langle\xi\rangle^{m_j+\delta|\beta|-\rho|\alpha|} 2^{-|\gamma|} \langle\xi\rangle^{m_k-m_j+\rho|\gamma|} 1_{B(2)\setminus B(1)}\left(\frac{\xi}{2^k}\right) \\
&\leqslant \langle\xi\rangle^{m_j+\delta|\beta|-\rho|\alpha|} 2^{-|\gamma|+m_k-m_j+\rho|\gamma|} \cdot 1_{B(2)\setminus B(1)}\left(\frac{\xi}{2^k}\right) \\
&\leqslant 2^{m_k-m_j} \langle\xi\rangle^{m_j+\delta|\beta|-\rho|\alpha|} 1_{B(2)\setminus B(1)}\left(\frac{\xi}{2^k}\right).
\end{aligned} \tag{43}
$$

Therefore, we deduce that

$$
\begin{aligned}
&\sup_{\substack{x\in\mathbb{R}^n \\ \xi\in\mathbb{R}^n}} \langle\xi\rangle^{-(m_j+\delta|\beta|-\rho|\alpha|)} \left|\sum_{k=0}^{\infty} \sum_{\substack{\gamma\in\mathbb{Z}_+^n \\ \alpha\geq\gamma\neq 0}} \binom{\alpha}{\gamma} \partial_\xi^\gamma \varphi\left(\frac{\xi}{2^k}\right) \cdot \partial_x^\beta \partial_\xi^{\alpha-\gamma} a_k(x,\xi)\right| \\
&\leqslant \sum_{k=1}^{K(j;\alpha)} \sup_{\xi\in\mathbb{R}^n} \left|2^{-|\gamma|} \langle\xi\rangle^{m_k-m_j+\rho|\gamma|} 1_{B(2)\setminus B(1)}\left(\frac{\xi}{2^k}\right)\right| + \sum_{k=1}^{\infty} 2^{m_k-m_j} < \infty.
\end{aligned} \tag{44}
$$

This finishes the proof of Lemma 11. □

7. Concludingremarks

When we think of extending those results obtained in this article, first of all we can list the extension or generalization of the definition of pseudodifferential operators of Kohn-Nirenberg type itself. In connection with this, we can list the generalization of the symbol class (5) or (7) itself. Next we may consider the extension of the approximation results, say, Theorem 2 or Theorem 4. In addition to that, we can list a generalization of the convergence topology in Theorem 2 or in Theorem 4. Moreover, related to the above-mentioned extension, the generalization of not only adjoint operator but also asymptotic expansion formula is considered and should be tried at a higher level. On the other hand, changing the basic distribution space $\mathcal{S}'$ into another space $\mathcal{F}'$, it is stimulating and exciting to extend the pseudodifferential operators on $\mathcal{F}'$, too.

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