A Remark on The Derivative Estimate of Entire Functions in A Class of Order q

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Summary

Various types of inequalities are used in the study of pseudo-differential operators (ΨDOs) and their applications to PDE theory. And also some inequalities are quite useful in the estimation of the ΨDOs. The inequality which we are going to introduce in this article is one of them, especially it is extremely useful and powerful, too, in the study of pseudo-differential equations whose symbols are formal series [7]. So that, the inequality stated in succeeding Proposition is not our original result, however, we are going to discuss the formulation of the inequality in the standpoint of the asymptotic behaviors of a specific pilot functions related to the inequality, which arise naturally in the discussion of extreme point problem.

Key Words: entire function, derivative estimate, a class of order q, asympotic behavior, specific pilot function.

1. The class of entire functions of order q

In this section we shall give the definition of a certain class of entire functions of order q, and introduce a result of the derivative estimate of those functions. For $0 < q < 1$ and $r > 0$ given, we define the norm

$$\|u\|_{q,r} := \sup\{\, |u(z)| \exp(-r|z|^q);\ z \in \mathbb{C}^n \,\}, \tag{1}$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$ with $z_k \in \mathbb{C}$ for $k = 1, 2, ..., n$, and $|z|^q = |z_1|^q + \cdots + |z_n|^q$. Let us define a class of entire functions of order $q < 1$ with weight index $r > 0$:

$$E_{q,r}(\mathbb{C}^n_z) := \{u(z) : \mathbb{C}^n \to \mathbb{C}^1 \text{ is a entire function such that } \|u\|_{q,r} < \infty\,\}. \tag{2}$$

Then it is easy to see that

Lemma 1. (a) *When $r_1 \leqslant r_2$ for any $r_1, r_2 > 0$, then we have the following inclusion*

$$E_{q,r_1}(\mathbb{C}^n_z) \subset E_{q,r_2}(\mathbb{C}^n_z). \tag{3}$$

(b) *If $r_1 < r_2$ holds, then the above inclusion is compact for any pair (r_1, r_2); namely, the inclusion map*

$$i : E_{q,r_1}(\mathbb{C}^n_z) \to E_{q,r_2}(\mathbb{C}^n_z) \tag{4}$$

is compact.

We shall give below a typical example for our target class.

EXAMPLE 2. Let $q{:=}m/p \in \mathbb{Q}$, $q < 1$ be a rational number with m and p coprime. When we define a function $f(z)$ as

$$f(z) := \sum_{k=1}^{p-1} \exp\left\{\left(\exp\frac{2\pi ik}{p}\right)\nu z\right\} \tag{5}$$

where $\nu=(\nu_1, ..., \nu_n)\in \mathbb{C}^n$ and $\nu z=\sum_{k=1}^{n} \nu_k z_k$, then its Taylor expansion is given by

$$f(z) = \sum_{|\alpha|=0}^{\infty} \frac{\nu^{p\alpha} z^{p\alpha}}{(p\alpha)!}, \tag{6}$$

where $(p\alpha)!=(p\alpha_1)!\cdots(p\alpha_n)!$, $\nu^{p\alpha}=\nu_1^{p\alpha 1}\cdots\nu_n^{p\alpha n}$, and $z^{p\alpha}=\prod_{k=1}^{n} z_k^{p\alpha k}$. On this account, it follows that the function

$$u(z) := f(z^q) \equiv f(z_1^q, \ldots, z_n^q) \tag{7}$$

is an entire function lying in $E_{q,r}(\mathbb{C}_z^n)$, where $r \geq \max\,(|\nu_1|,...,|\nu_n|)$. □

Now we are in a position to state the important proposition that provides us with the derivative estimate of entire functions lying in a certain class of order q.

PROPOSITION 3. [7] *For every* $u(z) \in E_{q,\,r}(\mathbb{C}_z^n)$, *the following inequality*

$$|D^\alpha u(z)| \leqslant \min\{\xi(u), \eta(u)\} \tag{8}$$

holds for any $z \in \mathbb{C}^n$ *and* $|\alpha|=0, 1, 2, ...,$ *where*

$$\xi(u) := \frac{\|u\|_{q,r} \exp\{r|z|^q\} \cdot (qr)^{|\alpha|}}{\prod_{i=1}^{n} (1+|z_i|)^{(1-q)\alpha_i}} \times \Phi_{1/2}^n(\alpha_i) \tag{9}$$

and

$$\eta(u) := \frac{\|u\|_{q,r} \exp\{r|z|^q\} \cdot (qr)^{|\alpha|/q}}{(\alpha!)^{1/q-1}} \times \Phi_{1/(2q)}^n(\alpha_i) \tag{10}$$

with $\Phi_k^n(\beta_i):=\prod_{i=1}^{n}(\beta_i^k)$.

2. Derivation of a primitive estimation

An application of Cauchy's formula in Complex Analysis reads

$$D^\alpha u(z) = \frac{\alpha!}{(2\pi i)^n} \int_{|\zeta-z|=|a|} \frac{u(\zeta)}{(\zeta-z)^{\alpha+1}} d\zeta \tag{11}$$

$$= \frac{\alpha!}{(2\pi i)^n} \int \cdots (n) \cdots \int_{|\zeta-z|=|a|} \frac{u(\zeta_1, \ldots \zeta_n)}{(\zeta-z)^{\alpha+1}} d\zeta_1 \cdots d\zeta_n, \tag{12}$$

where $a=(a_1,...,a_n)\in\mathbb{R}^n$, $a_1>0$, ..., $a_n>0$. Note that $|\zeta-z|=|a|$indicates

$$|\zeta_1 - z_1| = a_1, \ \ldots, \ |\zeta_n - z_n| = a_n. \tag{13}$$

By definition we easily get

$$|u(z)| \leqslant \|u\|_{q,r} \exp\{r|z|^q\}. \tag{14}$$

Paying attention to the above inequality, we readily obtain

$$|D^\alpha u(z)| \leqslant \frac{\alpha!}{(2\pi)^n} \|u\|_{q,r} \exp\left\{ r \sum_{k=1}^{n} (|z_k| + a_k)^q \right\} \times \frac{\prod_{k=1}^{n} 2\pi a_k}{\prod_{k=1}^{n} a_k^{\alpha_k+1}} \tag{15}$$

$$\leqslant \|u\|_{q,r} \exp\{r|z|^q\} \cdot \alpha! \times \prod_{k=1}^{n} \frac{\exp\{r\varepsilon_k\}}{\{(|z_k|^q + \varepsilon_k)^{1/q} - |z_k|\}^{\alpha_k}} \tag{16}$$

because we have put $(|z_k|+ a_k)^q= |z_k|^q+\varepsilon_k$(with $\varepsilon_k>0$) for simplicity.

3. Further estimation and asymptotic behaviors

For each k, we put

$$\mu_k(\varepsilon_k, z_k) := e^{r\varepsilon_k} \{(|z_k|^q + \varepsilon_k)^{1/q} - |z_k|\}^{-\alpha_k} \tag{17}$$

and let us consider the factors $\mu_k(\varepsilon_k, z_k)$. By virtue of finite limit property of the function μ_k not only as $|z_k|\to 0$ but also as $|z_k|\to\infty$, a simple inequality

$$\mu_k(\varepsilon_k, z_k) \leqslant \max_{z_k \in \mathbb{C}} \mu_k(\varepsilon_k, z_k) \tag{18}$$

leads to a new estimation result

$$|D^\alpha u(z)| \leqslant \|u\|_{q,r} \exp\{r|z|^q\} \cdot \alpha! \times \prod_{k=1}^{n} \min_{\varepsilon_k} \{\max_{z_k} \mu_k(\varepsilon_k, z_k)\}. \tag{19}$$

It is interesting to note that the maximum of the function

$$g_1(x) = \frac{1}{(x^q + \varepsilon)^{1/q} - x} \tag{20}$$

over the region $x>0$ is given by the value $1/\varepsilon^{(1/q)}$, and also that the minimum of the function $g_2(x)=e^{rx}x^{-(\alpha/\varepsilon)}$ over the region $x>0$ is attained by the value $(e/\alpha)^\alpha \cdot (qr)^{\alpha/q}$. By making use of the above-mentioned two results, we can deduce together with Stirling's formula and (16) that

$$|D^\alpha u(z)| \leqslant \|u\|_{q,r} \exp\{r|z|^q\} \cdot \frac{(qr)^{|\alpha|/q}}{(\alpha!)^{1/q-1}} \times \Phi^n_{1/(2q)}(\alpha_k). \tag{21}$$

This is nothing but (10) in Proposition 3.

4. Precise analysis of a specific function

In this section we shall investigate some interesting properties including asymptotic behaviors of a specific pilot function $h\,(x,\,y)$. We need the following auxiliary result.

LEMMA 4. *For every* $u \equiv u(z) \in E_{q,r}(\mathbb{C}_z^n)$, *the following inequality*

$$|D^{\alpha}u(z)| \leqslant \|u\|_{q,r}\exp\{r|z|^q\} \cdot \alpha!\Phi^n_{-(1-q)\alpha_k}(1+|z_k|)\times \tag{22}$$

$$\times\prod_{k=1}^{n}\frac{\exp\{r\varepsilon_k\}\cdot(1+|z_k|)^{(1-q)\alpha_k}}{\{(|z_k|^q+\varepsilon_k)^{1/q}-|z_k|\}^{\alpha_k}} \tag{23}$$

holds for $z \in \mathbb{C}^n$, $|\alpha|$=0,1,2,....

Proof. It suffices to rewrite (16) into the expression in the above-mentioned form. It is easy, hence the details omitted. □

Suggested by the aforementioned explicit representation in the right-hand side of (22)-(23) in Lemma 4, we need to consider a function in the specific form. So that, we define the specific pilot function $h\,(x,y)$ as

$$h(x,y) := \frac{(1+x)^{1-q}}{(x^q+y)^{1/q}-x} \tag{24}$$

for x≥0 and y≥ 0. Then we can readily obtain the following asymptotic behaviors.

LEMMA 5. (Key Lemma) [7] *The function* $h\,(x,y)$ *converges to* $\varepsilon^{-1/q}$ *in* $\mathbb{R}^1$ *as* $y\to 0$, *and* $h\,(x,y)$ *converges to* $^{\varepsilon-1}$ *in* $\mathbb{R}^1$ *as* $y \to +\infty$.
with the result that the quantity $\sup_y h\,(x,y)$ is finite.

5. Verification of the key lemma

In order to derive the results stated in Lemma 5, we have only to investigate the asymptotic behaviors of the derivative of the function $h\,(x,y)$. First of all, adopting the notation $(\partial/\partial x)\,h\,(x,\,y)$ $= h_x\,(x,\,y)$ for brevity's sake, we have

$$f_x(x,y) \tag{25}$$

$$= \frac{(1-q)x\{(1+yx^{-q})^{1/q}-1\}-(x+1)\{(1+yx^q)^{(1-q)/q}-1\}}{(x+1)^q\{(x^q+y)^{1/q}-x\}^2}. \tag{26}$$

An application of the Taylor expansion in Differential and Integral Calculus yields to

$$h_x(x,y) \sim -x^{q-1} = \frac{-1}{x^{1-q}} \quad (\text{as} \quad x \to 0) \tag{27}$$

and

$$h_x(x,y) \sim x^{-(1+q)} \quad (\text{as} \quad x \to +\infty). \tag{28}$$

Consequently, we can get

Lemma 6. *We have*

$$\lim_{x\to 0}\frac{\partial}{\partial x}h(x,y)=-\infty \qquad \text{and} \qquad \lim_{x\to +\infty}\frac{\partial}{\partial x}h(x,y)=0. \tag{29}$$

Moreover, it follows immediately that $h_x(x,y)>0$ holds for all x. A little computation with analysis related to the extreme points of function leads to a preferable result that the derivative $h_x(x,y)$ may possess only one zero. Hence, the desired asymptotic behaviors yield from the above result.

6. Concluding remarks

When we take the asymptotic behaviors in Lemma 5 into consideration, this only one zero proves to be the minimum point of $h(x,y)$. Hence the inequality

$$h(x,y)\leqslant M_0(q,y)\equiv \max(y^{-1/q},q/y) \tag{30}$$

is naturally derived for all $x>0$. To change variables here

$$x\to\rho, \qquad y\to\varepsilon \tag{31}$$

for a practical reason, we are going to consider the quantity $M_0(q,\varepsilon)$ in what follows. To proceed the discussion, we are required to split the interval of ε into two parts. We have $M_0(q,\varepsilon)=\varepsilon^{-1/q}$ since the inequality $\varepsilon^{1/q}\geq\rho/\varepsilon$ holds as far as $\varepsilon\in I_1=(0,q^{(q-1)/q})$. For α sufficiently large, it follows immediately that

$$\min_{\varepsilon\in I_1}e^{r\varepsilon}\{M_0(q,\varepsilon)\}^\alpha=\left(\frac{1}{q}\right)^{\frac{\alpha}{q}(1-\frac{1}{q})}\exp\left\{r\left(\frac{1}{q}\right)^{\frac{1}{q}-1}\right\}. \tag{32}$$

On the other hand, as far as $\varepsilon\in I_2=(q^{(q-1)/q},\infty)$, it turns out to be that

$$M_0(q,\varepsilon)=\frac{q}{\varepsilon}. \tag{33}$$

Therefore, we obtain

$$\min_{\varepsilon\in I_2}e^{r\varepsilon}\{M_0(q,\varepsilon)\}^\alpha=e^\alpha\left(\frac{qr}{\alpha}\right)^\alpha. \tag{34}$$

As a consequence, a little argument about the minimizing problem leads to the final conclusion

$$\min_{\varepsilon>0}e^{r\varepsilon}\{M(q,\varepsilon)\}^\alpha=\frac{1}{\alpha!}(qr)^\alpha e^\alpha. \tag{35}$$

Substituting (35) for (22)-(23) in Lemma 4, we may combine the resulting expression with (21) in Section 3 to derive the desired inequality (9), because we have employed Stirling's formula in the above calculation.

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